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OSCILLATION CRITERIA FOR A CLASS OF HIGHER ORDER NONLINEAR DIFFERENTIAL EQUATIONS

Abstract. The higher order nonlinear differential equation

$$\big(|x^{(n)}|^{\alpha}\operatorname{sgn} x^{(n)}\big)^{(n)} + q(t)|x|^{\beta}\operatorname{sgn} x = 0$$

is considered, where α and β are distinct positive constants and $q:[0,+\infty)\to[0,+\infty)$ is a continuous function. Necessary and sufficient conditions for oscillation of all proper solutions of this equation are established.

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რეზიუმე. განხილულია დიფერენციალური განტოლება

$$\big(|x^{(n)}|^{\alpha}\operatorname{sgn} x^{(n)}\big)^{(n)} + q(t)|x|^{\beta}\operatorname{sgn} x = 0,$$

სადაც α და β ერთმანეთისაგან განსხვავებული დადებითი მუდმივებია, ხოლო $q : [0,+\infty) \to [0,+\infty)$ უწყვეტი ფუნქციაა. დადგენილია ყოველი წესიერი ამონახსნის რხევადობის უცილებელი და საკმარისი პირობები.

1. Introduction

The classical Atkinson–Belohorec oscillation theory [1], [3] for the Emden–Fowler differential equation

$$x'' + q(t)|x|^{\gamma} \operatorname{sgn} x = 0, \tag{1.1}$$

where $\gamma > 0$ is a constant with $\gamma \neq 1$ and $q : [a, \infty) \to (0, \infty)$ is a continuous function, has been generalized in various directions. One remarkable generalization was made by Kiguradze [6]–[8] who established necessary and sufficient conditions for oscillation of all solutions of higher order nonlinear differential equations of the form

$$x^{(2n)} + q(t)|x|^{\gamma} \operatorname{sgn} x = 0. \tag{1.2}$$

Analogous results for the differential equations of the type $u^{(n)} = f(t, u, \dots, u^{(n-1)})$ are contained in [9]–[12].

Extension of the oscillation theorems of Atkinson and Belohorec for (1.1) to nonlinear differential equations involving nonlinear Sturm–Liouville operators of the type

$$\big(|x'|^{\alpha} \operatorname{sgn} x'\big)' + q(t)|x|^{\beta} \operatorname{sgn} x = 0 \tag{1.3}$$

was carried out by Elbert and Kusano [4] and Kusano, Ogata and Usami [5]. For related results the reader is referred to the book of Agarwal et al [2].

A question naturally arises as to the possibility of generalizing Kiguradze's oscillation theorems for (1.2) to higher order nonlinear differential equations of the type

$$\big(|x^{(n)}|^{\alpha} \operatorname{sgn} x^{(n)}\big)^{(n)} + q(t)|x|^{\beta} \operatorname{sgn} x = 0, \tag{1.4}$$

α and β being distinct positive constants, whose principal parts may well be called nonlinear Sturm–Liouville differential operators of order $2n$. To the best of the author's knowledge, no results characterizing the oscillation situation of (1.4) with general n has been found in the literature, though the fourth order case of (1.4) with $n = 2$ has been investigated by Wu [16] and Naito and Wu [13], [14]. We note that the asymptotic behavior of nonoscillatory solutions of (1.4) has been analyzed in detail in a recent paper of Tanigawa and Wu [15].

By a solution of (1.4) we mean a function $x : [T_x, \infty) \to \mathbb{R}$ which is n times continuously differentiable together with $|x^{(n)}|^{\alpha} \operatorname{sgn} x^{(n)}$ and satisfies the equation at every point $t \geq T_x$. We are concerned exclusively with proper solutions of (1.4), that is, those solutions $x(t)$ which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq T_x$. Such a solution is said to be oscillatory if it has an infinite sequence of zeros clustering at infinity and nonoscillatory if it has at most a finite number of zeros in its interval of existence.

The objective of this paper is to give an affirmative answer to the above question by showing that a necessary and sufficient condition for all proper solutions of (1.4) to be oscillatory can be established for the equation (1.4) which is strongly nonlinear in the sense that $\alpha \neq \beta$. Observing that the oscillation of all proper solutions is equivalent to the absence of nonoscillatory solutions, we derive the desired oscillation criteria as a consequence of thorough analysis of possible nonoscillatory solutions of (1.4) based on a generalization of Kiguradze's lemma which was crucial in the study of the equation (1.2). The generalized Kiguradze lemma, referred to as Lemma K, is proved at the beginning of Section 1 which is concerned with the oscillation of the strongly sublinear case ($\alpha > \beta$) of (1.4). The strongly superlinear case ($\alpha < \beta$) of (1.4) is considered in Sections 2. Our method used in this paper is an extended and elaborate adaptation of the one that was invented by Kiguradze [7], [8] for the study of (1.2).

2. Strongly sublinear equations

We introduce the notation L_i, $i = 0, 1, \ldots, 2n-1$ for the lower order (quasi-) derivatives associated with the Sturm–Liouville operator $L_{2n}x = (|x^{(n)}|^{\alpha} \operatorname{sgn} x^{(n)})^{(n)}$:

$$\begin{aligned} &L_i x(t) = x^{(i)}(t), \quad i = 0, 1, \ldots, n-1, \\ &L_i x(t) = \big(|x^{(n)}(t)|^{\alpha} \operatorname{sgn} x^{(n)}(t)\big)^{(i-n)}, \quad i = n, n+1, \ldots, 2n. \end{aligned} \tag{2.1}$$

Clearly, $L_i x(t) = (L_{i-1}x(t))'$ for $i = 1, 2, \ldots, \widehat{n}, \ldots, 2n$ (caret=omit), and $L_n x(t) = |(L_{n-1}x(t))'|^{\alpha} \operatorname{sgn}(L_{n-1}x(t))'$.

All our subsequent arguments essentially depend on the following lemma which is a generalization of the well-known Kiguradze's lemma [7].

Lemma K. *Let $x(t)$ be a nonoscillatory solution of* (1.4). *Then there exist an odd integer $k \in \{1, 3, \ldots, 2n-1\}$ and a $t_0 \geq a$ such that*

$$\begin{aligned} x(t)L_i x(t) > 0, \quad t \geq t_0, \quad \textit{for } i = 0, 1, \ldots, k-1, \\ (-1)^{i-k} x(t) L_i x(t) > 0, \quad t \geq t_0, \quad \textit{for } i = k, k+1, \ldots, 2n-1. \end{aligned} \tag{2.2}$$

Proof. We may assume without loss of generality that $x(t) > 0$ for $t \geq t_1$. Since $L_{2n}x(t) < 0$, $t \geq t_1$, by (1.4), it follows that each of the derivatives $L_i x(t)$, $i = 1, 2, \ldots, 2n-1$, is eventually of constant sign.

We first note that if there exist $c > 0$ and $T \geq t_1$ such that $L_i x(t) \geq c$, $t \geq T$, for some $i \in \{1, 3, \ldots, 2n-1\}$, then, integrating the inequality successively from T to t, we have

$$L_j x(\infty) = \lim_{t\to\infty} L_j x(t) = \infty, \quad j = 0, 1, \ldots, i-1.$$

We also note that it is impossible for any derivative $L_i x(t)$, $i \in \{1, 3, \ldots, 2n-1\}$, to satisfy the inequality $L_i x(t) \leq -c$, $t \geq T$, for some $c > 0$ and $T \geq t_1$, for otherwise integration of the inequality would imply that $L_0 x(\infty) = x(\infty) = -\infty$, which is impossible. From this fact it follows that none of the consecutive derivatives $L_i x(t)$ and $L_{i+1}x(t)$ can be eventually negative.

We claim that $L_{2n-1}x(t) > 0$ for $t \geq t_1$. In fact, if there is $T > t_1$ such that $L_{2n-1}x(t) < 0$ for $t \geq T$, then, since $L_{2n-1}x(t)$ is decreasing, we have $L_i x(t) \leq -c_1$, $t \geq T$, for some $c_1 > 0$, but this is impossible as remarked above. The positivity of $L_{2n-1}x(t)$ on $[t_1, \infty)$ then implies that $L_{2n-2}x(t)$ is increasing there, so that it is eventually one-signed. The two cases are possible: either $L_{2n-2}x(t) < 0$ on $[t_1, \infty)$ or $L_{2n-2}x(t) > 0$ on $[t_2, \infty)$ for some $t_2 \geq t_1$. In the latter case, since $L_{2n-2}x(t) \geq c_2$, $t \geq t_2$, for some constant $c_2 > 0$, from the above remark we have $L_i x(\infty) = \infty$ for $i = 1, \dots, 2n-3$, which shows that $L_i x(t)$, $i = 1, \dots, 2n-3$, are eventually positive. In the former case it is obvious that $L_{2n-1}x(\infty) = L_{2n-2}x(\infty) = 0$. In this case $L_{2n-3}x(t)$ must remain positive on $[t_2, \infty)$, since the simultaneous negativity of $L_{2n-2}x(t)$ and $L_{2n-3}x(t)$ is not allowed.

Applying the same arguments as above repeatedly, we conclude that all the odd order derivatives $L_i x(t)$, $i = 1, 3, \dots, 2n-1$, must be eventually positive, while the even order derivatives $L_i x(t)$, $i = 2, 4, \dots, 2n-2$, may be eventually positive or eventually negative, and that if $L_i x(t) < 0$ for some $i \in \{2, 4, \dots, 2n-2\}$, then $L_{i+1}x(\infty) = L_i x(\infty) = 0$. This completes the proof of Lemma K. □

We denote by P_k the set of all positive solutions of (1.4) that satisfy (2.2) on $[t_0, \infty)$ for some $k \in \{1, 3, \dots, 2n-1\}$. If $x(t)$ satisfies (1.4), then so does $-x(t)$, and so the analysis of nonoscillatory solutions of (1.4) is reduced to that of the union of all P_k.

One can characterize the oscillation situation of strongly sublinear equations of the form (1.4), which will be referred to as (A):

$$\left(|x^{(n)}|^\alpha \operatorname{sgn} x^{(n)}\right)^{(n)} + q(t)|x|^\beta \operatorname{sgn} x = 0, \quad \alpha > \beta. \tag{A}$$

Theorem 2.1. *All proper solutions of (A) are oscillatory if and only if*

$$\int_a^\infty t^{\beta(n+\frac{n-1}{\alpha})} q(t)\, dt = \infty. \tag{2.3}$$

The following lemma is crucial in the proof of Theorem 2.1.

Lemma 2.1. *Let $x(t)$ be a positive solution of (A). If $x(t) \in P_k$, for $k \in \{1, 3, \dots, 2n-1\}$, then*

$$x(t) \geq c(k, n, \alpha)(t - t_0)^{n+\frac{n-1}{\alpha}} \left[L_{2n-1}(2^{2n-k-1}t)\right]^{\frac{1}{\alpha}}, \quad t \geq t_0, \tag{2.4}$$

where $c(k, n, \alpha)$ is a positive constant depending only on n, k and α.

Proof. We distinguish the two cases (i) $n+1 \leq k \leq 2n-1$ and (ii) $1 \leq k \leq n$.

(i) Let $k = 2n-1$. Since $L_{2n-1}x(t) > 0$ is decreasing, we have $L_{2n-2}x(t) \geq (t - t_0)L_{2n-1}x(t)$, $t \geq t_0$. Integrating this inequality $n-1$ times from t_0 to t and using the decreasing property of $L_{2n-1}x(t)$, we obtain

$$L_n x(t) \geq \frac{(t - t_0)^{n-1}}{(n-1)!} L_{2n-1}x(t), \quad t \geq t_0,$$

or equivalently,

$$x^{(n)}(t) \geq \frac{(t - t_0)^{\frac{n-1}{\alpha}}}{[(n-1)!]^{\frac{1}{\alpha}}} \left[L_{2n-1}x(t)\right]^{\frac{1}{\alpha}}, \quad t \geq t_0,$$

from which, after integrating n times from t_0 to t, it follows that

$$x(t) \geq \frac{(t-t_0)^{n+\frac{n-1}{\alpha}}}{[(n-1)!]^{\frac{1}{\alpha}} \prod\limits_{i=1}^{n} \left(i+\frac{n-1}{\alpha}\right)} \left[L_{2n-1}x(t)\right]^{\frac{1}{\alpha}}, \quad t \geq t_0. \tag{2.5}$$

This shows that (2.4) holds for $k = 2n-1$.

Let $n+1 \leq k \leq 2n-3$. Then, noting that $L_{2n-1}x(t) > 0$ is decreasing and $L_{2n-2}x(t) < 0$, from the equation

$$L_{2n-2}x(2t) - L_{2n-2}x(t) = \int\limits_{t}^{2t} L_{2n-1}x(\tau)d\tau,$$

we see that $-L_{2n-2}x(t) \geq tL_{2n-1}x(2t)$ for $t \geq t_0$. Integrating the last inequality $2n-k-2$ times from t to $2t$ yields

$$(-1)^{2n-k-1}L_kx(t) \geq t^{2n-k-1}L_{2n-1}x(2^{2n-k-1}t), \quad t \geq t_0,$$

or

$$L_kx(t) \geq$$
$$\geq t^{2n-k-1}L_{2n-1}x(2^{2n-k-1}t) \geq (t-t_0)^{2n-k-1}L_{2n-1}x(2^{2n-k-1}t). \tag{2.6}$$

Using (2.6) and the decreasing nature of $L_kx(t) > 0$, we find

$$L_{k-1}x(t) \geq \int\limits_{t_0}^{t} L_kx(\tau)\,d\tau \geq \int\limits_{t_0}^{t} (\tau-t_0)^{2n-k-1}L_{2n-1}x(2^{2n-k-1}\tau)\,d\tau \geq$$
$$\geq \frac{(t-t_0)^{2n-k}}{2n-k}\,L_{2n-1}x(2^{2n-k-1}t), \quad t \geq t_0.$$

Further repeated integration of the above shows that

$$L_nx(t) \geq \frac{(t-t_0)^{n-1}}{(2n-k)(2n-k+1)\cdots(n-1)}L_{2n-1}(2^{2n-k-1}t), \quad t \geq t_0,$$

which is rewritten as

$$x^{(n)}(t) \geq$$
$$\geq \frac{(t-t_0)^{\frac{n-1}{\alpha}}}{[(2n-k)(2n-k+1)\cdots(n-1)]^{\frac{1}{\alpha}}} \left[L_{2n-1}x(2^{2n-k-1}t)\right]^{\frac{1}{\alpha}}, \quad t \geq t_0.$$

Integrating this n times, we obtain

$$x(t) \geq$$
$$\geq \frac{(t-t_0)^{n+\frac{n-1}{\alpha}}}{\left[\prod_{i=0}^{k-n-1}(2n-k+i)\right]^{\frac{1}{\alpha}} \prod_{i=1}^{n}\left(i+\frac{n-1}{\alpha}\right)} \left[L_{2n-1}x(2^{2n-k-1}t)\right]^{\frac{1}{\alpha}}. \tag{2.7}$$

(ii) Suppose that $1 \leq k \leq n$. In this case we start with the inequality

$$-L_{2n-2}x(t) \geq tL_{2n-1}x(2t) \quad \text{for} \quad t \geq t_0, \tag{2.8}$$

which can be obtained as in the second part of (i). First integrate this inequality $n-1$ times from t to $2t$, and then integrate the resulting inequality

$$(-1)^{n-1}x^{(n)}(t) \geq t^{\frac{n-1}{\alpha}}\left[L_{2n-1}x(2^{n-1}t)\right]^{\frac{1}{\alpha}} \tag{2.9}$$

$n-k$ times from t to $2t$, obtaining

$$\begin{aligned}(-1)^{2n-k-1}x^{(k)}(t) &\geq t^{n-k+\frac{n-1}{\alpha}}\left[L_{2n-1}x(2^{2n-k-1}t)\right]^{\frac{1}{\alpha}} \geq \\ &\geq (t-t_0)^{n-k+\frac{n-1}{\alpha}}\left[L_{2n-1}x(2^{2n-k-1}t)\right]^{\frac{1}{\alpha}}, \quad t \geq t_0. \end{aligned} \tag{2.10}$$

Note that $(-1)^{2n-k-1} = 1$ in (2.10). We combine (2.10) with the inequality $x^{(k-1)}(t) \geq (t-t_0)x^{(k)}(t)$, $t \geq t_0$, which is a consequence of the decreasing nature of $L_k x(t) > 0$ (cf. Lemma K). Then,

$$x^{(k-1)}(t) \geq (t-t_0)^{n-k+1+\frac{n-1}{\alpha}}\left[L_{2n-1}x(2^{2n-k-1}t)\right]^{\frac{1}{\alpha}}, \quad t \geq t_0, \tag{2.11}$$

and integrating (2.11) $k-1$ times from t_0 to t, we conclude that

$$x(t) \geq \frac{(t-t_0)^{n+\frac{n-1}{\alpha}}}{\prod_{i=2}^{k}\left(n-k+i+\frac{n-1}{\alpha}\right)}\left[L_{2n-1}x(2^{2n-k-1}t)\right]^{\frac{1}{\alpha}}. \tag{2.12}$$

Thus the proof of Lemma 2.1 is complete. □

Proof of Theorem 2.1. Suppose that the equation (A) possesses a nonoscillatory solution $x(t)$. We may assume that $x(t)$ is eventually positive. By Lemma K $x(t)$ satisfies (2.2) on $[t_0, \infty)$, that is, $x(t) \in P_k$ for some $k \in \{1, 3, \dots, 2n-1\}$. From Lemma 2.1 we have for $t \geq 2^{2n-k}t_0$

$$\begin{aligned}x(t) &\geq x(2^{1-2n+k}t) \geq c(k,n,\alpha)(2^{1-2n+k}t - t_0)^{n+\frac{n-1}{\alpha}}\left[L_{2n-1}x(t)\right]^{\frac{1}{\alpha}} \geq \\ &\geq c(k,n,\alpha)2^{-(2n-k)(n+\frac{n-1}{\alpha})}t^{n+\frac{n-1}{\alpha}}\left[L_{2n-1}x(t)\right]^{\frac{1}{\alpha}},\end{aligned}$$

which implies that there exists a constant $c_1(k,n,\alpha) > 0$ depending only on n, k and α such that

$$x(t) \geq c_1(k,n,\alpha)t^{n+\frac{n-1}{\alpha}}\left[L_{2n-1}x(t)\right]^{\frac{1}{\alpha}}, \quad t \geq t_1 = 2^{-2n+k}t_0. \tag{2.13}$$

Since $L_{2n-1}x(t) > 0$ is decreasing, integrating (A) over $[t, \infty)$, we see that

$$\left[L_{2n-1}x(t)\right]^{\frac{1}{\alpha}} \geq \left[\int_t^\infty q(s)(x(s))^\beta\, ds\right]^{\frac{1}{\alpha}}, \quad t \geq t_1. \tag{2.14}$$

Multiply both sides of (2.14) by $c_1(k,n,\alpha)t^{n+\frac{n-1}{\alpha}}$ and using (2.13), we obtain

$$x(t) \geq c_1(k,n,\alpha)t^{n+\frac{n-1}{\alpha}}\left[\int_t^\infty q(s)(x(s))^\beta\, ds\right]^{\frac{1}{\alpha}}, \quad t \geq t_1. \tag{2.15}$$

We now integrate the inequality

$$q(t)t^{\beta(n+\frac{n-1}{\alpha})} \le c_1(k,n,\alpha)^{-\beta} q(t)(x(t))^{\beta} \Big[\int_t^{\infty} q(s)(x(s))^{\beta}\, ds \Big]^{-\frac{\beta}{\alpha}}, \quad t \ge t_1,$$

following from (2.15), over $[t_1, \infty)$. This can be done because $\alpha > \beta$, and we conclude that

$$\int_{t_1}^{\infty} s^{\beta(n+\frac{n-1}{\alpha})} q(s)\, ds \le \frac{\alpha}{\alpha-\beta} c_1(k,n,\alpha)^{-\beta} \Big[\int_{t_1}^{\infty} q(t)(x(t))^{\beta}\, dt \Big]^{\frac{\alpha-\beta}{\alpha}} < \infty,$$

which contradicts (2.3). Therefore, the condition (2.3) generates the oscillation of all proper solutions of (A). This completes the proof of the "if part" of the theorem.

To prove the "only if part" it suffices to assume that

$$\int_a^{\infty} t^{\beta(n+\frac{n-1}{\alpha})} q(t)\, dt < \infty \tag{2.16}$$

and show the existence of a nonoscillatory solution of (A). This statement has been proved in the paper [10,Theorem I], but we give an outline of the proof for completeness.

Let $c > 0$ be an arbitrary constant and choose $T > a$ sufficiently large so that

$$\int_T^{\infty} t^{\beta(n+\frac{n-1}{\alpha})} q(t) dt \le 2^{-\frac{1}{2}} [(n-1)!]^{\frac{\beta}{\alpha}} \Big[\prod_{i=1}^{n} \Big(i + \frac{n-1}{\alpha}\Big) \Big]^{\beta} c^{1-\frac{\beta}{\alpha}}. \tag{2.17}$$

Define the set X_1 by

$$X_1 = \\ = \Big\{ x \in C[T,\infty) : \ k_1(t-T)^{n+\frac{n-1}{\alpha}} \le x(t) \le k_2(t-T)^{n+\frac{n-1}{\alpha}}, \ t \ge T \Big\} \tag{2.18}$$

which is a closed convex subset of the locally convex space $C[T,\infty)$ of continuous functions on $[T,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T,\infty)$, where k_1 and k_2, denote the positive constants

$$k_i = \frac{c_i}{[(n-1)!]^{\frac{1}{\alpha}} \prod_{m=1}^{n} \big(m + \frac{n-1}{\alpha}\big)}, \quad i=1,2, \quad c_1 = c^{\frac{1}{\alpha}}, \quad c_2 = (2c)^{\frac{1}{\alpha}}. \tag{2.19}$$

Consider the integral operator $\mathcal{F}$ defined by

$$\mathcal{F}x(t) = \int_T^{t} \frac{(t-s)^{n-1}}{(n-1)!} \Big[c \frac{(s-T)^{n-1}}{(n-1)!} + \\ + \int_T^{s} \frac{(s-r)^{n-2}}{(n-2)!} \int_r^{\infty} q(\sigma)(x(\sigma))^{\beta}\, d\sigma\, dr \Big]^{\frac{1}{\alpha}} ds \tag{2.20}$$

for $t > T$.

Using (2.17) and (2.19), we see that $\mathcal{F}$ maps X_1 into itself. If $\{x_\nu\}$ is a sequence in X_1 converging to x_0 in $C[T, \infty)$, then from the Lebesgue convergence theorem it follows that $\{\mathcal{F}x_\nu\}$ converges to $\mathcal{F}x_0$ in $C[T, \infty)$, so that $\mathcal{F}$ is a continuous mapping. Since $\mathcal{F}(X_1)$ and $\mathcal{F}'(X_1) = \{(\mathcal{F}x)'(t) : x \in X_1\}$ are locally bounded in $[T, \infty)$, the Ascoli-Arzelà theorem implies that $\mathcal{F}(X_1)$ is relatively compact in $C[T, \infty)$. Thus all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled, and so there exists an element $x \in X_1$ such that $x = \mathcal{F}x$. Differentiating the integral equation $x = \mathcal{F}x$, we conclude that $x = x(t)$ is a positive solution of (A) on $[T, \infty)$ such that $L_{2n-1}x(\infty) = c$. This sketches the proof of the "only if part" of the theorem. □

3. Strongly Superlinear Equations

We now turn to the oscillation problem for strongly superlinear equations of the form (1.4), which will be referred to as (B):

$$\left(|x^{(n)}|^\alpha \operatorname{sgn} x^{(n)}\right)^{(n)} + q(t)|x|^\beta \operatorname{sgn} x = 0, \quad \alpha < \beta, \tag{B}$$

where $q(t)$ is a positive continuous function on $[a, \infty)$.

Theorem 3.1. *All proper solutions of (B) are oscillatory if and only if either*

$$\int_a^\infty t^{n-1} q(t)\, dt = \infty \tag{3.1}$$

or

$$\int_a^\infty t^{n-1} q(t)\, dt < \infty \quad \text{and} \quad \int_a^\infty t^{n-1} \left[\int_t^\infty s^{n-1} q(s)\, ds \right]^{\frac{1}{\alpha}} dt = \infty. \tag{3.2}$$

The following lemma is needed in the proof of the theorem.

Lemma 3.1. *Let $x(t)$ be a positive solution of (B) on $[t_0, \infty)$ belonging to P_k for some $k \in \{1, 3, \dots, 2n-1\}$. Then, we have the following statements.*

(i) If $n+1 \le k \le 2n-1$, then, the following inequalities hold on $[t_0, \infty)$:

$$(t - t_0) L_{k-j} x(t) \le (1 + j) L_{k-j-1} x(t) \quad \text{for } j = 0, 1, \dots, k-n-1, \tag{3.3}$$

$$(t - t_0)[L_n x(t)]^{\frac{1}{\alpha}} \le \frac{k - n + \alpha}{\alpha} L_{n-1} x(t), \tag{3.4}$$

$$(t - t_0) L_{k-j} x(t) \le \frac{k - n + \{j + 1 - (k-n)\}\alpha}{\alpha} L_{k-j-1} x(t) \quad \text{for } j = k-n+1, \dots, k-1. \tag{3.5}$$

(ii) If $1 \le k \le n$, then (3.3) holds on $[t_0, \infty)$ for $j = 0, 1, \dots, k-1$.

Proof. (i) Let $n-1 \le k \le 2n-1$. Note that since $L_k x(t) > 0$ is decreasing, we have $(t-t_0)L_k x(t) \le L_{k-1}x(t), t \ge t_0$, which is (3.3) for $j = 0$. Combining the inequality with the relations

$$(t-t_0)L_{k-j}x(t) = (1+j)L_{k-j-1}x(t) - (1+j)L_{k-j-1}x(t_0)-$$
$$-\int_{t_0}^{t}\big[jL_{k-j}x(s)-(s-t_0)L_{k-j+1}x(s)\big]\,ds \ \text{ for } \ j=1,2,\dots,k-n-1, \tag{3.6}$$

we obtain (3.3) successively for $j = 1, 2, \dots, k-n-1$.

From (3.3) with $j = k-n-1$, which reads

$$(t-t_0)\big((x^{(n)}(t))^{\alpha}\big)' \le (k-n)(x^{(n)}(t))^{\alpha}, \ \ t \ge t_0,$$

it follows that

$$\alpha(t-t_0)x^{(n+1)}(t) \le (k-n)x^{(n)}(t), \ \ t \ge t_0. \tag{3.7}$$

Integrating (3.7) from t_0 to t yields

$$\alpha(t-t_0)x^{(n)}(t) \le \{(k-n)+\alpha\}x^{(n-1)}(t), \ \ t \ge t_0, \tag{3.8}$$

which is the inequality (3.4). If we combine (3.8) with the relations

$$(t-t_0)x^{(k-j)}(t) = \frac{k-n+\{j+1-(k-n)\}\alpha}{\alpha}x^{(k-j-1)}(t)-$$
$$-\frac{k-n+\{j+1-(k-n)\}\alpha}{\alpha}\,x^{(k-j-1)}x(t_0)-$$
$$-\int_{t_0}^{t}\Big[\frac{k-n+\{j-(k-n)\}\alpha}{\alpha}x^{(k-j)}(s)-(s-t_0)x^{(k-j+1)}(s)\Big]\,ds, \tag{3.9}$$

holding for $j = k-n+1, k-n+2, \dots, k-1$, then we can derive (3.5) successively for $j = k-n+1, \dots, k-1$. This finishes the proof of (i).

The proof of the satement (ii) for k such that $1 \le k \le n$ is similar to that of (i). In fact, using the decreasing nature of $x^{(k)}(t) > 0$, we obtain the inequality $(t-t_0)x^{(k)}(t) \le x^{(k-1)}(t), t \ge t_0$, which is (3.3) for $j = 0$. This combined with the relations

$$(t-t_0)x^{(k-j)}(t) = (1+j)x^{(k-j-1)}(t) - (1+j)x^{(k-j-1)}(t_0)-$$
$$-\int_{t_0}^{t}\big[jx^{(k-j)}(s)-(s-t_0)x^{(k-j+1)}(s)\big]\,ds \ \text{ for } \ j=1,2,\dots,k-1$$

shows successively that $(t-t_0)x^{(k-j)}(t) \le (1+j)x^{(k-j-1)}(t)$ for $t \ge t_0$. Thus (3.3) holds for $j = 0, 1, \dots, k-1$. □

Remark. Let $x(t)$ be a positive solution of (B) belonging to P_k for some odd k such that $n+1 \le k \le 2n-1$. Then, from (3.3)–(3.5) it can shown that $x(t)$ satisfies

$$L_k x(t) \le (k-n)!\Big[\prod_{i=1}^{n}\Big(\frac{k-n+\alpha i}{\alpha}\Big)\Big]^{\alpha}\frac{(x(t))^{\alpha}}{(t-t_0)^{k+(\alpha-1)n}},\ \ t\ge t_0. \quad (3.10)$$

Proof of Theorem 3.1. Suppose that (B) possesses an eventually positive solution $x(t)$. Then, $x(t)\in P_k$ for some $k\in\{1,3,\dots,2n-1\}$. Assume that (2.2) holds on the interval $[t_0,\infty)$, $t_0\ge a$.

We first consider the case where k satisfies $n+1\le k\le 2n-1$. We multiply (B) by $t^{n-1}(x(t))^{-\beta}$ and integrate it from $2t_0$ to t. Repeated application of integration by parts leads to the equation

$$w(t)+\beta\int_{2t_0}^{t} w(s)\,\frac{x'(s)}{x(s)}\,ds+\int_{2t_0}^{t} s^{n-1}q(s)\,ds=$$

$$=w(2t_0)+(n-1)(n-2)\cdots(k-n)\int_{2t_0}^{t} L_k x(s)\,\frac{s^{k-n-1}}{x(s)^{\beta}}\,ds,\ \ t\ge 2t_0, \quad (3.11)$$

where $w(t)$ is given by

$$w(t)=$$

$$=\Big[t^{n-1}L_{2n-1}x(t)-(n-1)t^{n-2}L_{2n-2}x(t)+(n-1)(n-2)t^{n-3}L_{2n-3}x(t)-$$

$$-\cdots+(n-1)(n-2)\cdots\{n-(2n-k-1)\}t^{n-(2n-k)}L_k x(t)\Big](x(t))^{-\beta}=$$

$$=\Big[t^{n-1}L_{2n-1}x(t)-(n-1)t^{n-2}L_{2n-2}x(t)+(n-1)(n-2)t^{n-3}L_{2n-3}x(t)-$$

$$-\cdots+(n-1)(n-2)\cdots(k-n+1)t^{k-n}L_k x(t)\Big](x(t))^{-\beta}. \quad (3.12)$$

Noting that $x'(t)\ge 0$ and $w(t)\ge 0$ on $[t_0,\infty)$ by Lemma K and using (3.10) we have

$$\int_{2t_0}^{t} s^{n-1}q(s)\,ds\le$$

$$\le w(2t_0)+c(k,n,\alpha)\int_{2t_0}^{t}\frac{s^{k-n-1}}{(s-t_1)^{k+(\alpha-1)n}}\,(x(s))^{\alpha-\beta}\,ds,\ \ t\ge 2t_0$$

for some constant $c(k,n,\alpha)>0$ depending only on k, n and α, from which it follows that

$$\int_{2t_0}^{\infty} t^{n-1}q(t)dt<\infty. \quad (3.13)$$

To proceed further we rewrite $w(t)$ as follows:

$$w(t) = t^{k-n-1}v(t)(x(t))^{-\beta} + \\ + (n-1)(n-2)\cdots(k-n)t^{k-n-1}L_{k-1}x(t)(x(t))^{-\beta}, \tag{3.14}$$

where $v(t)$ is defined by

$$v(t) = t^{2n-k}L_{2n-1}x(t) - (n-1)t^{2n-k-1}L_{2n-2}x(t) + \\ + \cdots + (n-1)(n-2)\cdots(k-n+2)(k-n+1)tL_kx(t) - \\ - (n-1)(n-2)\cdots(k-n+1)(k-n)L_{k-1}x(t). \tag{3.15}$$

As is easily verified $v'(t) \le 0$ for $t \ge t_0$, and so $v(t)$ is decreasing on $[t_0, \infty)$. Using this fact and the increasing nature of $L_{k-1}x(t) > 0$ (cf. Lemma K), we find from (3.14) that $w(t)$ satisfies

$$w(t) \le c_1(k,n)t^{k-n-1}L_{k-1}x(t)(x(t))^{-\beta}, \quad t \ge t_0, \tag{3.16}$$

for some constant $c_1(k,n) > 0$.

Let us now multiply (B) by $t^{n-1}(x(t))^{-\beta}$ and integrate it over $[t, \tau]$, $t \ge 2t_0$. Then, the same computation as in the beginning of the proof yields

$$w(\tau) + \beta\int_t^\tau w(s)\frac{x'(s)}{x(s)}\,ds + \int_t^\tau s^{n-1}q(s)\,ds = \\ = w(t) + (n-1)(n-2)\cdots(k-n)\int_t^\tau L_kx(s)\,\frac{s^{k-n-1}}{(x(s))^\beta}\,ds, \tag{3.17}$$

which implies

$$\int_t^\tau s^{n-1}q(s)\,ds \le w(t) + (n-1)\cdots(k-n)\int_t^\tau L_kx(s)\,\frac{s^{k-n-1}}{(x(s))^\beta}\,ds. \tag{3.18}$$

Since both integrals in (3.18) converge as $\tau \to \infty$ because of (3.13) and (3.10), we obtain

$$\int_t^\infty s^{n-1}q(s)\,ds \le w(t) + (n-1)\cdots(k-n)\int_t^\infty L_kx(s)\,\frac{s^{k-n-1}}{(x(s))^\beta}\,ds \le \\ \le c_1(k,n)L_{k-1}x(t)\,\frac{t^{k-n-1}}{(x(t))^\beta} + c_2(k,n)\int_t^\infty L_kx(s)\,\frac{s^{k-n-1}}{(x(s))^\beta}\,ds, \quad t \ge 2t_0$$

where $c_2(k,n)$ is a positive constant. A simple calculation with the aid of (3.10) and a similar inequality for $L_{k-1}x(t)$ leads to

$$\int_t^\infty s^{n-1}q(s)\,ds \le$$

$$\le c_3(k,n,\alpha)\,\frac{(x(t))^{\alpha-\beta}}{(t-t_0)^{\alpha n}} + c_4(k,n,\alpha)\int_t^\infty \frac{s^{k-n-1}}{(s-t_0)^{k-n+\alpha n}}\,(x(s))^{\alpha-\beta}\,ds \le$$

$$\le c_5(k,n,\alpha)\,\frac{(x(t))^{\alpha-\beta}}{(t-t_0)^{\alpha n}}, \quad t \ge t_0, \tag{3.19}$$

where $c_i(k,n,\alpha)$ $(i = 3,4,5)$ are positive constants, and the negativity of $\alpha-\beta$ has been used. Taking (3.19) into account, we compute

$$\int_{2t_0}^{\tau} t^{n-1}\Big[\int_t^\infty s^{n-1}q(s)\,ds\Big]^{\frac{1}{\alpha}}\,dt \le$$

$$\le c_6(k,n,\alpha)\int_{2t_0}^{\tau}\frac{t^{n-1}}{(t-t_0)^n}\,(x(t))^{1-\frac{\beta}{\alpha}}\,dt, \quad \tau \ge 2t_0. \tag{3.20}$$

We now combine (3.20) with the inequality

$$x(t) \ge c_k t^{n+\frac{k-n-1}{\alpha}}, \quad t \ge 2t_0,$$

$c_k > 0$ being a constant, (cf. Remark to Lemma K) to obtain

$$\int_{2t_0}^{\tau} t^{n-1}\Big[\int_t^\infty s^{n-1}q(s)\,ds\Big]^{\frac{1}{\alpha}}\,dt \le c_7(k,n,\alpha)\int_{2t_0}^{\tau} s^{-1}s^{(1-\frac{\beta}{\alpha})(n+\frac{k-n-1}{\alpha})}\,ds.$$

This clearly implies that

$$\int_{2t_0}^{\infty} t^{n-1}\Big[\int_t^\infty s^{n-1}q(s)\,ds\Big]^{\frac{1}{\alpha}}\,dt < \infty. \tag{3.21}$$

The inequalities (3.10) and (3.21) show that “if part” of Theorem 3.1 is true for k satisfying $n+1 \le k \le 2n-1$.

Let us turn to the case where $1 \le k \le n$. Since $L_i x(\infty) = 0, i = n, n+1, \ldots, 2n-1$, integrating (B) n times from t to ∞ and noting that $x(t)$ is increasing, we have

$$(-1)^{n-1}L_n x(t) \ge (x(t))^\beta \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}q(s)\,ds,$$

or

$$(-1)^{n-1}\frac{x^{(n)}(t)}{(x(t))^{\frac{\beta}{\alpha}}} \ge \Big[\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}q(s)\,ds\Big]^{\frac{1}{\alpha}}, \quad t \ge t_0. \tag{3.22}$$

Integrating (3.22) multiplied by t^{n-1} over $[2t_0, t]$ gives

$$\int_{2t_0}^{t} s^{n-1}\Big[\int_{s}^{\infty}\frac{(s-r)^{n-1}}{(n-1)!}q(r)\,dr\Big]^{\frac{1}{\alpha}}\,ds \le$$

$$\le (-1)^{n-1}w(t) + (-1)^{n}w(2t_0) + (-1)^{n-1}\,\frac{\beta}{\alpha}\int_{2t_0}^{t} w(s)\,\frac{x'(s)}{x(s)}\,ds +$$

$$+(-1)^{2n-k-1}(n-1)(n-2)\cdots(k+1)k\int_{2t_0}^{t} L_k x(s)\,\frac{s^{k-1}}{(x(s))^{\frac{\beta}{\alpha}}}\,ds,$$

where $w(t)$ is the function defined by (3.12). Since $x'(t) \ge 0$ and $(-1)^{n-1}w(t) \le 0$ by Lemma K and since $(t-t_0)^{k-1}L_k x(t) \le (k-1)!x'(t)$ by (ii) of Lemma 3.1, it follows that

$$\int_{2t_0}^{t} s^{n-1}\Big[\int_{s}^{\infty}\frac{(s-r)^{n-1}}{(n-1)!}q(r)\,dr\Big]^{\frac{1}{\alpha}}\,ds \le$$

$$\le (-1)^{n}w(2t_0) + (n-1)(n-2)\cdots(k+1)k\int_{2t_0}^{t} L_k x(s)\,\frac{s^{k-1}}{(x(s))^{\frac{\beta}{\alpha}}}\,ds \le$$

$$\le (-1)^{n}w(2t_0) + (n-1)(n-2)\cdots(k+1)k\cdot 2^{k-1}(k-1)!\int_{2t_0}^{t}\frac{x'(s)}{(x(s))^{\frac{\beta}{\alpha}}}\,ds$$

for $t \ge 2t_0$. Since $\alpha < \beta$ implies $\int_{2t_0}^{\infty} x'(t)/(x(t))^{\frac{\beta}{\alpha}}\,dt < \infty$, we conclude from the above that

$$\int_{2t_0}^{\infty} t^{n-1}\Big[\int_{t}^{\infty} s^{n-1}q(s)\,ds\Big]^{\frac{1}{\alpha}}\,dt < \infty. \tag{3.23}$$

Thus it has been shown that the "if part" of Theorem 3.1 is true also in the case where k satisfies $1 \le k \le n$.

The "only if" part of the theorem is proved as follows (cf. [10, Theorem I]). Let $c > 0$ be given arbitrarily and choose $T > a$ so that

$$\int_{T}^{\infty}\frac{t^{n-1}}{(n-1)!}\Big[\int_{t}^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}q(s)\,ds\Big]^{\frac{1}{\alpha}} \le 2^{-1}c^{1-\frac{\beta}{\alpha}}. \tag{3.24}$$

We define the set X_2 and the mapping $\mathcal{G}$ by

$$X_2 = \Big\{x \in C[T,\infty) : \ \frac{c}{2} \le x(t) \le c, \ t \ge T\Big\} \tag{3.25}$$

and

$$\mathcal{G}x(t) =$$
$$= c - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!} q(r)(x(r))^\beta \, dr \right]^{\frac{1}{\alpha}} ds, \quad t \geq T, \qquad (3.26)$$

respectively. Then it is routinely proved that $\mathcal{G}$ maps X_2 into itself, that $\mathcal{G}$ is a continuous mapping, and that $\mathcal{G}(X_2)$ is relatively compact in $C[T,\infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem there exists an element $x \in X_2$ such that $x = \mathcal{G}x$. It is clear that the fixed element $x = x(t)$ gives a positive solution of (B) on $[T,\infty)$ such that $x(\infty) = c$. This completes the proof. □

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