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ON THE SOLVABILITY AND WELL-POSEDNESS OF INITIAL PROBLEMS FOR NONLINEAR HYPERBOLIC EQUATIONS OF HIGHER ORDER

Abstract. For nonlinear hyperbolic equations of higher order with two independent variables sufficient conditions for the solvability and well-posedness of the initial problems are found.

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1. Statement of the Problem and Formulation of the Main Results

Let $0 < a, b < +\infty$ and $\gamma_1 : [0, b] \to \mathbb{R}$, $\gamma_2 : [0, a] \to \mathbb{R}$ be continuous functions such that

$$0 \le \gamma_1(y) < a \text{ for } 0 < y < b, \quad 0 \le \gamma_2(x) < b \text{ for } 0 < x < a. \tag{M_0}$$

Moreover, suppose that either

$$\gamma_1 \text{ and } \gamma_2 \text{ are non-decreasing}, \quad \begin{array}{l} \gamma_1(\gamma_2(x)) < x \text{ for } 0 < x < a, \\ \gamma_2(\gamma_1(y)) < y \text{ for } 0 < y < b, \end{array} \tag{M^*}$$

or

$$\gamma_1 \text{ and } \gamma_2 \text{ are non-increasing}, \quad \begin{array}{l} \gamma_1(\gamma_2(x)) \le x \text{ for } 0 \le x \le a, \\ \gamma_2(\gamma_1(y)) \le y \text{ for } 0 \le y \le b. \end{array} \tag{M_*}$$

Then the set $G = \{(x, y) : \ \gamma_1(y) < x < a, \ \gamma_2(x) < y < b\}$ is non-empty, and the curves $\Gamma_1 = \{(\gamma_1(y), y) : \ 0 \le y \le b\}$, $\Gamma_2 = \{(x, \gamma_2(x)) : \ 0 \le x \le a\}$ are parts of its boundary. In the present paper in the domain G we consider the nonlinear hyperbolic equations

$$u^{(m,n)} = \\ = f\big(x, y, u^{(0,0)}, ..., u^{(0,n-1)}, ..., u^{(m,0)}, ..., u^{(m,n-1)}, u^{(0,n)}, ..., u^{(m-1,n)}\big) \tag{1.1}$$

and

$$u^{(m,n)} = f_0\big(x, y, u^{(0,0)}, \dots, u^{(m-1,0)}, u^{(m,0)}, \dots, u^{(m,n-1)}\big) \tag{1.1$'$}$$

with the initial conditions on Γ_1 and Γ_2

$$\begin{array}{l} \lim\limits_{x \to \gamma_1(y)} u^{(i,0)}(x, y) = c_{1i}(y) \text{ for } 0 < y < b \ \ (i = 0, \dots, m-1), \\ \lim\limits_{y \to \gamma_2(x)} u^{(m,k)}(x, y) = c_{2k}(x) \text{ for } 0 < x < a \ \ (k = 0, \dots, n-1), \end{array} \tag{1.2}$$

where m and n are natural numbers and

$$u^{(0,0)}(x, y) = u(x, y), \quad u^{(i,0)}(x, y) = \frac{\partial^i u(x, y)}{\partial x^i}\,,$$
$$u^{(0,k)}(x, y) = \frac{\partial^k u(x, y)}{\partial y^k}\,, \quad u^{(i,k)}(x, y) = \frac{\partial^k}{\partial y^k}\Big(\frac{\partial^i u(x, y)}{\partial x^i}\Big).$$

In what follows, under $\overline{G}$ will be meant a closure of the set G, and the functions $f : \overline{G} \times \mathbb{R}^{mn+m+n} \to \mathbb{R}$, $f_0 : \overline{G} \times \mathbb{R}^{m+n} \to \mathbb{R}$, $c_{1i} : [0, b] \to \mathbb{R}$ $(i = 0, \dots, m-1)$, $c_{2k} : [0, a] \to \mathbb{R}$ $(k = 0, \dots, n-1)$, $\gamma_1 : [0, b] \to \mathbb{R}$ and $\gamma_2 : [0, a] \to \mathbb{R}$ will be assumed to be continuous.

Along with (1.1) and (1.1$'$), we consider the perturbed differential equations

$$v^{(m,n)} = f\big(x, y, v^{(0,0)}, ..., v^{(0,n-1)}, ..., v^{(m,0)}, ..., v^{(m,n-1)}, v^{(0,n)}, ..., v^{(m-1,n)}\big) + \\ + h(x, y), \tag{1.3}$$

$$v^{(m,n)} = f_0\big(x, y, v^{(0,0)}, \dots, v^{(m-1,0)}, v^{(m,0)}, \dots, v^{(m,n-1)}\big) + h(x, y) \tag{1.3$'$}$$

with the perturbed initial conditions

$$\begin{aligned}
&\lim_{x\to\gamma_1(y)} v^{(i,0)}(x,y)=c_{1i}(y)+e_{1i}(y) \ \text{ for } \ 0<y<b \ \ (i=0,\dots,m-1),\\
&\lim_{y\to\gamma_2(x)} v^{(m,k)}(x,y)=c_{2k}(x)+e_{2k}(x) \ \text{ for } \ 0<x<a \ \ (k=0,\dots,m-1).
\end{aligned} \tag{1.4}$$

For arbitrary continuous functions $h:\overline{G}\to\mathbb{R}$ and $e_{ek}:[0,a]\to\mathbb{R}$ $(k=0,\dots,n-1)$ and n-times continuously differentiable functions $e_{1i}:[0,b]\to\mathbb{R}$ $(i=0,\dots,m-1)$ we assume

$$\eta(e_{10},\dots,e_{1\,m-1};e_{20},\dots,e_{2\,n-1};h)=\max\Big\{\sum_{i=0}^{m-1}\sum_{j=0}^{n}|e_{1i}^{(j-1)}(y)|:0\le y\le b\Big\}+$$
$$+\max\Big\{\sum_{k=0}^{n-1}|e_{2k}(x)|:\ 0\le x\le a\Big\}+\max\big\{|h(x,y)|:\ (x,y)\in\overline{G}\big\}. \tag{1.5}$$

If, however, the functions e_{1i} $(i=0,\dots,m-1)$ are only continuous, then

$$\eta_0(e_{10},\dots,e_{1\,m-1};e_{20},\dots,e_{2\,n-1};h)=\max\Big\{\sum_{i=0}^{m-1}|e_{1i}(y)|:\ 0\le y\le b\Big\}+$$
$$+\max\Big\{\sum_{k=0}^{n-1}|e_{2k}(x)|:\ 0\le x\le a\Big\}+\max\big\{|h(x,y)|:\ (x,y)\in\overline{G}\big\}. \tag{1.6}$$

Definition 1.1. The function $u:G\to\mathbb{R}$ is said to be **a solution** (**a generalized solution**) **of the equation (1.1)** (**of the equation (1.1′)**), if it is uniformly continuous on G along with $u^{(i,k)}$ $(i=0,\dots,m;\ k=0,\dots,n)$ (along with $u^{(i,0)}$ $(i=0,\dots,m)$ and $u^{(m,k)}$ $(k=0,\dots,n)$) and at every point G satisfies the equation (1.1) (the equation (1.1′)). A solution of the equation (1.1) (a generalized solution of the equation (1.1′)), satisfying the initial conditions (1.2), is called **a solution of the problem (1.1), (1.2)** (**a generalized solution of the problem (1.1′), (1.2)**).

Definition 1.2. The problem (1.1), (1.2) is said to be **well-posed** if there exist positive constants r and ε such that for arbitrary continuous functions $h:\overline{G}\to\mathbb{R}$, $e_{2k}:[0,a]\to\mathbb{R}$ $(k=0,\dots,n-1)$ and n-times continuously differentiable functions $e_{1i}:[0,a]\to\mathbb{R}$ $(i=0,\dots,m-1)$ satisfying the condition

$$\eta(e_{10},\dots,e_{1\,m-1},e_{20},\dots,e_{2\,n-1},h)\le\varepsilon \tag{1.7}$$

the problem (1.3), (1.4) is uniquely solvable, and in the domain G the inequality

$$\sum_{i=0}^{m}\sum_{k=0}^{n-1}\big|u^{(i,k)}(x,y)-v^{(i,k)}(x,y)\big|+\sum_{i=0}^{m-1}\big|u^{(i,n)}(x,y)-v^{(i,n)}(x,y)\big|\le$$
$$\le r\eta(e_{10},\dots,e_{1\,m-1};e_{20},\dots,e_{2\,n-1};h) \tag{1.8}$$

is fulfilled, where u and v are, respectively, solutions of the problems (1.1), (1.2) and (1.3), (1.4).

Definition 1.3. The problem $(1.1'), (1.2)$ is said to be **well-posed in a generalized sense** if there exist positive constants r and ε such that for arbitrary continuous functions $h : \overline{G} \to \mathbb{R}$, $e_{1i} : [0,a] \to \mathbb{R}$ $(i = 0, \dots, m-1)$ and $e_{2k} : [0,a] \to \mathbb{R}$ $(k = 0, \dots, n-1)$ satisfying the condition

$$\eta_0(e_{10}, \dots, e_{1\,m-1}; e_{20}, \dots, e_{2\,n-1}; h) \le \varepsilon \tag{1.9}$$

the problem $(1.3'), (1.4)$ has a unique generalized solution, and in the domain G the inequality

$$\sum_{i=0}^{m-1} \left|u^{(i,0)}(x,y) - v^{(i,0)}(x,y)\right| + \sum_{k=0}^{n-1} \left|u^{(m,k)}(x,y) - v^{(m,k)}(x,y)\right| \le \\ \le r\eta_0(e_{10}, \dots, e_{1\,m-1}; e_{20}, \dots, e_{2\,n-1}; h) \tag{1.10}$$

is fulfilled, where u and v are, respectively, generalized solutions of the problems $(1.1'), (1.2)$ and $(1.1'), (1.3)$.

For $m = n = 1$, the problem $(1.1), (1.2)$ and its different particular cases have been investigated in [1]–[16]. For $m+n > 2$, this problem remains still studied insufficiently. In this paper, the conditions are found which ensure, respectively, the solvability and well-posedness of the problem $(1.1), (1.2)$ (the solvability and well-posedness in a generalized sense of the problem $(1.1'), (1{,}2)$).

Along with (1.5) and (1.6), below the use will be made of the following notation:

$$\mu = \sum_{i=0}^{m-1} \frac{a^i}{i!}\,, \quad \nu = \sum_{k=0}^{n-1} \frac{b^k}{k!}\,, \tag{1.11}$$

$$u_0(x,y) = \sum_{i=0}^{m-1} \frac{(x-\gamma_1(y))^i}{i!}\, c_{1i}(y) + \\ + \sum_{k=0}^{n-1} \int_{\gamma_1(y)}^{x} \frac{(x-s)^{m-1}(y-\gamma_2(s))^k}{(m-1)!k!}\, c_{2k}(s)\, ds, \tag{1.12}$$

$$\widetilde{u}_0(x,y) = \sum_{i=0}^{m-1} \frac{(x-\delta_1(y))^i}{i!}\, |c_{1i}(y)| + \\ + \sum_{k=0}^{n-1} \int_{\gamma_1(y)}^{x} \frac{(x-s)^{m-1}(y-\gamma_2(s))^k}{(m-1)!k!}\, |c_{2k}(s)|\, ds. \tag{1.13}$$

We investigate the problem $(1.1), (1.2)$ in the case where γ_1 and γ_2 satisfy one of the two conditions:

$$\gamma_1(y) \equiv 0, \quad \gamma_2 \text{ is non-decreasing}, \quad \gamma_2(0) = 0, \quad \gamma_2(x) < b \ \text{ for } \ 0 < x < a \tag{M_1}$$

and

$$\gamma_1 \text{ is continuously differentiable and decreasing}, \ \gamma_1(0) = a, \ \gamma_1(b) = 0, \\ \gamma_2(\gamma_1(y)) = y \text{ for } 0 \le y \le b. \tag{M_2}$$

Theorem 1.1. *Let γ_1 and c_{1i} $(i = 0, \dots, m-1)$ be n-times continuously differentiable functions and there exist a positive number δ and a continuous, non-decreasing function $\varphi : [0, +\infty[\to [0, +\infty[$ such that $\varphi(\tau) > 0$ for $\tau > 0$,*

$$\int_{\delta}^{+\infty} \frac{ds}{\varphi(s)} > (b\mu\nu + \mu + \nu)(a + b) \tag{1.14}$$

and, respectively, in the domains G and $G \times \mathbb{R}^{mn+m+n}$ the inequalities

$$\sum_{i=0}^{m} \sum_{k=0}^{n-1} \big|u_0^{(i,k)}(x,y)\big| + \sum_{i=0}^{m-1} \big|u_0^{(i,n)}(x,y)\big| \leq \delta \tag{1.15}$$

and

$$\begin{gathered} \big|f(x, y, z_{00}, \dots, z_{0\,n-1}, \dots, z_{m0}, \dots, z_{m\,n-1}, z_{0n}, \dots, z_{m-1\,n})\big| \leq \\ \leq \varphi\Big(\sum_{i=0}^{m} \sum_{k=0}^{n-1} |z_{ik}| + \sum_{i=0}^{m-1} |z_{in}|\Big) \end{gathered} \tag{1.16}$$

are fulfilled. Moreover, let the functions γ_1 and γ_2 satisfy either the condition (M_1) or the condition (M_2), and the function f satisfy the local Lipschitz condition with respect to the last $m + n$ variables (with respect to the last $mn + m + n$ variables). Then the problem (1.1), (1.2) *has at least one solution (is well-posed).*

Corollary 1.1. *Let there exist a positive number ℓ_0 such that in the domain $G \times \mathbb{R}^{mn+m+n}$ the inequality*

$$\begin{gathered} \big|f(x, y, z_{00}, \dots, z_{0\,n-1}, \dots, z_{m0}, \dots, z_{m\,n-1}, z_{0n}, \dots, z_{m-1\,n})\big| \leq \\ \leq \ell_0\Big(1 + \sum_{i=0}^{m} \sum_{k=0}^{n-1} |z_{ik}| + \sum_{i=0}^{m-1} |z_{in}|\Big) \end{gathered} \tag{1.17}$$

is fulfilled. Moreover, let the functions γ_1 and γ_2 satisfy either the condition (M_1), or the condition (M_2), and the function f satisfy the local Lipschitz condition with respect to the last $m + n$ variables (with respect to the last $mn+m+n$ variables). Then the problem (1.1), (1.2) *has at least one solution (is well-posed).*

Theorem 1.2. *Let there exist a positive number δ and a continuous, non-decreasing function $\varphi : [0, +\infty[\to [0, +\infty[$ such that $\varphi(\tau) > 0$ for $\tau > 0$,*

$$\int_{\delta}^{+\infty} \frac{ds}{\varphi(s)} > \Big(\frac{\mu b^n}{(n-1)!} + \nu\Big)(a + b) \tag{1.18}$$

and, respectively, in the domains G and $G \times \mathbb{R}^{m+n}$ the inequalities

$$\sum_{i=0}^{m-1} \widetilde{u}_0^{(i,0)}(x,y) + \sum_{k=0}^{n-1} \widetilde{u}_0^{(m,k)}(x,y) \leq \delta \tag{1.19}$$

and

$$f_0(x,y,z_1,\dots,z_{m+n})\operatorname{sgn} z_{m+n} \le \varphi\Big(\sum_{i=1}^{m+n}|z_i|\Big) \tag{1.20}$$

are fulfilled. Let, moreover, the functions γ_1 and γ_2 satisfy along with (M_0) one of the conditions (M_*) and (M^*), and the function f satisfy the local Lipschitz condition with respect to the last n variables (with respect to the last $m+n$ variables). Then the problem $(1.1'), (1.2)$ has at least one generalized solution (is well-posed in a generalized sense).

Corollary 1.2. *Let there exist a positive number ℓ_0 such that in the domain $G\times\mathbb{R}^{m+n}$ the inequality*

$$f_0(x,y,z_1,\dots,z_{m+n})\operatorname{sgn} z_{m+n} \le \ell_0\Big(1+\sum_{i=1}^{m+n}|z_i|\Big) \tag{1.21}$$

is fulfilled. Let, moreover, the functions γ_1 and γ_2 satisfy along with (M_0) one of the conditions (M_*) and (M^*), and the function f satisfy the local Lipschitz condition with respect to the last n variables (with respect to the last $m+n$ variables). Then the problem $(1.1'), (1.2)$ has at least one generalized solution (is well-posed in a generalized sense).

2. Lemmas on A Priori Estimates

In this section, in the domain G we consider the differential inequalities

$$\big|u^{(m,n)}(x,y)\big| \le \varphi\Big(\sum_{i=0}^{m}\sum_{k=0}^{n-1}\big|u^{(i,k)}(x,y)\big| + \sum_{i=0}^{m-1}\big|u^{(i,n)}(x,y)\big|\Big) \tag{2.1}$$

and

$$u^{(m,n)}(x,y)\operatorname{sgn} u^{(m,n-1)}(x,y) \le$$
$$\le \varphi\Big(\sum_{i=0}^{m-1}\big|u^{(i,0)}(x,y)\big| + \sum_{k=0}^{n-1}\big|u^{(m,k)}(x,y)\big|\Big) \tag{2.1$'$}$$

with the boundary conditions (1.2), where $\varphi : [0,+\infty[\,\to[0,+\infty[$ is a continuous non-decreasing function such that $\varphi(\tau)>0$ for $\tau>0$.

The function $u : G\to\mathbb{R}$ is said to be **a solution** (**a generalized solution**) **of the differential inequality (2.1)** (**of the differential inequality (2.1$'$)**) if it is uniformly continuous on G along with $u^{(i,k)}$ $(i=0,\dots,m;$ $k=0,\dots,n)$ (along with $u^{(i,0)}$ $(i=0,\dots,m-1)$ and $u^{(m,k)}$ $(k=0,\dots,n)$) and at every point of G satisfies the differential inequality (2.1) (the differential inequality (2.1$'$)). A solution of the differential inequality (2.1) (a generalized solution of the differential inequality (2.1$'$)) satisfying the initial conditions (1.2) is called a solution of the problem $(2.1), (1.2)$ (a generalized solution of the problem $(2.1'), (1.2)$).

If

$$\text{either } \delta>0, \text{ or } \delta=0 \text{ and } \int_0^1\frac{ds}{\varphi(s)}<+\infty, \tag{2.2}$$

we set

$$\psi_\delta(\tau) = \int_\delta^\tau \frac{ds}{\varphi(s)}$$

and denote by ψ_δ^{-1} the function, inverse to ψ_δ.

Lemma 2.1. *Let γ_1 and c_{1i} $(i = 0, \dots, m-1)$ be n-times continuously differentiable and the pair γ_1, γ_2 satisfy one of the conditions (M_1) and (M_2). If, moreover, the conditions (1.14), (1.15) and (2.2) are fulfilled, then an arbitrary solution u of the problem (2.1), (1.2) in the domain G admits the estimate*

$$\sum_{i=0}^{m}\sum_{k=0}^{n-1} \big|u^{(i,k)}(x,y)\big| + \sum_{i=0}^{m-1} \big|u^{(i,n)}(x,y)\big| \le \psi_\delta^{-1}\big((b\mu\nu + \mu + \nu)(x+y)\big). \quad (2.3)$$

Proof. By virtue of (1.2) and (1.12) we have

$$u^{(m,k)}(x,y) = u_0^{(m,k)}(x,y) + \frac{1}{(n-1-k)!} \int_{\gamma_2(x)}^{y} (y-t)^{n-1-k} u^{(m,n)}(x,t)\,dt \ \ (k = 0, \dots, n-1), \quad (2.4)$$

and

$$u^{(i,0)}(x,y) = u_0^{(i,0)}(x,y) + \frac{1}{(m-1-i)!(n-1)!}\int_{\gamma_1(y)}^{x} (x-s)^{m-1-i}\,ds \int_{\gamma_2(s)}^{y} (y-t)^{n-1} u^{(m,n)}(s,t)\,dt \quad (2.5)$$

$$(i = 0, \dots, m-1).$$

On the other hand, if we take into account that the pair γ_1, γ_2 satisfies one of the conditions (M_1) and (M_2), then from (2.5) we find

$$u^{(i,k)}(x,y) = u_0^{(i,k)}(x,y) + \frac{1}{(m-1-i)!(n-1-k)!}\int_{\gamma_1(y)}^{x} (x-s)^{m-1-i}\,ds \int_{\gamma_2(s)}^{y} (y-t)^{n-1-k} u^{(m,n)}(s,t)\,dt \quad (2.6)$$

$$(i = 0, \dots, m-1; \ \ k = 0, \dots, n-1),$$

and

$$u^{(i,n)}(x,y) = u_0^{(i,n)}(x,y) + \frac{1}{(m-1-i)!} \int_{\gamma_1(y)}^{x} (x-s)^{m-1-i} u^{(m,n)}(s,y)\,ds \ \ (i = 0, \dots, m-1). \quad (2.7)$$

Suppose

$$\rho(x,y) = \sum_{i=0}^{m}\sum_{k=0}^{n-1} \big|u^{(i,k)}(x,y)\big| + \sum_{i=0}^{m-1} \big|u^{(i,n)}(x,y)\big|.$$

Then according to the inequalities (1.15), (2.1) and the notation (1.11), from the identities (2.4), (2.6) and (2.7) we obtain

$$\rho(x,y) \le \delta + \mu\nu \int_{\gamma_1(y)}^{x} ds \int_{\gamma_2(s)}^{y} \varphi(\rho(s,t))\,dt + \\ + \mu \int_{\gamma_1(y)}^{x} \varphi(\rho(s,y))\,ds + \nu \int_{\gamma_2(x)}^{y} \varphi(\rho(x,t))\,dt.$$

The above inequality by virtue of Lemma 2.1 of [9] and the conditions (1.14) and (2.2) results in $\rho(x,y) \le \psi_\delta^{-1}((b\mu\nu + \mu + \nu)(x+y))$. Consequently, the estimate (2.3) is valid. □

Lemma 2.2. *Let* $c_{1i}(y) \equiv 0$ $(i = 0,\dots,m-1)$, $c_{2k}(x) \equiv 0$ $(k = 0,\dots,n-1)$, γ_1 *be n-times continuously differentiable and the pair* γ_1, γ_2 *satisfy one of the conditions* (M_1) *and* (M_2). *If, moreover,*

$$\int_0^1 \frac{ds}{\varphi(s)} = +\infty, \tag{2.8}$$

then the problem (2.1), (1.2) *has only a trivial solution.*

Proof. Let u be an arbitrary solution of the problem (2.1), (1.2). Then by Lemma 2.1 for an arbitrarily small $\delta > 0$ in the domain G the inequality (2.3) is fulfilled. On the other hand, by virtue of (2.8) we have

$$\lim_{\delta\to 0} \psi_\delta^{-1}(\tau) = 0 \ \text{ for } \ \tau > 0.$$

Therefore if in the inequality (2.3) we pass to the limit as $\delta \to 0$, then we get $u(x,y) \equiv 0$. □

Lemma 2.3. *Let the pair of functions* γ_1, γ_2 *along with* (M_0) *satisfy one of the conditions* (M_*) *and* (M^*). *If, moreover, the conditions* (1.18), (1.19) *and* (2.2) *are fulfilled, then an arbitrary generalized solution* u *of the problem* (2.1′), (1.2) *in the domain* G *admits the estimate*

$$\sum_{i=0}^{m-1} \left|u^{(i,0)}(x,y)\right| + \sum_{k=0}^{n-1} \left|u^{(m,k)}(x,y)\right| \le \psi_\delta^{-1}\Big(\Big(\frac{\mu b^n}{(n-1)!} + \nu\Big)(x,y)\Big). \tag{2.9}$$

Proof. For an arbitrarily fixed $(x,y) \in G$, almost everywhere on the interval $]\gamma_2(x), y[$ we have

$$\frac{\partial}{\partial t}\left|u^{(m,n-1)}(x,t)\right| = \left|u^{(m,n)}(x,t)\right| \operatorname{sgn} u^{(m,n-1)}(x,t),$$

whence by virtue of the conditions (1.2), (2.1′) and the notation (1.13) it follows that

$$\left|u^{(m,n-1)}(x,y)\right| \le \widetilde{u}_0^{(m,n-1)}(x,y) + \int_{\gamma_2(x)}^{y} \varphi(\rho(x,t))\,dt,$$

$$\left|u^{(m,k)}(x,y)\right| \le \widetilde{u}_0^{(m,k)}(x,y) +$$

$$+\frac{1}{(n-1-k)!}\int_{\gamma_2(x)}^{y}(y-t)^{n-k-1}\varphi(\rho(x,t))\,dt \ \ (k=0,\dots,n-1)$$

and

$$\big|u^{(i,0)}(x,y)\big| \le \widetilde{u}_0^{(i,0)}(x,y)+\frac{1}{(m-1-i)!(n-1)!}\times$$
$$\times\int_{\gamma_1(y)}^{x}(x-s)^{m-1-i}\,ds\int_{\gamma_2(s)}^{y}(y-t)^{n-1}\varphi(\rho(s,t))\,dt \ \ (i=0,\dots,m-1),$$

where

$$\rho(x,y)=\sum_{i=0}^{m-1}\big|u^{(i,0)}(x,y)\big|+\sum_{k=0}^{n-1}\big|u^{(m,k)}(x,y)\big|.$$

If, along with the above, we take into account the inequality (1.19), it becomes clear that the function ρ in the domain G satisfies the integral inequality

$$\rho(x,y)\le\delta+\mu\,\frac{b^{n-1}}{(n-1)!}\int_{\gamma_1(y)}^{x}ds\int_{\gamma_2(s)}^{y}\varphi(\rho(s,t))\,dt+\nu\int_{\gamma_2(x)}^{y}\varphi(\rho(x,t))\,dt.$$

This inequality, according to Lemma 2.1 of [9] and the conditions (1.18) and (2.2), results in

$$\rho(x,y)\le\psi_\delta^{-1}\Big(\Big(\frac{\mu b^n}{(n-1)!}+\nu\Big)(x+y)\Big).$$

Consequently, the estimate (2.9) is valid. □

From the above-proven lemma it immediately follows

Lemma 2.4. *Let $c_{1i}(y)\equiv 0$ $(i=0,\dots,m-1)$, $c_{2k}(x)\equiv 0$ $(k=0,\dots,n-1)$ and the pair of functions γ_1, γ_2 along with (M_0) satisfy one of the conditions (M_*) and (M^*). If, moreover, the condition (2.8) is fulfilled, then the problem $(2.1')$, (1.2) has only a trivial generalized solution.*

3. Lemmas on the Existence and Uniqueness of Solutions of the Problems (1.1), (1.2) and (1.1′), (1.2)

Lemma 3.1. *Let the functions γ_1 and c_{1i} $(i=0,\dots,m-1)$ be n-times continuously differentiable and the pair γ_1, γ_2 satisfy one of the conditions (M_1) and (M_2). Let, moreover, the function f satisfy the local Lipschitz condition with respect to the last $m+n$ variables and there exist a positive constant ρ_0 such that in the domain $G\times\mathbb{R}^{mn+m+n}$ the inequality*

$$\big|f(x,y,z_{00},\dots,z_{0\,n-1},\dots,z_{m0},\dots,z_{m\,n-1},z_{0n},\dots,z_{m-1\,n})\big|\le\rho_0 \qquad (3.1)$$

is fulfilled. Then the problem (1.1), (1.2) has at least one solution.

Proof. We choose a positive constant δ in such a way that the inequality (1.15) is fulfilled and assume that

$$\rho=\delta+(\mu\nu ab+\mu a+\nu b)\rho_0,\quad \mathbb{R}_\rho^{mn+m+n}=$$

$$= \Big\{ (z_{00}, \dots, z_{0\,n-1}, \dots, z_{m0}, \dots, z_{m\,n-1}, z_{0n}, \dots, z_{m-1\,n}) \in \mathbb{R}^{mn+m+n} : \sum_{i=0}^{m} \sum_{k=0}^{n-1} |z_{ik}| + \sum_{i=0}^{m-1} |z_{in}| \le \rho \Big\}.$$

Then because of the local Lipschitz property of f with respect to the last $m+n$ variables there exists a positive constant ℓ such that on the set $\overline{G} \times \mathbb{R}_\rho^{mn+m+n}$ the condition

$$\Big| f(x, y, z_{00}, \dots, z_{0\,n-1}, \dots, z_{m0}, \dots, z_{m\,n-1}, z_{0n}, \dots, z_{m-1\,n}) - \\ - f(x, y, z_{00}, \dots, z_{0\,n-1}, \dots, \overline{z}_{m0}, \dots, \overline{z}_{m\,n-1}, \overline{z}_{0n}, \dots, \overline{z}_{m-1\,n}) \Big| \le \\ \le \ell \Big(\sum_{k=0}^{n-1} |z_{mk} - \overline{z}_{mk}| + \sum_{i=0}^{m-1} |z_{in} - \overline{z}_{in}| \Big) \tag{3.2}$$

is satisfied.

By ω_0 we denote the modulus of continuity of the function f on the set $\overline{G} \times \mathbb{R}_\rho^{mn+m+n}$, and by ω we denote the function given by the equality

$$\omega(\tau) = (a\mu + b\nu)\omega_0((1+\rho)\tau) + \\ + a\mu\rho_0 \max \Big\{ \big|\gamma_1(y) - \gamma_1(\overline{y})\big| : \ 0 \le y, \overline{y} \le b, \ |y - \overline{y}| \le \tau \Big\} + \\ + \max \Big\{ \sum_{i=0}^{m-1} \sum_{k=0}^{n-1} \big| u_0^{(i,k)}(x,y) - u_0^{(i,k)}(x,\overline{y}) \big| : \ 0 \le x \le a, \\ 0 \le y, \overline{y} \le b, \ |y - \overline{y}| \le \tau \Big\} + \\ + b\nu\rho_0 \max \Big\{ \big|\gamma_2(x) - \gamma_2(\overline{x})\big| : \ 0 \le x, \overline{x} \le a, \ |x - \overline{x}| \le \tau \Big\} + \\ + \max \Big\{ \sum_{i=0}^{m-1} \sum_{k=0}^{n-1} \big| u_0^{(i,k)}(x,y) - u_0^{(i,k)}(\overline{x},y) \big| : \ 0 \le x, \overline{x} \le a, \ |x - \overline{x}| \le \tau \Big\}. \tag{3.3}$$

By B we will mean the Banach space of functions $u : G \to \mathbb{R}$ uniformly continuous along with $u^{(i,k)}$ ($i = 0, \dots, m$; $k = 0, \dots, n$; $i + k \le m + n - 1$), in which the norm is introduced by the equality

$$\|u\| = \sup \Big\{ \sum_{i=0}^{m} \sum_{k=0}^{n-1} \big| u^{(i,k)}(x,y) \big| + \sum_{i=0}^{m-1} \big| u^{(i,n)}(x,y) \big| : \ (x,y) \in G \Big\}.$$

Let B_0 be the set of all $u \in B$ satisfying in G the conditions

$$\|u\| \le \rho,$$

$$\sum_{i=0}^{m-1} \big| u^{(i,n)}(x,y) - u^{(i,n)}(x,\overline{y}) \big| \le \omega(|y - \overline{y}|) \exp(\mu\ell x), \tag{3.4}$$

$$\sum_{k=0}^{n-1} \big| u^{(m,k)}(x,y) - u^{(m,k)}(\overline{x},y) \big| \le \omega(|x-\overline{x}|)\exp(\nu\ell y). \tag{3.5}$$

It is obvious that an arbitrary $u \in B_0$ satisfies the condition

$$\sum_{i=0}^{m-1}\sum_{k=0}^{n-1} \big| u^{(i,k)}(x,y) - u^{(i,k)}(\overline{x},\overline{y}) \big| \le \rho\big(|x-\overline{x}| + |y-\overline{y}|\big) \tag{3.6}$$

and B_0 is a convex compact set of the space B.

On the set B_0 we consider the operator

$$p(u)(x,y) = u_0(x,y) + \frac{1}{(m-1)!(n-1)!}\times$$
$$\times\int_{\gamma_1(y)}^{x}(x-s)^{m-1}\,ds\int_{\gamma_2(s)}^{y}(y-t)^{n-1} f\big(s,t,u^{(0,0)}(s,t),\dots,u^{(m-1,n)}(s,t)\big)\,dt. \tag{3.7}$$

If for an arbitrarily fixed $u \in B_0$ we put $v(x,y) = p(u)(x,y)$ and take into account the fact that the pair γ_1, γ_2 satisfies one of the conditions (M_1) and (M_2), then (3.7) yields

$$v^{(i,k)}(x,y) = u_0^{(i,k)}(x,y) + \frac{1}{(m-1-i)!(n-1-k)!}\int_{\gamma_1(y)}^{x}(x-s)^{m-1-i}\,ds\times$$
$$\times\int_{\gamma_2(s)}^{y}(y-t)^{n-1-k} f\big(s,t,u^{(0,0)}(s,t),\dots,u^{(m-1,n)}(s,t)\big)\,dt \tag{3.8}$$
$$(i=0,\dots,m-1;\quad k=0,\dots,n-1),$$

$$v^{(i,n)}(x,y) = u_0^{(i,n)}(x,y) +$$
$$+\frac{1}{(m-1-i)!}\int_{\gamma_1(y)}^{x}(x-s)^{m-1-i} f\big(s,y,u^{(0,0)}(s,y),\dots,u^{(m-1,n)}(s,y)\big)\,ds \tag{3.9}$$
$$(i=0,\dots,m-1),$$

$$v^{(m,k)}(x,y) = u_0^{(m,k)}(x,y) +$$
$$+\frac{1}{(n-1-k)!}\int_{\gamma_2(x)}^{y}(y-t)^{n-1-k} f\big(x,t,u^{(0,0)}(x,t),\dots,u^{(m-1,n)}(x,t)\big)\,dt \tag{3.10}$$
$$(k=0,\dots,n-1).$$

By the conditions (3.1)–(3.6) from equalities (3.8)–(3.10) it follows that

$$\sum_{i=0}^{m}\sum_{k=0}^{n-1}\big|v^{(i,k)}(x,y)\big| + \sum_{i=0}^{m-1}\big|v^{(i,n)}(x,y)\big| \le \delta + \mu\nu\rho_0 + \mu\rho_0 + \nu\rho_0 = \rho,$$

$$\sum_{i=0}^{m-1}\big|v^{(i,n)}(x,y) - v^{(i,n)}(x,\overline{y})\big| \le \sum_{i=0}^{m-1}\big|u_0^{(i,n)}(x,y) - u_0^{(i,n)}(x,\overline{y})\big| +$$
$$+a\mu\rho_0\big|\gamma_1(y)-\gamma_1(\overline{y})\big| + a\mu\omega_0\big((1+\rho)|y-\overline{y}|\big) +$$

$$+\mu\ell\omega(|y-\overline{y}|)\int_{\gamma_1(y)}^{x}\exp(\mu\ell s)\,ds\le$$
$$\le\omega(|y-\overline{y}|)+\mu\ell\omega(|y-\overline{y}|)\int_{0}^{x}\exp(\mu\ell s)\,ds=\omega(|y-\overline{y}|)\exp(\mu\ell x)$$

and

$$\sum_{k=0}^{n-1}\big|v^{(m,k)}(x,y)-v^{(m,k)}(\overline{x},y)\big|\le\sum_{k=0}^{n-1}\big|u_0^{(m,k)}(x,y)-u_0^{(m,k)}(\overline{x},y)\big|+$$
$$+b\nu\rho_0\big|\gamma_2(x)-\gamma_2(\overline{x})\big|+b\nu\omega_0\big((1+\rho)|x-\overline{x}|\big)+$$
$$+\nu\ell\omega(|x-\overline{x}|)\int_{\gamma_2(x)}^{y}\exp(\nu\ell t)\,dt\le$$
$$\le\omega(|x-\overline{x}|)+\nu\ell\omega(|x-\overline{x}|)\int_{0}^{y}\exp(\nu\ell t)\,dt=\omega(|x-\overline{x}|)\exp(\nu\ell y).$$

Consequently, $v\in B_0$. Thus we have proved that the operator p transforms the convex compact set B_0 into itself. On the other hand, because of the fact that f is continuous, from the equalities (3.7)–(3.10) it follows that p is the continuous operator. By Schauder's principle, there exists $u\in B_0$ such that $p(u)(x,y)=u(x,y)$ for $(x,y)\in G$. If we again take into account the equalities (3.7)–(3.10), then it will become clear that u is a solution of the problem (1.1), (1.2). □

The following lemma can be proved analogously to Lemma 3.1.

Lemma 3.2. *Let the pair of functions γ_1, γ_2 along with (M_0) satisfy one of the conditions (M_*) and (M^*). Moreover, let the function f_0 satisfy the local Lipschitz condition with respect to the last n variables, and let there exist a positive constant ρ_0 such that in the domain $G\times\mathbb{R}^{m+n}$ the inequality $|f_0(x,y,z_1,\dots,z_{m+n})|\le\rho_0$ is fulfilled. Then the problem $(1.1')$, (1.2) has at least one generalized solution.*

Lemma 3.3. *Let γ_1 be n-times continuously differentiable and the pair γ_1, γ_2 satisfy one of the conditions (M_1) and (M_2). If, moreover, the function f satisfies the local Lipschitz condition with respect to the last $mn+m+n$ variables, then the problem (1.1), (1.2) has at most one solution.*

Proof. Let u and $\overline{u}$ be arbitrary solutions of the problem (1.1), (1.2). Since f possesses the local Lipschitz property with respect to the last $mn+m+n$ variables, there exists a positive constant ℓ such that in the domain G the inequality

$$\Big|f\big(x,y,u^{(0,0)}(x,y),\dots,u^{(m-1,n)}(x,y)\big)-$$
$$-f\big(x,y,\overline{u}^{\,(0,0)}(x,y),\dots,\overline{u}^{\,(m-1,n)}(x,y)\big)\Big|\le$$
$$\le\ell\bigg(\sum_{i=0}^{m}\sum_{k=0}^{n-1}\big|u^{(i,k)}(x,y)-\overline{u}^{\,(i,k)}(x,y)\big|+\sum_{i=0}^{m-1}\big|u^{(i,n)}(x,y)-\overline{u}^{\,(i,n)}(x,y)\big|\bigg)$$

is fulfilled.

Consequently, the function $v(x,y)=u(x,y)-\overline{u}(x,y)$ is a solution of the problem

$$\big|v^{(m,n)}(x,y)\big|\le \ell\bigg(\sum_{i=0}^{m}\sum_{k=0}^{n-1}\big|v^{(i,k)}(x,y)\big|+\sum_{i=0}^{m-1}\big|v^{(i,n)}(x,y)\big|\bigg),$$
$$\lim_{x\to\gamma_1(y)} v^{(i,0)}(x,y)=0 \ \text{ for } \ 0<y<b \ \ (i=0,\dots,m-1),$$
$$\lim_{y\to\gamma_2(x)} v^{(m,k)}(x,y)=0 \ \text{ for } \ 0<x<a \ \ (k=0,\dots,n-1).$$

This by virtue of Lemma 2.2 implies that $v(x,y)\equiv 0$, i.e. $u(x,y)\equiv\overline{u}(x,y)$. □

Lemma 3.4. *Let the pair of functions γ_1, γ_2 along with (M_0) satisfy one of the conditions (M_*) and (M^*), and the function f_0 satisfy the local Lipschitz condition with respect to the last $m{+}n$ variables. Then the problem $(1.1')$, (1.2) has at most one generalized solution.*

This lemma is proved analogously to Lemma 3.3. The only difference is that instead of Lemma 2.2 we use Lemma 2.4.

4. Proof of the Main Results

Proof of Theorem 1.1. By virtue of (1.14), there exists a positive number ε such that

$$\int_{\delta+\varepsilon}^{+\infty}\frac{ds}{\varepsilon+\varphi(s)}>(b\mu\nu+\mu+\nu)(a+b). \tag{4.1}$$

Let

$$\psi(\tau)=\int_{\delta+\varepsilon}^{\tau}\frac{ds}{\varepsilon+\varphi(s)}\,,$$

and let ψ^{-1} be the function inverse to ψ. Assume

$$\rho=\psi^{-1}\big((b\mu\nu+\mu+\nu)(a+b)\big), \tag{4.2}$$

$$\sigma(\tau)=\begin{cases}1 & \text{for } \ \tau\le\rho\\ 2-\dfrac{\tau}{\rho} & \text{for } \ \rho<\tau\le 2\rho\\ 0 & \text{for } \ \tau>2\rho\end{cases}, \tag{4.3}$$

$$\widetilde{f}(x,y,z_{00},\dots,z_{m-1\,n})=$$
$$=\sigma\bigg(\sum_{i=0}^{m}\sum_{k=0}^{n-1}|z_{ik}|+\sum_{i=0}^{m-1}|z_{in}|\bigg)f(x,y,z_{00},\dots,z_{m-1\,n}) \tag{4.4}$$

and consider the differential equations

$$u^{(m,n)}=$$
$$=\widetilde{f}\big(x,y,u^{(0,0)},\dots,u^{(0,n-1)},\dots,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)}\big) \tag{4.5}$$

and

$$v^{(m,n)} = \\ = \widetilde{f}\big(x, y, v^{(0,0)}, ..., v^{(0,n-1)}, ..., v^{(m,0)}, ..., v^{(m,n-1)}, v^{(0,n)}, ..., v^{(m-1,n)}\big) + \\ + h(x,y) \tag{4.6}$$

with the initial conditions (1.2) and (1.4), where $h : \overline{G} \to \mathbb{R}$ and $e_{2k} : [0,a] \to \mathbb{R}$ $(k = 0, \dots, n-1)$ are continuous, while $e_{1i} : [0,b] \to \mathbb{R}$ $(i = 0, \dots, m-1)$ are n-times continuously differentiable functions satisfying the inequality (1.7).

First, let us show that the problems (1.1), (1.2) and (1.3), (1.4) are equivalent to the problems (4.5), (1.2) and (4.6), (1.2), respectively. We introduce the function

$$v_0(x,y) = \sum_{i=0}^{m-1} \frac{(x-\gamma_1(y))^i}{i!}\,(c_{1i}(y) + e_{1i}(y)) + \\ + \sum_{k=0}^{n-1} \int_{\gamma_1(y)}^{x} \frac{(x-s)^{m-1}(y-\gamma_2(s))^k}{(m-1)!k!}\,(c_{2k}(s) + e_{2k}(s))\,ds.$$

By the conditions (1.7), (1.15), (1.16), (4.2) and (4.3) respectively in the domains G and $G \times \mathbb{R}^{mn+m+n}$ the inequalities

$$\sum_{i=0}^{m}\sum_{k=0}^{n-1} \big|v_0^{(i,k)}(x,y)\big| + \sum_{i=0}^{m-1} \big|v_0^{(i,n)}(x,y)\big| \le \delta + \varepsilon, \tag{4.7}$$

$$\big|\widetilde{f}(x, y, z_{00}, \dots, z_{0\,n-1}, \dots, z_{m0}, \dots, z_{m\,n-1}, z_{0n}, \dots, z_{m-1\,n})\big| + \\ + |h(x,y)| \le \varepsilon + \varphi\Big(\sum_{i=0}^{m}\sum_{k=0}^{n-1} |z_{ik}| + \sum_{i=0}^{m-1} |z_{in}|\Big), \tag{4.8}$$

$$\big|\widetilde{f}(x, y, z_{00}, \dots, z_{0\,n-1}, \dots, z_{m0}, \dots, z_{m\,n-1}, z_{0n}, \dots, z_{m-1\,n})\big| + \\ + |h(x,y)| \le \rho_0 \tag{4.9}$$

are fulfilled, where $\rho_0 = \varepsilon + \varphi(2\rho)$.

Let u be a solution of the problem (1.1), (1.2) (of the problem (4.5), (1.2)). Then by the condition (1.16) (by the conditions (1.16), (4.3) and (4.4)) it likewise is a solution of the problem (2.1), (1.2). By virtue of Lemma 2.1, the inequalities (1.14) and (1.15) guarantee the validity of the estimate (2.3). According to (4.2), from (2.3) it follows that

$$\sum_{i=0}^{m}\sum_{k=0}^{n-1} \big|u^{(i,k)}(x,y)\big| + \sum_{i=0}^{m-1} \big|u^{(i,n)}(x,y)\big| \le \rho. \tag{4.10}$$

If along with the above we take into account the equalities (4.3) and (4.4), then it will become clear that u is a solution of the problem (4.5), (1.2) (of the problem (1.1), (1.2)). Consequently, the sets of solutions of the problems

$(1.1),(1.2)$ and $(4.5),(1.2)$ coincide, and every solution of these problems in the domain G admits the estimate (4.10).

Assume now that v is a solution of the problem $(1.3),(1.4)$ (of the problem $(4.6),(1.4)$). Then by the conditions (1.7) and (1.16) (by the condition (4.8)) the function v is a solution of the differential inequality

$$\big|v^{(m,n)}(x,y)\big| \leq \varepsilon+\varphi\bigg(\sum_{i=0}^{m}\sum_{k=0}^{n-1}\big|v^{(i,k)}(x,y)\big|+\sum_{i=0}^{m-1}\big|v^{(i,n)}(x,y)\big|\bigg).$$

By virtue of Lemma 2.1, the conditions (4.1), (4.2) and (4.7) guarantee the validity of the estimate

$$\sum_{i=0}^{m}\sum_{k=0}^{n-1}\big|v^{(i,k)}(x,y)\big|+\sum_{i=0}^{m-1}\big|v^{(i,n)}(x,y)\big| \leq \rho. \tag{4.11}$$

If along with the above we take into account the equalities (4.3) and (4.4), then it will become clear that v is a solution of the problem $(4.6),(1.4)$ (of the problem $(1.3),(1.4)$). Consequently, the set of solutions of the problems $(1.3),(1.4)$ and (4.6), (1.4) coincide, and every solution of these problems in the domain G admits the estimate (4.11).

If the function f satisfies the local Lipschitz condition with respect to the last $m+n$ variables (with respect to the last $mn+m+n$ variables), then, obviously, the function $\widetilde{f}$ satisfies the same condition. In this case, by virtue of the condition (4.9) and Lemma 3.1 (of Lemmas 3.1 and 3.2), the problem $(4.5),(1.2)$, as well as the problem (4.6),(1.4) has at least one solution (one and only one solution). This, according to the above-proven, implies that the problem $(1.1),(1.2)$, as well as the problem $(1.3),(1.4)$ has at least one solution (one and only one solution), and solutions of these problems admit the estimates (4.10) and (4.11).

To complete the proof of the theorem, it remains to show that in the case where the function f satisfies the local Lipschitz condition with respect to the last $mn+m+n$ variables the difference of solutions of the problems $(1.1),(1.2)$ and $(1.3),(1.4)$ admits the estimate (1.8), where r is a positive constant not depending on h, e_{1i} $(i=0,\dots,m-1)$ and e_{2k} $(k=0,\dots,n-1)$.

In the above-mentioned case, there exists a positive constant ℓ such that in the domain $G\times\mathbb{R}_\rho^{mn+m+n}$ the condition

$$\begin{aligned}\Big|f\big(x,y,z_{00},\dots,z_{0\,n-1},\dots,z_{m0},\dots,z_{m\,n-1},z_{0n},\dots,z_{m-1\,n}\big)-\\ -f\big(x,y,\overline{z}_{00},\dots,\overline{z}_{0\,n-1},\dots,\overline{z}_{m0},\dots,\overline{z}_{m\,n-1},\overline{z}_{0n},\dots,\overline{z}_{m-1\,n}\big)\Big|\leq\\ \leq\ell\bigg(\sum_{i=0}^{m}\sum_{k=0}^{n-1}|z_{ik}-\overline{z}_{ik}|+\sum_{i=0}^{m-1}|z_{in}-\overline{z}_{in}|\bigg)\end{aligned} \tag{4.12}$$

is fulfilled.

Let u and v be, respectively, the solutions of the problems (1.1), (1.2) and (1.3), (1.4). Suppose

$$w(x,y)=u(x,y)-v(x,y),\quad w_0(x,y)=\sum_{i=0}^{m-1}\frac{(x-\gamma_1(y))^i}{i!}\,e_{1i}(y)+ \\ +\sum_{k=0}^{n-1}\int_{\gamma_1(y)}^{x}\frac{(x-s)^{m-1}(y-\gamma_2(s))^k}{(m-1)!k!}\,e_{2k}(s)\,ds.$$

Then by the notation (1.5) and the inequalities (4.10)–(4.12), the function w is a solution of the problem

$$\big|w^{(m,n)}(x,y)\big|\le\eta(e_{10},\dots,e_{1\,m-1};e_{20},\dots,e_{2\,n-1};h)+ \\ +\ell\bigg(\sum_{i=0}^{m}\sum_{k=0}^{n-1}\big|w^{(i,k)}(x,y)\big|+\sum_{i=0}^{m-1}\big|w^{(i,n)}(x,y)\big|\bigg), \tag{4.13}$$

$$\begin{aligned}&\lim_{x\to\gamma_1(y)}w^{(i,0)}(x,y)=e_{1i}(y)\ \text{ for }\ 0<y<b\ \ (i=0,\dots,m-1),\\ &\lim_{y\to\gamma_2(x)}w^{(m,k)}(x,y)=e_{2k}(y)\ \text{ for }\ 0<x<a\ \ (k=0,\dots,n-1).\end{aligned} \tag{4.14}$$

As for w_0, it in the domain G satisfies the inequality

$$\sum_{i=0}^{m}\sum_{k=0}^{n-1}\big|w_0^{(i,k)}(x,y)\big|+\sum_{i=0}^{m-1}\big|w_0^{(i,n)}(x,y)\big|\le \\ \le r_0\eta(e_{10},\dots,e_{1\,m-1};e_{20},\dots,e_{2\,n-1};h), \tag{4.15}$$

where $r_0>0$ is a constant not depending on h, e_{1i} $(i=0,\dots,m-1)$ and e_{2k} $(k=0,\dots,n-1)$.

By virtue of the conditions (4.13)–(4.15) and Lemma 2.1, the function w in the domain G admits the estimate

$$\sum_{i=0}^{m}\sum_{k=0}^{n-1}\big|w^{(i,k)}(x,y)\big|+\sum_{i=0}^{m-1}\big|w^{(i,n)}(x,y)\big|\le \\ \le r\eta(e_{10},\dots,e_{1\,m-1};e_{20},\dots,e_{2\,n-1};h),$$

where $r=(1+r_0)\exp((a+b)(b\mu\nu+\mu+\nu))-1$. Consequently, the estimate (1.8) is valid, where r is a positive constant not depending on h, e_{1i} $(i=0,\dots,m-1)$ and e_{2k} $(k=0,\dots,n-1)$. ☐

Proof of Corollary 1.1. By (1.17), in the domain $G\times\mathbb{R}^{mn+m+n}$ the condition (1.16) is fulfilled, where $\varphi(\tau)=\ell_0(1+\tau)$.

We choose $\delta>0$ in such a way that the inequality (1.15) in G be fulfilled. Obviously, the condition (1.14) is likewise fulfilled. If now we apply Theorem 1.1, the validity of Corollary 1.1 becomes evident. ☐

Theorem 1.2 can be proved analogously to Theorem 1.1. The only difference in the proof is that instead of Lemmas 2.1, 2.2, 3.1 and 3.3 we apply Lemmas 2.3, 2.4, 3.2 and 3.4.

In the case $\varphi(\tau) = \ell_0(1+\tau)$, from Theorem 1.2 we obtain Corollary 1.2.

References

1. A. Alexiewicz and W. Orlicz, Some remarks on the existence and uniqueness of solutions of the hyperbolic equation $\frac{\partial^2 z}{\partial x \partial y} = f\big(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\big)$. *Stud. Math.* **15**(1956), No. 2, 201–215.
2. A. V. Bitsadze, To the question on the statement of a characteristic problem for second order hyperbolic systems. (Russian) *Dokl. AN SSSR* **223**(1976), No. 6, 1289–1291.
3. K. Deimling, A Carathéodory theory for systems of integral equations. *Ann. di Mat. Pura ed Appl.* **86**(1970), 217–260.
4. M. Grigolia, On the uniqueness of a solution of the Goursat problem for a quasilinear hyperbolic system. (Russian) *Differentsial'nye Uravneniya* **11**(1975), No. 12, 2210–2219; English transl.: *Differ. Equations* **11**(1975), No. 12, 1641–1647.
5. M. Grigolia, On the solvability of the Goursat problem. (Russian) *Trudy Tbiliss. Gos. Univ.* **204**(1978), 91–105.
6. M. Grigolia, On the well-posedness of the Goursat problem. (Russian) *Trudy Tbiliss. Gos. Univ.* **232–233**(1982), 30–48.
7. M. Grigolia, On a generalization of a characteristic problem for hyperbolic systems. (Russian) *Differentsial'nye Uravneniya* **21**(1985), No. 4, 678–686; English transl.: *Differ. Equations* **21**(1985), No. 4, 458–465.
8. M. Grigolia, On the existence and uniqueness of solutions of the Goursat problem for systems of functional partial differential equations of hyperbolic type. *Mem. Differential Equations Math. Physics* **16**(1999), 154–158.
9. M. Grigolia, Some remarks on the initial problems for nonlinear hyperbolic systems. *Mem. Differential Equations Math. Physics* **34**(2005), 77–95.
10. P. Hartman and A. Wintner, On hyperbolic partial differential equations. *Amer. J. Math.* **74**(1952), No. 4, 834–864.
11. M. Meredov, On the Goursat and Darboux problem for a class of hyperbolic systems. (Russian) *Differentsial'nye Uravneniya* **9**(1973), No. 7, 1326–1333.
12. V. N. Vragov, On the Goursat and Darboux problems for a class of hyperbolic equations. (Russian) *Differentsial'nye Uravneniya* **8**(1972), No. 1, 7–16.
13. W. Walter, Über die differentialgleichung $u_{xy} = f(x, y, u, u_x, u_y)$. I. Eindeutigkeitssätze für die charakteristische anfangswertproblem. *Math. Z.* **71**(1959), 308–324.
14. W. Walter, Über die differentialgleichung $u_{xy} = f(x, y, u, u_x, u_y)$. II. Existenzsätze für das charakteristische anfangswertproblem. *Math. Z.* **71**(1959), 436–453.
15. W. Walter, Über die differentialgleichung $u_{xy} = f(x, y, u, u_x, u_y)$. III. Die nichtcharakteristische anfangswertaufgabe. *Math. Z.* **73**(1960), 268–279.
16. W. Walter, Differential and integral inequalities. *Springer-Verlag, Berlin, Heidelberg, New York*, 1970.

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