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ON QUASIINTEGRALS OF NON-STATIONARY THREE-DIMENSIONAL LINEAR DIFFERENTIAL SYSTEMS WITH ANTISYMMETRIC COEFFICIENT MATRIX

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Consider a three-dimensional linear differential system

$$\dot{x} = A(t)x, \quad x \in R^3, \quad t \geq 0, \tag{1_A}$$

with a continuous piecewise differentiable coefficient matrix $A(t) = -A^\top(t)$ for all $t \geq 0$.

Those systems have applications [1] in designing automated equipment and modelling its work using research methods of holonomic systems for building parametric programmed movements of executive devices in the three-dimensional physical space.

It is well known that the n-dimensional linear differential system has a stationary quadratic integral $\|x\|^2 = x_1^2 + \ldots + x_n^2 = c^2$, and in the two-dimensional case it has non-stationary integrals

$$x_k \cos \int\limits_0^t a_{12}(\tau)d\tau + (-1)^k x_{3-k} \sin \int\limits_0^t a_{12}(\tau)d\tau = C_k, \quad k = 1, 2, \quad t \geq 0, \tag{2}$$

which imply the integral $x_1^2 + x_2^2 = C^2$.

In the trivial case $\alpha(t) \equiv a_{12}^2(t) + a_{13}^2(t) \equiv 0$, $t \geq 0$, the three-dimensional system (1_A) disintegrates into two independent antisymmetric systems: a two-dimensional one with respect to the vector-function (x_2, x_3), two independent integrals of which could be represented as (2), and the one-dimensional system $\dot{x}_1 = 0$. We, respectively, will not take into consideration this trivial case and hereafter will consider only the non-trivial case $\alpha(t) \not\equiv 0$, $t \geq 0$.

Using the elements of the coefficient matrix $A(t)$ of the system (1_A), we define the vector-functions

$$a(t) = \begin{pmatrix} a_{23}(t) \\ -a_{13}(t) \\ a_{12}(t) \end{pmatrix}, \quad v_A(t,\eta) = \begin{pmatrix} -\alpha(t)C_{\|a\|}(\eta) \\ -a_{13}(t)a_{23}(t)C_{\|a\|}(\eta) + a_{12}(t)\|a(t)\|S_{\|a\|}(\eta) \\ a_{12}(t)a_{23}(t)C_{\|a\|}(\eta) + a_{13}(t)\|a(t)\|S_{\|a\|}(\eta) \end{pmatrix} \equiv$$

$$\equiv v_{c,s}(t,\eta), \quad w_A(t,\eta) = v_{s,-c}(t,\eta), \quad t, \eta \in [0, +\infty], \tag{3}$$

where the notation

$$S_f(\eta) = \sin \int\limits_0^\eta f(\tau)d\tau, \quad C_f(\eta) = \cos \int\limits_0^\eta f(\tau)d\tau$$

is used for an arbitrary continuous function $f : [0, +\infty) \to R^1$.

The following lemma holds.

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Lemma 1. *The three-dimensional stationary system* (1_A) *with a constant antisymmetric matrix* $A(t) = A = \mathrm{const}$ *for all* $t \geq 0$ *has linear integrals: the stationary integral*

$$a_{23}x_1 - a_{13}x_2 + a_{12}x_3 = C_1 \tag{4_1}$$

and the non-stationary integrals

$$(x, v_A(0,t)) = C_2, \quad (x, w_A(0,t)) = C_3. \tag{4_2}$$

Therefore the integrals (4_1) and (4_2) turn into identities

$$(x(t), a) - (x(0), a) \equiv 0, \quad (x(t), v_A(0,t) - (x(0), v_A(0,0)) \equiv 0, \tag{4_3}$$

$$(x(t), w_A(0,t)) - (x(0), w_A(0,0)) \equiv 0, \quad t \geq 0,$$

on every solution $x : [0, +\infty] \to R^3$ to the stationary system (1_A). Obviously the identities (4_3) and those analogous to them are not valid for the non-stationary system (1_A). Therefore there arises the problem of obtaining estimations for the absolute values of the deviations from the identical zero of the linear non-stationary forms

$$(x(t), a(t)) - (x(0), a(0)), \quad (x(t), v_A(t,t)) - (x(0), v_A(0,0)),$$

$$(x(t), w_A(t,t)) - (x(0), w_A(0,0)), \quad t \geq 0,$$

of the components of the solutions $x : [0, +\infty) \to R^3$ to the non-stationary system (1_A) in the stationary case and of the proof of the accuracy of these estimations. They will make possible to find a strip region on the sphere $\|x\|^2 = \|x(0)\|^2$ in the space R^3, containing on a not small time interval a trajectory of the solution $x(t)$ of the non-stationary system (1_A) with given initial solution $x(0)$, and being sufficiently narrow under small values of the norm of the coefficient matrix derivative $\|dA(t)/dt\|$ of the considered system. Building such strip regions containing the trajectories of the solutions could be used in the appropriate mechanical applications.

To obtain the mentioned estimates we shall require the following

Lemma 2. *For every solution* $x : [0, +\infty) \to R^3$ *to the three-dimensional system* (1_A) *the following identities are valid*

$$L_1(x(t), t) \equiv (x(t), a(t)) - (x(0), a(0)) \equiv \int_0^t (x(\tau), \dot{a}(\tau)) d\tau, \quad t \geq 0, \tag{5_1}$$

$$L_2(x(t), t) \equiv (x(t), v_A(t,t)) - (x(0), v_A(0,0)) \equiv \int_0^t \left(x(\tau), \frac{\partial v_A(\tau, t)}{\partial \tau} \right) d\tau, \quad t \geq 0, \tag{5_2}$$

$$L_3(x(t), t) \equiv (x(t), w_A(t,t)) - (x(0), w_A(0,0)) \equiv \int_0^t \left(x(\tau), \frac{\partial w_A(\tau, t)}{\partial \tau} \right) d\tau, \quad t \geq 0. \tag{5_3}$$

Proof. Obviously, for the proof of the identity (5_1) it is sufficient to verify the equality $(\dot{x}(t), a(t)) = 0, \quad t \geq 0$. Indeed, for the linear form $(\dot{x}, a(t))$, $x \in R^3$, evaluated under the conditions of the system (1_A), the equalities

$$(\dot{x}, a(t)) = a_{23}(t)\dot{x}_1 - a_{13}(t)\dot{x}_2 + a_{12}(t)\dot{x}_3 = a_{23}(t)[a_{12}(t)x_2 + a_{13}(x_3)] -$$

$$-a_{13}(t)[-a_{12}(t)x_1 + a_{23}(t)x_3] + a_{12}(t)[-a_{13}(t)x_1 - a_{23}(t)x_2] \equiv 0, \quad \forall x \in R^3, \quad \forall t \geq 0,$$

are valid. We will note for the proof of the identity (5_2) that the components of the vector-function $v_A(t, \eta)$, as well as of the vector-function $w_A(t, \eta)$, are the sums of products of two functions of the single variable t or η. Therefore its derivative $dv_A(t,t)/dt$ can be represented as follows

$$\frac{dv_A(t,t)}{dt} = \left. \frac{\partial v_A(t,\eta)}{dt} \right|_{\eta=t} + \left. \frac{\partial v_A(t,\eta)}{\partial \eta} \right|_{\eta=t}.$$

Accordingly the derivative of the left-hand side $L_2(x(t),t)$ of the equality (5_2) admits the representation

$$\frac{dL_2(x(t),t)}{dt} = (\dot{x}(t), v_A(t,t)) + \left(x(t), \frac{\partial v_A(t,\eta)}{\partial \eta}\bigg|_{\eta=t}\right) + \\ + \left(x(t), \frac{\partial v_A(\eta,t)}{\partial \eta}\bigg|_{\eta=t}\right), \quad t \geq 0. \tag{6}$$

We evaluate the sum $S_2(x(t),t)$ of the first two addends of the above representation using for the derivative $\dot{x}(t)$ its value from the (1_A) in an arbitrary point $x \in R^3$:

$$S_2(x,t) = -(a_{12}x_2 + a_{13}x_3)(a_{12}^2 + a_{13}^2)C_{\|a\|} + (-a_{12}x_1 + a_{23}x_3)\times \\ \times(-a_{13}a_{23}C_{\|a\|} + a_{12}\|a\|S_{\|a\|}) - (a_{13}x_1 + a_{23}x_2)(a_{12}a_{23}C_{\|a\|} + a_{13}\|a\|S_{\|a\|}) + \\ + x_1(a_{12}^2 + a_{13}^2)\|a\|S_{\|a\|} + x_2(a_{13}a_{23}\|a\|S_{\|a\|} + a_{12}\|a\|^2C_{\|a\|}) + \\ + x_3(-a_{12}a_{23}\|a\|S_{\|a\|} + a_{13}\|a\|^2C_{\|a\|} \equiv 0, \quad x \in R^3, \\ a_{ij} = a_{ij}(t), \quad S_{\|a\|} = S_{\|a\|}(t), \quad C_{\|a\|} = C_{\|a\|}(t), \quad i = 1,2, \quad j = 1,2,3, \quad t \geq 0.$$

Now it is enough to note for the proof of the identity (5_2) that the third addend of the equality (6) coincides with the derivative of the right-hand side of the identity (5_2), evaluated at the moment t.

The proof of the identity (5_3) is analogous. □

The following theorem can be proved using this lemma.

Theorem 1. *For every solution $x : [0,+\infty) \to R^3$ to the three-dimensional system (1_A) the double-sided estimates are valid*

$$|(x(t),a(t)) - (x(0),a(0))| \leq \|x(0)\| \int_0^t \|\dot{a}(\tau)\| d\tau, \quad t \geq 0, \tag{7_1}$$

$$\left.\begin{array}{r} |(x(t),v_A(t,t)) - (x(0),v_A(0,0))| \quad (7_2) \\ |(x(t),w_A(t,t)) - (x(0),w_A(0,0))| \quad (7_3) \end{array}\right\} \leq 2\sqrt{3}\|x(0)\| \int_0^t \|a(\tau)\|\|\dot{a}(\tau)\| d\tau, \quad t \geq 0.$$

Proof. The estimate (7_1) obviously follows from the identities (5_1) and $\|x(t)\| \equiv \|x(0)\|$, $t \geq 0$.

For the proof of the estimate (7_2), first we estimate the norm of the derivative $\partial v_A(t,\eta)/\partial t$ of the vector-function $v_A(t,\eta)$ for arbitrary fixed values $t,\eta \in [0,+\infty)$. The following obvious estimate for the first component $\partial v_A^{(1)}(t,\eta)/\partial t$ of that derivative

$$|\partial v_A^{(1)}(t,\eta)/\partial t| \leq 2\|a(t)\| \cdot \|\dot{a}(t)\|, \quad t \geq 0, \tag{8}$$

is valid for every $\eta \in [0,+\infty)$. For the second of its components

$$\partial v_A^{(2)}(t,\eta)/\partial t = [\dot{a}_{12}(t)\|a(t)\| + a_{12}(t)\|a(t)\|']S_{\|a\|}(\eta) - \\ - [a_{23}(t)\dot{a}_{13}(t) - a_{13}(t)\dot{a}_{23}(t)]C_{\|a\|}(\eta),$$

using the inequality $|\|a(t)\|'| = |(a(t),\dot{a}(t))|/\|a(t)\| \leq \|\dot{a}(t)\|$, $t \geq 0$, we get the intermediate estimate

$$\left|\frac{\partial v_A^{(2)}(t,\eta)}{\partial t}\right| \leq \|\dot{a}(t)\|\left[|a_{12}(t)| + \sqrt{a_{13}^2(t) + a_{23}^2(t) + \|a(t)\|^2}\right] \equiv \\ \equiv \|\dot{a}(t)\|b_2(t), \quad \eta \in [0,+\infty), \quad t \geq 0. \tag{9}$$

Since the inequality

$$a_{12}^2(a_{12}^2 + 2a_{13}^2 + 2a_{23}^2) \leq (a_{12}^2 + a_{13}^2 + a_{23}^2)^2 \tag{10}$$

holds for any real numbers a_{ij}, the estimate

$$b_2^2 = 2|a_{12}|\sqrt{a_{12}^2 + 2a_{13}^2 + 2a_{23}^2} + 2\|a\|^2 \le 4\|a\|^2 \tag{11}$$

based on the inequality (10) is valid for the quadratic bracket $b_2(t)$ from (9).

From the above estimate and (9) we obtain the final inequality

$$|\partial v_A^{(k)}(t,\eta)/\partial t| \le 2\|a(t)\|\|\dot a(t)\|, \quad t,\eta \in [0,+\infty), \tag{12_k}$$

when $k = 2$.

Proving the estimate (12_3) we use another form $a_{13}^2(a_{13}^2 + 2a_{12}^2 + 2a_{23}^2) \le \|a\|^4$ of the inequality (10), which allows to obtain the similar to (11) inequality $b_3^2(t) \le 4\|a(t)\|^2$, $t \ge 0$, for the function $b_3(t) = |a_{13}(t)| + \sqrt{a_{12}^2(t) + a_{23}^2(t) + \|a(t)\|^2}$, $t \ge 0$, giving the estimate $|\partial v_A^{(3)}(t,\eta)/\partial t| \le \|\dot a(t)\|\|b_3(t)\|$, $t,\eta \in [0,+\infty)$.

The inequalities (8), (12_2) and (12_3) make possible to establish the final estimate

$$|\partial v_A(t,\eta)/\partial t| \le 2\sqrt{3}\|a(t)\|\|\dot a(t)\|, \quad t,\eta \in [0,+\infty). \tag{13_1}$$

Estimating the absolute components of the derivative $\partial v_A(t,\eta)/\partial t$, we have used the obvious inequalities $|S_{\|a\|}(\eta)|, |C_{\|a\|}(\eta)| \le 1, \quad \eta \in [0,+\infty)$. Therefore it is easy to notice that the estimation of the component modules of the derivative $\partial w_A(t,\eta)/\partial t$ is identical to already obtained estimations of the component modules of the partial derivative in t of the vector-function $v_A(t,\eta)$. Hence the inequality

$$\|\partial w_A(t,\eta)/\partial t\| \le 2\sqrt{3}\|a(t)\|\|\dot a(t)\|, \quad t,\eta \in [0,+\infty), \tag{13_2}$$

is proved as well.

From the estimates (13_1) and (13_2), the identities (5_2) and (5_3), and the equality $\|x(t)\| = \|x(0)\|$, $t \ge 0$, now follow the inequalities (7_2) and (7_3). $\square$

A natural question about accuracy of the proved estimates (7_1)–(7_3) arises. It is established by the following theorem, stated without proof.

Theorem 2. *There exists an antisymmetric matrix $A_\omega(t)$ of the third order with arbitrary primarily given nontrivial* (*in the sense of the validity of the inequality $\alpha \equiv a_{12}^2 + a_{13}^2 > 0$*) *initial value $A = A_\omega(0) = A_0(t)$ such that it is continuously differentiable in the time $t \ge 0$ and the parameter $\omega \ge 0$, stationary when $\omega = 0$ and periodical of the period $2\pi/\omega$ when $\omega > 0$, and the non-stationary system* (1_{A_ω}) *with that coefficient matrix has solutions*

$$u_\omega : [0,+\infty) \to R^3 \setminus \{0\}, \quad y_\omega : [0,+\infty) \to R^3 \setminus \{0\}, \quad z_\omega : [0,+\infty) \to R^3 \setminus \{0\},$$

for which the inequalities

$$|(u_\omega(t), a(t)) - (u_\omega(0), a(0))| \ge c\|u_\omega(0)\| \int_0^t \|\dot a(\tau)\| d\tau, \quad t \in [0, t_c(A)], \tag{14_1}$$

$$|(y_\omega(t), v_{A_\omega}(t,t)) - (y_\omega(0), v_{A_\omega}(0,0))| \ge$$
$$\ge cd(\omega)\|y_\omega(0)\| \int_0^t \|a_\omega(\tau)\|\|\dot a_\omega(\tau)\| d\tau, \quad t \in [0, t_c(A_\omega)], \tag{14_2}$$

$$|(z_\omega(t), w_{A_\omega}(t,t)) - (z_\omega(0), w_A(0,0)| \ge$$
$$\ge cd(0)\|z_\omega(0)\| \int_0^t \|a_\omega(\tau)\| \cdot \|\dot a_\omega(\tau)\| d\tau, \quad t \in [0, t_c(A_\omega)], \tag{14_3}$$

hold when $\omega > 0$ for any constant $c \in (0,1)$ and the constant $d(\omega) = |\omega + a_{23}|/\sqrt{a_{12}^2 + a_{13}^2 + (\omega + a_{23})^2}$.

Comparing the estimates (7_1)–(7_3) and (14_1)–(14_3) of Theorems 1 and 2 respectively, we get their

Corollary. *The estimate* (7_1) *of Theorem* 1 *is exact, its estimates* (7_2) *and* (7_3) *are unimprovable up to a constant multiplier* $c \in [1, 2\sqrt{3}]$ *at the right-hand sides.*

References

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