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M. Basheleishvili and K. Svanadze

INVESTIGATION OF BASIC PLANE BOUNDARY VALUE PROBLEMS OF STATICS OF ELASTIC MIXTURES FOR PIECEWISE HOMOGENEOUS ISOTROPIC MEDIA

Abstract. Using potentials with complex densities, we reduce solution of basic plane boundary-value and boundary-contact problems of statics of elastic mixtures for piecewise homogeneous isotropic media to solution of systems of Fredholm linear integral equations of second kind.

The solvability of the integral equations is proved, and the uniqueness and existence theorems are proved for the above-mentioned boundary-contact problems.

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რეზიუმე. ნაშრომში კომპლექსურ სიმკვრივეებიანი პოტენციალების გამოყენებით, უბნობრივ ერთგვაროვანი იზოტროპული სხეულებისათვის დრეკად ნარევთა სტატიკის ძირითადი ბრტყელი სასაზღვრო ამოცანების ამოხსნა მიყვანილია ფრედჰოლმის მეორე გვარის წრფივი ინტეგრალური განტოლების ამოხსნამდე.

მტკიცდება აღნიშნული ინტეგრალური განტოლებების ამოხსნადობა და ამით დადგენილია სასაზღვრო-საკონტაქტო ამოცანების ამონახსნთა არსებობის და ერთადერთობის თეორემები.

1. Some Auxiliary Formulas and Operators

The homogeneous equation of statics of the theory of elastic mixture in the complex form is written as [5]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + K \frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0, \tag{1.1}$$

where $U = \{u_1 + iu_2, u_3 + iu_4\}^T$, $u' = \{u_1, u_2\}^T$ and $u'' = \{u_3, u_4\}^T$ are partial displacements, $\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big)$, $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big)$,

$$\begin{gathered} k = -\frac{1}{2} e m^{-1}, \quad e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \\ m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \quad m^{-1} = \begin{bmatrix} m_3 & -m_2 \\ -m_2 & m_1 \end{bmatrix}, \end{gathered} \tag{1.2}$$

$$\begin{gathered} \Delta_0 = \det m, \ \ m_k = e_k + \frac{1}{2} e_{3+k}, \ \ k = 1,2,3, \ \ e_1 = a_2/d_2, \ \ e_2 = -c/d_2, \\ e_3 = a_1/d_2, \ \ a_1 = \mu_1 - \lambda_5, \ \ a_2 = \mu_2 - \lambda_5, \ \ c = \mu_3 + \lambda_5, \ \ d_2 = a_1 a_2 - c^2, \end{gathered} \tag{1.3}$$

$$\begin{gathered} e_1 + e_4 = b/d_1, \ \ e_2 + e_5 = -c_0/d_1, \ \ e_3 + e_6 = a/d_1, \ \ a = a_1 + b_1, \ \ b = a_2 + b_2, \\ c_0 = c + d, \ \ b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2 \rho_2/\rho, \ \ b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2 \rho_1/\rho, \\ d = \mu_3 + \lambda_3 - \lambda_5 - \alpha_2 \rho_1/\rho \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2 \rho_2/\rho, \ \ \alpha_2 = \lambda_3 - \lambda_4, \\ \rho = \rho_1 + \rho_2, \ \ d_1 = ab - c_0^2. \end{gathered} \tag{1.4}$$

Here μ_1, μ_2, μ_3, λ_p, $p = \overline{1,5}$ and α_2 are elastic moduli characterizing mechanical properties of the mixture, ρ_1 and ρ_2 are partial densities of the mixture (positive constants).

It is assumed that the elastic constants μ_1, μ_2, μ_3, λ_p, $p = \overline{1,5}$, and the partial densities ρ_1 and ρ_2 satisfy the following conditions [2]:

$$\begin{gathered} \mu_1 > 0, \quad \Delta_1 = \mu_1 \mu_2 - \mu_3^2 > 0, \quad \lambda_1 - \alpha_2 \rho_2/\rho + \frac{2}{3}\mu_1 > 0, \quad \lambda_5 < 0, \\ \Big(\lambda_1 - \alpha_2 \rho_2/\rho + \frac{2}{3}\mu_1\Big)\Big(\lambda_2 + \alpha_2 \rho_1/\rho + \frac{2}{3}\mu_2\Big) > \\ > \Big(\lambda_3 - \alpha_2 \rho_1/\rho + \frac{2}{3}\mu_3\Big)^2 > 0. \end{gathered} \tag{1.5}$$

The above conditions guarantee both positive definiteness of the potential energy and fulfillment of the following inequalities for $\lambda_5 \le 0$ [1]:

$$\begin{gathered} a_1 > 0, \quad a_2 > 0, \quad a > 0, \quad b > 0, \quad a_0 = \mu_1 + \mu_2 + 2\mu_3 > 0, \\ b_0 = (b_1 - \lambda_5)(b_2 - \lambda_5) - (d + \lambda_5)^2 > 0, \\ p_0 = \mu_1(b_2 - \lambda_5) + \mu_2(b_1 - \lambda_5) - 2\mu_3(d + \lambda_5) > 0, \\ q_0 = b_1 + b_2 + 2d > 0, \quad d_2 = a_1 a_2 - c^2 = \Delta_1 - a_0 \lambda_5 > 0, \\ d_1 = ab - c_0^2 = \Delta_1 + p_0 + b_0 > 0, \quad m_1 > 0, \quad m_3 > 0, \\ \Delta_0 = m_1 m_3 - m_2^2 > 0. \end{gathered} \tag{1.6}$$

Using the analogues of the general representations established by Kolosov–Muskhelishvili [3], we can write

$$U = m\varphi(z) + \frac{1}{2}ez\overline{\varphi}'(z) + \overline{\psi}(z), \tag{1.7}$$

$$\begin{aligned} TU &= \{(Tu)_2 - i(Tu)_1,\ (Tu)_4 - i(Tu)_3)\}^T = \\ &= \frac{\partial}{\partial s(x)}[(A-2E)\varphi(z) + Bz\overline{\varphi}'(z) + 2\mu\overline{\psi}(z)], \end{aligned} \tag{1.8}$$

where E is the unit matrix, and $\varphi(z) = \{\varphi_1, \varphi_2\}^T$ and $\psi(z) = \{\psi_1, \psi_2\}^T$ are arbitrary analytic vector functions, $\frac{\partial}{\partial s(x)} = n_1\frac{\partial}{\partial x_2} - n_2\frac{\partial}{\partial x_2} - n_2\frac{\partial}{\partial x_1}$, $n = \{n_1, n_2\}^T$ is an arbitrary unit vector, m and e are the matrices defined by (1.2),

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = 2\mu m, \quad \mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \mu e, \tag{1.9}$$

$(Tu)_p$, $p = \overline{1,4}$, are the components of stresses:

$$\begin{aligned} (TU)_1 &= (a\theta' + b\theta'')n_1 - (a_1\omega' + c\omega'')n_2 - 2\frac{\partial}{\partial s(x)}(\mu_1 u_2 + \mu_3 u_4), \\ (TU)_2 &= (a_0\theta' + c_0\theta'')n_2 + (a_1\omega' + c\omega'')n_1 + 2\frac{\partial}{\partial s(x)}(\mu_1 u_1 + \mu_3 u_3), \\ (TU)_3 &= (c_0\theta' + b\theta'')n_1 - (c\omega' + a_2\omega'')n_2 - 2\frac{\partial}{\partial s(x)}(\mu_3 u_2 + \mu_2 u_4), \\ (TU)_4 &= (c_0\theta' + b\theta'')n_2 + (c\omega' + a_2\omega'')n_1 + 2\frac{\partial}{\partial s(x)}(\mu_3 u_1 + \mu_2 u_3), \\ &\theta' = \operatorname{div} u', \quad \theta'' = \operatorname{div} u'', \quad \omega' = \operatorname{rot} u', \quad \omega'' = \operatorname{rot} u''. \end{aligned} \tag{1.10}$$

Let D^+ (D^-) be a finite (infinite) two-dimensional domain bounded by the contour $S \in C^{2,\beta}$, $0 < \beta < 1$, $\overline{D}^+ = D^+ \cup S$, $D^- = R^2 \backslash \overline{D}^+$, $\overline{D}^- = D^- \cup S$.

A vector $U = \{U_1, U_2\}^T = \{u_1 + iu_2, u_3 + iu_4\}^T$ defined in the domain D^+ is called regular if $U \in C^2(D^+) \cap C^{1,\alpha}(\overline{D}^+)$, $0 < \alpha < \beta \le 1$. A vector regular in D^- is defined analogously. In this case it is required of the vector $U = \{U_1 U_2\}^T$ that along with the smoothness $U \in C^2(D^-) \cap C^{1,\alpha}(\overline{D}^-)$ the conditions

$$U = O(1), \quad |x|^2\frac{\partial U}{\partial x_k} = O(1), \quad k = 1, 2, \quad |x|^2 = x_1^2 + x_2^2, \tag{1.11}$$

be fulfilled at infinity [1].

To investigate boundary-contact problems, the use will be made of the following vectors [5]:

$$V = \{v_1 + iv_2, v_3 + iv_4\}^T = i[-m\varphi(z) + \frac{1}{2}ez\overline{\varphi}'(z) + \overline{\psi}(z)], \tag{1.12}$$

$$TV = i\frac{\partial}{\partial s(x)}[-(A-2E)\varphi(z) + Bz\overline{\varphi}'(z) + 2\mu\overline{\psi}(z)] =$$

$$= i[2\varphi(z) - 2i\mu v(x)], \tag{1.13}$$

$$NU = TU - (2\mu - m^{-1})\frac{\partial U}{\partial s(x)} = \frac{\partial}{\partial s(x)}[-2\varphi(z) + m^{-1}U(x)], \tag{1.14}$$

$$NV = i\frac{\partial}{\partial s(x)}[2\varphi(z) - im^{-1}V(x)], \tag{1.15}$$

where N is the pseudo-stress operator.

It is not difficult to prove that (1.12) satisfies (1.1); moreover,

$$U(x) + iV(x) = 2m\varphi(z), \tag{1.16}$$

$$NU = -im^{-1}\frac{\partial V}{\partial s(x)}, \quad NV = im^{-1}\frac{\partial U}{\partial s(x)}\,. \tag{1.17}$$

Here we introduce the following

Definition 1.1. If U and V satisfy the relations (1.17), then they are mutually associated.

If $U = \{u_1 + iu_2, u_3 + iu_4\}^T$ and $W = \{w_1 + iw_2, w_3 + iw_4\}^T$ are regular solutions of the equation (1.1) in the domain D^+ (D^-), then

$$\int\limits_{D^\pm} T(u,w)dy_1dy_2 = \pm \operatorname{Im}\int\limits_S U^\pm \overline{TW}^\pm ds, \tag{1.18}$$

where Im is the imaginary part, $TW = \{(Tw)_2 - i(Tw)_1, (Tw)_4 - i(Tw)_3\}^T$, $T(u,w)$ is a symmetric function with respect to the derivatives of u_p and w_p, $p = \overline{1,4}$, $(T(u,w) = T(w,u))$ [1].

If $w = u$, we have

$$\int\limits_{D^\pm} T(u,u)dy_1dy_2 = \pm \operatorname{Im}\int\limits_S U^\pm (\overline{TU})^\pm ds, \tag{1.19}$$

$$\int\limits_{D^\pm} N(u,u)dy_1dy_2 = \pm \operatorname{Im}\int\limits_S U^\pm (\overline{NU})^\pm = \pm \operatorname{Im}\int\limits_S V^\pm (\overline{NV})^\pm ds, \tag{1.20}$$

where $T(u,u)$ and $N(u,u)$ defined in [1] (see pp. 5–6) are positive definite quadratic forms. The formulas (1.18)–(1.20) will be called generalized Green's formulas in the theory of elastic mixtures for the equation of statics (1.1).

We have the following

Lemma 1.2. *The solutions of the equations* $T(u,u) = 0$ *and* $N(u,u) = 0$ *have, respectively, the form*

$$U = a^* + ib^*\begin{pmatrix}1\\1\end{pmatrix}z, \quad a^* = \{a_1^*, a_2^*\}^T, \tag{1.21}$$

$$U = C^*, \quad C^* = \{C_1^*, C_2^*\}^T, \tag{1.22}$$

where a_1^*, a_2^*, c_1^*, c_2^* *and* b^* *are arbitrary constants.*

2. Problems for Inhomogeneous Media and the Uniqueness Theorems

Let an isotropic body with the constants $\mu_k^{(0)}$, $\lambda_p^{(0)}$, $\rho^{(0)}$, $k=1,2,3$, $p=\overline{1,5}$, $q=1,2$, and the boundary S_0 contain entirely an inclusion of a different isotropic material with the constants $\mu_k^{(1)}$, $\lambda_p^{(1)}$, $\rho^{(0)}$, $k=1,2,3$, $p=\overline{1,5}$, $q=1,2$, and bounded by a curve S_1. By D_1 we denote the domain occupied by the inclusion, and the remaining domain is denoted by D_0. We denote by D_0^- the complement of the domain $D_1\cup S_1\cup D_0\cup S_0$ with respect to the whole plane. The counterclockwise direction on each of the contours is selected as positive, and the positive direction of the normal is the direction of the outer normal with respect to D_1 and D_0, respectively. A point belonging to the domain D_0 will be taken as the origin of the coordinates.

Consider the following boundary value problems. Find the vector $\overset{(j)}{U}$, $j=0,1$, satisfying the conditions ([4], [6]):

1^0. For $x\in D_j$, $\dfrac{\partial^2\overset{(j)}{U}}{\partial z\partial\overline{z}}+\overset{(j)}{K}\dfrac{\partial^2\overset{(j)}{U}}{\partial\overline{z}^2}=0$, $j=1,0$.

2^0. For $t\in S_1$, $\big(\overset{(1)}{U}(t)\big)^+-\big(\overset{(0)}{U}(t)\big)^-=f(t)$, $\big(\overset{(1)}{T}\overset{(1)}{U}(t)\big)^+-\big(\overset{(0)}{T}\overset{(0)}{U}(t)\big)^-=F(t)$.

3^0. For $t\in S_0$,

(a) $\big(\overset{(0)}{U}(t)\big)^+=\phi(t)$, or

(b) $\big(\overset{(0)}{T}\overset{(0)}{U}(t)\big)^+=\phi(t)$,

where f, F and ϕ are given vector functions satisfying certain smoothness conditions. In the case (a) we have the first boundary-value problem while in the case (b) we have the second boundary-contact problem.

The following boundary-value problem is of great importance. Find the vector $\overset{(j)}{U}$, $j=0,1$, satisfying the conditions

$$
\begin{aligned}
&1^0.\ \text{for}\quad x\in D_j \quad \frac{\partial^2\overset{(j)}{U}}{\partial z\partial\overline{z}}+\overset{(j)}{K}\frac{\partial^2\overset{(j)}{U}}{\partial\overline{z}^2}=0;\\
&2^0.\ \text{for}\quad t\in S_1 \quad \big(\overset{(1)}{U}(t)\big)^+-\big(\overset{(0)}{U}(t)\big)^-=f(t),\\
&\qquad\qquad\qquad\quad \big(\overset{(1)}{T}\overset{(1)}{U}(t)\big)^+-\big(\overset{(0)}{T}\overset{(0)}{U}(t)\big)^-=F(t).
\end{aligned}
\tag{2.1}
$$

In this case D_0 is an infinite domain. This problem will be called the basic contact problem, or problem (A).

The following theorems are valid.

Theorem 2.1. *The general solution of the problem* $(A)^0$ *satisfying the condition* (1.5) *is given by the formula* $\overset{(j)}{U}=a^*$, $x\in D_j$, $j=0,1$, *where* a^* *is an arbitrary constant vector.*

Corollary 2.2. *If the vector* $\overset{(0)}{U}(x)$ *tends at infinity to zero, then the problem* $(A)^0$ *has only the trivial solution.*

Theorem 2.3. *The first homogeneous boundary-contact problem has only the trivial solution.*

Theorem 2.4. *The general solution of the second homogeneous boundary-contact problem is given by the formula*

$$\overset{(j)}{U} = a^* + ib^* \tbinom{1}{1} z, \quad z = x_1 + ix_2, \quad x \in D_j, \quad j = 0, 1,$$

where a^* *is an arbitrary constant vector, and* b^* *is an arbitrary constant.*

The proof of these theorems is based on Green's formulas and on the boundary conditions.

3. Integral Equations of the Basic Contact Problem

The basic contact problem (see (2.1)) can be formulated as follows. Find in the domains D_1 and $D_0 = R^2 \setminus \overline{D}_1$ regular vectors $\overset{(1)}{U}(x)$ and $\overset{(0)}{U}(x)$ satisfying the equation

$$\frac{\partial^2 \overset{(j)}{U}}{\partial z \partial \overline{z}} + \overset{(j)}{K} \frac{\partial^2 \overline{\overset{(j)}{U}}}{\partial \overline{z}^2} = 0, \quad x \in D_j, \quad j = 0, 1,$$

and on S_1 the contact conditions

$$\begin{gathered} \big(\overset{(1)}{U}(t)\big)^+ - \big(\overset{(0)}{U}(t)\big)^- = f(t), \quad t \in S_1, \\ \int\limits_0^{S(t)} \big(\overset{(1)}{T}\overset{(1)}{U}\big)^+ ds - \int\limits_0^{S(t)} \big(\overset{(0)}{T}\overset{(0)}{U}\big)^- ds = F_0(t) + \text{const}, \quad t \in S_1, \end{gathered} \tag{3.1}$$

where $F_0(t) = \int_0^{S(t)} F ds$, $f \in C^{1,\alpha}(S_1)$, $F \in C^{0,\alpha}(S_1)$, $S_1 \in C^{2,\beta}$, $0 < \alpha < \beta \leq 1$, are given vectors, s is the arc of the contour S_1, and the constant in the right-hand side must be arbitrarily fixed.

First, for the basic contact problem we set up a system of Fredholm integral equations of second kind. Towards this end, we use the formulas (1.7) and (1.8) and choose $\varphi(z)$ and $\overline{\psi}(z)$ as follows:

$$\begin{aligned} \overset{(j)}{\varphi}(z) &= \frac{1}{2\pi i} \int\limits_{S_1} \frac{\partial \ln \sigma}{\partial S(y)} \big(\overset{(j)}{\alpha} g(y) + \beta^j h(y)\big) dS, \\ \overset{(j)}{\psi}(z) &= \frac{\overset{(j)}{m}}{2\pi i} \int\limits_{S_1} \frac{\partial \ln \overline{\sigma}}{\partial S(y)} \big(\overset{(j)}{\gamma} g(y) + \overset{(j)}{\delta} h(y)\big) dS + \\ &\quad + \frac{\overset{(j)}{e}}{4\pi i} \int\limits_{S_1} \frac{\partial}{\partial S(y)} \frac{\zeta}{\overline{\sigma}} \big(\overset{(j)}{\alpha} \overline{g}(y) + \overset{(j)}{\beta} \overline{h(y)}\big) dS, \quad j = 0, 1, \end{aligned} \tag{3.2}$$

where $\sigma = z-\zeta$ and $\overline{\sigma}=\overline{z}-\overline{\zeta}$, $z=x_1+ix_2$, $\overline{z}=x_1-ix_2$, $\zeta=y_1+iy_2$, $\zeta=y_1-iy_2$, $g=\{g_1,g_2\}^T$ and $h=\{h_1,h_2\}^T$ are smooth complex vectors; matrices $\overset{(j)}{m}$, $j=0,1$, are defined by (1.2), $\overset{(j)}{\alpha}$, $\overset{(j)}{\beta}$, $\overset{(j)}{\gamma}$ and $\overset{(j)}{\delta}$, $j=0,1$, are arbitrary real matrices of dimension 2×2 which will be defined below.

After not difficult transformations we get

$$
\begin{aligned}
\overset{(j)}{U}(x) &= \frac{\overset{(j)}{m}}{2\pi}\int\limits_{S_1}\frac{\partial\ln(\sigma)}{\partial n(y)}\big[\big(\overset{(j)}{\alpha}-\overset{(j)}{\gamma}\big)g(y)+\big(\overset{(j)}{\beta}-\overset{(j)}{\delta}\big)h(y)\big]dS+\\
&+\frac{\overset{(j)}{m}}{2\pi i}\int\limits_{S_1}\frac{\partial\ln(\sigma)}{\partial S(y)}\big[\big(\overset{(j)}{\alpha}+\overset{(j)}{\gamma}\big)g(y)+\big(\overset{(j)}{\beta}+\overset{(j)}{\delta}\big)n(y)\big]dS-\\
&-\frac{\overset{(j)}{e}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(j)}{\alpha}\,\overline{g}(y)+\overset{(j)}{\beta}\,\overline{h}(y)\big)dS,\quad j=0,1;
\end{aligned}\tag{3.3}
$$

$$
\begin{aligned}
\int\limits_0^{S(x)}\big(\overset{(j)}{T}\overset{(j)}{U}\big)dS+\overset{(j)}{C} &= \frac{1}{2\pi}\int\limits_{S_1}\frac{\partial\ln(\sigma)}{\partial n(y)}\Big(\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\alpha}-\overset{(j)}{A}\overset{(j)}{\gamma}\big]g(y)+\\
&+\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\beta}-\overset{(j)}{A}\overset{(j)}{\delta}\big]h(y)\Big)dS+\\
+\frac{1}{2\pi i}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial S(y)}&\Big(\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\alpha}-\overset{(j)}{A}\overset{(j)}{\gamma}\big]g(y)+\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\beta}+\overset{(j)}{A}\overset{(j)}{\delta}\big]h(y)\Big)dS-\\
-\frac{\overset{(j)}{B}}{2\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}&\big(\overset{(j)}{\alpha}\,\overline{g}(y)+\beta^{(j)}\overline{h}(y)\big)dS,\quad j=0,1,
\end{aligned}\tag{3.4}
$$

where $\overset{(j)}{C}=\{\overset{(j)}{C}_1,\overset{(j)}{C}_2\}^T$ are arbitrary constant vectors ($j=0,1$).

In these formulas, $\overset{(j)}{m}$, $\overset{(j)}{e}$, $\overset{(j)}{A}$ and $\overset{(j)}{B}$, $j=0,1$, are the known matrices (see (1.2) and (1.4)), $g=\{g_1,g_2\}^T$, $h=\{h_1,h_2\}^T$ are unknown complex vectors with certain properties of smoothness, $\overset{(j)}{\alpha}$, $\overset{(j)}{\beta}$, $\overset{(j)}{\gamma}$ and $\overset{(j)}{\delta}$, $j=0,1$, are arbitrary real matrices which will be defined below so that the integral equations of the basic contact problem must necessarily be Fredholm equations of second kind.

Let

$$
\begin{gathered}
\overset{(1)}{m}\big(\overset{(1)}{\alpha}-\overset{(1)}{\gamma}\big)+\overset{(0)}{m}\big(\overset{(0)}{\alpha}-\overset{(0)}{\gamma}\big)=2E;\quad \overset{(1)}{m}\big(\overset{(1)}{\beta}-\overset{(1)}{\delta}\big)+\overset{(0)}{m}\big(\overset{(0)}{\beta}-\overset{(0)}{\delta}\big)=0;\\
\overset{(1)}{m}\big(\overset{(1)}{\alpha}+\overset{(1)}{\gamma}\big)+\overset{(0)}{m}\big(\overset{(0)}{\alpha}+\overset{(0)}{\gamma}\big)=0;\quad \overset{(1)}{m}\big(\overset{(1)}{\beta}+\overset{(1)}{\delta}\big)-\overset{(0)}{m}\big(\overset{(0)}{\beta}+\overset{(0)}{\delta}\big)=0;\\
\big(\overset{(1)}{A}-2E\big)\overset{(1)}{\alpha}+\overset{(1)}{A}\overset{(1)}{\gamma}+\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\alpha}-\overset{(0)}{A}\overset{(0)}{\gamma}=0,
\end{gathered}\tag{3.5}
$$

$$\big(\overset{(1)}{A}+2E\big)\overset{(1)}{\beta}-\overset{(1)}{A}\overset{(1)}{\delta}+\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\beta}-\overset{(0)}{A}\overset{(0)}{\delta}=2E,$$
$$\big(\overset{(1)}{A}-2E\big)\overset{(1)}{\alpha}+\overset{(1)}{A}\overset{(1)}{\gamma}-\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\alpha}-\overset{(0)}{A}\overset{(0)}{\gamma}=0,$$
$$\big(\overset{(1)}{A}-2E\big)\overset{(1)}{\beta}+\overset{(1)}{A}\overset{(1)}{\delta}-\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\beta}-\overset{(0)}{A}\overset{(0)}{\delta}=0,$$

Simple calculations yield

$$\begin{gathered}\overset{(1)}{\alpha}=\overset{(1)}{X}\overset{(0)}{\mu},\quad \overset{(0)}{\alpha}=\overset{(0)}{X}\overset{(1)}{\mu};\quad \overset{(1)}{X}=\big[E+\big(\overset{(0)}{\mu}-\overset{(1)}{\mu}\big)\overset{(1)}{m}\big]^{-1},\\ \overset{(0)}{X}=\big[E+\big(\overset{(1)}{\mu}-\overset{(0)}{\mu}\big)\overset{(0)}{m}\big]^{-1},\quad \overset{(1)}{\beta}=-\frac{1}{2}\overset{(1)}{X},\quad \overset{(0)}{\beta}=-\frac{1}{2}\overset{(0)}{X},\\ \overset{(1)}{\delta}=-\frac{1}{2}\big(\overset{(1)}{m}\big)^{-1}\overset{(0)}{m}\overset{(0)}{X},\quad \overset{(0)}{\delta}=-\frac{1}{2}\big(\overset{(0)}{m}\big)^{-1}\overset{(1)}{m}\overset{(1)}{X},\\ \overset{(1)}{\gamma}=\big(\overset{(1)}{m}\big)^{-1}\big(\overset{(0)}{m}\overset{(0)}{X}\overset{(1)}{\mu}-E\big),\quad \overset{(0)}{\gamma}=\big(\overset{(0)}{m}\big)^{-1}\big(\overset{(1)}{m}\overset{(1)}{X}\overset{(0)}{\mu}-E\big).\end{gathered}\tag{3.6}$$

After cumbersome, but evident transformations we can prove [7] that $\det\big(\overset{(j)}{X}\big)^{-1}>0$, $j=1,0$.

Taking into account (3.1) and the properties of the potential appearing in (3.3) and (3.4), on the basis of (3.5) after simple transformations for the determination of g and h we obtain the following system of Fredholm integral equations of second kind:

$$\begin{gathered}g(t)+\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\big[\big(\overset{(1)}{m}\overset{(1)}{\alpha}-\overset{(0)}{m}\overset{(0)}{\alpha}\big)g(y)+\big(\overset{(1)}{m}\overset{(1)}{\beta}-\overset{(0)}{m}\overset{(0)}{\beta}\big)h(y)\big]dS+\\ +\frac{1}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big[\big(\overset{(0)}{e}\overset{(0)}{\alpha}-\overset{(1)}{e}\overset{(1)}{\alpha}\big)\overline{g}(y)+\big(\overset{(0)}{e}\overset{(0)}{\beta}-\overset{(1)}{e}\overset{(1)}{\beta}\big)\overline{h}(y)\big]dS=f(t),\quad t\in S_1,\\ h(t)+\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln(t-\zeta)}{\partial n(y)}\Big(\big[\big(\overset{(1)}{A}-2E\big)\overset{(1)}{\alpha}-\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\alpha}\big]g(y)+\\ +\big[\big(\overset{(1)}{A}-2E\big)\overset{(1)}{\beta}-\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\beta}\big]h(y)\Big)dS+\frac{1}{2\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\times\\ \times\big[\big(\overset{(0)}{B}\overset{(0)}{\alpha}-\overset{(1)}{B}\overset{(1)}{\alpha}\big)\overline{g}(y)+\big(\overset{(0)}{B}\overset{(0)}{\beta}-\overset{(1)}{B}\overset{(1)}{\beta}\big)\overline{h}(y)\big]dS-C^*=F_0(t),\quad t\in S_1,\end{gathered}\tag{3.7}$$

where $(f,F)\in C^{1,\alpha}(S_1)$, $S_1\in C^{2,\beta}$, $0<\alpha<\beta\le 1$, $C^*=\{C_1^*,C_2^*\}^T$ is an arbitrary constant vector.

We tie the unknown constant vector C^* and the unknown vector functions g and h by the relation

$$C^*=\int\limits_{S_1}(g(t)+h(t))dS.\tag{3.8}$$

If now in the left-hand side of the second integral equation in (3.7) under the vector C^* is meant the expression (3.8), then the system (3.7) will turn into a system of integral equations containing no unknowns except the vectors g and h.

Since $(f,F_0)\in C^{1,\alpha}(S_1)$, $S_1\in C^{2,\beta}$, and the vector C^* is defined from (3.8), we can conclude that $(g,h)\in C^{1,\alpha}(S_1)$, $0<\alpha<\beta\le 1$ and hence the displacement and stress vectors exist and they are Hölder continuous up to the boundary.

4. Solution of the Basic Contact Problem

Let us prove that the system

$$
\begin{gathered}
g(t)+\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\big[\big(\overset{(1)}{m}\overset{(1)}{\alpha}-\overset{(0)}{m}\overset{(0)}{\alpha}\big)g(y)+\big(\overset{(1)}{m}\overset{(1)}{\beta}-\overset{(0)}{m}\overset{(0)}{\beta}\big)h(y)\big]dS+\\
+\frac{1}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big[\big(\overset{(0)}{e}\overset{(0)}{\alpha}-\overset{(1)}{e}\overset{(1)}{\alpha}\big)\overline{g}(y)+\big(\overset{(0)}{e}\overset{(0)}{\beta}-\overset{(1)}{e}\overset{(1)}{\beta}\big)\overline{h}(y)\big]dS=f(t),\ t\in S_1\\
h(t)+\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\Big(\big[\big(\overset{(1)}{A}-2E\big)\overset{(1)}{\alpha}-\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\alpha}\big]g(y)+\\
+\big[\big(\overset{(1)}{A}-2E\big)\overset{(1)}{\beta}-\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\beta}\overset{(0)}{\beta}\big]h(y)\Big)dS-\\
-\int\limits_{S_1}(g(y)+h(y))dS+\frac{1}{2\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\times\\
\times\big[\big(\overset{(0)}{B}\overset{(0)}{\alpha}-\overset{(1)}{B}\overset{(1)}{\alpha}\big)\overline{g}(y)+\big(\overset{(0)}{B}\overset{(0)}{\beta}-\overset{(1)}{B}\overset{(1)}{\beta}\big)h(y)\big]dS=F_0(t),\ t\in S_1,
\end{gathered}
\tag{4.1}
$$

is always solvable.

Towards this end, we consider the homogeneous system $(4.1)^0$ which is obtained from (4.1) for $f=0$, $F=0$, $(F_0=0)$ and prove that this system has no different from zero solutions. Let g_0 and h_0 be a solution of that (homogeneous) system. We denote the corresponding vectors by $(\overset{(j)}{U})_0$, $j=0,1$;

$$
\begin{gathered}
(\overset{(j)}{U})_0(x)=\frac{\overset{(j)}{m}}{2\pi}\int\limits_{S_1}\frac{\partial\ln(\sigma)}{\partial n(y)}\big[\big(\overset{(j)}{\alpha}-\overset{(j)}{\gamma}\big)g_0(y)+\big(\overset{(j)}{\beta}-\overset{(j)}{\delta}\big)h_0(y)\big]dS+\\
+\frac{\overset{(j)}{m}}{2\pi}\int\limits_{S_1}\frac{\partial\ln(\sigma)}{\partial S(y)}\big[\big(\overset{(j)}{\alpha}+\overset{(j)}{\gamma}\big)g_0(y)+\big(\overset{(j)}{\beta}+\overset{(j)}{\delta}\big)h_0(y)\big]dS-\\
-\frac{\overset{(j)}{e}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(j)}{\alpha}\overline{g}_0(y)+\overset{(j)}{\beta}\overline{h}_0(y)\big)dS,\ \ x\in D_j,\ \ j=0,1.
\end{gathered}
\tag{4.2}
$$

Obviously, $\overset{(j)}{U}_0(x)$, $j = 0,1$, on S_1 satisfy the following homogeneous contact conditions:

$$\begin{aligned} &\big(\overset{(1)}{U}_0(t)\big)^+ - \big(\overset{(0)}{U}_0(t)\big)^- = 0, \\ &\big(\overset{(1)}{T}\overset{(1)}{U}_0(t)\big)^+ - \big(\overset{(0)}{T}\overset{(0)}{U}_0(t)\big)^- = 0, \end{aligned} \qquad t \in S_1. \tag{4.3}$$

Taking into account (4.3) and using Corollary 2.2, we obtain

$$\overset{(0)}{U}_0(x) = 0, \quad x \in D_0, \quad \overset{(1)}{U}_0(x) = 0, \quad x \in D_1, \tag{4.4}$$

whence on the basis of (1.14) and (1.17) we can write

$$O = \overset{(j)}{N}\overset{(j)}{U}_0 = -i\big(\overset{(j)}{m}\big)^{-1}\frac{\partial \overset{(j)}{V}_0}{\partial S(x)}, \quad x \in D_j, \quad j = 1,0, \tag{4.5}$$

where by (3.2) and (1.12)

$$\begin{aligned} \overset{(j)}{V}_0(x) = {} & -\frac{i\overset{(j)}{m}}{2\pi}\int\limits_{S_1}\frac{\partial \ln|\sigma|}{\partial n(y)}\big[\big(\overset{(j)}{\alpha} + \overset{(j)}{\gamma}\big)g_0(y) + \big(\overset{(j)}{\beta} + \overset{(j)}{\delta}\big)h_0(y)\big]dS - \\ & -\frac{\overset{(j)}{m}}{2\pi}\int\limits_{S_1}\frac{\partial \ln|\sigma|}{\partial S(y)}\big[\big(\overset{(j)}{\alpha} - \overset{(j)}{\gamma}\big)g_0(y) + \big(\overset{(j)}{\beta} - \overset{(j)}{\delta}\big)h_0(y)\big)\big]dS - \\ & -\frac{\overset{(j)}{e}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(j)}{\alpha}\,\overline{g}_0(y) + \overset{(j)}{\beta}\,\overline{h}_0(y)\big)dS, \quad x \in D_j, \quad j = 0,1. \end{aligned} \tag{4.6}$$

By virtue of (4.5) and the fact that $\overset{(0)}{V}_0(\infty) = 0$, we find that $\overset{(0)}{V}_0(x) = 0$, $x \in D_0$. Since according to (4.5) $\overset{(1)}{V}_0(x)$ is defined to within constant summands, we can adopt that $\overset{(1)}{V}_0(\overset{(1)}{x}) = 0$, where $\overset{(1)}{x}$ is an arbitrary point in D_1, and hence we can assume that $\overset{(1)}{V}_0(x) = 0$, $x \in D_1$.

Thus

$$\overset{(j)}{V}_0(x) = 0, \quad x \in D_j, \quad j = 0,1. \tag{4.7}$$

Consider the vectors

$$\begin{aligned} \overset{(j)}{V}_0(x) = {} & -\frac{i}{2\pi i}\int\limits_{S_1}\frac{\partial \ln|\sigma|}{\partial n(y)}\overset{(j)}{m}\big[\big(\overset{(j)}{\alpha} + \overset{(j)}{\gamma}\big)g_0(y) + \big(\overset{(j)}{\beta} + \overset{(j)}{\delta}\big)h_0(y)\big]dS - \\ & -\frac{1}{2\pi}\int\limits_{S_1}\frac{\partial \ln|\sigma|}{\partial S(y)}\overset{(j)}{m}\big[\big(\overset{(j)}{\alpha} - \overset{(j)}{\gamma}\big)g_0(y) + \big(\overset{(j)}{\beta} - \overset{(j)}{\delta}\big)h_0(y)\big]dS - \end{aligned}$$

$$-\frac{\overset{(j)}{e}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(j)}{\alpha}\overline{g}_0(y)+\overset{(j)}{\beta}\overline{h}_0(y)\big)dS, \tag{4.8}$$

$$x\in(\delta_{0j}D_1+\delta_{1j}D_0),\quad j=1,0,$$

where δ_{kj} is the Kronecker symbol.

From (4.6) and (4.8) we have

$$\big(\overset{(1)}{V}_0(t)\big)^+-\big(\overset{(1)}{V}_0(t)\big)^-=-i\overset{(1)}{m}\big[\big(\overset{(1)}{\alpha}+\overset{(1)}{\gamma}\big)g_0(t)+\big(\overset{(1)}{\beta}+\overset{(1)}{\delta}\big)h_0(t)\big],$$
$$\big(\overset{(0)}{V}_0(t)\big)^+-\big(\overset{(0)}{V}_0(t)\big)^-=-i\overset{(0)}{m}\big[\big(\overset{(0)}{\alpha}+\overset{(0)}{\gamma}\big)g_0(t)+\big(\overset{(0)}{\beta}+\overset{(0)}{\delta}\big)h_0(t)\big].$$

According to (4.7) and (3.5), we can write

$$\big(\overset{(0)}{V}_0(t)\big)^+-\big(\overset{(1)}{V}_0(t)\big)^-=0,\quad t\in S_1. \tag{4.9}$$

Define now the vectors $\int_0^{S(x)}\overset{(j)}{T}\overset{(j)}{V}_0dS$, $j=0,1$. By (1.3) and (3.2) we get

$$\int\limits_0^{S(x)}\big(\overset{(j)}{T}\overset{(j)}{V}_0\big)dS+\overset{(j)}{L}=$$
$$=-\frac{i}{2\pi}\int\limits_{S_1}\frac{\partial\ln(\sigma)}{\partial n(y)}\Big(\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\alpha}+\overset{(j)}{A}\overset{(j)}{\gamma}\big]g_0(y)+$$
$$+\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\beta}+\overset{(j)}{A}\overset{(j)}{\delta}\big]h_0(y)\Big)dS-$$
$$-\frac{i}{2\pi}\int\limits_{S_1}\frac{\partial\ln(\sigma)}{\partial n(y)}\Big(\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\alpha}-\overset{(j)}{A}\overset{(j)}{\gamma}\big]g_0(y)+$$
$$+\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\beta}-\overset{(j)}{A}\overset{(j)}{\delta}\big]h_0(y)\Big)dS-$$
$$-\frac{\overset{(j)}{B}}{2\pi}\int\limits_{S_1}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(j)}{\alpha}\overline{g}_0(y)+\overset{(j)}{\beta}\overline{h}_0(y)\big)dS,\quad x\in D_j,\quad j=1,0, \tag{4.10}$$

where $\overset{(j)}{L}$, $j=0,1$, are arbitrary constant vectors.

Consider the following vectors:

$$\int\limits_0^{S(x)}\big(\overset{(j)}{T}\overset{(j)}{V}\big)dS+\overset{(j)}{L}=-\frac{i}{2\pi}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial n(y)}\Big(\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\alpha}+\overset{(j)}{A}\overset{(j)}{\gamma}\big]g_0(y)+$$
$$+\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\beta}+\overset{(j)}{A}\overset{(j)}{\delta}\big]h_0(y)\Big)dS-$$

$$-\frac{i}{2\pi}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial n(y)}\Big(\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\alpha}-\overset{(j)}{A}\overset{(j)}{\gamma}\big]g_0(y)+$$

$$+\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\beta}-\overset{(j)}{A}\overset{(j)}{\delta}\big]h_0(y)\Big)dS-$$

$$-\frac{\overset{(j)}{B}}{2\pi}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\big(\overset{(j)}{\alpha}\overline{g}_0(y)+\overset{(j)}{\beta}\overline{h}_0\big)dS,\quad x\in(\delta_{oj}D_1+\delta_{1j}D_0),\ \ j=0,1. \tag{4.11}$$

Taking into account the equality (4.7), from (4.10) and (4.11) by virtue of (3.5) we can conclude that

$$\int\limits_0^{S(t)}\big(\overset{(0)}{T}\overset{(0)}{V}_0\big)^+dS-\int\limits_0^{S(t)}\big(\overset{(1)}{T}\overset{(1)}{V}_0\big)^-dS=$$

$$=-i\Big(\big[\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\alpha}+\overset{(0)}{A}\overset{(0)}{\gamma}-\big(\overset{(1)}{A}-2E\big)\overset{(1)}{\alpha}-\overset{(1)}{A}\overset{(1)}{\gamma}\big]g_0(t)+$$

$$+\big[\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\beta}+\overset{(0)}{A}\overset{(0)}{\delta}-\big(\overset{(1)}{A}-2E\big)\overset{(1)}{\beta}-\overset{(1)}{A}\overset{(1)}{\delta}\big]\Big)=0,\quad t\in S_1. \tag{4.12}$$

Thus we have

$$\big(\overset{(0)}{V}_0(t)\big)^+=\big(\overset{(1)}{V}_0(t)\big)^-\quad\text{and}\quad\overset{(0)}{T}\big(\overset{(0)}{V}_0(t)\big)^+=\big(\overset{(1)}{T}\overset{(1)}{V}_0(t)\big)^-\quad t\in S_1. \tag{4.13}$$

We will use the formulas

$$\begin{gathered}\int\limits_{D_1}\overset{(0)}{T}\big(\overset{(0)}{v}_0,\overset{(0)}{v}_0\big)dy_1dy_2=\operatorname{Im}\int\limits_{S_1}\overset{(0)}{V}{}^+\,\overline{\big(\overset{(0)}{T}\overset{(0)}{V}\big)}{}^+dS,\\ \int\limits_{D_0}\overset{(1)}{T}\big(\overset{(1)}{v}_0,\overset{(1)}{v}_0\big)dy_1dy_2=-\operatorname{Im}\int\limits_{S_1}\overset{(1)}{V}{}^-_0\,\overline{\big(\overset{(1)}{T}\overset{(1)}{V}\big)}{}^-dS,\end{gathered} \tag{4.14}$$

where $\overset{(j)}{T}\big(\overset{(1)}{v}_0,\overset{(j)}{v}_0\big)$, $j=0,1$, are positive quadratic forms.

Taking into account the equality (4.13) and the fact that $\int_{S_1}\big(\overset{(0)}{T}\overset{(0)}{V}\big)^+dS=0$, from (4.14) we arrive at $\overset{(j)}{T}\big(\overset{(0)}{v}_0,\overset{(0)}{v}_0\big)=0$, $x\in D_1$ and $\overset{(1)}{T}\big(\overset{(1)}{v}_0,\overset{(1)}{v}_0\big)=0$, $x\in D_0$, and hence by Lemma 1.1 we have

$$\overset{(0)}{V}_0(x)=d_0+\gamma_0\begin{pmatrix}1\\1\end{pmatrix}z,\quad x\in D_1,\ \overset{(1)}{V}_0(x)=d_1+i\gamma_1\begin{pmatrix}1\\1\end{pmatrix}z,\quad x\in D_0, \tag{4.15}$$

where d_0 and d_1 are arbitrary constant vectors, and γ_0 and γ_1 are arbitrary constants.

Since $\overset{(1)}{V}_0(\infty)=0$, we have $d_1=\gamma_1=0$, i.e.,

$$\overset{(1)}{V}_0(x)=0,\quad x\in D_0. \tag{4.16}$$

This, by virtue of (4.13), allows one to write that

$$\overset{(0)}{V}_0(x)=0,\quad x\in D_1. \tag{4.17}$$

From (4.4), (4.7), (4.16) and (4.17), by the relation (4.16), where $\varphi(z)$ is an analytic vector function, we can conclude that the associate with (4.8) vectors are also equal to zero, i.e.,

$$\begin{aligned}\overset{(j)}{U}_0(x)=\frac{\overset{(j)}{m}}{2\pi}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial n(y)}\big[\big(\overset{(j)}{\alpha}-\overset{(j)}{\gamma}\big)g_0(y)+\big(\overset{(j)}{\beta}-\overset{(j)}{\delta}\big)h_0(y)\big]dS+\\ +\frac{\overset{(j)}{m}}{2\pi i}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial S(y)}\big[\big(\overset{(j)}{\alpha}+\overset{(j)}{\gamma}\big)g_0(y)+\big(\overset{(j)}{\beta}+\overset{(j)}{\delta}\big)h_0(y)\big]dS-\\ -\frac{\overset{(j)}{e}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(j)}{\alpha}\overline{g}_0(y)+\overset{(j)}{\beta}\overline{h}_0(y)\big)dS=0,\\ x\in(\delta_{0j}D_1+\delta_{1j}D_0),\quad j=0,1.\end{aligned} \tag{4.18}$$

The expressions (4.2), (4.4) and (4.18) yield

$$\big(\overset{(1)}{U}_0(t)\big)^+-\big(\overset{(1)}{U}_0(t)\big)^-=+\overset{(1)}{m}\big[\big(\overset{(1)}{\alpha}-\overset{(1)}{\gamma}\big)g_0(t)+\big(\overset{(1)}{\beta}-\overset{(1)}{\delta}\big)h_0(t)\big]=0$$

$$\big(\overset{(0)}{U}_0(t)\big)^+-\big(\overset{(0)}{U}_0(t)\big)^-=+\overset{(0)}{m}\big[\big(\overset{(0)}{\alpha}-\overset{(0)}{\gamma}\big)g_0(t)+\big(\overset{(0)}{\beta}-\overset{(0)}{\delta}\big)h_0(t)\big]=0,$$

which, according to (3.5), make it possible to conclude that $g_0(t)=0$.

Let us now consider the vectors

$$\begin{aligned}\int\limits_0^{S(x)}\overset{(j)}{T}\overset{(j)}{U}_0(x)dS+\overset{(j)}{C}=\frac{1}{2\pi}\int\limits_{S_i}\frac{\partial\ln|\sigma|}{\partial n(y)}\Big(\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\alpha}-\overset{(j)}{A}\overset{(j)}{\gamma}\big]g_0(y)+\\ +\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\beta}-\overset{(j)}{A}\overset{(j)}{\delta}\big]h_0(y)\Big)dS+\\ +\frac{1}{2\pi i}\int\limits_{S_i}\frac{\partial\ln|\sigma|}{\partial S(y)}\Big(\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\alpha}+\overset{(j)}{A}\overset{(j)}{\gamma}\big]g_0(y)+\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\beta}+\overset{(j)}{A}\overset{(j)}{\delta}\big]h_0(y)\Big)dS-\\ -\frac{\overset{(j)}{B}}{2\pi i}\int\limits_{S_i}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(j)}{\alpha}\overline{g}_0(y)+\overset{(j)}{\beta}\overline{h}_0(y)\big)dS,\\ x\in(\delta_{0j}D_1+\delta_1,D_0),\quad j=0,1.\end{aligned} \tag{4.19}$$

Taking into account the formulas (3.4) (for $g=g_0$ and $h=h_0$), (4.19), (4.4), (4.18) and also the relation $\big(\overset{(1)}{A}-2E\big)\overset{(1)}{\beta}-\overset{(1)}{A}\overset{(1)}{\delta}+\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\alpha}-\overset{(0)}{A}\overset{(0)}{\delta}=2E$, we obtain $h_0=0$.

Thus we have proved that the homogeneous system $(4.1)^0$ has only the trivial solution, i.e., the system (4.1) is uniquely solvable. Consequently, we have shown that the basic contact problem always has the solution which is represented in the form (3.3) $(j = 0, 1)$.

5. The First Boundary-Contact Problem

The boundary and contact conditions in the case under consideration can be written in the form (see § 2)

$$\begin{gathered}
\big(\overset{(0)}{U}(t)\big)^+ = \phi(t), \ \ t \in S_0, \quad \big(\overset{(1)}{U}(t)\big)^+ - \big(\overset{(0)}{U}(t)\big)^- = f(t), \ \ t \in S_1, \\
\int\limits_0^{S(t)} \big(\overset{(1)}{T}\overset{(1)}{U}\big)^+ dS - \int\limits_0^{S(t)} \big(\overset{(0)}{T}\overset{(0)}{U}\big)^- dS = F_0(t) + \text{const}, \quad t \in S_1,
\end{gathered} \tag{5.1}$$

where ϕ, f and F_0 are given vectors, $\phi \in C^{1,\alpha}(s_0)$, $f \in C^{1,\alpha}(s_1)$, $F_0 \in C^{1,\alpha}(s_1)$, $0 < \alpha < \beta \le 1$, $j = 0, 1$.

To reduce the problem (5.1) to a system of Fredholm integral equations of second kind, we choose $\overset{(j)}{\varphi}(z)$ and $\overset{(j)}{\overline{\varphi}}(\psi)(z)$, $j = 0, 1$, as follows:

$$\overset{(j)}{\varphi}(z) = \delta_{0j} \frac{(\overset{(0)}{m})^{-1}}{2\pi i} \int\limits_{S_0} \frac{\partial \ln \sigma}{\partial S(y)} \chi(y) dS + \frac{1}{2\pi i} \int\limits_{S_1} \frac{\partial \ln \sigma}{\partial S(y)} \big(\overset{(j)}{\alpha} g(y) + \overset{(j)}{\beta} h(y)\big) dS,$$

$$\begin{aligned}
\overset{(j)}{\psi}(z) = -\delta_{0j} \Bigg(\frac{1}{2\pi i} \int\limits_{S_0} \frac{\partial \ln \overline{\sigma}}{\partial S(y)} \chi(y) dS + \frac{\overset{(0)}{K}}{2\pi i} \int\limits_{S_0} \frac{\partial}{\partial S(y)} \frac{\zeta}{\overline{\sigma}} \overline{\sigma \chi}(y) dS \Bigg) + \\
+ \frac{\overset{(j)}{m}}{2\pi i} \int\limits_{S_1} \frac{\partial \ln \overline{\sigma}}{\partial S(y)} \big(\overset{(j)}{\gamma} g(y) + \overset{(j)}{\delta} h(y)\big) dS + \\
+ \frac{\overset{(j)}{l}}{4\pi i} \int\limits_{S_1} \frac{\partial}{\partial S(y)} \frac{\zeta}{\overline{\sigma}} \big(\overset{(j)}{\alpha} \overline{g}(y) + \overset{(j)}{\beta} \overline{h}(y)\big) dS, \quad j = 0, 1.
\end{aligned} \tag{5.2}$$

After not complicated transformations, taking into account (1.7) and (1.8), we obtain

$$\begin{aligned}
\overset{(j)}{U}(x) = \delta_{oj} \Bigg(\frac{1}{\pi} \int\limits_{S_0} \frac{\partial \ln |\sigma|}{\partial n(y)} \chi(y) dS + \frac{\overset{(0)}{K}}{2\pi i} \int\limits_{S_0} \frac{\partial}{\partial S(y)} \frac{\sigma}{\overline{\sigma}} \overline{\chi}(y) dS + \\
+ \frac{\overset{(j)}{m}}{2\pi} \int\limits_{S_1} \frac{\partial \ln |\sigma|}{\partial n(y)} \big[\big(\overset{(j)}{\alpha} - \overset{(j)}{\gamma}\big) g(y) + \big(\overset{(j)}{\beta} - \overset{(j)}{\delta}\big) h(y) \big] dS +
\end{aligned}$$

$$+\frac{\overset{(j)}{m}}{2\pi i}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial S(y)}\big[\big(\overset{(j)}{\alpha}+\overset{(j)}{\gamma}\big)g(y)+\big(\overset{(j)}{\beta}+\overset{(j)}{\delta}\big)h(y)\big]dS-$$

$$-\frac{\overset{(j)}{e}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(j)}{\alpha}\,\overline{g}(y)+\overset{(j)}{\beta}\,\overline{h}(y)\big)dS,\quad x\in D_j,\quad j=0,1, \tag{5.3}$$

$$\int\limits_0^{S(x)}\overset{(j)}{T}\overset{(j)}{U}dS+\overset{(j)}{d_0}=\delta_{oj}\bigg(\frac{2\overset{(0)}{\mu}-(\overset{(0)}{m})^{-1}}{\pi}\int\limits_{S_0}\frac{\partial\ln|\sigma|}{\partial n(y)}\chi(y)dS-$$

$$-\frac{(\overset{(0)}{m})^{-1}}{\pi i}\int\limits_{S_0}\frac{\partial\ln|\sigma|}{\partial S(y)}\chi(y)dS-\frac{\overset{(0)}{B}(\overset{(0)}{m})^{-1}}{2\pi i}\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\overline{\chi}(y)dS\bigg)+$$

$$+\frac{1}{2\pi}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial n(y)}\Big(\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\alpha}-\overset{(j)}{A}\overset{(j)}{\gamma}\big]g(y)+\Big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\beta}-\overset{(j)}{A}\overset{(j)}{\delta}\Big]h(y)\Big)dS+$$

$$+\frac{1}{2\pi i}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial S(y)}\Big(\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\alpha}+\overset{(j)}{A}\overset{(j)}{\gamma}\big]g(y)+\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\beta}+\overset{(j)}{A}\overset{(j)}{\delta}\big]h(y)\Big)dS-$$

$$-\frac{B}{2\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(j)}{\alpha}\,\overline{g}(y)+\overset{(j)}{\beta}\,\overline{h}(y)\big)dS,\quad j=0,1, \tag{5.4}$$

where χ, g and h are unknown complex vectors, $\overset{(j)}{d}$, $j=0,1$, are arbitrary constant vectors, and the remaining quantities appearing in (5.3) and (5.4) have been defined in §1 and §3.

Taking into account (5.1) and the properties of potentials appearing in (5.3) and (5.4), after some calculations for the determination of the vectors χ, g and h, we obtain the following system:

$$\chi(t)+\frac{1}{\pi}\int\limits_{S_0}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\chi(y)dS+\frac{\overset{(0)}{K}}{2\pi i}\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\overline{\chi}(y)dS+$$

$$+\frac{\overset{(0)}{m}}{2\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\big[\big(\overset{(0)}{\alpha}-\overset{(0)}{\gamma}\big)g(y)+\big(\overset{(0)}{\beta}-\overset{(0)}{\delta}\big)h(y)\big]dS+$$

$$+\frac{\overset{(0)}{m}}{2\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial S(y)}\big[\big(\overset{(0)}{\alpha}+\overset{(0)}{\gamma}\big)g(y)+\big(\overset{(0)}{\beta}+\overset{(0)}{\delta}\big)h(y)\big]dS-$$

$$-\frac{\overset{(0)}{l}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big(\overset{(0)}{\alpha}\,\overline{g}(y)+\overline{g}(y)+\overset{(0)}{\beta}\,\overline{h}(y)\big)dS=\phi(t),\quad t\in S_0,$$

$$g(t)+\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\big[\big(\overset{(1)}{m}\overset{(1)}{\alpha}-\overset{(0)}{m}\overset{(0)}{\alpha}\big)g(y)+\big(\overset{(1)}{m}\overset{(1)}{\beta}-\overset{(0)}{m}\overset{(0)}{\beta}\big)h(y)\big]dS+$$

$$+\frac{1}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big[\big(\overset{(0)}{e}\overset{(0)}{\alpha}-\overset{(1)}{e}\overset{(1)}{\alpha}\big)\overline{g}(y)\big)+\big(\overset{(0)}{e}\overset{(0)}{\beta}-\overset{(1)}{e}\overset{(1)}{\beta}\big)h(y)\big]dS-$$

$$-\frac{1}{\pi}\int\limits_{S_0}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\chi(y)dS-$$

$$-\frac{\overset{(0)}{K}}{2\pi i}\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\overline{\chi}(y)dS=f(t),\quad t\in S_1, \tag{5.5}$$

$$h(t)+\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\Big(\big[\big(\overset{(1)}{A}-2E\big)\overset{(1)}{\alpha}-\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\alpha}\big]g(y)+$$

$$+\big[\big(\overset{(1)}{A}-2E\big)\overset{(1)}{\beta}-\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\beta}\big]h(y)\Big)dS-$$

$$-\int\limits_{S_1}(g(y)+h(y))ds+\frac{1}{2\pi i}\int\limits_{S_1}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big[\big(\overset{(0)}{B}\overset{(0)}{\alpha}-\overset{(1)}{B}\overset{(1)}{\alpha}\overline{g}(y)\big)+$$

$$+\big(\overset{(0)}{B}\overset{(0)}{\beta}-\overset{(1)}{B}\overset{(1)}{\beta}\big)\overline{h}(y)\big]dS-\frac{2\overset{(0)}{\mu}-(\overset{(0)}{m})^{-1}}{\pi}\int\limits_{S_0}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\chi(y)dS+$$

$$+\frac{(\overset{0}{m})^{-1}}{\pi i}\int\limits_{S_0}\frac{\partial\ln|t-\zeta|}{\partial S(y)}\chi(y)dS+\frac{\overset{(0)}{B}(\overset{(0)}{m})^{-1}}{2\pi i}\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\overline{\chi}(y)dS=F_0(t),\ t\in S_1.$$

Let us show that the system (5.5) is always solvable. To this end, we have to consider the homogeneous system which is obtained from (5.5) for $\phi=f=F_0=0$ and then to prove that it has no different from zero solution.

Let χ_0, g_0 and h_0 be any solution of that (homogeneous) system. Then $\overset{(j)}{U}_0(x)$, $j=0,1$, defined by the formulas (5.3) for $\chi=\chi_0$, $g=g_0$ and $h=h_0$, is a regular solution of the first homogeneous boundary-contact problem, and hence

$$\overset{(1)}{U}_0(x)=0,\quad x\in D_1,\quad \overset{(0)}{U}_0(x)=0,\quad x\in D_0, \tag{5.6}$$

by Theorem 2.3.

From (5.6) it follows (see (1.17)) that

$$O=\overset{(0)}{N}\overset{(0)}{U}_0(x)=-i(\overset{(0)}{m})^{-1}\frac{\partial\overset{(0)}{V}_0(x)}{\partial S(x)}=0,\quad x\in D_0, \tag{5.7}$$

where

$$\overset{(0)}{V}_0(x)=-\frac{1}{\pi}\int\limits_{S_0}\frac{\partial\ln|\sigma|}{\partial S(y)}\chi_0(y)dS+\frac{\overset{(0)}{K}}{2\pi}\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\overline{\chi}_0(y)dS+$$
$$+\frac{\overset{(0)}{m}}{2\pi i}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial n(y)}\big[\big(\overset{(0)}{\alpha}+\overset{(0)}{\gamma}\big)g_0(y)+\big(\overset{(0)}{\beta}+\delta_0\big)h_0(y)\big]dS-$$
$$-\frac{\overset{(0)}{m}}{2\pi}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial S(y)}\big[\big(\overset{(0)}{\alpha}-\overset{(0)}{\gamma}\big)g_0(y)+\big(\overset{(0)}{\beta}-\overset{(0)}{\delta}\big)h_0(y)\big]dS-$$
$$-\frac{\overset{(0)}{l}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(0)}{\alpha}\,\overline{g}_0(y)+\overset{(0)}{\beta}\,\overline{h}_0(y)\big)dS,\quad x\in D_0. \tag{5.8}$$

Since $\overset{(0)}{V}_0(\infty)=0$, by virtue of (5.7) we have

$$\overset{(0)}{V}_0(x)=0,\quad x\in D_0. \tag{5.9}$$

As far as $\overset{(0)}{V}_0(x)$ passes continuously through the boundary S_0, the equality (5.9) allows one to write

$$\overset{(0)}{V}_0(t)=0,\quad t\in S_0. \tag{5.10}$$

Consider in the domain $D_0^-=R^2\backslash\overline{D}_1\cup D_0$ the following Green's formulas:

$$\int\limits_{D_0^-}\overset{(0)}{N}\big(\overset{(0)}{U}_0,\overset{(0)}{U}_0\big)dy_1dy_2=-\operatorname{Im}\int\limits_{S_0}\overset{(0)}{V}_0^{\,-}\,\overline{(\overset{(0)}{N}\overset{(0)}{V})^-}dS, \tag{5.11}$$

where $\overset{(0)}{N}\big(\overset{(0)}{U}_0,\overset{(0)}{U}_0\big)$ is a positive definite function.

Taking into account (5.10), we find that $\overset{(0)}{N}\big(\overset{(0)}{U}_0,\overset{(0)}{U}_0\big)=0$, and hence $\overset{(0)}{U}_0(x)=\delta_0=\text{const}$, $x\in D_0^-$, by Lemma 1.1.

Since $\overset{(0)}{U}_0(\infty)=0$, we have $\delta_0=0$, and

$$\overset{(0)}{U}_0(x)=0,\quad x\in D_0^-. \tag{5.12}$$

Taking into consideration that $\big(\overset{(0)}{U}_0(t)\big)^+-\big(\overset{(0)}{U}_0(t)\big)^-=2\chi_0(t)$, $t\in S_0$, and $\big(\overset{(0)}{U}_0(t)\big)^+=\big(\overset{(0)}{U}_0(t)\big)^-=0$, $t\in S_0$ (see (5.12) and (5.6)), we obtain

$$\chi_0(t)=0,\quad t\in S_0. \tag{5.13}$$

Comparing the formulas (5.6), (5.12) and (5.13), we obtain

$$\overset{(j)}{U}_0(x)=\frac{\overset{(j)}{m}}{2\pi}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial n(y)}\big[\big(\overset{(j)}{\alpha}-\overset{(j)}{\gamma}\big)g_0(y)+\big(\overset{(j)}{\beta}-\overset{(j)}{\delta}\big)h_0(y)\big]dS+$$
$$+\frac{\overset{(j)}{m}}{2\pi i}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial S(y)}\big[\big(\overset{(j)}{\alpha}+\overset{(j)}{\gamma}\big)g_0(y)+\big(\overset{(j)}{\beta}+\overset{(j)}{\delta}\big)h_0(y)\big]dS-$$
$$-\frac{\overset{(j)}{l}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(j)}{\alpha}\overline{g}_0(y)+\overset{(j)}{\beta}\overline{h}_0(y)\big)dS=0, \tag{5.14}$$
$$x\in(\delta_1,D_1+\delta_0;D_0\cup S_0\cup D_0^-),\quad j=0,1.$$

Obviously, the vectors (5.14) (for $j=0$ and $j=1$) satisfy the same conditions as (4.3) (for $j=0,1$) (see §4).

Repeating word by word the reasoning we have used for (4.3), $j=0,1$, we obtain $g_0=0$, $h_0=0$.

Thus the system $(5.5)^0$ has only the zero solution, i.e., the system (5.5) is uniquely solvable.

Consequently, we have proved that the first boundary-value problem has a unique solution which is representable in the form (5.3).

6. The Second Boundary-Contact Problem

The boundary and contact conditions in the case under consideration can be written as follows (see §2):

$$\int\limits_0^{S(t)}\big(\overset{(0)}{T}\overset{(0)}{U}\big)^+dS=\phi_0(t)+\text{const},\quad t\in S_0$$
$$\big(\overset{(1)}{U}(t)\big)^+-\big(\overset{(0)}{U}(t)\big)^-=f(t),\quad t\in S_1, \tag{6.1}$$
$$\int\limits_0^{S(t)}\big(\overset{(1)}{T}\overset{(1)}{U}\big)^+dS-\int\limits_0^{S(t)}\big(\overset{(0)}{T}\overset{(0)}{U}\big)^-dS=f_0(t)+\text{const},\quad t\in S_1,$$

where $\phi_0(t)=\int_0^{S(t)}\phi dS$, $F_0(t)=\int_0^{S(t)}FdS$, $\phi\in C^{0,\alpha}(S_0)$, $f\in C^{1,\alpha}(S_1)$, $F\in C^{0,\alpha}(S_1)$, $S_j\in C^{2,\beta}$, $0<\alpha<\beta\leq 1$, $j=0,1$, are given vectors.

To reduce the problem (6.1) to a Fredholm system of second kind, we choose $\overset{(j)}{\varphi}(z)$ and $\overset{(j)}{\overline{\psi}}$ (for $j=0,1$) as

$$\overset{(j)}{\varphi}(z)=\delta_{0j}\frac{\big(\overset{(0)}{A}-2E\big)^{-1}}{2\pi i}\int\limits_{S_0}\frac{\partial\ln\sigma}{\partial S(y)}\chi(y)dS+$$

$$+\frac{1}{2\pi i}\int\limits_{S_1}\frac{\partial\ln\sigma}{\partial S(y)}\big(\overset{(j)}{\alpha} g(y)+\overset{(j)}{\beta} h(y)\big)dS,$$

$$\overset{(j)}{\psi}(z)=-\delta_{0j}\Bigg(\frac{(2\overset{(0)}{\mu})^{-1}}{2\pi i}\int\limits_{S_0}\frac{\partial\ln\overline{\sigma}}{\partial S(y)}\chi(y)dS- \tag{6.2}$$

$$-\frac{\overset{(0)}{e}(\overset{(0)}{A}-2E)^{-1}}{4\pi i}\int\limits_{S_0}^{-1}\frac{\partial}{\partial S(y)}\frac{\zeta}{\overline{\sigma}}\overline{\chi}(y)dS\Bigg)+$$

$$+\frac{\overset{(j)}{m}}{2\pi i}\int\limits_{S_1}\frac{\partial\ln\overline{\sigma}}{\partial S(y)}\big(\overset{(j)}{\gamma} g(y)+\overset{(j)}{\delta} h\big)dS+$$

$$+\frac{\overset{(j)}{e}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\zeta}{\overline{\sigma}}\big(\overset{(j)}{\alpha}\overline{g}+\overset{(j)}{\beta}\overline{h}(y)\big)dS,\quad j=0,1,$$

where χ, g and h are unknown complex vectors, and the remaining quantities appearing in (6.2) have been defined in §1 and §3.

Taking into account (6.2), from (1.7) and (1.8) we get

$$\overset{(j)}{U}(x)=\delta_{oj}\Bigg(\frac{\overset{(0)}{m}(\overset{(0)}{A}-2E)^{-1}+(2\overset{(0)}{\mu})^{-1}}{2\pi}\int\limits_{S_0}\frac{\partial\ln|\sigma|}{\partial n(y)}\chi(y)dS+$$

$$+\frac{\overset{(0)}{m}(\overset{(0)}{A}-2E)^{-1}-(2\overset{(0)}{\mu})^{-1}}{2\pi i}\int\limits_{S_0}\frac{\partial\ln|\sigma|}{\partial S(y)}\chi(y)dS-$$

$$-\frac{\overset{(0)}{e}(\overset{(0)}{A}-2E)^{-1}}{4\pi i}\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\overline{\chi}(y)dS\Bigg)+\frac{\overset{(j)}{(m)}}{2\pi}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial n(y)}\big[\big(\overset{(j)}{\alpha}-\overset{(j)}{\gamma}\big)g(y)+$$

$$+\big(\overset{(j)}{\beta}-\overset{(j)}{\delta}\big)h(y)\big]dS+\frac{\overset{(j)}{(m)}}{2\pi i}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial S(y)}\big[\big(\overset{(j)}{\alpha}+\overset{(j)}{\gamma}\big)g(y)+\big(\overset{(j)}{\beta}+\overset{(j)}{\delta}\big)h(y)\big]dS-$$

$$-\frac{\overset{(j)}{e}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(j)}{\alpha}\overline{g}(y)+\overset{(j)}{\beta}\overline{h}(y)\big)dS,\quad x\in D_j,\quad j=0,1, \tag{6.3}$$

$$\int\limits_0^{S(x)}\overset{(j)}{T}\overset{(j)}{U}dS+d_j=\delta_{oj}\Bigg(\frac{1}{\pi}\int\limits_{S_0}\frac{\partial\ln|\sigma|}{\partial n(y)}\chi(y)dS-\frac{\overset{(0)}{H}}{2\pi i}\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\overline{\chi}(y)dS\Bigg)+$$

$$+\frac{1}{2\pi i}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial S(y)}\Big(\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\alpha}-\overset{(j)}{A}\overset{(j)}{\gamma}\big]g(y)+$$

$$+\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\beta}-\overset{(j)}{A}\overset{(j)}{\delta}\big]h(y)\Big)dS+$$

$$+\bigg(\frac{1}{2\pi i}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial S(y)}\Big(\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\alpha}+\overset{(j)}{A}\overset{(j)}{\gamma}\big]g(y)+\big[\big(\overset{(j)}{A}-2E\big)\overset{(j)}{\beta}+\overset{(j)}{A}\overset{(j)}{\delta}\big]h(y)\Big)dS\bigg)-$$

$$-\frac{\overset{(j)}{B}}{2\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(j)}{\alpha}\overline{g}(y)+\overset{(j)}{\beta}\overline{h}(y)\big)dS,\quad x\in D_j,\ \ j=0,1, \tag{6.4}$$

where d_j, $j=0,1$, are arbitrary constant vectors,

$$\overset{(j)}{H}=\overset{(j)}{B}\big(\overset{(j)}{A}-2E\big)^{-1},\quad j=0,1. \tag{6.5}$$

With regard for the boundary conditions of the second boundary-contact problem, after some calculations for determination of the vectors χ, g and h we obtain the following Fredholm integral equations of second kind:

$$\chi(t)+\frac{1}{\pi}\int\limits_{S_0}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\chi(y)dS-\frac{\overset{(0)}{H}}{2\pi i}\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\overline{\chi}(y)dS+$$

$$+\frac{1}{2\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\Big(\big[\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\alpha}-\overset{(0)}{A}\overset{(0)}{\gamma}\big]g(y)+\big[\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\beta}-\overset{(0)}{A}\overset{(0)}{\delta}\big]h(y)\Big)dS+$$

$$+\frac{1}{2\pi i}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial S(y)}\Big(\big[\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\alpha}+\overset{(0)}{A}\overset{(0)}{\gamma}\big]g(y)+\big[\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\beta}+\overset{(0)}{A}\overset{(0)}{\delta}\big]h(y)\Big)dS-$$

$$-\frac{\overset{(0)}{B}}{2\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{i-\zeta}{\overline{t}-\overline{\zeta}}(\overset{(0)}{\alpha}\overline{g}(y)+\overset{(0)}{\beta}\overline{h}(y))dS-\overset{(0)}{d}=\phi_0(t),\quad t\in S_0,$$

$$g(t)+\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\big[\big(\overset{(1)}{m}\overset{(1)}{\alpha}-\overset{(0)}{m}\overset{(0)}{\alpha}\big)g(y)+\big(\overset{(1)}{m}\overset{(1)}{\beta}-\overset{(0)}{m}\overset{(0)}{\beta}\big)h(y)\big]dS+$$

$$+\frac{1}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big[\big(\overset{(0)}{e}\overset{(0)}{\alpha}-\overset{(1)}{e}\overset{(1)}{\alpha}\big)\overline{g}(y)+\big(\overset{(0)}{e}\overset{(0)}{\beta}-\overset{(1)}{e}\overset{(1)}{\beta}\big)\overline{h}(y)\big]dS-$$

$$-\frac{\overset{(0)}{m}(\overset{(0)}{A}-2E)^{-1}+(2\overset{(0)}{\mu})^{-1}}{2\pi}\int\limits_{S_0}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\chi(y)dS+$$

$$+\frac{\overset{(0)}{m}(\overset{(0)}{A}-2E)^{-1}-(2\overset{(0)}{\mu})^{-1}}{2\pi i}\int\limits_{S_0}\frac{\partial\ln|t-\zeta|}{\partial S(y)}\chi(y)dS-$$

$$-\frac{\overset{(0)}{e}(\overset{(0)}{A}-2E)^{-1}}{2\pi i}\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\overline{\chi}(y)dS=f(t),\quad t\in S_1,\tag{6.6}$$

$$h(t)+\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\Big(\big[\big(\overset{(1)}{A}-2E\big)\overset{(1)}{\alpha}-\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\alpha}\big]g(y)+$$

$$+\big[\big(\overset{(0)}{A}-2E\big)\overset{(1)}{\beta}-\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\beta}\big]h(y)\Big)dS+$$

$$+\frac{1}{2\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big[\big(\overset{(0)}{B}\overset{(0)}{\alpha}-\overset{(1)}{B}\overset{(1)}{\alpha}\big)\overline{g}(y)+\big(\overset{(0)}{B}\overset{(0)}{\beta}-\overset{(1)}{B}\overset{(1)}{\beta}\big)\overline{h}(y)\big]dS-$$

$$-\frac{1}{\pi}\int\limits_{S_0}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\chi(y)dS+\frac{\overset{(0)}{H}}{2\pi i}\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\overline{\chi}(y)dS-C^*=F_0(t),\quad t\in S_1.$$

We tie now the unknown constant vectors $\overset{(0)}{d}$ and C^* with $\chi(t)$, $g(t)$ and $h(t)$ as follows:

$$\overset{(0)}{d}=\int\limits_{S_0}\chi dS,\quad C^*=\int\limits_{S_1}(g+h)dS+\int\limits_{S_0}\chi dS.\tag{6.7}$$

If in the left-hand side of the system (6.6) under $\overset{(0)}{d}$ and C^* we mean the expressions (6.7), then this system will transform into a system containing no unknowns except $\chi(t)$, $g(t)$ and $h(t)$.

Investigate these systems. For this we have to consider those which can be obtained from the given systems if to the first equation in the left-hand side we add the expression $\frac{i}{4\pi}\binom{1}{1}M\frac{1}{\overline{t}}$, where $\overline{t}=t_1-it_2$,

$$M=-i\Big[\frac{\partial}{\partial z}\Big(\overset{(0)}{U}_1+\overset{(0)}{U}_2\Big)-\frac{\partial}{\partial\overline{z}}\Big(\overset{(0)}{U}_1+\overset{(0)}{U}_2\Big)\Big]_{\substack{x_1=0\\x_2=0}}.\tag{6.8}$$

After elementary transformations the above system can be written as

$$\chi(t)+\frac{1}{\pi}\int\limits_{S_0}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\chi(y)dS-\frac{\overset{(0)}{H}}{2\pi i}\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\overline{\chi}(y)dS-\int\limits_{S_0}\chi(y)dS+$$

$$+\frac{1}{2}\Big[\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\big(\overset{(0)}{A}-2E\big)\big(\overset{(0)}{\alpha}g(y)+\overset{(0)}{\beta}h(y)\big)dS-$$

$$-\frac{\overset{(0)}{H}}{2\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big(\overset{(0)}{A}-2E\big)\big(\overset{(0)}{\alpha}\overline{g}(y)+\overset{(0)}{\beta}\overline{h}(y)\big)dS\Big]-$$

$$-\frac{1}{2}\Big[\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\big(\overset{(1)}{A}-2E\big)\big(\overset{(1)}{\alpha}g(y)+\overset{(1)}{\beta}h(y)\big)dS-$$

$$-\frac{\overset{(0)}{H}}{2\pi i}\int\limits_{S}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big(\overset{(1)}{A}-2E\big)\big(\overset{(1)}{\alpha}\overline{g}(y)+\overset{(1)}{\beta}\overline{h}(y)\big)dS\Big]+$$

$$+\frac{1}{2}\Big[\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}h(y)dS-\frac{\overset{(0)}{H}}{2\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\overline{h}(y)dS\Big]+$$

$$+\frac{1}{2i}\Big[\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial S(y)}\big(\overset{(0)}{A}-2E\big)\big(\overset{(0)}{\alpha}g(y)+\overset{(0)}{\beta}h(y)\big)dS-$$

$$-\frac{\overset{(0)}{H}}{2\pi}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big(\overset{(0)}{A}-2E\big)\big(\overset{(0)}{\alpha}\overline{g}(y)+\overset{(0)}{\beta}\overline{h}(y)\big)dS\Big]+$$

$$+\frac{1}{2i}\Big[\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial S(y)}\big(\overset{(1)}{A}-2E\big)\big(\overset{(1)}{\alpha}g(y)+\overset{(1)}{\beta}h(y)\big)dS-$$

$$-\frac{\overset{(0)}{H}}{2\pi}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big(\overset{(1)}{A}-2E\big)\big(\overset{(1)}{\alpha}\overline{g}(y)+\overset{(1)}{\beta}\overline{h}(y)\big)dS\Big]-$$

$$-\frac{1}{2}\Big[\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial S(y)}h(y)dS-\frac{\overset{(0)}{H}}{2\pi}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\overline{h}(y)dS\Big]+$$

$$+\frac{i}{4\pi}\begin{pmatrix}1\\1\end{pmatrix}M\frac{1}{\overline{t}}=\phi_0(t),\quad \phi=\{\phi_{01},\phi_{02}\}^T,\quad t\in S_0;$$

$$g(t)+\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\big[\big(\overset{(1)}{m}\overset{(1)}{\alpha}-\overset{(0)}{m}\overset{(0)}{\alpha}\big)g(y)+\big(\overset{(1)}{m}\overset{(1)}{\beta}-\overset{(0)}{m}\overset{(0)}{\beta}\big)h(y)\big]dS+$$

$$+\frac{1}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big[\big(\overset{(0)}{e}\overset{(0)}{\alpha}-\overset{(1)}{e}\overset{(1)}{\alpha}\big)\overline{g}(y)+\big(\overset{(0)}{e}\overset{(0)}{\beta}-\overset{(1)}{e}\overset{(1)}{\beta}\big)\overline{h}(y)\big]dS-$$

$$-\frac{\overset{(0)}{m}\big(\overset{(0)}{A}-2E\big)^{-1}+(2\overset{(0)}{\mu})^{-1}}{2\pi}\int\limits_{S_0}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\chi(y)dS-$$

$$-\frac{\overset{(0)}{m}\big(\overset{(0)}{A}-2E\big)^{-1}-(2\overset{(0)}{\mu})^{-1}}{2\pi i}\int\limits_{S_0}\frac{\partial\ln|t-\zeta|}{\partial S(y)}\chi(y)dS+\frac{\overset{(0)}{e}\big(\overset{(0)}{A}-2E\big)^{-1}}{4\pi i}\times$$

$$\times\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\overline{\chi}(y)dS=f(t),\quad f=\{f_1,f_2\}^T,\quad t\in S_1;$$

$$h(t)+\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\big(\overset{(1)}{A}-2E\big)\big[\overset{(1)}{\alpha}g(y)+\overset{(1)}{\beta}h(y)\big]dS-$$

$$-\frac{\overset{(1)}{H}}{2\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big(\overset{(1)}{A}-2E\big)\big[\overset{(1)}{\alpha}\overline{g}(y)+\overset{(1)}{\beta}\overline{h}(y)\big]dS+$$

$$-\frac{1}{\pi}\int\limits_{S_1}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\big(\overset{(0)}{A}-2E\big)\big[\overset{(0)}{\alpha}g(y)+\overset{(0)}{\beta}h(y)\big]dS+$$

$$+\frac{\overset{(0)}{H}}{2\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\big(\overset{(0)}{A}-2E\big)\big[\overset{(0)}{\alpha}\overline{g}(y)+\overset{(0)}{\beta}\overline{h}(y)\big]dS-$$

$$-\int\limits_{S_0}\chi dS-\int\limits_{S_1}(g(y)+h(y))dS-\frac{1}{\pi}\int\limits_{S_0}\frac{\partial\ln|t-\zeta|}{\partial n(y)}\chi(y)dS+$$

$$+\frac{\overset{(0)}{H}}{2\pi i}\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}\chi(y)dS=F_0(t),\quad F_0=\{F_{01,F_{02}}\},\quad t\in S_1.\qquad(6.9)$$

Obviously, the system (5.9) is a Fredholm system of second kind.

Let us show that if the system (6.9) has a regular solution, then necessarily

$$M=0,\qquad(6.10)$$

if only the conditions for the principal vector and the principal moment to be equal to zero are satisfied.

Indeed, performing elementary calculations, from (6.9) we obtain

$$M=\operatorname{Re}\int\limits_{S_0}(\phi_{01}+\phi_{02})d\overline{t}+\operatorname{Re}\int\limits_{S_1}(F_{01}+F_{02})d\overline{t}=$$

$$=-\operatorname{Re}\int\limits_{S_0}\overline{t}(\phi_1+\phi_2)dS-\operatorname{Re}\int\limits_{S_1}\overline{t}(F_1+F_2)dS,\quad \overline{t}=t_1-it_2.\qquad(6.11)$$

Thus if

$$\operatorname{Re}\int\limits_{S_0}\overline{t}(\phi_1+\phi_2)dS+\operatorname{Re}\int\limits_{S_1}\overline{t}(F_1+F_2)dS=0,\qquad(6.12)$$

then the condition (6.10) is satisfied.

Note now that the condition that the principal vector and the principal moment of external stresses acting at the boundary of the domain D_0 are equal to zero can be expressed by the formula (6.12).

It is evident from (6.11) and (6.12) that if the equality (6.12) is fulfilled, then the system (6.10) coincides with the system (6.6).

Let us now show that the system (6.9) is solvable. Consider the homogeneous system which is obtained from (6.9) for $\phi_0 = f = F_0 = 0$ or $\phi = f = F = 0$, and prove that it has no different from zero solution. Let χ_0, g_0 and h_0 be any solution of that (homogeneous) system.

Since $\phi = F = 0$, from (6.11) we find that $M^0 = 0$. In this case $(6.9)^0$ coincides with the system $(6.6)^0$ which corresponds to the boundary-contact conditions

$$\int\limits_0^{S(t)} \big(\overset{(0)}{T}\overset{(0)}{U}\big)^+ dS = 0, \quad t \in S_0, \quad \text{or} \quad \big(\overset{(0)}{T}\overset{(0)}{U}_0(t)\big)^+ = 0, \quad t \in S_0,$$

$$\big(\overset{(1)}{U}_0(t)\big)^+ - \big(\overset{(0)}{U}_0(t)\big)^- = 0, \quad t \in S_1, \tag{6.13}$$

$$\int\limits_0^{S(t)} \big(\overset{(1)}{T}\overset{(1)}{U}_0\big)^+ dS - \int\limits_0^{S(t)} \big(\overset{(0)}{T}\overset{(0)}{U}_0\big)^- dS = \text{const}, \quad \text{or}$$

$$\big(\overset{(1)}{T}\overset{(1)}{U}_0(t)\big)^+ - \big(\overset{(0)}{T}\overset{(0)}{U}_0(t)\big)^- = 0, \quad t \in S_1,$$

where $\overset{(j)}{U}_0(x)$, $j = 0, 1$, are obtained from (6.3) when $\chi = \chi_0$, $g = g_0$ and $h = h_0$.

Taking into account (6.13) and using Theorem 2.4, we obtain

$$\overset{(j)}{U}_0(x) = a^* + ib^* \begin{pmatrix} 1 \\ 1 \end{pmatrix} z, \quad x \in D_j, \quad j = 0, 1, \quad a^* = \{a_1^*, a_2^*\}^T, \tag{6.14}$$

where a_1^*, a_2^* and b^* are arbitrary constants.

If now we take into account that

$$M^0 = -i\Big[\frac{\partial}{\partial z}\big(\overset{(0)}{U}_{01} + \overset{(0)}{U}_{02}\big) - \frac{\partial}{\partial \overline{z}}\big(\overset{(0)}{\overline{U}}_{01} + \overset{(0)}{\overline{U}}_{02}\big)\Big]_{,\substack{x_1=0\\ x_2=0}} = 0,$$

then we write $b^* = 0$, and hence

$$\overset{(j)}{U}_0(x) = a^*, \quad x \in D_j, \quad j = 0, 1. \tag{6.15}$$

Since $\overset{(0)}{U}_0(\infty) = 0$ (see (6.3)), we have $a^* = 0$,

$$\overset{(1)}{U}_0(x) = 0, \quad x \in D_1, \quad \overset{(0)}{U}_0(x) = 0, \quad x \in D_0. \tag{6.16}$$

From (6.16) it follows (see (1.17)) that

$$O = \overset{(0)}{N}\overset{(0)}{U}_0(x) = -i(\overset{(0)}{m})^{-1} \frac{\partial \overset{(0)}{V}_0}{\partial S(x)} = 0, \quad x \in D_0, \tag{6.17}$$

where

$$\overset{(0)}{V}_0(x)=-\frac{\overset{(0)}{m}(\overset{(0)}{A}-2E)^{-1}+(2\mu)^{-1}}{2\pi}\int\limits_{S_0}\frac{\partial\ln|\sigma|}{\partial S(y)}\chi_0(y)dS+$$

$$+\frac{\overset{(0)}{m}(\overset{(0)}{A}-2E)^{-1}-(2\mu)^{-1}}{2\pi i}\int\limits_{S_0}\frac{\partial\ln|\sigma|}{\partial n(y)}\chi_0(y)dS-$$

$$-\frac{\overset{(0)}{e}(A-2E)^{-1}}{4\pi}\int\limits_{S_0}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\overline{\chi}_0(y)dS+$$

$$+\frac{\overset{(0)}{m}}{2\pi i}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial n(y)}\big[\big(\overset{(0)}{\alpha}+\overset{(0)}{\gamma}\big)g_0(y)+\big(\overset{(0)}{\beta}+\overset{(0)}{\delta}\big)h_0(y)\big]dS-$$

$$-\frac{\overset{(0)}{m}}{2\pi}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial S(y)}\big[\big(\overset{(0)}{\alpha}-\overset{(0)}{\gamma}\big)g_0(y)+\big(\overset{(0)}{\beta}-\overset{(0)}{\delta}\big)h_0(y)\big]dS-$$

$$-\frac{\overset{(0)}{l}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(0)}{\alpha}\,\overline{g}_0(y)+\overset{(0)}{\beta}\,\overline{h}_0(y)\big)dS,\quad x\in D_0. \tag{6.18}$$

As far as $\overset{(0)}{V}_0(\infty)=0$, by virtue of (6.17) we have

$$\overset{(0)}{V}_0(\infty)=0,\quad x\in D_0, \tag{6.19}$$

whence

$$\overset{(0)}{T}\overset{(0)}{V}_0(x)=0,\quad x\in D_0, \tag{6.20}$$

where

$$\overset{(0)}{T}\overset{(0)}{V}_0(x)=-\frac{1}{\pi}\int\limits_{S_0}\frac{\partial^2\ln|\sigma|}{\partial S(x)\partial S(y)}\chi_0(y)dS-\frac{\overset{(0)}{H}}{2\pi}\int\limits_{S_0}\frac{\partial^2}{\partial S(x)\partial S(y)}\frac{\sigma}{\overline{\sigma}}\overline{\chi}_0(y)dS+$$

$$+\frac{1}{2\pi i}\int\limits_{S_1}\frac{\partial^2\ln|\sigma|}{\partial S(x)\partial n(y)}\Big(\big[\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\alpha}+\overset{(0)}{A}\overset{(0)}{\gamma}\big]g_0(y)+$$

$$+\big[\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\beta}+\overset{(0)}{A}\overset{(0)}{\delta}\big]h_0(y)\Big)dS-$$

$$-\frac{1}{2\pi}\int\limits_{S_1}\frac{\partial^2\ln|\sigma|}{\partial S(x)\partial S(y)}\Big(\big[\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\alpha}-\overset{(0)}{A}\overset{(0)}{\gamma}\big]g_0(y)+$$

$$+\big[\big(\overset{(0)}{A}-2E\big)\overset{(0)}{\beta}-\overset{(0)}{A}\overset{(0)}{\delta}\big]h_0(y)\Big)dS-$$

$$-\frac{\overset{(0)}{B}}{2\pi}\int\limits_{S_1}\frac{\partial^2\frac{\sigma}{\overline{\sigma}}}{\partial S(x)\partial S(y)}\big(\overset{(0)}{\alpha}\,\overline{g}_0(y)+\overset{(0)}{\overline{\beta}}\,\overline{h}_0(y)\big)dS,\quad x\in D_0. \tag{6.21}$$

The vector $\overset{(0)}{T}\overset{(0)}{V}_0(x)$ is continuous when the point x passes through the contour S_0, therefore (see (6.20))

$$\big(\overset{(0)}{T}\overset{(0)}{V}_0(t)\big)^-=0,\quad t\in S_0. \tag{6.22}$$

On the basis of the uniqueness theorem for the second external problem of statics of elastic mixtures [5] we can conclude that

$$\overset{(0)}{V}_0(x)=0,\quad x\in D_0^-=R^2\backslash D_1\cup S_1\cup D_0\cup_0\,. \tag{6.23}$$

From (6.26), for the associate to $\overset{(0)}{V}_0(x)$ vector $\overset{(0)}{U}_0(x)$ in the domain D_0^- we can write

$$\overset{(0)}{U}_0(x)=0,\quad x\in D_0^-\,. \tag{6.24}$$

Taking into account (1.16), (6.16), (6.19), (6.23), (6.24) and $(6.2)_1$ for $\chi=\chi_0$, $g=g_0$ and $h=h_0$, we get

$$0=\big(\overset{(0)}{U}_0(t)+i\overset{(0)}{V}_0(t)\big)^+-\big(\overset{(0)}{U}_0(t)+i\overset{(0)}{V}_0(t)\big)^-=2\overset{(0)}{m}\big(\overset{(0)}{A}-2E\big)^{-1}\chi_0(t),\ \ t\in S_0,$$

whence it follows $\det\overset{(0)}{m}>0$, $\det\big(\overset{(0)}{A}-2E\big)>0$ since $\chi_0(t)=0$, $t\in S_0$ [1].

Comparing the formulas (6.19), (6.24) and (6.3) (for $\chi=\chi_0=0$, $g=g_0$, $h=h_0$), we have

$$\overset{(1)}{U}_0(x)=\frac{\overset{(1)}{m}}{2\pi}\int\limits_{S_1}\big[\big(\overset{(1)}{\alpha}-\overset{(1)}{\gamma}\big)g_0(y)+\big(\overset{(1)}{\beta}-\overset{(1)}{\delta}\big)h_0(y)\big]\frac{\partial\ln|\sigma|}{\partial n(y)}dS+$$

$$+\frac{\overset{(1)}{m}}{2\pi i}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial S(y)}\big[\big(\overset{(1)}{\alpha}+\overset{(1)}{\gamma}\big)g_0(y)+\big(\overset{(1)}{\beta}+\overset{(1)}{\delta}\big)h_0(y)\big]dS-$$

$$-\frac{\overset{(1)}{e}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(1)}{\alpha}\,\overline{g}_0(y)+\overset{(1)}{\overline{\beta}}\,\overline{h}_0(y)\big)dS=0,\quad x\in D_1, \tag{6.25}$$

$$\overset{(0)}{U}_0(x)=\frac{\overset{(0)}{m}}{2\pi}\int\limits_{S_1}\frac{\partial\ln|\sigma|}{\partial n(y)}\big[\big(\overset{(0)}{\alpha}-\overset{(0)}{\gamma}\big)g_0(y)+\big(\overset{(1)}{\beta}-\overset{(0)}{\delta}\big)h_0(y)\big]dS+$$

$$+\frac{\overset{(0)}{m}}{2\pi i}\int\limits_{S_1}\Big[\frac{\partial\ln|\sigma|}{\partial S(y)}(\overset{(0)}{\alpha}+\overset{(0)}{\gamma})g_0(y)+(\overset{(0)}{\beta}+\overset{(0)}{\delta})h_0(y)\Big]ds-$$

$$-\frac{\overset{(0)}{e}}{4\pi i}\int\limits_{S_1}\frac{\partial}{\partial S(y)}\frac{\sigma}{\overline{\sigma}}\big(\overset{(0)}{\alpha}\,\overline{g}_0(y)+\overset{(0)}{\beta}\,\overline{h}_0(y)\big)dS=0,\ x\in D_0\cup S_0\cup D_0^-. \qquad (6.26)$$

Obviously, the vectors $\overset{(1)}{U}_0(x)$ and $\overset{(0)}{U}_0(x)$ satisfy the same conditions as (4.2) (for $j=1$ and $j=0$) (see §4). Repeating word by word the reasoning we have used above for (4.2) (for $j=1$, $j=0$), we find that $g_0=h_0=0$.

Thus the system $(6.9)^0$ has only the zero solution, and hence, the inhomogeneous system (6.9) has the unique solution. In case the principal vector and the principal moment of external stresses are equal to zero, the system (6.9) transforms into the system (6.5).

From the above reasoning it is obvious that if the principal vector and the principal moment of external stresses are equal to zero, then the second boundary-contact problem is solvable. The displacements are defined to within the rigid displacement and the stresses are defined exactly.

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Authors' addresses:

M. Basheleishvili
I. Vekua Inst. Appl. Math.
I. Javakhishvili Tbilisi State Univ.
2, University St., Tbilisi 0143
Georgia

K. Svanadze
A. Tsereteli Kutaisi State University
59, Queen Tamar Ave., Kutaisi 4600
Georgia