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A COMPLEX METHOD FOR SOLVING BOUNDARY VALUE PROBLEMS WITH BOUNDARY DATA PIECEWISE BELONGING TO FRACTIONAL ORDER SPACES

Abstract. Very general boundary data for which boundary value problems are solvable, belong to fractional order spaces. Suppose, e.g., that the boundary data g belongs to the fractional order space $\mathcal{W}_p^s(\partial\Omega)$, where $p > 2$, $s = 1 - \frac{1}{p}$ and $\partial\Omega$ is sufficiently smooth. Then the solution of the boundary value problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial\Omega$$

belongs to $\mathcal{W}_p^1(\Omega)$. On the other hand, piecewise Hölder continuous functions do not belong to a fractional order space, in general. However, sufficiently general boundary data g can be split up into the sum $g = \tilde{g} + \hat{g}$, where $\tilde{g}$ belongs to a fractional order space and $\hat{g}$ is piecewise Hölder continuous.

The paper constructs the solution of the Dirichlet boundary value problem with such boundary data for non-linear partial complex differential equations of the type $\frac{\partial w}{\partial \bar{z}} = F\left(z, w, \frac{\partial w}{\partial z}\right)$ provided the right hand side satisfies the global Lipschitz condition with respect to w and $\frac{\partial w}{\partial z}$, and the Lipschitz constants are small enough.

The method which will be applied to the Dirichlet boundary value problem can also be used in order to solve the modified Dirichlet boundary value problem and the Riemann–Hilbert boundary value problem with Hölder continuous coefficients.

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Key words and phrases. Dirichlet problem, fractional order space, integral operator, weighted Sobolev spaces.

რეზიუმე. მეტად ზოგადი სახის მქონე სასაზღვრო მონაცემები, რომელთათვისაც სასაზღვრო ამოცანები ამოხსნადია, წილადი რიგის სივრცეებს ეკუთვნის. მაგალითად, დავუშვათ, რომ g სასაზღვრო მონაცემი წილადი რიგის $\mathcal{W}_p^s(\partial\Omega)$ სივრცეს ეკუთვნის, სადაც $p > 2$, $s = 1 - \frac{1}{p}$ და $\partial\Omega$ საკმაოდ გლუვია. მაშინ

$$\Delta u = 0 \quad \Omega\text{-ში}, \quad u = g \quad \partial\Omega\text{-ზე}$$

სასაზღვრო ამოცანის ამონახსნი $\mathcal{W}_p^1(\Omega)$-ს ეკუთვნის. მეორეს მხრივ, ჰელდერის აზრით უბან-უბან უწყვეტი ფუნქციები საზოგადოდ წილადი რიგის სივრცეს არ ეკუთვნის. თუმცა საკმაოდ ზოგადი სახის სასაზღვრო g მონაცემი შეიძლება გაიხლიჩოს $g = \tilde{g} + \hat{g}$ ჯამად, სადაც $\tilde{g}$ რომელიმე წილადი რიგის სივრცის ელემენტია, ხოლო $\hat{g}$ ჰელდერის აზრით უბან-უბან უწყვეტია.

ნაშრომში $\frac{\partial w}{\partial \bar{z}} = F\left(z, w, \frac{\partial w}{\partial z}\right)$ ტიპის კომპლექსური დიფერენციალური განტოლებებისათვის, სადაც მარჯვენა მხარე აკმაყოფილებს ლიფშიცის გლობალურ პირობას w-სა და $\frac{\partial w}{\partial z}$-ს მიმართ საკმაოდ მცირე ლიფშიცის მუდმივებით, აგებულია დირიხლეს სასაზღვრო ამოცანის ამონახსნი ასეთ სასაზღვრო პირობებში.

მეთოდი, რომელიც გამოიყენება დირიხლეს სასაზღვრო ამოცანისათვის, შეიძლება გამოყენებულ იქნეს აგრეთვე დირიხლეს მოდიფიცირებული ამოცანის და რიმან–ჰილბერტის სასაზღვრო ამოცანების ამოსახსნელად ჰელდერის აზრით უწყვეტი კოეფიციენტებით.

1. Goal of the Paper*

Solving boundary value problems for non-linear partial differential equations using fixed-point methods entails the construction of an associated operator. This construction often involves the use of the solution of an auxiliary boundary value problem for a linear partial differential equation. For instance, if $w = w(z)$ is a solution of the partial complex differential equation

$$\frac{\partial w}{\partial \overline{z}} = F(z, w, \partial w/\partial z), \tag{1}$$

then (w, h) is a fixed point of the operator

$$W = \Psi + \Phi_{(w,h)} + T_\Omega F(\cdot, w, h), \tag{2}$$

$$H = \Psi' + \Phi'_{(w,h)} + \Pi_\Omega F(\cdot, w, h), \tag{3}$$

where $h = \partial w/\partial z$. The operators T_Ω and Π_Ω are the well-known integral operators defined by

$$(T_\Omega f)\,[z] = -\frac{1}{\pi}\iint\limits_{\Omega} \frac{f(\zeta)}{\zeta - z} d\xi d\eta, \quad (\Pi_\Omega f)\,[z] = -\frac{1}{\pi}\iint\limits_{\Omega} \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta,$$

where $\zeta = \xi + i\eta$. Further, Ψ is the holomorphic solution of the boundary value problem under consideration, and $\Phi_{(w,h)}$ is another holomorphic function such that the functor $\Phi_{(w,h)} + T_\Omega F(\cdot, w, \partial w/\partial z)$ has zero boundary values. Since the definition of the operator (2), (3) contains first order derivatives, one has to ensure that the auxiliary functions Ψ and $\Phi_{(w,h)}$ have integrable first order derivatives Ψ' and $\Phi'_{(w,h)}$.

In the sequel the boundary values are understood as the traces of the desired solutions. If, for instance, the real part of a holomorphic function has a trace belonging to the fractional order space $\mathcal{W}_p^s(\partial\Omega)$, $s = 1 - \frac{1}{p}$, $p > 1$, then the complex derivative of the holomorphic function belongs to $L_p(\Omega)$.

The boundary value problems with boundary data belonging to fractional order spaces are quite general. However, they do not contain the cases where the boundary values are piecewise Hölder continuous. In what follows, we will consider boundary value problems for the differential equation (1) with boundary data piecewise belonging to fractional order spaces. This will cover the cases where the boundary data for the differential equation (1) is piecewise Hölder continuous.

The Dirichlet boundary value problem for the differential equation (1) is the problem of finding a solution satisfying the condition

$$\operatorname{Re} w = g \quad \text{on} \quad \Gamma, \tag{4}$$

$$\operatorname{Im} w\,[z_0] = c, \tag{5}$$

where z_0 is a given point in Ω.

* The statement of the problem and first approaches to its solution were outlined in 1997 when G. F. Manjavidze and H. L. Vasudeva visited Graz University of Technology as visiting professors.

The right-hand side $F(z, w, h)$ of the differential equation (1) is supposed to satisfy a global Lipschitz condition of the type

$$|F(z, w_1, h_1) - F(z, w_2, h_2)| \leq L_1|w_1 - w_2| + L_2|h_1 - h_2| \tag{6}$$

for each $z \in \Omega$. Further, $F(z, 0, 0)$ is supposed to be integrable. Since

$$F(z, w(z), h(z)) = F(z, 0, 0) + \Big(F(z, w(z), h(z)) - F(z, 0, 0)\Big),$$

we have

$$|F(z, w(z), h(z))| \leq |F(z, 0, 0)| + \Big|F(z, w(z), h(z)) - F(z, 0, 0)\Big| \leq$$
$$\leq |F(z, 0, 0)| + L_1|w(z)| + L_2|h(z)|.$$

Thus the composition $F(z, w(z), h(z))$ turns out to be integrable provided $w(z)$ and $h(z)$ are integrable.

2. Statement of the Problem

To be specific, we will restrict the consideration of boundary value problems in Ω, where Ω is the unit disk in the complex plane. Moreover, the given boundary data g can be split up into the sum $\tilde{g} + \hat{g}$, where $\tilde{g}$ belongs to the fractional order space $\mathcal{W}_p^s(\partial\Omega)$ while $\hat{g}$ is piecewise Hölder continuous.

Denote the (oriented) boundary $\partial\Omega$ by Γ. Suppose Γ can be decomposed into finitely many arcs Γ_j whose initial [resp. end] points will be denoted by c_j [resp. c_{j+1}], $j = 1, \dots, r$, $c_{r+1} = c_1$. As usual, denote by $\mathcal{H}^\mu(\Gamma; c_1, \dots, c_r)$ the class of boundary data whose restrictions to Γ_j belong to $\mathcal{H}^\mu(\Gamma_j)$, that is, they are Hölder continuous on Γ_j with exponent μ.

Definition 1. A boundary data g belongs to $\mathcal{W}_p^s(\Gamma; \mu; c_1, \dots, c_r)$ in case g can be decomposed into the sum $\tilde{g} + \hat{g}$, where $\tilde{g} \in \mathcal{W}_p^s(\Gamma)$ and $\hat{g} \in \mathcal{H}^\mu(\Gamma; c_1, \dots, c_r)$.

Since a piecewise Hölder continuous function can be represented as a sum of a Hölder continuous function and a piecewise linear function, we get the following statement:

Lemma 1. *Suppose g belongs to $\mathcal{W}_p^s(\Gamma; \mu; c_1, \dots, c_r)$. Then $g = g_1 + g_2 + g_3$, where*

$$g_1 \in \mathcal{W}_p^s(\Gamma), \quad g_2 \in \mathcal{H}^\mu(\Gamma) \quad \textit{and} \quad g_3 \textit{ is piecewise linear.}$$

Note that the boundary data g_1 belongs to $\mathcal{W}_p^s(\Gamma)$ with $s = 1 - \frac{1}{p}$ if

$$\int_\Gamma \int_\Gamma \frac{|g_1(x) - g_1(y)|^p}{|x - y|^{p+1}} dx dy < +\infty.$$

Now suppose g_2 is Hölder continuous with exponent μ, i.e.,

$$|g_2(x) - g_2(y)| \leq H \cdot |x - y|^\mu.$$

Then
$$\frac{|g_2(x)-g_2(y)|^p}{|x-y|^{p+1}} \le H \cdot |x-y|^{\mu p-p-1}$$
and, consequently, g_2 belongs to $\mathcal{W}_p^s(\Gamma)$ if
$$\mu > 1-\frac{1}{p}. \tag{7}$$
In particular, piecewise linear functions (without any jumps) belong to $\mathcal{W}_p^s(\partial\Omega)$.

The following lemma formulates a sufficient condition under which the boundary data g belong to $\mathcal{W}_p^s(\Gamma; c_1, \ldots, c_r)$.

Lemma 2. *Let g be defined on Γ. Suppose*
a) *the restriction of g to Γ_j belongs to $\mathcal{W}_p^s(\Gamma_j)$, $j = 1, \ldots, r$, and*
b) *g has one-sided limits l at each c_j such that*
$$|g(x)-l| \le C \cdot |x-c_j|^\eta$$
with $\eta > 1-\frac{1}{p}$.

Then g has the form $g = g_1 + g_3$, where $g_1 \in \mathcal{W}_p^s(\Gamma)$ and g_3 is piecewise linear.

Proof. Choose a piecewise linear function g_3 such that $g_1 = g - g_3$ has two-sided limits (which are again denoted by l) at c_j. It remains to prove that $g_1 \in \mathcal{W}_p^s(\Gamma)$. For this purpose consider a pair of points x and y in a neighbourhood of c_j which are located on different sides of c_j. Then one has
$$\begin{aligned} |g_1(x)-g_1(y)|^p &\le (|g_1(x)-l| + |g_1(y)-l|)^p \le \\ &\le 2^p\left(|g_1(x)-l|^p + |g_1(y)-l|^p\right) \le \\ &\le 2^p C^p \left(|x-c_j|^{\eta p} + |y-c_j|^{\eta p}\right) \le \\ &\le 2^{p+1} C^p M^{\eta p} \cdot |x-y|^{\eta p} \end{aligned}$$
because $|x-c_j|,\ |y-c_j| \le M \cdot |x-y|$ for smooth curves ($M = 1$ in case of the unit circle). Since
$$\frac{|g_1(x)-g_1(y)|^p}{|x-y|^{p+1}} \le 2^{p+1} C^p M^{\eta p} \cdot |x-y|^{\eta p-p-1},$$
we see, finally, that g_1 belongs to $\mathcal{W}_p^s(\Gamma)$. □

3. An Application of Sobolev's Imbedding Theorem

Let Ω be a bounded domain in the plane whose boundary curve has locally Lipschitz-continuous representations. Then the general Sobolev Imbedding Theorem (see [1], pp. 97-98) contains as a special case the following statement (Part II, case C' on p. 98 of [1], see also C. B. Morrey [6]):

The Sobolev space $\mathcal{W}_p^1(\Omega)$ is imbedded into $\mathcal{H}^\lambda(\Omega)$ with
$$\lambda = \frac{p-2}{p}. \tag{8}$$

Later on (see Section 7) we will give a new proof of the above special case of Sobolev's imbedding theorem using methods of Complex Analysis. This proof can also be used in order to estimate the imbedding constant by the norm of the T_Ω-operator.

4. The Holomorphic Solution of the Dirichlet Boundary Value Problem

Again suppose that Ω is the unit disk. Note first that a holomorphic function Ψ in Ω whose real part has the boundary values g is given by the Schwarz Integral Formula

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{g(\zeta)d\zeta}{\zeta - z} - \frac{1}{4\pi i} \int\limits_{\Gamma} \frac{g(\zeta)d\zeta}{\zeta} + iC, \tag{9}$$

where C is an arbitrary real constant.

Now let g be a given boundary data belonging to $\mathcal{W}_p^s(\Gamma;\mu;c_1,\dots,c_r)$. In view of Lemma 1 the given g can be split up into the sum $g_1+g_2+g_3$. Let Ψ_j, $j=1,2,3$, be a holomorphic function whose real part has the boundary values g_j. Note that Ψ_j is uniquely determined up to a constant which can be chosen in accordance with (5).

First we show that Ψ_1 belongs to $\mathcal{W}_p^1(\Omega)$. Indeed, since the boundary values g_1 of Re Ψ_1 belong to the fractional order space $\mathcal{W}_p^s(\Gamma)$, the real part Re Ψ_1 belongs to this space in view of the following lemma (see H. Triebel [10]; cf. also A. S. A. Mshimba [7]):

Lemma 3. *In case the boundary values g_1 belong to $\mathcal{W}_p^s(\Gamma)$, the solution u of the boundary value problem*

$$\Delta u = 0 \quad in \quad \Omega,$$
$$u = g_1 \quad on \quad \Gamma$$

belongs to $\mathcal{W}_p^1(\Omega)$, and its norm can be estimated by an estimate of the form

$$\|u\|_{\mathcal{W}_p^1(\Omega)} \le K \, \|g_1\|_{\mathcal{W}_p^s(\partial\Omega)} \, .$$

The above imbedding theorem (see Section 3) implies that Re Ψ_1 and thus Im Ψ_1, too, belongs to $\mathcal{H}^\lambda(\Omega)$. Lemma 3 shows, especially, that the first order derivatives of Re Ψ_1 belong to $L_p(\Omega)$. Using Cauchy-Riemann equations, we see that the same is true for the first order derivatives of Im Ψ_1. To sum up, we have proved the following statement:

The pair (Ψ_1, Ψ_1') belongs to $\big(\mathcal{H}^\lambda(\Omega),\ L_p(\Omega)\big)$.

By the way, the above Lemma 3 can also be used in order to give a new proof of the Hardy-Littlewood Theorem. The Hardy-Littlewood Theorem is the following statement:

Suppose the boundary values of the real part of a holomorphic function Φ are Hölder continuous with exponent μ. Then Φ' belongs to $L_p(\Omega)$ in case $p < \frac{1}{1-\mu}$.

This statement can be proved as follows:

Using (7), we see that the boundary values under consideration belong to $\mathcal{W}_p^s(\Gamma)$ provided $p < 1/(1-\mu)$. In view of Lemma 3 the holomorphic function Φ belongs to $\mathcal{W}_p^s(\Omega)$, i.e., in particular, Φ' belongs to $L_p(\Omega)$.

Next we consider the holomorphic function Ψ_2 with Hölder continuous boundary values g_2. If g_2 is Hölder continuous with exponent μ, then Ψ_2 belongs to $\mathcal{H}^\mu(\Omega)$, and in view of Hardy-Littlewood's Theorem its derivative Ψ_2' belongs to $L_p(\Omega)$ provided $p < \frac{1}{1-\mu}$. Hence μ has to satisfy the inequality $\mu > 1 - \frac{1}{p}$. Then the pair (Ψ_2, Ψ_2') belongs to $\big(\mathcal{H}^\mu(\Omega),\ L_p(\Omega)\big)$. Since $\lambda = 1 - \frac{2}{p} < 1 - \frac{1}{p} < \mu$, the pair (Ψ_2, Ψ_2') belongs also to $\big(\mathcal{H}^\lambda(\Omega),\ L_p(\Omega)\big)$.

Consider, finally, the holomorphic function Ψ_3 whose real part has piecewise linear boundary values g_3. The Schwarz Integral Formula (9) shows that Ψ_3 has logarithmic singularities of the type $\log(z - c_j)$ at the jump discontinuities c_j of the given boundary data. Consequently, Ψ_3' has singularities of the type

$$\frac{1}{z - c_j}$$

at those points and thus $|\Psi_3'|^p$ is not integrable if $p > 2$. However,

$$\frac{1}{|z - c_j|^p} \cdot |z - c_j|^\gamma$$

is integrable provided

$$\gamma - p > -2. \tag{10}$$

Let $\varrho(z) = \prod_j |z - c_j|$. A function f is said to belong to the weighted L_p-space with the weight ϱ^γ in case $|f|^p \cdot \varrho^\gamma$ is integrable. The totality of all such functions is denoted by $L_p(\Omega, \varrho^\gamma)$. The space $L_p(\Omega, \varrho^\gamma)$ is a Banach space equipped with the norm

$$\|f\|_{L_p(\Omega,\varrho^\gamma)} = \left(\iint_\Omega |f|^p \cdot \varrho^\gamma dxdy \right)^{\frac{1}{p}}.$$

In view of (10) Ψ_3 belongs to $L_p(\Omega, \varrho^\gamma)$ if

$$\gamma > p - 2. \tag{11}$$

Since Ψ_3 has logarithmic singularities at the jump discontinuities c_j, Ψ_3 belongs to the weighted Hölder space $\mathcal{H}^\kappa(\Omega, \varrho^\beta)$, where both exponents κ and β are arbitrary positive numbers. Choose $\kappa = \lambda$ and $\beta = \gamma/p$. This choice of β implies that f belongs to $L_p(\Omega, \varrho^\gamma)$ if f belongs to $\mathcal{H}^\lambda(\Omega, \varrho^\beta)$. Indeed, since $f \in \mathcal{H}^\lambda(\Omega, \varrho^\beta)$, we have

$$f = \frac{f_1}{\varrho^\beta},$$

where $f_1 \in \mathcal{H}^\lambda(\Omega)$ and therefore

$$|f|^p \cdot \varrho^\gamma = |f_1|^p$$

is integrable.

Note that the norm of f in the weighted Hölder space $\mathcal{H}^\lambda(\Omega, \varrho^\beta)$ is nothing but the ordinary Hölder norm of $f \cdot \varrho^\beta$.

Summarizing the considerations concerning Ψ_3, we see that the pair (Ψ_3, Ψ_3') belongs to $\big(\mathcal{H}^\lambda(\Omega, \varrho^\beta), L_p(\Omega, \varrho^\gamma)\big)$, where

$$\beta = \frac{\gamma}{p}. \tag{12}$$

5. Boundary Data Defined by the T_Ω-Operator

The construction of the operator (2), (3) requires the calculation of a holomorphic function Φ such that the real part of $\Phi + T_\Omega f$ has boundary values zero. If $f(z) = F(z, w(z), h(z))$, then the desired Φ will be denoted by $\Phi_{(w,h)}$. In case Γ is the boundary of the unit disk Ω, the function Φ can be represented explicitly by a modified T_Ω-operator. Indeed, the above boundary condition can be rewritten in the form

$$\text{Re } \Phi = -\text{Re } \left[T_\Omega f\right] = -\text{Re } \left[\overline{T_\Omega f}\right]. \tag{13}$$

On Γ we have $z = \frac{1}{\bar{z}}$, i.e.,

$$\overline{\left(T_\Omega f\right)[z]} = -\frac{1}{\pi} \iint\limits_{|\zeta|<1} \frac{\overline{f(\zeta)}}{\bar{\zeta} - \frac{1}{z}} d\xi d\eta = +\frac{z}{\pi} \iint\limits_{|\zeta|<1} \frac{\overline{f(\zeta)}}{1 - \bar{\zeta} z} d\xi d\eta.$$

Consequently,

$$\Phi(z) = -\frac{z}{\pi} \iint\limits_{|\zeta|<1} \frac{\overline{f(\zeta)}}{1 - \bar{\zeta} z} d\xi d\eta \tag{14}$$

is a holomorphic function in the unit disk satisfying the boundary condition (13).

Obviously, the derivative of Φ has the integral representation

$$\Phi(z) = -\frac{z}{\pi} \iint\limits_{|\zeta|<1} \frac{\overline{f(\zeta)}}{(1 - \bar{\zeta} z)^2} d\xi d\eta. \tag{15}$$

The integral operators in (13) and (14) have similar properties as the T_Ω- and the Π_Ω-operators (see B. Bojarski [2]). The next section proves some lemmas describing the behaviour of these operators in weighted function spaces.

6. Some Lemmas

In order to estimate the operator (2), (3), we have to investigate mapping properties of the operators T_Ω and Π_Ω and of their modifications in the weighted spaces $\mathcal{H}^\lambda(\Omega, \varrho^\beta)$ and $L_p(\Omega, \varrho^\gamma)$. For this purpose we need the following lemmas. To be short, consider, without any restriction of generality, only one jump discontinuity c, i.e., $\varrho(z)$ is supposed to have the form

$$\varrho(z) = |z - c|.$$

Lemma 4. *Let*

$$\varphi(u,z)=\iint_{\Omega}\frac{K(u,\zeta)f(\zeta)}{|\zeta-z|^{\alpha}}d\xi d\eta,$$

where $0<\alpha<1$ *and* Ω *is a finite domain,* $f\in L_p(\Omega)$*, the function* $K=K(u,\zeta)$ *is defined on* $\Omega\times\Omega$*,*

$$|K(u_1,\zeta)-K(u_2,\zeta)|\le A|u_1-u_2|^{\mu},$$

A *and* μ *are positive constants. If* $p>\frac{2}{2-\alpha}$*, then*

$$|\varphi(u_1,z)-\varphi(u_2,z)|\le A_1|u_1-u_2|^{\mu},$$
$$|\varphi(u,z_1)-\varphi(u,z_2)|\le A_2|z_1-z_2|^{\beta},$$

where A_1,A_2 *are positive constants and* $\beta=\min\left(1,2-\alpha-\dfrac{2}{p}\right)$.

Proof. Using Hölder's inequality, one gets

$$|\varphi(u_1,z)-\varphi(u_2,z)|\le A|u_1-u_2|^{\mu}\iint_{\Omega}\frac{|f(\zeta)|}{|\zeta-z|^{\alpha}}d\xi d\eta\le$$
$$\le A|u_1-u_2|^{\mu}\left(\iint_{\Omega}|\zeta-z|^{-\alpha\delta}d\xi d\eta\right)^{\frac{1}{\delta}}\left(\iint_{\Omega}|f(\zeta)|^{\frac{\delta}{\delta-1}}d\xi d\eta\right)^{\frac{\delta-1}{\delta}}.$$

Choosing

$$\frac{p}{p-1}\le\delta<\frac{2}{\alpha},\tag{16}$$

one has

$$-\alpha\delta>-2\quad\text{and}\quad\frac{\delta}{\delta-1}\le p$$

and, therefore, one gets an estimate of the type

$$|\varphi(u_1,z)-\varphi(u_2,z)|\le A_1|u_1-u_2|^{\mu},$$

where A_1 is a positive constant. Notice that the choice (16) implies

$$p>\frac{2}{2-\alpha}.\tag{17}$$

One has, further,

$$|\varphi(u,z_1)-\varphi(u,z_2)|\le\iint_{\Omega}\frac{\left||\zeta-z_1|^{\alpha}-|\zeta-z_2|^{\alpha}\right|}{|\zeta-z_1|^{\alpha}|\zeta-z_2|^{\alpha}}|f(\zeta)|d\xi d\eta\le$$
$$\le|z_1-z_2|^{\alpha}\iint_{\Omega}|f(\zeta)|\frac{1}{|\zeta-z_1|^{\alpha}|\zeta-z_2|^{\alpha}}d\xi d\eta\le$$
$$\le|z_1-z_2|^{\alpha}\left(\iint_{\Omega}|f(\zeta)|^{p}d\xi d\eta\right)^{\frac{1}{p}}\left(\iint_{\Omega}\frac{1}{|\zeta-z_1|^{\alpha q}|\zeta-z_2|^{\alpha q}}d\xi d\eta\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$. Since (16) implies $\alpha q<2$, the above inequality yields

$$|\varphi(u,z_1)-\varphi(u,z_2)| \le B|z_1-z_2|^{\alpha}\,|z_1-z_2|^{\frac{2-2\alpha q}{q}} \le B\,|z_1-z_2|^{\frac{2}{q}-\alpha}.$$

Once more taking into account the estimate (17), we see that

$$\frac{2}{q}-\alpha = 2\left(1-\frac{1}{p}\right)-\alpha > 2-(2-\alpha)-\alpha > 0$$

and thus Lemma 4 has been proved. □

Lemma 5. *Suppose* $f \in L_p(\Omega, |z-c|^{\gamma})$, *where* $p>2$ *and* $1-\frac{\gamma}{p}<\alpha<1$, *where* α *is any number with* $0<\alpha<1$. *Then* $T_{\Omega}f$ *belongs to* $\mathcal{H}^{\lambda}(\Omega, |z-c|^{\beta})$, *where*

$$\lambda = \min\left(\frac{p-2}{p},\ \frac{\gamma}{p}+\alpha-1\right)$$

and $\beta = \dfrac{\gamma}{p}$.

Proof. Consider the function φ defined by

$$\varphi(z) = \varrho^{\beta}\iint\limits_{\Omega}\frac{f(\zeta)}{\zeta-z}d\xi d\eta.$$

In order to prove Lemma 5, we have to show that φ belongs to $\mathcal{H}^{\lambda}(\Omega)$. We have

$$\begin{aligned}\varphi(z) &= \iint\limits_{\Omega}\frac{|z-c|^{\beta}-|\zeta-c|^{\beta}}{\zeta-z}f(\zeta)d\xi d\eta + \iint\limits_{\Omega}\frac{|\zeta-c|^{\beta}}{\zeta-z}f(\zeta)d\xi d\eta = \\ &= \iint\limits_{\Omega}\frac{K(z,\zeta)}{|\zeta-z|^{\alpha}}f(\zeta)d\xi d\eta + \iint\limits_{\Omega}\frac{f_1(\zeta)}{\zeta-z}d\xi d\eta,\end{aligned} \tag{18}$$

where

$$f_1(\zeta) = |\zeta-c|^{\beta}f(\zeta) \in L_p(D) \quad \text{and}$$
$$K(z,\zeta) \in H_{\delta}(D\times D), \quad \delta = \frac{\gamma}{p}+\alpha-1.$$

One has $\delta>0$ because $1-\frac{\gamma}{p}<\alpha<1$. The second term in (18) belongs to $H_{\nu}(D)$ with $\nu=\frac{p-2}{p}$, while in view of Lemma 4 the first term belongs to

$$H_s(D) \quad \text{with} \quad s = \min\left(\frac{\gamma}{p}+\alpha-1, 2-\alpha-\frac{2}{p}\right).$$

This proves Lemma 5. □

Next we investigate the behaviour of the T_{Ω}- and the Π_{Ω}- operators in weighted Lebesgue spaces.

Lemma 6. *The* T_{Ω}*- and the* Π_{Ω}*-operators are bounded operators mapping* $L_p(\Omega, |z-c|^{\gamma})$ *into itself provided* $\gamma=\sigma(p-1)$ *and* $0<\sigma<1$.

Proof. The following proof is a modification of the proof of a theorem on the behaviour of Cauchy type integrals in weighted Lebesgue spaces (see the proof of Theorem 4 in Section 3 of Chapter I in B. V. Khedelidze's trend-setting book [3]).

Denote by Ω_z and Ω_ζ the domain Ω as domains in the z- and the ζ-planes, respectively. Then

$$(S_k\varphi)[z] = -\frac{1}{\pi}\iint\limits_{\Omega_\zeta} \frac{\varphi(\zeta)}{(\zeta - z)^{1+k}} d\xi d\eta$$

defines the T_Ω- and the Π_Ω-operators for $k = 0$ and $k = 1$, respectively. One has to show the existence of a constant C such that

$$\iint\limits_{\Omega_z} |S_k\varphi|^p |z - c|^{\sigma(p-1)} dxdy \leq C \iint\limits_{\Omega_z} |\varphi|^p |z - c|^{\sigma(p-1)} dxdy$$

for each $\varphi \in L_p(\Omega, |z - c|^{\sigma(p-1)})$.

The elements φ under consideration can be represented in the form

$$\varphi(z) = f(z)|z - c|^{-\frac{\sigma}{q}}, \tag{19}$$

where f belongs to $L_p(\Omega)$ and q means again the exponent conjugate to p, that is, $\frac{1}{p} + \frac{1}{q} = 1$ and thus $\frac{p-1}{p} = \frac{1}{q}$. Note that

$$|z - c|^{\frac{\sigma}{q}} = \left(|z - c|^{\frac{\sigma}{q}} - |\zeta - c|^{\frac{\sigma}{q}}\right) + |\zeta - c|^{\frac{\sigma}{q}}.$$

Since

$$\varphi(\zeta)|\zeta - c|^{\frac{\sigma}{q}} = f(\zeta),$$

in view of (19) we get

$$|S_k\varphi|^p |z - c|^{\sigma(p-1)} = |J_k\varphi + S_k f|^p \leq 2^{p-1}\Big(|J_k\varphi|^p + |S_k f|^p\Big),$$

where

$$(J_k\varphi)[z] = -\frac{1}{\pi}\iint\limits_{\Omega_\zeta} \frac{|z - c|^{\frac{\sigma}{q}} - |\zeta - c|^{\frac{\sigma}{q}}}{(\zeta - z)^{1+k}} \varphi(\zeta) d\xi d\eta.$$

Hence it follows

$$\iint\limits_{\Omega_z} |S_k\varphi|^p |z - c|^{\sigma(p-1)} dxdy \leq$$

$$\leq 2^{p-1} \iint\limits_{\Omega_z} |J_k\varphi|^p dxdy + 2^{p-1} \iint\limits_{\Omega_z} |S_k f|^p dxdy. \tag{20}$$

Since the operators S_k are bounded in $L_p(\Omega)$, the second integral on the right-hand side of (20) can (up to a constant factor) be estimated by

$$\iint\limits_{\Omega_z} |f|^p dxdy = \iint\limits_{\Omega_z} |\varphi|^p |\zeta - c|^{\sigma(p-1)} dxdy.$$

It remains to verify an analogous estimate for the first integral on the right-hand side of (20). Since

$$\Big||z-c|^{\frac{\sigma}{q}} - |\zeta - c|^{\frac{\sigma}{q}}\Big| \le |\zeta - z|^{\frac{\sigma}{q}},$$

it follows

$$|J_k\varphi| \le \frac{1}{\pi} \iint\limits_{\Omega_\zeta} \frac{|\varphi(\zeta)|}{|\zeta - z|^{\nu+k}} d\xi d\eta,$$

where $\nu = 1 - \frac{\sigma}{q}$. Choose $\varepsilon > 0$ such that $\sigma_0 = \sigma + \varepsilon < 1$. Hölder's inequality implies

$$|J_k\varphi|^p \le \frac{1}{\pi^p}\left(\iint\limits_{\Omega_\zeta} \frac{|\varphi(\zeta)| \cdot |\zeta - c|^{\frac{\sigma_0}{q}}}{|\zeta - z|^{(\nu+k)\frac{1}{p}}} \cdot \frac{1}{|\zeta - z|^{(\nu+k)\frac{1}{q}} \cdot |\zeta - c|^{\frac{\sigma_0}{q}}} d\xi d\eta\right)^p \le$$

$$\le \iint\limits_{\Omega_\zeta} \frac{|\varphi(\zeta)|^p \cdot |\zeta - c|^{\sigma_0(p-1)}}{|\zeta - z|^{\nu+k}} d\xi d\eta \cdot \left(\iint\limits_{\Omega_\zeta} \frac{d\xi d\eta}{|\zeta - z|^{\nu+k} \cdot |\zeta - c|^{\sigma_0}}\right)^{p-1}. \quad (21)$$

Note that

$$\iint\limits_{\Omega} \frac{d\xi d\eta}{|\zeta - z_1|^s \cdot |\zeta - z_2|^t} \le \begin{cases} C_1 + C_2|z_1 - z_2|^{2-s-t} \text{ if } s+t \ne 2 \\ C_3 + 4\pi\Big|\ln|z_1 - z_2|\Big| \text{ if } s+t = 2 \end{cases}$$

provided Ω is a domain of finite measure and s and t are less than 2. If Ω is bounded, denote its diameter by d_0. Then $|z_1 - z_2| \le d_0$ for any two points in Ω and thus

$$1 \le d_0^{s+t-2} \quad \text{in case} \quad s+t > 2.$$

This implies for any two points z_1, z_2 of the bounded domain Ω the estimate

$$\iint\limits_{\Omega} \frac{d\xi d\eta}{|\zeta - z_1|^s \cdot |\zeta - z_2|^t} \le \begin{cases} C_4|z_1 - z_2|^{2-s-t} \text{ if } s+t > 2 \\ C_3 + 4\pi\Big|\ln|z_1 - z_2|\Big| \text{ if } s+t = 2 \\ C_5 \text{ if } s+t < 2 \end{cases}$$

(where $C_5 = C_1 + C_2 d_0^{2-s-t}$ and so on).

Since $\nu + k = 1 - \frac{\alpha}{q} + k < 2$ and $\sigma_0 < 1$, this estimate is applicable to the second integral in (21). Suppose, first, that $\nu + k + \sigma_0 > 2$. Then the second factor in (21) can be estimated by

$$C_4^{p-1}|z - c|^{(2-\nu-k-\sigma_0)(p-1)}.$$

The exponent of $|z - c|$ equals

$$\left(2 - 1 + \frac{\sigma}{q} - k - \sigma - \varepsilon_0\right)(p-1) = (1-k)(p-1) - \varepsilon_0(p-1) - \frac{\sigma}{q}\ .$$

Interchanging the order of integration, the estimate (21) leads to

$$\iint\limits_{\Omega_z} |J_k\varphi|^p dxdy \le$$

$$\leq C_6 \iint\limits_{\Omega_\zeta} |\varphi(\zeta)|^p \cdot |\zeta - c|^{\sigma_0(p-1)} \left(\iint\limits_{\Omega_z} \frac{dxdy}{|z-\zeta|^{\nu+k} \cdot |z-c|^n} \right) d\xi d\eta, \quad (22)$$

where the exponent

$$n = \varepsilon_0(p-1) + \frac{\sigma}{q} - (1-k)(p-1)$$

is less than 2 provided ε_0 is small enough.

Additionally assume now that also

$$\nu + k + \varepsilon_0(p-1) + \frac{\sigma}{q} - (1-k)(p-1) > 2.$$

Then the second integral in (22) can be estimated by

$$C_4|\zeta - c|^{-\varepsilon_0(p-1)+p(1-k)} \leq C_4|\zeta - c|^{-\varepsilon_0(p-1)} d_0^{p(1-k)}. \quad (23)$$

Finally, since $\sigma_0 - \varepsilon_0 = \sigma$, we get the desired estimate

$$\iint\limits_{\Omega_z} |J_k\varphi|^p dxdy \leq C_7 \iint\limits_{\Omega_\zeta} |\varphi(\zeta)|^p \cdot |\zeta - c|^{\sigma_0(p-1)} d\xi d\eta. \quad (24)$$

If $\nu + k + \varepsilon_0(p-1) + \frac{\sigma}{q} - (1-k)(p-1) = 2$, then the bound (23) is to be replaced by $C_3 + 4\pi \Big| \ln |z_1 - z_2| \Big|$, and we obtain

$$\iint\limits_{\Omega_z} |J_k\varphi|^p dxdy \leq$$

$$\leq C_6 \iint\limits_{\Omega_\zeta} |\varphi(\zeta)|^p \cdot |\zeta - c|^{\sigma(p-1)} \cdot |\zeta - c|^{\varepsilon_0(p-1)} \cdot \left(C_3 + 4\pi \Big| \ln |z_1 - z_2| \Big| \right) d\xi d\eta.$$

Since $|\zeta - c|^{\varepsilon_0(p-1)} \cdot (C_3 + 4\pi \, |\ln |z_1 - z_2|| \,)$ is bounded, it follows an estimate of the type (24) in this case too. Similar arguments lead to analogous estimates in the remaining cases. This completes the proof of Lemma 6. □

The integral operators in (13) and (14) are modifications of the T_Ω- and the Π_Ω-operators. They will be denoted by T_Ω^* and Π_Ω^*, respectively.

Analogously to Lemma 5, the following statement is true:

Lemma 7. *The T_Ω^*-operator is a bounded operator mapping $L_p(\Omega, |z-c|^\gamma)$ into $\mathcal{H}^\lambda(\Omega, \varrho^\beta)$.*

By analogy with Lemma 6, one has

Lemma 8. *The T_Ω^*- and the Π_Ω^*-operators map $L_p(\Omega, |z-c|^\gamma)$ into itself and are bounded ($\gamma = \sigma(p-1)$ and $0 < \sigma < 1$).*

Remark. Sobolev's imbedding theorem led to the choice $\lambda = \frac{p-2}{p}$ for the Hölder exponent. Lemma 5 gives the same value for λ if γ satisfies the inequality $\frac{\gamma}{p} + \alpha - 1 > \frac{p-2}{p}$, i.e., γ has to satisfy the condition $\gamma > p(2-\alpha) - 2$.

In order to obtain a small γ we choose α near to 1, i.e., we take $\alpha = 1 - \varepsilon$, where ε is small, $0 < \varepsilon < 1$. This gives $\gamma > (p-2) + p\varepsilon$. This condition is satisfied if we choose

$$\gamma = (p-2) + (p+1)\varepsilon. \tag{25}$$

Note that the choice (25) for γ implies that (11) is also satisfied. Moreover, this choice of γ allows to apply Lemma 6.

7. Proof of an Embedding Theorem for Weighted Sobolev Spaces Using Complex Analysis

The weighted Sobolev space $\mathcal{W}_p^1(\Omega, \varrho^\gamma)$ is the set of all functions which together with their first order derivatives belong to the weighted Lebesgue space $L_p(\Omega, \varrho)$. As a side result of the above considerations, we show that $\mathcal{W}_p^1(\Omega, \varrho)$ is imbedded in the weighted Hölder space $\mathcal{H}^\lambda(\Omega, \varrho^\beta)$ $(p > 2,\ \lambda = (p-2)/p,\ \beta = \gamma/p)$. In particular, $\mathcal{W}_p^1(\Omega)$ is imbedded in $\mathcal{H}^\lambda(\Omega)$ $(\gamma = \beta = 0)$. The proof which will be given below is based on the boundedness of the T_Ω- and Π_Ω-operators, and the imbedding constants can be estimated by the norms of the latter operators.

Suppose $w = w(z)$ belongs to $\mathcal{W}_p^1(\Omega, \varrho^\gamma)$. Define

$$\Phi = w - T_\Omega \left[\frac{\partial w}{\partial \overline{z}} \right]. \tag{26}$$

Since $\frac{\partial \Phi}{\partial \overline{z}} = 0$, the complex version of the Weyl Lemma shows that Φ is holomorphic. Rewriting (26) in the form

$$w = \Phi + T_\Omega \left[\frac{\partial w}{\partial \overline{z}} \right], \tag{27}$$

and taking into account that T_Ω maps $\mathcal{W}_p^1(\Omega, \varrho^\gamma)$ into $\mathcal{H}^\lambda(\Omega, \varrho^\beta)$, it follows that $w = w(z)$ is Hölder continuous. Although the values $w(z)$ are defined only almost everywhere, the representation (27) implies, in particular, that $w = w(z)$ has a continuous representative.

In order to estimate the imbedding constant, we need the following lemma:

Lemma 9. *If Φ is holomorphic in the unit disk Ω and $\Phi' \in L_p(\Omega, \varrho^\gamma)$, then*

$$\overline{\Phi} = T_\Omega \overline{\Phi'} + c, \tag{28}$$

where c is a suitably chosen constant.

Proof. Denote $T_\Omega \overline{\Phi'}$ by h and $h - \overline{\Phi}$ by h_0. Then one has

$$\frac{\partial h_0}{\partial \overline{z}} = \overline{\Phi'} - \overline{\Phi'} = 0,$$

i.e., h_0, too, turns out to be holomorphic. Expanding Φ and h_0 into power series, we have

$$\Phi(z) = \sum_{k=0}^{\infty} a_k z^k, \quad h_0(z) = \sum_{k=0}^{\infty} b_k z^k, \quad h(z) = \sum_{k=0}^{\infty} \overline{a}_k \overline{z}^k + \sum_{k=0}^{\infty} b_k z^k. \tag{29}$$

Let C_r be the circumference with radius r represented by

$$z = r\exp(i\vartheta), \quad 0 \leq \vartheta \leq 2\pi.$$

Multiplying (29) by $z^{-s}, s = 2, 3, \ldots$, and integrating over C_r, one gets

$$\int_{C_r} z^{-s}h(z)dz = \sum_{k=0}^{\infty} \overline{a}_k \int_{C_r} \overline{z}^k z^{-s}dz + \sum_{k=0}^{\infty} b_k \int_{C_r} z^{k-s}dz. \tag{30}$$

Since $dz = ri\exp(i\vartheta)d\vartheta$ on C_r and since

$$\int_0^{2\pi i} \exp(im\vartheta)d\vartheta = \begin{cases} 2\pi & \text{if} \quad m = 0 \\ 0 & \text{if} \quad m \neq 0 \end{cases},$$

we obtain

$$\int_{C_r} z^{-s}h(z)dz = 2\pi i \cdot b_{s-1} \quad \text{for} \quad s = 2, 3, \ldots .$$

On the other hand, one has

$$\int_{C_r} z^{-s}h(z)dz \to 0 \quad \text{as} \quad r \to 1$$

because h is holomorphic outside the unit disk and vanishes at ∞. This shows $b_{s-1} = 0$ for $s = 2, 3, \ldots$ and Lemma 9 has been proved. □

Let Φ be holomorphic in the unit disk and $\Phi' \in L_p(\Omega, \varrho^\gamma)$. Then the representation (28) is true with

$$c = \overline{\Phi(0)} - \left(T_\Omega \overline{\Phi'}\right)[0]. \tag{31}$$

Since T_Ω is a bounded operator mapping $\mathcal{W}_p^1(\Omega, \varrho^\gamma)$ into $\mathcal{H}^\lambda(\Omega, \varrho^\beta)$. Lemma 9 yields the following statement:

Proposition 1. *If Φ' belongs to $\mathcal{W}_p^1(\Omega, \varrho^\gamma)$, then Φ belongs to $\mathcal{H}^\lambda(\Omega, \varrho^\beta)$ and the estimate*

$$\|\Phi - \overline{c}\|_{\mathcal{H}^\lambda(\Omega,\varrho^\beta)} \leq K \|\Phi'\|_{\mathcal{W}_p^1(\Omega,\varrho^\gamma)}$$

is true, where $c = \overline{\Phi(0)} - \left(T_\Omega \overline{\Phi'}\right)[0]$ and K is the norm of the T_Ω-operator as operator mapping $\mathcal{W}_p^1(\Omega, \varrho^\gamma)$ into $\mathcal{H}^\lambda(\Omega, \varrho^\beta)$.

Next suppose that $w = w(z)$ is a given element belonging to $\mathcal{W}_p^1(\Omega, \varrho^\gamma)$. Define

$$\Phi = w - T_\Omega \left[\frac{\partial w}{\partial \overline{z}}\right]. \tag{32}$$

Then Φ is holomorphic, and one has

$$\Phi' = \frac{\partial w}{\partial z} - \Pi_\Omega \left[\frac{\partial w}{\partial \overline{z}}\right]. \tag{33}$$

The formula (33) shows that Φ' belongs to $L_p^1(\Omega, \varrho^\gamma)$ because the Π-operator maps this space into itself. Moreover, (33) leads to an estimate of the

$L_p^1(\Omega, \varrho^\gamma)$-norm of Φ' by the $\mathcal{W}_p^1(\Omega, \varrho^\gamma)$-norm of the given element $w = w(z)$. Rewriting (32) in the form

$$w = (\Phi - \overline{c}) + \overline{c} + T_\Omega \left[\frac{\partial w}{\partial \overline{z}}\right],$$

where c is again given by (31), and applying Proposition 1 we get the following statement:

Proposition 2. *The weighted Sobolev space $\mathcal{W}_p^1(\Omega, \varrho^\gamma)$ is imbedded in the weighted Hölder space $\mathcal{H}^\lambda(\Omega)$, where $\lambda = (p-2)/p$ and $\beta = \gamma/p$. The imbedding constant can be estimated by the norms of the T_Ω- and T_Ω-operators.*

The above method for solving the Dirichlet boundary value problem for holomorphic functions leads also to imbedding theorems for fractional order spaces and, in addition, the corresponding imbedding constant can be estimated by the norm of the T_Ω-operator.

Suppose g belongs to $\mathcal{W}_p^s(\Gamma)$. Then in view of Lemma 3 the solution u of the Dirichlet boundary value problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in} \quad \Omega, \\ u &= g \quad \text{on} \quad \Gamma \end{aligned}$$

belongs to $\mathcal{W}_p^1(\Omega)$ and its $\mathcal{W}_p^1$-norm can be estimated by $\|g\|_{\mathcal{W}_p^s(\Gamma)}$. This leads, in particular, to an estimate of the L_p-norm of the first order derivatives. Introduce

$$\Phi' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

Then one has also an estimate of the L_p-norm of Φ' by the $\mathcal{W}_p^s(\Gamma)$-norm of the boundary data g. Applying the above Proposition 1, one gets an analogous estimate of the $\mathcal{H}^\lambda$-norm of Φ. Since g are the boundary values of the real part of Φ, we see that g is Hölder continuous with exponent λ and, further, that the $\mathcal{H}^\lambda$-norm of the boundary values g can be estimated by $\|g\|_{\mathcal{W}_p^s(\Gamma)}$.

8. Main Result

In order to solve the Dirichlet boundary value problem (4), (5) for the partial complex differential equation (1), we construct fixed points (w, h) of the operator (2), (3). This will be done using contraction-mapping principle in

$$\Big(\mathcal{H}^\lambda(\Omega, \varrho^\beta), L_p(\Omega, \varrho^\gamma)\Big). \tag{34}$$

The norm in this space is defined by

$$\|(w, h)\|_* = \max\left(\|w\|_{\mathcal{H}^\lambda(\Omega,\varrho^\beta)}, \|h\|_{L_p(\Omega,\varrho^\gamma)}\right).$$

Let (W_j, H_j), $j = 1, 2$, be the image of (w_j, h_j) defined by the operator (2), (3). Then we have

$$\begin{aligned} W_1 - W_2 &= \Big(\Phi_{(w_1,h_1)} - \Phi_{(w_2,h_2)}\Big) + T_\Omega\Big(F(\cdot, w_1, h_1) - F(\cdot, w_2, h_2)\Big), \\ H_1 - H_2 &= \Big(\Phi'_{(w_1,h_1)} - \Phi'_{(w_2,h_2)}\Big) + \Pi_\Omega\Big(F(\cdot, w_1, h_1) - F(\cdot, w_2, h_2)\Big). \end{aligned}$$

Notice that both differences $W_1 - W_2$ and $H_1 - H_2$ do not depend on the holomorphic solution Ψ and its derivative Ψ' resp. According to the above definition of the norm in the space (34), we see

$$\|(W_1, H_1) - (W_2, H_2)\|_* = \max\Big(\|W_1 - W_2\|_{\mathcal{H}^\lambda(\Omega,\varrho^\beta)}, \|H_1 - H_2\|_{L_p(\Omega,\varrho^\gamma)}\Big).$$

Choosing λ, γ and β according to the relations (8), (25) and (12), and taking into consideration the Lipschitz condition (6), the lemmas of Section 6 lead to an estimate of the type

$$\|(W_1, H_1) - (W_2, H_2)\|_* \leq (K_1 L_1 + K_2 L_2)\, \|(w_1, h_1) - (w_2, h_2)\|_*,$$

where the constants K_1 and K_2 can be expressed by the norms of the operators T_Ω, Π_Ω and their modifications whose properties are described by Lemmas 7 and 8.

Summarizing these arguments, the following theorem has been proved:

Theorem 1. *Suppose the right-hand side of the differential equation is Lipschitz continuous with sufficiently small Lipschitz constants. Choose an arbitrary ε, $0 < \varepsilon < 1$. Suppose the given boundary values can piecewise be decomposed into a part belonging to a fractional order space $\mathcal{W}_p^s$, $s = 1 - \dfrac{1}{p}$, and a Hölder continuous part with Hölder exponent μ, where $\mu > 1 - \frac{1}{p}$. Then the boundary value problem under consideration can be solved in $\big(\mathcal{H}^\lambda(\Omega, \varrho^\beta),\ L_p(\Omega, \varrho^\gamma)\big)$, where*

$$\gamma = (p-2) + (p+1)\varepsilon, \quad \beta = \frac{p-2}{p} + \frac{p+1}{p} \cdot \varepsilon \quad \textit{and} \quad \varrho(z) = \prod_j |z - c_j|.$$

Notice that both β and γ are arbitrarily small if ε is sufficiently small and, at the same time, p is close enough to 2.

In the special case of Hölder continuous boundary data g, the decomposition $g = g_1 + g_2 + g_3$ existing in view of Lemma 1 is true with $g_1 \equiv 0$ and $g_3 \equiv 0$. Under this assumption the Dirichlet boundary value problem for the differential equation (1) has been solved already in the papers [4, 5]. In accordance with the above theorem, in this case $(w, \partial w/\partial z)$ belongs to $\Big(\mathcal{H}^\mu(\Omega),\ L_p(\Omega)\Big)$ where μ is the Hölder exponent of the given boundary data and $p < \dfrac{1}{1-\mu}$.

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