Memoirs on Differential Equations and Mathematical Physics
Volume 39, 2006, 105–140

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SOLUTION OF SPATIAL AXIALLY SYMMETRIC PROBLEMS OF THE THEORY OF FILTRATION WITH PARTIALLY UNKNOWN BOUNDARIES

Abstract. In the present work we consider spatial axially symmetric stationary motions of incompressible liquid in a porous medium with partially unknown boundaries. The domain of liquid motion is bounded by an unknown depression curve and by known segments of lines, half-lines and lines. The liquid motion is subjected to the Darcy law. The porous medium is assumed to be undeformable, isotropic and homogeneous.

First, we prove that to the domain of the liquid motion, on the plane of complex velocity there corresponds a circular polygon of particular type. Then we construct an algorithm for solution of spatial axially symmetric problems of filtration with partially unknown boundaries. We construct an algorithm for finding three analytic functions, by means of which the half-plane is conformally mapped on a circular polygon, on the domain of liquid motion and on the domain of complex potential.

Finally, the construction of the solutions is reduced to the construction of solutions of integral and integro-differential equations, which are solved by the method of successive approximations. Here, the use is made of ordinary and generalized analytic functions. The systems of equations are set up for determination of unknown parameters of the problem of filtration, and equations are found for determination of unknown segments of boundaries.

2000 Mathematics Subject Classification. 35J55, 76S05.

Key words and phrases. Filtration, analytic functions, conformal mapping, differential equation.

რეზიუმე. ნაშრომში განიხილება ფოროვან არეში არაკუმშვადი სითხის სივრცული ღერძსიმეტრიული სტაციონარული მოძრაობა. სითხის მოძრაობის არე შემოსაზღვრულია უცნობი დეპრესიის მრუდით და ცნობილი წრფის მონაკვეთებით, ნახევარწრფეებით და წრფეებით. სითხის მოძრაობა ემორჩილება დარსის კანონს. ფორებიანი არე არადეფორმადია, იზოტროპულია და ერთგვაროვანი. ნაშრომში მტკიცდება, რომ სითხის მოძრაობის არეს კომპლექსური სიჩქარის სიბრტყეზე შეესაბამება კერძო სახის წრიული მრავალკუთხედი. აგებულია ფილტრაციის თეორიის ნაწილობრივ უცნობ საზღვრიანი სივრცული ღერძსიმეტრიული ამოცანების ამონახსნების მოძებნის ალგორითმი. აგებულია სამი ანალიზური ფუნქციის მოძებნის მეთოდი, რომელთა დახმარებით ნახევარი სიბრტყე კონფორმულად გადაისახება წრიულ მრავალკუთხედზე, სითხის მოძრაობის არეზე და კომპლექსური პოტენციალის არეზე.

ამოცანების ამონახსნების აგება საბოლოოდ დაიყვანება ინტეგრალური და ინტეგრო-დიფერენციალური განტოლებების ამონახსნების აგებამდე, ისინი კი მიმდევრობითი მიახლოების მეთოდით ამოიხსნებიან. აქ გამოყენებულია ჩვეულებრივი და განზოგადებული ანალიზური ფუნქციები. უცნობი პარამეტრების მოსაძებნად შედგენილია განტოლებათა სისტემები, საზღვრების უცნობი უბნების მოსაძებნად მიღებულია განტოლებები.

1. Liquid Motion with Axial Symmetry

In the present work we suggest an algorithm allowing one to construct solutions of spatial axially symmetric problems of the theory of filtration with partially unknown boundaries.

Let us consider a spatial axially symmetric problem with partially unknown boundaries of the theory of stationary motion of incompressible liquid in a porous medium, subject to the Darcy law. The porous medium is assumed to be non-deformable, isotropic and homogeneous [1]–[37].

The liquid motion is said to be axially symmetric, if all velocity vectors lie in half-planes which pass through a straight line called usually the axis of symmetry, and in all such half-planes the liquid motion is the same. The field of velocities of axially symmetric motion is completely described by a plane field on every such a half-plane. By z we denote the axis of symmetry which is directed vertically downwards, the distance to the axis Oz is denoted by $\rho = \sqrt{x^2+y^2}$, and V_z and V_ρ denote respectively the coordinates of the velocity vector $\vec{V}(V_z, V_\rho)$ which is connected with the velocity potential as follows: $\vec{V}(V_z, V_\rho) = \operatorname{grad}\varphi(z,\rho)$ [1]–[37].

Of the infinite set of half-planes we choose arbitrarily one, passing through the axis of symmetry; the moving liquid on it occupies a certain simply connected domain $S(\sigma)$ a part of whose boundary is unknown and should be defined, where $\sigma = z + i\rho$, $i = \sqrt{-1}$.

Below, we will need definition of the surface of revolution.

A surface F is said to be the surface of revolution if it is formed upon rotation of a curve γ around the axis of symmetry Oz. Let in the plane (z,ρ) the curve γ (the generatrix) be given by the parametric equations

$$z = \gamma(u), \quad x = \rho(u), \quad \text{where} \quad \rho(u) \geq 0.$$

We rotate it, as a rigid body, around the axis z. If we denote by v the angle of rotation, then the equation of the obtained surface of revolution can be written in the form

$$x = \rho(u)\cos v, \quad y = \rho(u)\sin v, \quad z = \gamma(u), \quad 0 \leq v \leq 2\pi.$$

The lines of intersection of the surface and the planes passing through the Oz-axis of rotation are called the meridians, while the lines of intersection with planes, perpendicular to the Oz-axis are called parallels. The lines $v = const$ are the surface meridians, and $u = const$ are the parallels.

The equation of continuity of the steady axially symmetric flow of incompressible liquid and the potentiality condition of its motion have respectively the form

$$\frac{\partial(\rho V_z)}{\partial z} + \frac{\partial(\rho V_\rho)}{\partial \rho} = 0, \tag{1.1}$$

$$\frac{\partial V_z}{\partial \rho} - \frac{\partial V_\rho}{\partial z} = 0. \tag{1.2}$$

From the condition (1.2) it follows that the expression $V_z\,dz + V_\rho\,d\rho$ is the exact differential $d\varphi(z,\rho) = \frac{\partial\varphi}{\partial z}\,dz + \frac{\partial\varphi}{\partial\rho}\,d\rho$ of the function $\varphi(z,\rho)$. The differential equation for every streamline of the spatial axially symmetric flow has the form

$$-V_\rho\,dz + V_z\,d\rho = 0. \tag{1.3}$$

According to (1.1), the expression

$$-\rho V_\rho\,dz + \rho V_z\,d\rho \tag{1.4}$$

is the exact differential of the stream function $\psi(z,\rho)$,

$$d\psi(z,\rho) = -\rho V_\rho\,dz + \rho V_z\,d\rho. \tag{1.5}$$

From (1.5) we have

$$V_z = \frac{1}{\rho}\,\frac{\partial\psi}{\partial\rho}\,, \quad V_\rho = -\frac{1}{\rho}\,\frac{\partial\psi}{\partial z}\,, \tag{1.6}$$

and (1.2) and (1.6) imply that

$$\frac{\partial}{\partial z}\left(\frac{1}{\rho}\,\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\,\frac{\partial\psi}{\partial\rho}\right) = 0. \tag{1.7}$$

Recall that for any harmonic function $\varphi(x,y,z)$ satisfying the Laplace condition $\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$ (which depends only on z and $\rho = \sqrt{x^2+y^2}$) we can find its conjugate function in the cylindrical coordinates $\psi(z,\rho)$, such that $\varphi(z,\rho)$ and $\psi(z,\rho)$ together satisfy according to $\vec{V}(V_z,V_\rho) = \operatorname{grad}\varphi(z,\rho)$ and (1.6) the system of differential equations ([1]–[10])

$$\rho\,\frac{\partial\varphi}{\partial z} = \frac{\partial\psi}{\partial\rho}\,, \quad \rho\,\frac{\partial\varphi}{\partial\rho} = -\frac{\partial\psi}{\partial z}\,. \tag{1.8}$$

However, it should be noted that unlike the plane case, the function $\psi(z,\rho)$ ceases now to be harmonic, but satisfies the equation (1.7).

We can see from (1.8) and (1.1) that the vector lines of the field of velocities coincide with tangential lines $\psi(z,\rho) = const$, hence $\psi(z,\rho)$ is, just as in the plane case, the stream function. It is seen from (1.8) that $\frac{\partial\varphi}{\partial z}\,\frac{\partial\psi}{\partial z} + \frac{\partial\varphi}{\partial\rho}\,\frac{\partial\psi}{\partial\rho} = 0$, hence the lines $\varphi(z,\rho) = const$ and $\psi(z,\rho) = const$ are orthogonal. However, the mapping $f(z,\rho) = \varphi(z,\rho) + i\psi(z,\rho)$ is not conformal; it transforms infinitely small squares into rectangles. Consequently, the mappings $f(z,\rho) = \varphi(z,\rho) + i\psi(z,\rho)$ under consideration constitute a class of quasi-conformal mappings.

The coefficient whose positivity determines the ellipticity of the system (1.8) in our case is equal to ρ^2. Therefore the system (1.8) is strongly elliptic only in the domains not adjacent to the axis of revolution $\rho = 0$ on which the system degenerates ([1]–[6]]).

As it has been noted above, $\psi(z,\rho)$ is the stream function. On every stream line the function $\psi(z,\rho)$ is constant, and hence it will be constant on the surface which is obtained by rotation of the given stream-line around the axis of symmetry. If on one of such surfaces $\psi(z,\rho) = c_0$ we fix an arbitrary point $A(z_0, \rho_0)$, and on the other surface $\psi(z, \rho) = c$ we take a point $P(z, \rho)$ and draw an arbitrary surface S_0 resting on the coaxial circumferences Γ_0 and Γ that lie on the given surfaces and pass through the points A and P, then the liquid volume Π flowing through the surface S_0 per unit of time can be expressed as the difference of the stream function values multiplied by 2π ([1]):

$$\Pi = 2\pi \left[\psi(z, \rho) - \psi(z_0, \rho_0)\right] = (c - c_0)2\pi. \tag{1.9}$$

It is well-known that the meridians and parallels of the surface of revolution form an orthogonal net.

The methods of solution of the problems of filtration in hydrodynamics are significantly simplified in the plane case owing to the fact that the velocity potential and the stream function form an analytic function, the so-called complex potential, and the theory of such functions is developed satisfactorily. Solution of the corresponding plane problems with partially unknown boundaries becomes more complicated in the general case, but despite this fact, the methods of their solving do exist ([1]–[37]). As for the methods of solution of spatial axially symmetric problems with partially unknown boundaries, we can with confidence say that because of great mathematical difficulties there are practically no problems solved in this area. But as regards spatial axially symmetric problems with known boundaries, we can find them in the works of T. Carleman, I. N. Vekua, M. A. Lavrent'ev, B. V. Shabat, G. N. Polozhiĭ, B. V. Boyarskiĭ, L. Bers, S. Bergman, and so on [1]–[19].

In the present work we will not concern ourselves with the method of springs and drains.

Of the whole class of functions $\varphi(z, \rho)$ and $\psi(z, \rho)$ satisfying the system of equations (1.8), we distinguish only such $\varphi(z, \rho)$ and $\psi(z, \rho)$ which depend on ρ^2, i.e. they are even with respect to ρ.

We can rewrite the system (1.7) as follows ([1]–[5]):

$$\frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^2 \varphi}{\partial \rho^2} + \frac{1}{\rho}\,\frac{\partial \varphi}{\partial \rho} = 0, \tag{1.10}$$

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho}\,\frac{\partial \psi}{\partial \rho} = 0, \tag{1.11}$$

and the system (1.10), (1.11) can be replaced by ([10])

$$\frac{\partial^2 \varphi}{\partial z^2} + 4\alpha\,\frac{\partial^2 \varphi}{\partial \alpha^2} + 4\,\frac{\partial \varphi}{\partial \alpha} = 0, \tag{1.12}$$

$$\frac{\partial^2 \psi}{\partial z^2} + 4\alpha\,\frac{\partial^2 \psi}{\partial \alpha^2} = 0, \tag{1.13}$$

where $\alpha = \rho^2$.

It is almost impossible to make use of the system (1.12), (1.13). If we have the initial system in the form (1.12), (1.13), then we can reduce it to the system (1.10), (1.11).

It is easily seen from (1.12), (1.13) that for $\alpha = \rho^2 \neq 0$ the system (1.12), (1.13) is strongly elliptic, and on the axis Oz, as $\alpha \to 0$, from (1.12) and (1.13) we obtain

$$\frac{\partial^2 \varphi}{\partial z^2} + 4\,\frac{\partial \varphi}{\partial \alpha} = 0, \tag{1.14}$$

$$\frac{\partial^2 \psi}{\partial z^2} = 0. \tag{1.15}$$

Along the axis of symmetry Oz we have

$$\begin{gathered}
\lim_{\rho\to 0} \frac{\partial \varphi}{\partial \rho} = 0, \quad \lim_{\rho\to 0} \frac{\partial \psi}{\partial \rho} = 0, \quad \lim_{\rho\to 0} \frac{\partial \psi}{\partial z} = 0, \\
\lim_{\rho\to 0} \frac{1}{\rho}\,\frac{\partial \varphi}{\partial \rho} = \frac{\partial^2 \varphi}{\partial \rho^2}\,, \quad \lim_{\rho\to 0} \frac{1}{\rho}\,\frac{\partial \psi}{\partial \rho} = \frac{\partial^2 \psi}{\partial \rho^2}\,.
\end{gathered} \tag{1.16}$$

The system (1.10), (1.11) by means of the transformations

$$\varphi(z,\rho) = \rho^{-1/2}\varphi_1(z,\rho), \quad \psi(z,\rho) = \rho^{1/2}\psi_1(z,\rho), \tag{1.17}$$

can be reduced to the form ([9]–[10])

$$\Delta\varphi_1(z,\rho) + \frac{1}{4}\,\rho^{-2}\varphi_1(z,\rho) = 0, \tag{1.18}$$

$$\Delta\psi_1(z,\rho) - \frac{3}{4}\,\rho^{-2}\psi_1(z,\rho) = 0. \tag{1.19}$$

The equations (1.18) and (1.19) are incompatible with the system (1.8). We have to take one of them, for example (1.18), solve it with respect to $\varphi_1(z,\rho)$ and then determine $\psi(z,\rho)$ from the system (1.8) with regard for $\rho^{-1/2}\varphi_1(z,\rho)$.

We map conformally the half-plane $\mathrm{Im}(\zeta) > 0$ (or $\mathrm{Im}(\zeta) < 0$) of the plane $\zeta = \xi + i\eta$ onto the domains

$$S(\sigma), \quad \sigma(\zeta) = z(\xi,\eta) + i\rho(\xi,\eta), \quad f(\zeta) = f(\xi + i\eta). \tag{1.20}$$

The system (1.8) on the plane $\zeta = \xi + i\eta$ takes the form

$$\frac{\partial \varphi}{\partial \xi} = \frac{1}{\rho(\xi,\eta)}\,\frac{\partial \psi}{\partial \eta}\,, \tag{1.21}$$

$$\frac{\partial \varphi}{\partial \eta} = -\frac{1}{\rho(\xi,\eta)}\,\frac{\partial \psi}{\partial \xi}\,, \tag{1.22}$$

that is,

$$\frac{\partial}{\partial \xi}\left[\rho(\xi,\eta)\,\frac{\partial \varphi(\xi,\eta)}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[\rho(\xi,\eta)\,\frac{\partial \varphi(\xi,\eta)}{\partial \eta}\right] = 0, \tag{1.23}$$

$$\frac{\partial}{\partial \xi}\left[\frac{1}{\rho(\xi,\eta)}\,\frac{\partial \psi(\xi,\eta)}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[\frac{1}{\rho(\xi,\eta)}\,\frac{\partial \psi(\xi,\eta)}{\partial \eta}\right] = 0. \tag{1.24}$$

In turn, the system (1.23), (1.24) can be written as

$$\Delta\varphi(\xi,\eta) + \frac{1}{\rho(\xi,\eta)}\left(\frac{\partial\rho}{\partial\xi}\frac{\partial\varphi}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\frac{\partial\varphi}{\partial\eta}\right) = 0, \tag{1.25}$$

$$\Delta\psi(\xi,\eta) - \frac{1}{\rho(\xi,\eta)}\left(\frac{\partial\rho}{\partial\xi}\frac{\partial\psi}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\frac{\partial\psi}{\partial\eta}\right) = 0. \tag{1.26}$$

On the axis of symmetry Oz, the equalities

$$\begin{gathered}\rho(\xi,\eta) = 0, \quad \frac{\partial\rho(\xi,\eta)}{\partial\xi} = 0, \quad \frac{\partial\rho(\xi,\eta)}{\partial\eta} = 0, \quad \psi(\xi,\eta) = 0, \\ \frac{\partial\psi(\xi,\eta)}{\partial\xi} = 0, \quad \frac{\partial\psi(\xi,\eta)}{\partial\eta} = 0, \quad \frac{\partial\varphi}{\partial\rho} = 0, \quad \frac{\partial\varphi}{\partial\xi}\frac{\partial\xi}{\partial\rho} + \frac{\partial\varphi}{\partial\eta}\frac{\partial\eta}{\rho} = 0\end{gathered} \tag{1.27}$$

hold.

Introducing new unknown functions by the formulas

$$\varphi(\xi,\eta) = \rho^{-1/2}(\xi,\eta)\varphi(\xi,\eta), \quad \psi(\xi,\eta) = \rho^{1/2}(\xi,\eta)\psi(\xi,\eta), \tag{1.28}$$

we reduce the system (1.25), (1.26) to the form

$$\Delta\varphi_1(\xi,\eta) + \frac{1}{4}\left[\left(\frac{1}{\rho(\xi,\eta)}\frac{\partial\rho}{\partial\xi}\right)^2 + \left(\frac{1}{\rho(\xi,\eta)}\frac{\partial\rho}{\partial\eta}\right)^2\right]\varphi_1(\xi,\eta) = 0, \tag{1.29}$$

$$\Delta\psi_1(\xi,\eta) - \frac{3}{4}\left[\left(\frac{1}{\rho(\xi,\eta)}\frac{\partial\rho}{\partial\xi}\right)^2 + \left(\frac{1}{\rho(\xi,\eta)}\frac{\partial\rho}{\partial\eta}\right)^2\right]\psi_1(\xi,\eta) = 0. \tag{1.30}$$

But (1.29) and (1.30) are incompatible with (1.21) and (1.22). From the equation (1.29) (or from (1.30)) we can find $\varphi_1(\xi,\eta)$ by means of rapidly converging series, and then multiplying $\varphi_1(\xi,\eta)$ by $\rho^{-1/2}(\xi,\eta)$, we can determine $\varphi(\xi,\eta)$. After such an operation, we can determine $\psi(\xi,\eta)$ from the system of equations (1.21), (1.22). We can use the system of equations (1.18), (1.19) analogously.

2. The Boundary Conditions ([14]–[19])

(a) The boundary conditions on a free unknown surface (on a depression curve) have the form

$$\varphi(z,\rho) - kz = const, \tag{2.1}$$

$$\psi(z,\rho) = const, \tag{2.2}$$

where k is the coefficient of filtration;

(b) along the boundaries of water basins,

$$\varphi(z,\rho) = const, \tag{2.3}$$

$$a_1 z + b_1\rho + c_1 = 0, \quad a_1, b_1, \text{ and } c_1 = const; \tag{2.4}$$

(c) along the leakage interval,

$$\varphi(z,\rho) - kz = const, \tag{2.5}$$

$$a_2 z + b_2\rho + c_2 = 0, \quad a_2, b_2, \text{ and } c_2 = const; \tag{2.6}$$

(d) along the axis of symmetry, when the segment of the axis Oz coincides with some part of the boundary of $S(\sigma)$, the boundary conditions have the form

$$\rho = 0, \tag{2.7}$$

$$\psi(z, \rho) = 0, \tag{2.8}$$

while if the axis of symmetry does not coincide with a part of the boundary of $S(\sigma)$, then

$$\rho \neq 0, \quad \rho = const, \tag{2.9}$$

$$\psi(z, \rho) = const. \tag{2.10}$$

For example, in the case (a) we may have a circular axially symmetric basin with water filtration into the earth, while in the case (b) we may assume that we have a single perfect circular well of finite radius, which is constructed on a nonpermeable base to which the liquid flows symmetrically, and there is a free surface and a leakage interval.

(e) Along nonpermeable boundaries there take place the following boundary conditions:

$$\psi(z, \rho) = const, \tag{2.11}$$

$$a_3 z + b_3 \rho + c_3 = 0, \quad a_3, b_3, \text{ and } c_3 = const. \tag{2.12}$$

Along a nonpermeable boundary, the vector of velocity is directed along that boundary.

Suppose that the nonpermeable boundary forms with the Ox-axis the angle $(\pi\alpha)$. Then differentiating (2.11) and (2.12), we obtain

$$\rho(V_\rho \cos(s, z) - V_z \cos(s, \rho)) = 0, \tag{2.13}$$

$$a_3 \cos(s, z) + b_3 \cos(s, \rho) = 0, \tag{2.14}$$

where $\cos(s, z)$ and $\cos(s, \rho)$ are the direction cosines. Then it follows from (2.13) and (2.14) that $V_\rho / V_z = \operatorname{tg}(\pi\alpha)$. Hence on the plane (V_z, V_ρ) we have the straight line passing through the origin and parallel to the boundary.

The boundary of the water basin (2.3), (2.4) is an equipotential line, and therefore the vector of velocity is perpendicular to that boundary. Suppose that the angular coefficient of the equipotential line is of the form $\operatorname{tg}(\pi\beta) = -a_1/b_1$. Then differentiating (2.3) and (2.4), we obtain

$$V_z \cos(s, z) + V_\rho \cos(s, \rho) = 0, \tag{2.15}$$

$$a_1 \cos(s_1, z) + b_1 \cos(s_1, \rho) = 0. \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$V_\rho / V_z = -\operatorname{ctg}(\pi\beta). \tag{2.17}$$

Thus we have obtained the equation of the straight line passing through the origin and perpendicular to the segment of the boundary under consideration.

Along the free surface we differentiate (2.1) and (2.2) to obtain

$$\frac{\partial\varphi}{\partial z}\cos(s,z)+\frac{\partial\varphi}{\partial\rho}\cos(s,\rho)-k\cos(s,z)=0, \tag{2.18}$$

$$\frac{\partial\psi}{\partial z}\cos(s,\rho)+\frac{\partial\psi}{\partial z}\cos(s,z)=0. \tag{2.19}$$

We rewrite (2.19) as follows:

$$-\rho V_z\cos(s,\rho)+\rho V_\rho\cos(s,z)=0. \tag{2.20}$$

Then dividing both parts of the equality (2.20) by ρ ($\rho\neq 0$) and excluding $\cos(s,\rho)/\cos(s,z)$ from the system (2.18),(2.20), we obtain ([19])

$$V_z^2+V_\rho^1-kV_z=0. \tag{2.21}$$

(2.21) is the equation of the circumference of radius $k/2$ passing through the points $(0,0)$ and $(0,k)$, with the center at the point $(0,k/2)$. The formulas (2.13)–(2.21) imply that on the plane (V_z,V_ρ) we obtain a circular polygon of particular type. The boundary conditions (2.1)–(2.10) can be written in a more general form,

$$a_{k1}\varphi(z,\rho)+a_{k2}\psi(z,\rho)+a_{k2}\rho+a_{k4}z=f_k,\quad k=1,2, \tag{2.22}$$

where a_{kj}, f_k, $k=1,2$, $j=\overline{1,n}$, are piecewise constant functions.

If we take arbitrarily a part of the boundary $\ell(\sigma)$ of the domain $S(\sigma)$ and then differentiate the condition (2.22) along the chosen segment of the boundary with respect to the parameter s, we will obtain

$$\begin{aligned}a_{k1}\varphi'_z\cos(s,z)+a_{k1}\varphi'_\rho\cos(s,\rho)+a_{k2}\psi'_z\cos(s,z)+a_{k2}\psi'_\rho\cos(s,\rho)+&\\ +a_{k3}\cos(s,\rho)+a_{k4}\cos(s,z)=0,\quad k=1,2.&\end{aligned} \tag{2.23}$$

The conditions (2.23) can be rewritten as

$$(a_{k1}\varphi'_z+a_{k2}\psi'_z+a_{k4})\cos(s,z)+(a_{k1}\varphi'_\rho+a_{k2}\psi'_\rho+a_{k3})\cos(s,\rho)=0, \tag{2.24}$$

where $\frac{dz}{ds}=\cos(s,z)$, $\frac{d\rho}{ds}=\cos(s,\rho)$.

For a nontrivial solution of the system (2.24) with respect to $\cos(s,z)$ and $\cos(s,\rho)$ to exist, it is necessary and sufficient that the determinant of the system be equal to zero,

$$\begin{aligned}\rho(a_{11}a_{22}-a_{21}a_{12})[(\varphi'_z)^2+(\varphi'_\rho)^2]+[(a_{11}a_{23}-a_{21}a_{13})+&\\ +\rho(a_{14}a_{22}-a_{24}a_{12})]\varphi'_z+[\rho(a_{22}a_{13}-a_{12}a_{23})+&\\ +(a_{14}a_{21}-a_{24}a_{11})]\varphi'_\rho+a_{14}a_{23}-a_{24}a_{13}=0.&\end{aligned} \tag{2.25}$$

On the coefficients a_{kj}, $k=1,2$, $j=\overline{1,4}$, we imply the following restrictions:

$$a_{11}a_{23}-a_{21}a_{13}=0,\quad a_{14}a_{21}-a_{24}a_{11}=0,\quad a_{14}a_{23}-a_{24}a_{13}=0. \tag{2.26}$$

It follows from (2.26) that

$$\frac{a_{11}}{a_{21}}=\frac{a_{13}}{a_{23}}=\frac{a_{14}}{a_{24}}\,. \tag{2.27}$$

Taking into account (2.27), from (2.25) we obtain

$$\rho \neq 0, \quad A_{11}[(\varphi_z')^2 + (\varphi_\rho')^2] + A_{12}\varphi_z' + A_{13}\varphi_\rho' = 0, \tag{2.28}$$

where

$$\begin{gathered} A_{11} = a_{11}a_{22} - a_{21}a_{12}, \quad A_{12} = a_{14}a_{22} - a_{24}a_{12}, \\ A_{13} = a_{22}a_{13} - a_{13}a_{23}. \end{gathered} \tag{2.29}$$

With regard for (2.27) we can conclude that the system (2.22) depends on eight parameters, a_{kj}, f_k, $k = 1, 2$, $j = \overline{1, 4}$, while the equation of the circular polygon (2.28) depends on two parameters. This means that we obtain a circular polygon of particular type. The equation of the circular polygon (2.28) can be rewritten in the form

$$w = \frac{\overline{w}\overline{B}}{iA_{11}\overline{w} + B}, \quad B = \frac{1}{2}(A_{13} + iA_{12}), \quad \overline{B} = \frac{1}{2}(A_{13} - iA_{12}), \tag{2.30}$$

$$w = \varphi_z' - i\varphi_\rho', \quad \overline{w} = \varphi_z' + i\varphi_\rho'. \tag{2.31}$$

Using the boundary conditions, we have established that on the plane of complex velocity the domain of velocity variation is a circular polygon of particular type. There exist axially symmetric spatial problems with partially unknown boundaries, when the boundary of the domain is free from the axis of symmetry, but there are also problems, when, as is said in Section 2, the boundary of the domain involves the axis of symmetry.

For circular polygons we are able to solve plane problems of filtration with partially unknown boundaries. Statement and solution of such problems is given in Sections 3–4.

Assume that we have solved the plane problem, i.e., constructed analytic functions by which the half-plane $\mathrm{Im}(\zeta) \geq 0$ (or $\mathrm{Im}(\zeta) < 0$) of the plane $\zeta = \xi + i\eta$ is mapped conformally onto the circular polygon. For general reasoning, assume that there is a circular polygon with m vertices. To find such an analytic function, we have to solve the nonlinear Schwartz differential equation of third order whose solution is reduced to solution of the Fuchs differential equation. The Schwartz, and hence the corresponding Fuchs equations contain $2(m-3)$ unknown parameters. After integration of the Schwartz equation there appear additionally six parameters of integration. Thus we set up a system of $2(m-3)$ higher transcendent equations and also a system of six equations for finding the parameters of integration. The boundary conditions for the problems of filtration contain additional unknown parameters. Further, by means of solutions of the plane problems we construct solutions $\varphi(\xi, \eta)$ and $\psi(\xi, \eta)$ for the systems (1.21), (1.22) of differential equations of the spatial axially symmetric problems and then by means of the latter we construct functions which map quasi-conformally the plane $\mathrm{Im}(\zeta) \geq 0$ both onto the domain of a complex potential and onto the domains of complex velocity, i.e., onto $S(\omega)$, $S(\omega'(\zeta)/\sigma'(\zeta))$.

Here we introduce the notation for three analytic functions

$$\begin{gathered}\sigma_0(\zeta) = z_0(\xi,\eta) + i\rho_0(\xi,\eta), \quad \omega_0(\zeta) = \varphi_0(\xi,\eta) + i\psi_0(\xi,\eta), \\ \chi_0(\zeta) = \omega_0'(\zeta)/\sigma_0'(\zeta), \quad \mathrm{Im}(\zeta) \geq 0,\end{gathered} \tag{2.32}$$

which respectively map conformally the plane $\mathrm{Im}(\zeta) \geq 0$ onto the domain $S(\sigma)$ of liquid motion, onto the domains of complex potential $S(\omega)$, and onto the domain of complex velocity $S(\omega'(\zeta)/\sigma'(\zeta))$. Below, we will need the Dirichlet problem for the half-plane.

Let on the real axis a bounded function $u(\zeta)$ be given with a finite number of points of discontinuity. To find at the point $\zeta = \xi + i\eta$ the value of the harmonic in the upper half-plane function, taking the given values on the real axis, we have to use the Poisson integral for the upper half-plane:

$$u(\zeta) = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} u(t)\, \frac{\eta\, dt}{(t-\xi)^2 + \eta^2}\,, \quad \zeta = \xi + i\eta. \tag{2.33}$$

Statement of the problem. Find in the domain $\mathrm{Im}(\zeta) > 0$ functions $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$, solutions of the system of equations (1.25), (1.28), connected with each other by (1.21), (1.22), which satisfy the boundary conditions $\varphi(\xi,\eta)|_{\eta=0} = \varphi_0(\xi)$, $\psi(\xi,\eta)|_{\eta=0} = \psi_0(\xi)$ and certain conditions at infinity ($\xi \to \infty$).

We write the system of equations (1.25), (1.26) as follows:

$$\Delta\varphi(\xi,\eta) + L[\varphi(\xi,\eta)] = 0, \tag{2.34}$$

$$\Delta\psi(\xi,\eta) - L[\psi(\xi,\eta)] = 0, \tag{2.35}$$

where the function

$$Lf(\xi,\eta) = \frac{1}{\rho(\xi,\eta)} \left[\frac{\partial\rho(\xi,\eta)}{\partial\xi}\,\frac{\partial f(\xi,\eta)}{\partial\xi} + \frac{\partial\rho(\xi,\eta)}{\partial\eta}\,\frac{\partial f(\xi,\eta)}{\partial\eta}\right] \tag{2.36}$$

is bounded in the domain $\mathrm{Im}(\zeta) \geq 0$.

A solution of the system (2.34), (2.35) is sought in the form

$$\varphi(\xi,\eta) = u(\xi,\eta) + u^*(\xi,\eta), \tag{2.37}$$

$$\psi(\xi,\eta) = v(\xi,\eta) + v^*(\xi,\eta), \tag{2.38}$$

where $u^*(\xi,\eta)$ and $v^*(\xi,\eta)$ are conjugate harmonic functions satisfying the conditions

$$\begin{gathered}\Delta u^*(\xi,\eta) = 0, \quad \Delta v^*(\xi,\eta) = 0, \quad \Delta = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}, \\ \frac{\partial u^*(\xi,\eta)}{\partial\xi} = \frac{\partial v^*(\xi,\eta)}{\partial\eta}, \quad \frac{u^*(\xi,\eta)}{\partial\eta} = -\frac{\partial v^*(\xi,\eta)}{\partial\xi}, \\ u^*(\xi,\eta)|_{\eta=0} = \varphi(\xi), \quad v^*(\xi,\eta)|_{\eta=0} = \psi(\xi).\end{gathered} \tag{2.39}$$

The system of equations (2.34), (2.35) takes with respect to $u(\xi,\eta)$ and $v(\xi,\eta)$ the form

$$\Delta u(\xi,\eta) + L[u(\xi,\eta) + u^*(\xi,\eta)] = 0, \tag{2.40}$$

$$\Delta v(\xi,\eta) - L[v(\xi,\eta) + v^*(\xi,\eta)] = 0, \tag{2.41}$$

while the boundary conditions can be written as

$$u(\xi,\eta)|_{\eta=0} = 0, \quad v(\xi,\eta)|_{\eta=0} = 0. \tag{2.42}$$

The system (2.40), (2.41) with regard for (2.42) is reduced to the system of integro-differential equations

$$u(\xi,\eta) = \frac{1}{2\pi} \iint\limits_{\operatorname{Im}(\zeta)\geq 0} G(\xi,\eta;x,y) L[u(x,y) + u^*(x,y)]\,dx\,dy, \tag{2.43}$$

$$v(\xi,\eta) = \frac{1}{2\pi} \iint\limits_{\operatorname{Im}(\zeta)\geq 0} G(\xi,\eta;x,y) L[v(x,y) + v^*(x,y)]\,dx\,dy, \tag{2.44}$$

where $G(\xi,\eta;x,y)$ is Green's function of the Dirichlet problem for harmonic functions in the domain $\operatorname{Im}(\zeta) \geq 0$,

$$G(\xi,\eta;x,y) = \frac{1}{4\pi} \ln \frac{(\xi-x)^2 + (\eta-y)^2}{(\xi-x)^2 + (\eta+y)^2}\,. \tag{2.45}$$

The system of integro-differential equations (2.43), (2.44) can be solved by the method of successive approximations according to the following scheme:

$$u_n(\xi,\eta) = \frac{1}{2\pi} \iint\limits_{\operatorname{Im}(\zeta)\geq 0} G(\xi,\eta;x,y) L[u_{n-1}(x,y) + u^*(x,y)]\,dx\,dy, \tag{2.46}$$

$$v_n(\xi,\eta) = \frac{1}{2\pi} \iint\limits_{\operatorname{Im}(\zeta)\geq 0} G(\xi,\eta;x,y) L[v_{n-1}(x,y) + v^*(x,y)]\,dx\,dy, \tag{2.47}$$

$$n = 1, 2, \dots ;$$

$$u_0(\xi,\eta) = 0, \quad v_0(\xi,\eta) = 0, \tag{2.48}$$

$$u_1(\xi,\eta) = \frac{1}{2\pi} \iint\limits_{\operatorname{Im}(\zeta)\geq 0} G(\xi,\eta;x,y) L[u^*(x,y)]\,dx\,dy, \tag{2.49}$$

$$v_1(\xi,\eta) = \frac{1}{2\pi} \iint\limits_{\operatorname{Im}(\zeta)\geq 0} G(\xi,\eta;x,y) L[v^*(x,y)]\,dx\,dy. \tag{2.50}$$

To construct the system of functions $u^*(\xi,\eta)$ and $v^*(\xi,\eta)$, we have to use the results stated in Sections 3–4, where we construct first the system of functions (2.39) on the boundary $\eta = 0$ by the given conditions, then by means of the latter we reestablish $u^*(\xi,\eta)$ and $v^*(\xi,\eta)$ in the domain $\operatorname{Im}(\zeta) > 0$ by the formula (2.33). Using the above-mentioned functions, we construct $u_n(\xi,\eta)$, $v_n(\xi,\eta)$, $n = 1,2,\dots$, and then $\varphi_n(\xi,\eta) = u_n(\xi,\eta) + u^*(\xi,\eta)$, $\psi_n(\xi,\eta) = v_n(\xi,\eta) + v^*(\xi,\eta)$.

Thus we have constructed the analytic functions $\sigma_0(\zeta) = z_0(\xi,\eta) + i\sigma_0(\xi,\eta)$, $\omega_0(\zeta) = \varphi_0(\xi,\eta) + i\psi_0(\xi,\eta)$ and $w_0(\xi) = \omega_0'(\zeta)/\sigma_0'(\zeta)$ which allow us to map conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ onto the domains $S(\sigma_0)$, $S(\omega_0)$ and $S(w_0)$. For construction of these functions, the use is made of general solution of the Schwartz equation, the boundary conditions and the general Riemann theory; these conditions ensure the existence and uniqueness of the functions. To prove the existence and uniqueness of the functions $\omega(\zeta) = \varphi(\xi,\eta) + i\psi(\xi,\eta)$, $\chi(\zeta) = \omega'(\zeta)/\sigma_0'(\zeta)$ which map quasi-conformally the half-plane $\mathrm{Im}(\zeta) > 0$ respectively onto the domains $S(\omega)$ and $S(w)$, it is sufficient to prove the uniqueness of the functions $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$. Suppose there exist two pairs of distinct sequences u^1, u^2 and v^1, v^2. Then the differences $u_n^1 - u_n^2$ and $v_n^1 - v_n^2$, $n = 1,2,\dots$, are equal to zero. Indeed,

$$u_1^1(\xi,\eta) - u_1^2(\xi,\eta) = \\ = \frac{1}{2\pi} \iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y) L[u^*(x,y) - u^*(x,y)]\,dx\,dy = 0, \tag{2.51}$$

$$\begin{aligned} &u_n^1(\xi,\eta) - u_n^2(\xi,\eta) = 0, \quad n = 2,3,\dots\,; \\ &v_n^1(\xi,\eta) - v_n^2(\xi,\eta) = 0, \quad n = 2,3,\dots\,, \end{aligned} \tag{2.52}$$

and hence

$$\lim_{n\to\infty} \left\{u_n^1(\xi,\eta) - u_n^2(\xi,\eta)\right\} = 0, \quad \lim_{n\to\infty} \left\{v_n^1(\xi,\eta) - v_n^2(\xi,\eta)\right\} = 0, \tag{2.53}$$

$$\lim_{n\to 0} \left\{\varphi_n^1(\xi,\eta) - \varphi_n^2(\xi,\eta)\right\} = 0, \quad \lim_{n\to 0} \left\{\psi_n^1(\xi,\eta) - \psi_n^2(\xi,\eta)\right\} = 0. \tag{2.54}$$

The uniqueness of the analytic function $\sigma_0(\zeta)$ have been proved by us above. As for the function $\frac{d\omega(\zeta)}{d\zeta} = \omega'(\zeta)$, its uniqueness follows from (2.53) and (2.54).

Another way of Solving the Problem. We can write the system (1.25), (1.26) as follows:

$$\Delta[\sqrt{\rho(\xi,\eta)}\,\varphi(\xi,\eta)] + \frac{1}{4\rho^2}\left[\left(\frac{\partial\rho(\xi,\eta)}{\partial\xi}\right)^2 + \left(\frac{\partial\rho(\xi,\eta)}{\partial\eta}\right)^2\right] \times \\ \times [\sqrt{\rho(\xi,\eta)}\,\varphi(\xi,\eta)] = 0, \tag{2.55}$$

$$\Delta[\psi(\xi,\eta)/\sqrt{\rho(\xi,\eta)}\,] - \frac{3}{4\rho^2}\left[\left(\frac{\partial\rho(\xi,\eta)}{\partial\xi}\right)^2 + \left(\frac{\partial\rho(\xi,\eta)}{\partial\eta}\right)^2\right] \times \\ \times [\psi(\xi,\eta)/\sqrt{\rho(\xi,\eta)}\,] = 0. \tag{2.56}$$

The system (2.55), (2.56) is reduced to the system of integral equations:

$$\sqrt{\rho(\xi,\eta)}\,\varphi(\xi,\eta) = u_0(\xi,\eta) +$$

$$+\frac{1}{8\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\bigg\{\frac{1}{\rho^2(x,y)}\left[\left(\frac{\partial\rho(x,y)}{\partial x}\right)^2+\left(\frac{\partial\rho(x,y)}{\partial y}\right)^2\right]\times$$

$$\times\left[\sqrt{\rho(x,y)}\,\varphi(x,y)+u_0(x,y)\right]\bigg\}dx\,dy, \tag{2.57}$$

$$\psi(\xi,\eta)/\sqrt{\rho(\xi,\eta)}=v_0(\xi,\eta)-$$

$$-\frac{3}{8\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\bigg\{\frac{1}{\rho^2(x,y)}\left[\left(\frac{\partial\rho(x,y)}{\partial x}\right)^2+\left(\frac{\partial\rho(x,y)}{\partial y}\right)^2\right]\times$$

$$\times\left[\psi(\xi,\eta)/\sqrt{\rho(x,y)}+v_0(x,y)\right]\bigg\}dx\,dy, \tag{2.58}$$

where $u_0(\xi,\eta)$, $v_0(\xi,\eta)$ satisfy the conditions

$$\begin{gathered}\Delta u_0(\xi,\eta)=0,\quad \Delta v_0(\xi,\eta)=0,\\ \frac{\partial u_0(\xi,\eta)}{\partial\xi}=\frac{\partial v_0(\xi,\eta)}{\partial\eta},\quad \frac{\partial u_0(\xi,\eta)}{\partial\eta}=-\frac{\partial v_0(\xi,\eta)}{\partial\xi}\,;\end{gathered} \tag{2.59}$$

$G(\xi,\eta;x,y)$ is Green's function of the Dirichlet problem for harmonic functions in the domain $\mathrm{Im}(\zeta)\geq 0$.

The functions $u_0(\xi,\eta)$ and $v_0(\xi,\eta)$ satisfy the boundary conditions

$$u_0(\xi,\eta)|_{\eta=0}=u_0(\xi),\quad v_0(\xi,\eta)|_{\eta=0}=v_0(\xi). \tag{2.60}$$

A solution of the system (2.57) and (2.58) is sought by the method of successive approximations according to the following scheme:

$$\sqrt{\rho(\xi,\eta)}\,\varphi_n(\xi,\eta)=u_0(\xi,\eta)+$$

$$+\frac{1}{8\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\bigg\{\frac{1}{\rho^2(x,y)}\left[\left(\frac{\partial\rho(x,y)}{\partial x}\right)^2+\left(\frac{\partial\rho(x,y)}{\partial y}\right)^2\right]\times$$

$$\times\left[\sqrt{\rho(x,y)}\,\varphi_{n-1}(x,y)+u_0(x,y)\right]\bigg\}dx\,dy,\quad n=1,2,\dots\,, \tag{2.61}$$

$$\psi_n(\xi,\eta)/\sqrt{\rho(\xi,\eta)}=v_0(\xi,\eta)-$$

$$-\frac{3}{8\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\bigg\{\frac{1}{\rho^2(x,y)}\left[\left(\frac{\partial\rho(x,y)}{\partial x}\right)^2+\left(\frac{\partial\rho(x,y)}{\partial y}\right)^2\right]\times$$

$$\times\left[\psi_{n-1}(\xi,\eta)/\sqrt{\rho(x,y)}+v_0(x,y)\right]\bigg\}dx\,dy,\quad n=1,2,\dots\,. \tag{2.62}$$

Assume that $\varphi_0(\xi,\eta)=0$, $\psi_0(\xi,\eta)=0$,

$$\sqrt{\rho(\xi,\eta)}\,\varphi_1(\xi,\eta)=u_0(\xi,\eta)+$$

$$+\frac{1}{8\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\bigg\{\frac{1}{\rho^2(x,y)}\left[\left(\frac{\partial\rho(x,y)}{\partial x}\right)^2+\left(\frac{\partial\rho(x,y)}{\partial y}\right)^2\right]\times$$

$$\times\, u_0(x,y)\Bigg\}dx\,dy, \tag{2.63}$$

$$\psi_1(\xi,\eta)/\sqrt{\rho(\xi,\eta)} = v_0(\xi,\eta)-$$

$$-\,\frac{3}{8\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\Bigg\{\frac{1}{\rho^2(x,y)}\left[\left(\frac{\partial\rho(x,y)}{\partial x}\right)^2+\left(\frac{\partial\rho(x,y)}{\partial y}\right)^2\right]\times$$

$$\times\, v_0(x,y)\Bigg\}dx\,dy. \tag{2.64}$$

The sequences

$$\varphi_n(\xi,\eta),\quad \psi_n(\xi,\eta),\quad n=0,1,2,\dots\,, \tag{2.65}$$

converge, since there take place the inequalities

$$|\lambda| = 1/(8\pi) < 1;\quad |\lambda| = 3/(8\pi) < 1. \tag{2.66}$$

The uniqueness of the functions $u_0(\xi,\eta)$ and $v_0(\xi,\eta)$ follows from Sections 3–4, while that of the functions $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$ follows from Sections 1–2.

3. Solution of the Plane Problem of Filtration with Partially Unknown Boundaries

The plane of steady incompressible liquid motion in a porous medium, subject to the Darcy law, coincides with the plane of the complex variable $\sigma = z + i\rho$.

The porous medium is assumed to be isotropic, homogeneous and nondeformable. The boundary $\ell(\sigma)$ of the domain $S(\sigma)$ of liquid motion consists of a depression curve to be defined, and of the known segments of straight lines, half-lines and lines.

In the domain $S(\sigma)$ with the boundary $\ell(\sigma)$ we seek for a complex potential $\omega_0(\sigma) = \varphi_0(z,\rho) + i\psi_0(z,\rho)$, where $\varphi_0(z,\rho)$ is the velocity potential and $\psi_0(z,\rho)$ is the flow function, satisfying the Cauchy–Riemann conditions and the following boundary conditions:

$$a^0_{k1}\varphi_1(z,\rho)+a^0_{k2}\psi(z,\rho)+a^0_{k3}z+a^0_{k4}\rho = f_{0k},\quad k=1,2,\quad (z,\rho)\in\ell(\sigma), \tag{3.1}$$

where a^0_{kj}, f_{0k}, $k=1,2$, $j=\overline{1,4}$, are given piecewise constant real functions ([7], [14]–[19], [32]–[37]).

The boundary conditions (3.1) allow one to determine a part of the boundary $\ell(\omega_0)$ of the domain $S(\omega_0)$, as well as the boundary $\ell(w_0)$ of the domain of the complex velocity $w_0(\sigma) = \omega_0'(\sigma) = d\omega_0(\sigma)/d\sigma_0$, with the exception of some vertex coordinates of the circular polygons $S(w_0)$ ([7], [14]–[19], [32]–[37]). By the functions $\omega_0(\sigma)$ and $w_0(\sigma)$, the domain $S(\sigma)$ with the boundary $\ell(\sigma)$ is mapped conformally respectively onto the domains $S(\omega_0)$ and $S(w_0)$ with the boundaries $\ell(\omega_0)$ and $\ell(w_0)$, where $S(w_0)$ is a circular polygon with the boundary $\ell(w_0)$ consisting of a finite number of circular arcs which, in particular, may be segments of straight line, half-lines and lines.

We denote by A_k, $k=\overline{1,n}$, the angular points of the boundaries $\ell(\sigma_0)$, $\ell(\omega_0)$ and $\ell(w_0)$ encountered on them when going round in the positive direction.

For solving the problem of filtration, the half-plane $\mathrm{Im}(\zeta)>0$ (or $\mathrm{Im}(\zeta)<0$) of the plane $\zeta=\xi+i\eta$, $i=\sqrt{-1}$, is mapped conformally onto the domains $S(\sigma_0$, $S(\omega_0)$ and $S(w_0)$. We denote the corresponding mapping functions by $\sigma_0(\zeta)$, $\omega_0(\zeta)$ and $w_0(\zeta)=\omega_0(\zeta)/\sigma_0'(\zeta)$, $d\omega_0(\zeta)/d\zeta=\omega_0'(\zeta)$, $d\sigma_0(\zeta)/d\zeta=\sigma_0'(\zeta)$.

To the angular points A_k, $k=\overline{1,4}$, along the axis ξ there correspond the points $\xi=e_k$, $k=\overline{1,n+1}$, where $-\infty<e_1<e_2<\cdots<e_n<+\infty$. The point $t=e_{n+1}=\infty$ is mapped into a nonangular point A_∞ of the boundary $\ell(\sigma_0)$, which lies between the points A_n and A_1.

We denote by $\sigma_0(\xi)=z_0(\xi)+i\rho(\xi)$, $\omega_0(\xi)=\varphi_0(\xi)+i\psi_0(\xi)$, $w_0(\xi)=u_0(\xi)-iv_0(\xi)$ the boundary values of the functions $\sigma_0(\zeta)$, $\omega_0(\zeta)$ and $w_0(\zeta)$, as $\zeta\to\xi$, $\zeta\in\mathrm{Im}(\zeta)>0$, and by $\overline{\sigma_0(\xi)}$, $\overline{\omega_0(\xi)}$ and $\overline{w_0(\xi)}$ we denote the functions complex-conjugate to the functions $\sigma_0(\xi)$, $\omega_0(\xi)$ and $w_0(\xi)$, respectively.

Introduce the vectors $\Phi(\xi)=[\omega_0(\xi),\sigma_0(\xi)]$, $\overline{\Phi}(\xi)=[\overline{\omega_0(\xi)},\overline{\sigma_0(\xi)}]$, $\Phi'(\xi)=[\omega_0'(\xi),\sigma_0'(\xi)]$, $\overline{\Phi'(\xi)}=[\overline{\omega_0'(\xi)},\overline{\sigma_0'(\xi)}]$, $f(\xi)=[f_1(\xi),f_2(\xi)]$. Then the boundary conditions can be written as follows:

$$\Phi(\xi)=g(\xi)\overline{\Phi(\xi)}+i2G^{-1}(\xi)f(\xi),\quad -\infty<\xi<+\infty, \tag{3.2}$$

where $g(\xi)=G^{-1}(\xi)\overline{G(\xi)}$ is a piecewise constant nonsingular matrix of the second order with the points $\xi=e_k$, $k=\overline{1,n}$, of discontinuity, G^{-1} is the inverse matrix to $G(t)$, $\overline{G(\xi)}$ is the matrix complex-conjugate to the matrix $G(\xi)$.

We determine the matrices G and G^{-1} as follows:

$$\begin{gathered}G=\begin{pmatrix}a_{12}+ia_{11}, & a_{14}+ia_{13}\\ a_{22}+ia_{21}, & a_{24}+ia_{23}\end{pmatrix},\\ G^{-1}=\frac{1}{\det G}\begin{pmatrix}a_{24}+ia_{23}, & -(a_{14}+ia_{13})\\ -(a_{22}+ia_{21}), & a_{12}+ia_{11}\end{pmatrix}.\end{gathered} \tag{3.3}$$

In the interval (a_j,a_{j+1}), the matrix $g(\xi)$ is determined as

$$g_i(\xi)=G_j^{-1}G_j=[\det G_j]^{-1}\begin{pmatrix}A_{11}^{*j}, & iA_{12}^{*j}\\ iA_{21}^{*j}, & iA_{11}^{*j}\end{pmatrix},\quad a_j<\xi<a_{j+1}. \tag{3.4}$$

Differentiation of (3.2) along the boundary ξ results in

$$\Phi'(\xi)=g(\xi)\overline{\Phi'(\xi)},\quad -\infty<\xi<+\infty. \tag{3.5}$$

It is not difficult to verify that the equality $\overline{g(\xi)}=g^{-1}(\xi)=\overline{G}^{-1}G$ holds. For the points $t=e_j$, $j=\overline{1,n}$, we consider the equations ([7], [32]–[37])

$$\det\big(g_{j+1}^{-1}(e_j+0)g_j(e_j-0)-\lambda E\big)=0 \tag{3.6}$$

with respect to the parameter λ, where E is the unit matrix, $g_{j+1}^{-1}(e_j+0)$ and $g_j(e_j-0)$ are the limiting values of the matrices $g_{j+1}^{-1}(\xi)$ and $g_j(\xi)$ at the point $t=e_j$ from the right and from the left, respectively.

By means of the roots λ_{kj} of the equation (3.6), we determine uniquely the numbers ([32]–[37])

$$\alpha_{kj} = (2\pi i)^{-1} \ln \lambda_{kj}, \quad k = 1, 2, \quad k = \overline{1, n}. \tag{3.7}$$

Assume that among the points A_k, $k = \overline{1, k}$, of the boundaries $\ell(\sigma_0)$ and $\ell(\omega_0)$ there exist removable angular points to which on the boundary $\ell(w_0)$ of the domain $S(w_0)$ there correspond ordinary nonangular points. Such angular singular points of the boundaries $\ell(\sigma_0)$ and $\ell(w_0)$ are usually called removable singular points ([13]).

For the sake of simplicity, we assume that the number of removable singular points is equal to 2. Suppose that the removable singular points coincide with the points $\xi = e_j$, $\xi = e_{j+k}$. To these points there correspond on the boundaries $\ell(\sigma_0)$ and $\ell(\omega_0)$ the angles $\pi/2$, and on the boundary $\ell(w_0)$ the angle π. To remove the singular points from the boundary conditions (3.5), we introduce a new unknown vector $\Phi_1(\xi)$ by the formula

$$\Phi'(\xi) = \chi_{01}(\xi)\Phi_1(\xi), \quad -\infty < \xi < +\infty, \tag{3.8}$$

where

$$\chi_{01}(\xi) = \sqrt{(\xi - e_{j-1})(\xi - e_{j+k-1})/[(\xi - e_j)(\xi - e_{j+k})]} > 0, \tag{3.9}$$
$$t > e_{j+k-1}.$$

After passage from the vector $\Phi'(\xi)$ to the vector $\Phi_1(\xi)$, the matrices $g_{j-1}(\xi)$ and $g_{j+k}(t)$ in the intervals (e_{j-1}, e_j) and (e_{j+k}, e_{j+k-1}) are multiplied by (-1).

The boundary condition with respect to $\Phi_1(\xi)$ takes now the form

$$\Phi_1(\xi) = g^*(\xi)\overline{\Phi}_1(\xi), \quad -\infty < \xi < +\infty, \tag{3.10}$$

where

$$g^*(\xi) = [\chi_{01}^+(\xi)]^{-1} g(\xi)[\chi_{01}^-(\xi)]. \tag{3.10$_1$}$$

Next, we again enumerate singular points on the contour $\ell(w)$ and denote them by B_j, $j = \overline{1, m}$; the corresponding points along the axis ξ we denote by a_j, $j = \overline{1, m}$. The uniquely determined characteristic numbers corresponding to $\xi = a_j$ we again denote by α_{kj}, $k = 1, 2$, $j = \overline{1, m}$. They satisfy the Fuchs condition, hence $\alpha_{1j} - \alpha_{2j} = \nu_j$, $j = \overline{1, m+1}$, $\alpha_{1(m+1)} = 3$, $\alpha_{2(m+1)} = 2$, $\xi = a_{m+1} = \infty$, $\sum\limits_{k=1}^{m} [1 - \alpha_{1j} - \alpha_{2j}] = 6$.

Now we compose the Fuchs class equation ([13]–[37])

$$u''(\xi) + p(\xi)u'(\xi) + q(\xi)u(\xi) = 0, \tag{3.11}$$

where

$$p(\xi) = \sum_{j=1}^{m} (1 - \alpha_{1j} - \alpha_{2j})(\xi - a_j)^{-1}, \tag{3.12}$$

$$q(\xi) = \sum_{j=1}^{m} [\alpha_{1j}\alpha_{2j}(\xi - a_j)^{-2} + c_j(\xi - a_1)^{-1}], \tag{3.13}$$

c_j are unknown accessory parameters which satisfy as yet the condition

$$\sum_{j=1}^{m} c_j = 0. \tag{3.14}$$

We write the equation (3.11) in the form of the system:

$$\chi'(\xi) = \chi(\xi)\mathcal{P}(\xi), \tag{3.15}$$

where

$$\mathcal{P}(\xi) = \begin{pmatrix} 0, & -q(\xi) \\ 1, & -p(\xi) \end{pmatrix}, \quad \chi(\xi) = \begin{pmatrix} u_1(\xi), & u_1'(\xi) \\ u_2(\xi), & u_2'(\xi) \end{pmatrix}. \tag{3.16}$$

Using linearly independent solutions $u_1(\xi)$ and $u_2(\xi)$ of the equation (3.11), we construct

$$w_0(\xi) = [Au_1(\xi) + Bu_2(\xi)]\,[Cu_1(\xi) + Du_2(\xi)]^{-1}, \tag{3.17}$$

the general solution of the Schwartz equation ([23]–[37]),

$$\{w_0, \xi\} \equiv w_0'''(\xi)/w_0'(\xi) - 1,5[w_0''(\xi)/w_0'(\xi)]^2 = R(\xi), \tag{3.18}$$

where

$$\begin{aligned} R(\xi) &= 2q(\xi) - p'(\xi) - 0,5[p(\xi)]^2 = \\ &= \sum_{j=1}^{m} \left\{0,5[1 - (\alpha_{1j} - \alpha_{2j})^2](\zeta - a_j)^{-1} + c_j^*(\xi - a_j)^{-1}\right\}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} &\alpha_{1j} - \alpha_{2j} = \nu_j, \;\; j = \overline{1,m}, \quad \beta_k = 1 - \alpha_{1k} - \alpha_{2k}, \;\; k = \overline{1,m}, \\ &c_j^* = 2c_j - \beta_j \sum_{k=1,\,k\neq j}^{m} \beta_k (a_j - a_k)^{-1}, \end{aligned} \tag{3.20}$$

A, B, C, and D are constants of integration of (3.18) satisfying the condition

$$AD - BC \neq 0. \tag{3.21}$$

It is seen from (3.19) that the condition (3.18) depends on $\alpha_{1j} - \alpha_{2j} = \nu_j$, $j = \overline{1,m}$. By (3.17), the half-plane $\mathrm{Im}(\zeta) > 0$ (or $\mathrm{Im}(\zeta) < 0$) is mapped conformally onto the domain $S(w_0)$ with the boundary $\ell(w_0)$.

Expanding the function $R(\zeta)$ near $\zeta = \infty$ in powers of $1/\zeta$, we obtain

$$R(\zeta) = \sum_{k=1}^{\infty} M_k \zeta^{-k}. \tag{3.22}$$

Since the point $\zeta = \infty$ is the image of a nonangular point of the boundary $\ell(w_0)$, the conditions ([23], [32]–[37])

$$\begin{aligned} &M_1 = \sum_{k=1}^{m} c_k^* = 0, \quad M_2 = \sum_{k=1}^{m} [a_k c_k^* + 0,5(1 - \nu_k)^2] = 0, \\ &M_3 = \sum_{k=1}^{m} [a_k^2 c_k^* + a_k(1 - \nu_k)^2] = 0 \end{aligned} \tag{3.23}$$

should be fulfilled.

The condition $M_1 = 0$ implies the condition (3.14), and vice versa, from (3.14) follows $M_1 = 0$.

The conditions (3.23) will be obtained below in somewhat different way. These conditions allow one to determine three parameters c_j^*, $j = \overline{1,3}$. Moreover, of the parameters $\xi = a_k$, $k = \overline{1,m}$, we choose three arbitrarily and fix them (according to Riemann theorem). Then $R(\zeta)$ is defined by the formula (3.19) which depends on $2(m-3)$ unknown parameters a_j, c_j, $j = \overline{1, m-3}$. We rewrite the equation (3.11) near the point $t = a_j$ as follows:

$$(\xi - a_j)^2 u''(\xi) + (\xi - a_j) p_j(\xi) u'(\xi) + q_j(\xi) u(\xi) = 0, \tag{3.24}$$

where

$$\begin{aligned}
& p_j(\xi) = p_{0j} + \sum_{j=1}^{m} p_{nj} (\xi - a_j)^n, \\
& p_{nj} = (-1)^{n-1} \sum_{k=1,\, k\neq j}^{m} \beta_k (a_j - a_k)^{-n}, \\
& p_{0j} = \beta_j, \quad \beta_k = 1 - \alpha_{1k} - \alpha_{2k}, \\
& q_j(\xi) = \alpha_{1j}\alpha_{2j} + c_j(\xi - a_j) + \sum_{n=2}^{\infty} a_{nj} (\xi - a_j)^n, \\
& q_{nj} = (-1)^{n-2} \sum_{k=2,\, k\neq j}^{m} [\alpha_{1k}\alpha_{2k}(n-1) + c_k(a_j - a_k)](a_j - a_k)^{-n}, \\
& \qquad\qquad\qquad\qquad n = 2, 3, \dots\,, \\
& q_{0j} = \alpha_{1j}\alpha_{2j}, \quad q_{1j} = c_j, \quad j = \overline{1,m}, \quad n = 0, 1.
\end{aligned} \tag{3.25}$$

The local conditions (3.24) for the points $t - a_j$, $j = \overline{1,m}$, are sought in the form $u_j(\xi) = (\xi - a_j)^{\alpha_j} \widetilde{u}_j(\xi)$, $\widetilde{u}_j(\xi) = 1 + \sum\limits_{n=1}^{\infty} \gamma_{nj} (\xi - a_j)^n$, where γ_{nj}, $n = \overline{1,\infty}$, $j = \overline{1,m}$, are defined by the recurrence formulas

$$f_{0j}(\alpha_j) = \alpha_j(\alpha_j - 1) + p_{0j}\alpha_j + q_{0j} = 0, \tag{3.26}$$

$$\gamma_{1j} f_{0j}(\alpha_j + 1) + f_{1j}(\alpha_j) = 0, \tag{3.27}$$

$$\gamma_{2j} f_{0j}(\alpha_j + 2) + \gamma_{1j} f_{1j}(\alpha_j + 1) + f_{2j}(\alpha_j) = 0, \tag{3.28}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\begin{aligned}
& \gamma_{nj} f_{0j}(\alpha_j + n) + \gamma_{(n-1)j} f_{1j}(\alpha_j + n - 1) + \gamma_{(n-2)j} f_{2j}(\alpha_j + n - 2) + \\
& \quad + \dots + \gamma_{1j} f_{(n-1)j}(\alpha_j + 1) + f_{nj}(\alpha_j) = 0,
\end{aligned} \tag{3.29}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

where

$$f_n(\alpha_j) = \alpha_j p_{nj} + q_{nj}. \tag{3.30}$$

If the differences $\alpha_{1j} - \alpha_{2j}$, $j = \overline{1,m}$, are nonintegers, then by means of the formulas (3.26)–(3.30) we construct the linearly independent solutions

(3.11),

$$u_{kj}(\xi) = (\xi - a_j)^{\alpha_{kj}} \widetilde{u}_{kj}(\xi),$$
$$\widetilde{u}_{kj}(\xi) = 1 + \sum_{n=1}^{\infty} \gamma_{nj}^k (\xi - a_j)^n, \quad k = 1, 2, \quad j = \overline{1, m}. \tag{3.31}$$

If, however, $\alpha_{1j} - \alpha_{2j} = n$, $n = 0, 1, 2$, then $u_{1j}(\xi)$, $j = \overline{1, m}$, can be constructed by the formulas (3.26)–(3.30), and $u_{2j}(\xi)$ is constructed by the Frobenius method ([22]–[26], [32]–[37]). Moreover, when $\alpha_{1j} - \alpha_{2j} = 0$, $u_{2j}(\xi)$ has the form

$$u_{2j}(\xi) = u_{1j}(\xi) \ln(\xi - a_j) + (\xi - a_j)^{\alpha_{1j}} \sum_{n=1}^{\infty} \gamma_{nj}^2 (\xi - a_j)^n, \tag{3.32}$$

where

$$\gamma_{nj}^2 = \left\{ \frac{d\gamma_{1j}(\alpha_j)}{d\alpha_j} \right\}_{\alpha_j = \alpha_{2j}} .$$

If $\alpha_{1j} - \alpha_{2j} = n$, $n = 1, 2$, then for constructing $u_{2j}(\xi)$ we have to differentiate the equality

$$u_{2j}(\xi) = (\xi - a_j)^{\alpha_j} \left[\alpha_j - \alpha_{2j} + \sum_{n=1}^{\infty} \gamma_{nj}(\alpha_j)(\xi - a_j)^n \right] \tag{3.33}$$

with respect to α_j and let $\alpha_j \to \alpha_{2j}$. Thus we will obtain

$$u_{2j}(\xi) = (\xi - a_j)^{\alpha_{2j}} \left[\sum_{n=1}^{\infty} \lim_{\alpha_j \to \alpha_{2j}} \gamma_{nj}(\alpha_j)(\xi - a_j)^n \right] \ln(\xi - a_j) +$$
$$+ (\xi - a_j)^{\alpha_{2j}} \left\{ 1 + \sum_{n=1}^{\infty} \left[\frac{d\gamma_n(\alpha_j)}{d\alpha_j} \right]_{\alpha_j = \alpha_{2j}} (\xi - a_j)^n \right\}. \tag{3.34}$$

P. Ya. Polubarinova-Kochina has proved that the solution for the cut end $u_{2j}(\xi)$, where $\alpha_{1j} - \alpha_{2j} = 2$, is free from the logarithmic term. Moreover, for such points she has obtained an equation connecting the parameters a_j, c_j, $j = \overline{1, m}$. To the points $t = a_j$ on the contour $\ell(w)$ there correspond cut ends (circular or rectilinear) with the angle 2π. To construct $u_{2j}(\xi)$ uniquely, in our work ([32]–[37]) we suggested the following method. For such point $\xi = a_j$, the equality (3.28) fails to be fulfilled, since

$$f_{0j}(\alpha_j + 2) = 0, \quad \alpha_j \to \alpha_{2j}. \tag{3.35}$$

For the equality (3.28) to take place as $\alpha_j \to \alpha_{2j}$, it is necessary and sufficient that the condition

$$\gamma_{1j} f_1(\alpha_j + 1) + f_2(\alpha_j) = 0, \quad \alpha_{1j} \to \alpha_{2j} + 2, \tag{3.36}$$

be fulfilled. After transformation, (3.36) takes the form ([7], [32]–[37])

$$q_{2j} + q_{1j}^2 + q_{1j} p_{1j} = 0. \tag{3.37}$$

To construct $u_{2j}(\xi)$ uniquely, it suffices to construct $\gamma^2_{2j}(\alpha_{2j})$ uniquely, and the remaining $\gamma^2_{nj}(\alpha_{2j})$, $n = 1, 3, 4, \dots$, are calculated by virtue of (3.27) and (3.29). Indeed, assume $\alpha_{1j} \neq \alpha_{2j}$. Then from (3.28) it follows that

$$\gamma_{2j}(\alpha_j) = -[\gamma_{1j}(\alpha_j) f_{1j}(\alpha_j + 1) + f_{2j}(\alpha_j)]/f_0(\alpha_j + 2). \tag{3.38}$$

After solving indeterminacy in (3.38) as $\alpha_j \to \alpha_{2j}$, we uniquely obtain

$$\gamma^2_{2j} = -0,5[p_{1j}(p_{1j} + 2q_{1j}) + p_{2j}]. \tag{3.39}$$

Further, we determine local solutions in the vicinity of the point $t = \infty$ and represent the functions $p(\xi)$ and $q(\xi)$ in the vicinity of $\zeta = \infty$ as

$$p(\xi) = \xi^{-1} \sum_{n=0}^{\infty} p_{n\infty} \xi^{-n}, \quad q(\xi) = \xi^{-2} \sum_{n=0}^{\infty} q_{n\infty} \xi^{-n}, \tag{3.40}$$

where

$$p_{n\infty} = \sum_{k=1}^{m} \beta_k a_k^n, \quad p_{0\infty} = 6, \tag{3.41}$$

$$q_{n\infty} = \sum_{k=1}^{m} [a_{1k} \alpha_{2k}(n + 1) + c_k a_k] a_k^n, \tag{3.42}$$

$$q_{0\infty} = \sum_{k=1}^{m} [a_{1k} \alpha_{2k} + c_k a_k], \tag{3.43}$$

$$q_{1\infty} = \sum_{k=1}^{m} [2a_{1k} \alpha_{2k} + c_k a_k] a_k. \tag{3.44}$$

Local solutions in the vicinity of the point $\xi = \infty$ will be sought in the form

$$u_\infty(t) = \xi^{-\alpha_\infty} + \sum_{n=1}^{\infty} \gamma_{n\infty} \xi^{-(\alpha_\infty + n)}, \tag{3.45}$$

where $\gamma_{n\infty}$, $n = \overline{1, \infty}$, are defined by the formulas

$$f_{0\infty}(\alpha_\infty) = \alpha_\infty(\alpha_\infty + 1) - p_{0\infty}\alpha_\infty + q_{0\infty} = 0, \tag{3.46}$$

$$\gamma_{1\infty} f_{0\infty}(\alpha_\infty + 1) - p_{1\infty}(\alpha_\infty) + q_{1\infty} = 0, \tag{3.47}$$

$$\gamma_{2\infty} f_{0\infty}(\alpha_\infty + 2) + \gamma_{1\infty}(\alpha_\infty + 1) - p_{2\infty}(\alpha_\infty) + q_{2\infty} = 0, \tag{3.48}$$

. . . .

$$\gamma_{n\infty} f_{0\infty}(\alpha_\infty + n) + \gamma_{(n-1)\infty} f_{1\infty}(\alpha_\infty + n - 1) + \gamma_{(n-2)\infty} f_{2\infty}(\alpha_\infty + n - 2) + \\ + \dots + \gamma_{1\infty} f_{(n-1)\infty}(\alpha_\infty + 1) - p_{n\infty}(\alpha_j) + q_{n\infty} = 0, \tag{3.49}$$

. . . .

where

$$f_{k\infty} = q_{k\infty} - (\alpha_\infty + k) p_{k\infty}. \tag{3.50}$$

Because of the fact that $\xi = \infty$ is the image of a nonangular point, the equation (3.46) should have the roots $\alpha_{1\infty} = 3$ and $\alpha_{2\infty} = 2$. Consequently,

$$q_{0\infty} = \sum_{k=1}^{m} [\alpha_{1k}\alpha_{2k} + a_k c_k] = 6. \tag{3.51}$$

Since $\alpha_{1\infty} - \alpha_{2\infty} = 1$, the equality (3.47) fails to be fulfilled. Therefore the formulas (3.46)–(3.49) allow us to define only one solution $u_{1\infty}(\xi)$. To define $u_{2\infty}(\xi)$, we will act as follows ([25], [32]–[37]). For (3.47) to take place as $\alpha_\infty \to \alpha_{2\infty}$, it is necessary and sufficient that the condition

$$q_{1\infty} - p_{1\infty}\alpha_{2\infty} = 0 \tag{3.52}$$

be valid. To determine $\gamma^2_{1\infty}$, we will act as follows. First determine $\gamma_{1\infty}$ from (3.47) for $\alpha_\infty \neq \alpha_{2\infty}$. We have

$$\gamma_{1\infty}[p_{1\infty}\alpha_\infty - q_{1\infty}]/f_{0\infty}(\alpha_\infty + 1). \tag{3.53}$$

Since the numerator and denominator in (3.53) vanish as $\alpha_\infty \to \alpha_{2\infty}$, after solving indeterminacy we uniquely obtain ([32]–[37])

$$\gamma^2_{1\infty} = p_{1\infty}. \tag{3.54}$$

Using the formulas (3.54), (3.48) and (3.49), we define $\gamma^2_{n\infty}$, $n = \overline{1,\infty}$, and hence the solutions $u_{2\infty}(\xi)$.

Finally, we obtain

$$u_{k\infty}(\xi) = \xi^{-\alpha_{k\infty}} + \sum_{n=1}^{\infty} \gamma^k_{n\infty}\xi^{-\alpha_{k\infty}-n}, \quad k = 1, 2. \tag{3.55}$$

In our work we have formulated without proof, as an obvious statement, that the system (3.23) for $M_k = 0$, $k = 1, 2, 3$, coincides respectively with the system (3.14),(3.51) and (3.52). Here we present the proof of that statement.

From the equation $M_1 = \sum\limits_{k=1}^{m} c^*_k = 0$ follows the equality (3.14). Taking into account (3.20), we can rewrite the equation $M_1 = 0$ as follows:

$$2\sum_{j=1}^{m} c_j - \sum_{j=1}^{m}\beta_j \sum_{k=1,\, k\neq j}^{m} \beta_k(a_j - a_k)^{-1} = 0, \tag{3.56}$$

where $\beta_k = 1 - \alpha_{1k} - \alpha_{2k}$.

It can be verified that the equality

$$\sum_{j=1}^{m}\beta_j \sum_{k=1,\, k\neq j}^{m} \beta_k(a_j - a_k)^{-1} = 0 \tag{3.57}$$

holds. Consequently, the equality (3.14) follows from (3.56).

From the equality $M_2 = 0$ follows (3.51). Indeed, the equation $M_2 = 0$ with regard for (3.20) can be rewritten as

$$\sum_{j=1}^{m}[2a_jc_j + 0,5(1-\nu_j^2)] - \sum_{j=1}^{m} a_j\beta_j \sum_{k=1,\, k\neq j}^{m} \beta_k(a_j-a_k)^{-1} = 0, \qquad (3.58)$$

The following obvious equality

$$\begin{aligned}
&\sum_{j=1}^{m} a_j\beta_j \sum_{k=1,\, k\neq j}^{m} \beta_k(a_j-a_k)^{-1} = \\
&\quad = a_1\beta_1\left(\frac{\beta_2}{a_1-a_2}+\frac{\beta_3}{a_1-a_3}+\cdots+\frac{\beta_m}{a_1-a_m}\right)+ \\
&\quad + a_2\beta_2\left(\frac{\beta_1}{a_2-a_1}+\frac{\beta_3}{a_2-a_3}+\cdots+\frac{\beta_m}{a_2-a_m}\right)+ \\
&\quad + a_3\beta_3\left(\frac{\beta_1}{a_3-a_1}+\frac{\beta_2}{a_3-a_2}+\cdots+\frac{\beta_m}{a_3-a_m}\right)+\cdots+ \\
&\quad + a_m\beta_m\left(\frac{\beta_1}{a_m-a_1}+\frac{\beta_2}{a_m-a_2}+\cdots+\frac{\beta_{m-1}}{a_m-a_{m-1}}\right) = \\
&\quad = \beta_1(\beta_2+\beta_3+\cdots+\beta_m)+\beta_2(\beta_3+\beta_4+\cdots+\beta_m)+ \\
&\quad + \beta_3(\beta_4+\beta_5+\cdots+\beta_m)+\cdots = \\
&\quad = \frac{1}{2}\left\{(\beta_1+\beta_2+\cdots+\beta_m)^2 - \sum_{j=1}^{m}\beta_j^2\right\} = \frac{1}{2}\left[6^2 - \sum_{j=1}^{m}\beta_j^2\right] \qquad (3.59)
\end{aligned}$$

holds.

Taking into account (3.59), we can rewrite the equation (3.58) as follows:

$$\begin{aligned}
\sum_{j=1}^{m}\Big\{2a_jc_j + \frac{1}{2}\big[1-\alpha_{1j}^2-\alpha_{2j}^2+2\alpha_{1j}\alpha_{2j}+1+\alpha_{1j}^2+\alpha_{2j}^2 - \\
- 2(\alpha_{1j}+\alpha_{2j})+2\alpha_{1j}\alpha_{2j}\big] - 18\Big\} = 0. \qquad (3.60)
\end{aligned}$$

The equation (3.60) can also be rewritten as

$$2\left\{\sum_{j=1}^{m}[a_jc_j+\alpha_{1j}\alpha_{2j}] - 9 + 3\right\} = 0, \quad \sum_{j=1}^{m}\beta_j = 6. \qquad (3.61)$$

Consequently, from the equality $M_2 = 0$ we obtain (3.51). Finally, from $M_3 = 0$ follows (3.52), and we obtain

$$\sum_{k=1}^{m}\,[2\alpha_{1k}\alpha_{2k}a_k + c_ka_k^2] - p_{1\infty}\alpha_{2\infty} = 0, \qquad (3.62)$$

where $\alpha_{1\infty} = 2$, $p_{1\infty} = \sum\limits_{k=1}^{m} a_k\beta_k$.

Bearing in mind (3.56), we rewrite the equality $M_3 = 0$ as

$$\sum_{j=1}^{m} [2a_j^2 c_j + \alpha_j(1-\nu_j^2)] - \sum_{j=1}^{m} a_j^2 \beta_j \sum_{k=1,\, k\neq j}^{m} \beta_k (a_j - a_k)^{-1} = 0. \qquad (3.63)$$

There takes place the following equality:

$$\begin{aligned}
&\sum_{j=1}^{m} a_j^2 \beta_j \sum_{k=1,\, k\neq j}^{m} \beta_k (a_j - a_k)^{-1} = \\
&\quad = a_1^2\beta_1 \left(\frac{\beta_2}{a_1 - a_2} + \frac{\beta_3}{a_1 - a_3} + \cdots + \frac{\beta_m}{a_1 - a_m} \right) + \\
&\quad + a_2^2\beta_2 \left(\frac{\beta_1}{a_2 - a_1} + \frac{\beta_3}{a_2 - a_3} + \frac{\beta_4}{a_2 - a_4} + \cdots + \frac{\beta_m}{a_2 - a_m} \right) + \\
&\quad + a_3^2\beta_3 \left(\frac{\beta_1}{a_3 - a_1} + \frac{\beta_2}{a_3 - a_2} + \frac{\beta_4}{a_3 - a_4} + \frac{\beta_m}{a_3 - a_m} \right) + \cdots + \\
&\quad + a_4^2\beta_4 \left(\frac{\beta_1}{a_4 - a_1} + \frac{\beta_2}{a_4 - a_2} + \frac{\beta_3}{a_4 - a_3} + \frac{\beta_5}{a_4 - a_5} + \cdots + \frac{\beta_m}{a_4 - a_m} \right) + \\
&\quad + \cdots + a_m^2\beta_m \left(\frac{\beta_1}{a_m - a_1} + \frac{\beta_2}{a_m - a_2} + \cdots + \frac{\beta_{m-1}}{a_m - a_{m-1}} \right) = \\
&\quad = \beta_1\beta_2(a_1+a_2) + \beta_1\beta_3(a_1+a_3) + \cdots + \beta_1\beta_m(a_1+a_m) + \\
&\quad + \beta_2\beta_3(a_2+a_3) + \beta_2\beta_4(a_2+a_4) + \cdots + \beta_2\beta_3(a_2+a_3) + \cdots = \\
&\quad = a_1\beta_1\{(\beta_2+\beta_3+\cdots+\beta_m) + \beta_1(a_2\beta_2 + a_3\beta_3 + \cdots + a_m\beta_m) + \\
&\quad + a_2\beta_2(\beta_3+\beta_4+\cdots+\beta_m) + \beta_2(a_3\beta_3 + a_4\beta_4 + \cdots + a_m\beta_m)\} + \\
&\quad + a_3\beta_3\{(\beta_4+\beta_5+\cdots+\beta_m) + \beta_3(a_4\beta_4 + a_5\beta_5 + \cdots + a_m\beta_m)\} + \\
&\quad + \cdots + a_{m-1}\beta_{m-1}\beta_m = \\
&\quad = a_1\beta_1 \sum_{j=1}^{m} \beta_j - a_1\beta_1^2 + a_2\beta_2 \sum_{j=1}^{m}\beta_j + \cdots + a_m\beta_m \sum_{j=1}^{m}\beta_j - \\
&\quad - a_m\beta_m^2 = p_{0\infty} p_{1\infty} - \sum_{j=1}^{m} a_j\beta_j^2, \ \ p_{0\infty} = 6, \ \ p_{1\infty} = \sum_{j=1}^{m} a_j\beta_j.
\end{aligned}$$

The equation (3.63) can be rewritten as

$$\begin{aligned}
&\sum_{j=1}^{m} [2a_j^2 c_j + a_j(1 - (\alpha_{1j} - \alpha_{2j})^2) + \beta_j^2] - 6p_{1\infty} = 0, \\
&1 - \alpha_{1j}^2 - \alpha_{2j}^2 + 2\alpha_{1j}\alpha_{2j} + 1 + \alpha_{1j}^2 + \alpha_{2j}^2 - 2(\alpha_{1j} + \alpha_{2j}) + 2\alpha_{1j}\alpha_{2j} = \\
&\qquad = 2[1 - \alpha_{1j} - \alpha_{2j} + 2\alpha_{1j}\alpha_{2j}], \\
&2\left\{ \left[\sum_{k=1}^{m} a_k^2 c_k + 2a_k \alpha_{1k}\alpha_{2k} \right] + p_{1\infty} - 3p_{1\infty} \right\} = 0,
\end{aligned}$$

$$2\left\{\left[\sum_{k=1}^{m} a_k^2 c_k + 2a_k\alpha_{1k}\alpha_{2k}\right] - \alpha_{2\infty}p_{1\infty}\right\} = 0.$$

Consequently, from the system (3.23), $M_k = 0$, $k = 1, 2, 3$, we have obtained the systems (3.14), (3.51), (3.52). Conversely, from the systems (3.14),(3.51),(3.52) we can obtain the system (3.25), $M_k = 0$, $k = 1, 2, 3$.

4. Local Matrices

The local solutions $u_{kj}(\xi)$, $k = 1, 2$, $j = \overline{1, m}$, contain multi-valued functions from which we select the single-valued branches

$$\begin{aligned}
&\exp[\beta_{kj}(\xi - a_j)] > 0, \quad \xi > a_j,\\
&\{\exp[p\alpha_{kj}\ln(\xi - a_j)\}^{+} = \exp[i\pi\alpha_{kj}]\times\\
&\qquad \times[\exp[\alpha_{kj}\ln(a_j - \xi)]], \quad a_j - \xi > 0,\\
&\{\exp[p\alpha_{kj}\ln(\xi - a_j)\}^{-} = \exp[-i\pi\alpha_{kj}]\times\\
&\qquad \times[\exp[\alpha_{kj}\ln(a_j - \xi)]], \quad a_j - \xi > 0.
\end{aligned}$$

For the equation (3.11), in the vicinity of every singular point $\xi - a_j$, $j = \overline{1, m+1}$, and in the vicinity of the points $\xi - a_j^* = (a_j + a_{j+1})/2$, $j = \overline{1, n-1}$, we construct respectively $u_{kj}(\xi)$, $k = 1, 2$, $j = \overline{1, m}$, and $\sigma_{kj}(\xi)$, $k = 1, 2$, $j = \overline{1, n-1}$.

We seek for a solution of (3.17) by means of the matrix $T\chi(\xi)$, where $\chi(\xi)$ is a solution of (3.15). If $\chi(\xi)$ is a solution of (3.15), then $T\cdot\chi(\xi)$ is likewise a solution of (3.15), where the matrix T is defined as

$$T = \begin{pmatrix} p, & q\\ r, & s\end{pmatrix}, \quad \det T \neq 0, \tag{4.1}$$

p, q, r, s are constants of integration of the equation (3.18).

The local fundamental matrices $\theta_j(\xi)$, $\sigma_j(\xi)$, $\theta_j^*(\xi)$, $\theta_j^{\pm}(\xi)$ are defined as follows:

$$\theta_j(\xi) = \begin{pmatrix} u_{1j}(\xi), & u'_{1j}(\xi)\\ u_{2j}(\xi), & u'_{2j}(\xi)\end{pmatrix}, \quad a_j < \xi < a_{j+1}, \quad j = \overline{1, n-1}, \tag{4.2}$$

$$\xi = a_j, \quad j = \overline{1, n},$$

$$\theta_j^*(\xi) = \begin{pmatrix} u_{1j}^*(\xi), & u_{1j}'^*(\xi)\\ u_{2j}^*(\xi), & u_{2j}'^*(\xi)\end{pmatrix}, \quad a_{j-1} < \xi < a_j, \tag{4.3}$$

$$\xi = a_j, \quad j = \overline{1, n}, \quad \xi = a_{m+1} = \infty,$$

$$\sigma_j(\xi) = \begin{pmatrix} \sigma_{1j}(\xi), & \sigma'_{1j}(\xi)\\ \sigma_{2j}(\xi), & \sigma'_{2j}(\xi)\end{pmatrix}, \quad \xi = (a_j + a_{j+1})/2 = a_j^*, \quad j = \overline{1, n-1}, \tag{4.4}$$

$$\theta_j^*(\xi) = \vartheta_j^{+}\theta_j^*(\xi), \quad a_{j-1} < \xi < a_j, \tag{4.5}$$

$$\theta_\infty(\xi) = \begin{pmatrix} u_{1\infty}(\xi), & u'_{1\infty}(\xi)\\ u_{2\infty}(\xi), & u'_{2\infty}(\xi)\end{pmatrix}, \tag{4.6}$$

where the matrices ϑ_j^+ for $\alpha_{1j}-\alpha_{2j}\neq n$, $n=0,1,2$, are defined as

$$\vartheta_j^+ = \begin{pmatrix} \exp(\pm i\pi\alpha_{1j}), & 0 \\ 0, & \exp(\pm i\pi\alpha_{2j}) \end{pmatrix}, \tag{4.7}$$

while for $\alpha_{1j}-\alpha_{2j}=n$, $n=0,1,2$, they are defined by the equalities

$$\vartheta_j^\pm = \exp(\pm i\pi\alpha_{2j}) \begin{pmatrix} 1, & 0 \\ \pm i\pi, & 1 \end{pmatrix}, \quad n=0,2, \tag{4.8}$$

$$\vartheta_j^\pm = \exp(\pm i\pi\alpha_{2j}) \begin{pmatrix} -1, & 0 \\ \mp i\pi, & 1 \end{pmatrix}, \quad n=1. \tag{4.9}$$

Here it should be noted that the series $u_{kj}(\xi)$, $k=1,2$, $j=\overline{1,m}$, converge slowly, making thus the process of calculation very difficult. To eliminate this drawback, we act as follows ([32]–[37]). We replace the series $u_{kj}(\xi)$, $k=1,2$, $j=\overline{1,m+1}$, by rapidly and uniformly convergent series. To this end, it suffices to write the series $u_{kj}(\xi)$, $k=1,2$, $j=\overline{1,m+1}$, in the form

$$\begin{aligned} &u_{kj}(\xi) = (\xi-a_j)^{\alpha_{kj}}\widetilde{u}_{kj}(\xi-a_j), \\ &\widetilde{u}_{kj}(\xi-a_j) = 1+\sum_{n=1}^{\infty}\gamma_{nj}^k(\xi-a_j), \quad k=1,2, \quad j=\overline{1,m}, \end{aligned} \tag{4.10}$$

$$u_{k\infty} = \xi^{-\alpha_{k\infty}}\bigg(1+\sum_{n=1}^{\infty}\gamma_{n\infty}^k(\xi)\bigg), \tag{4.11}$$

where γ_{nj}^k, $\gamma_{n\infty}^k$ are defined through $f_{nj}(\alpha_j)$ and $f_{n\infty}(\alpha_j)$ as follows:

$$f_{nj}[(\xi-a_j),\beta_k] = \alpha_{kj}p_{nj}(\xi-a_j)+q_{nj}(\xi-a_j), \tag{4.12}$$

$$p_{nj}(\xi-a_j) = (-1)^{n-1}\sum_{k=1,\,k\neq j}^{m}\beta_i[(\xi-a_j)/(a_i-a_k)]^n, \quad n=1,2,\dots, \tag{4.13}$$

$$\begin{aligned} &q_{1j}(\xi-a_j) = c_j(\xi-a_j), \quad q_{nj}(\xi-a_j) = \\ &\quad = (-1)^{n-2}\sum_{k=1,\,k\neq j}^{m}[\alpha_{1k}\alpha_{2k}(n-1)+c_k(a_j-a_k)]\left(\frac{\zeta-a_j)}{a_j-a_k}\right)^n, \end{aligned} \tag{4.14}$$

$$\left|\frac{\xi-a_j}{a_j-a_k}\right|<1, \quad k\neq j,$$

$$\begin{aligned} &p_{n\infty}(\xi) = \sum_{k=1}^{m}\beta_k(a_k/\xi)^n, \quad q_{n\infty} = \\ &\quad = \sum_{k=1}^{n}[\alpha_{1j}\alpha_{2j}(n+1)+c_ka_k](a_k/t)^n, \quad n=0,1,2,\dots. \end{aligned} \tag{4.15}$$

The local matrix $\theta_j^-(\xi)$ is the complex-conjugate to the matrix $\theta_j^+(\xi)$. The real matrices $\theta_j(\xi)$ and $\theta_j^*(\xi)$ are local solutions of the system of equations (3.15) in the vicinity of the points $\xi=a_{j-1}$, $\xi>a_{j-1}$, $\xi=a_j$, $\xi<a_j$. Assume that the elements of these matrices converge in some part of the

interval $a_{j-1} < \xi < a_j$, where the matrices $\theta_j^*(\xi)$ and $\theta_{j-1}(\xi)$ are connected by the matrix identity ([32]–[37])

$$\theta_j^*(\xi) = T_{j-1}\theta_{j-1}(\xi), \quad j = m, m-1, \dots, 2, \tag{4.16}$$

from which we can determine the matrices T_{j-1} uniquely.

Assume that the domains of convergence of the matrices $\theta_j^*(\xi)$, $\theta_{j-1}(\xi)$ do not intersect. In this case we construct at the point $\xi = a_j^* = (a_{j-1} + a_j)/2$ a fundamental local matrix $\sigma_j(\xi)$ which converges in the interval $a_{j-1} < \xi < a_j$. Thus it becomes obvious that one can pass from the matrix $\theta_j^*(\xi)$ to the matrix $\theta_{j-1}(\xi)$ in the following sequence:

$$\theta_j^*(\xi) = T_{a_j^*}\sigma_j(\xi), \tag{4.17}$$

$$\sigma_j(\xi) = T_{j-1}^*\theta_{j-1}(\xi). \tag{4.18}$$

From (4.17) and (4.18) we can determine T_{a^*} and T_{j-1}^* uniquely. It follows from the above-said that $\theta_m(\xi)$ can be extended analytically along the whole axis ξ.

To find the functions $\omega_0'(\xi)$ and $z_0'(\xi)$ in the interval $(-\infty, +\infty)$, we consider the matrices ([32]–[37])

$$\chi^{\pm}(\xi) = T\theta_m^{\pm}(\xi), \quad \xi > a_m; \quad \theta_m^+(\xi) = \theta_m^-(\xi), \quad a_m < \xi < +\infty, \tag{4.19}$$

where the matrix T is defined by the formula (4.1). It follows from (4.19) that $T = \overline{T}$.

The matrices $\chi^{\pm}(\xi)$ are solutions of (3.15), where the signs "+" and "−" denote, respectively, limiting values of the matrix $\chi(\xi)$ from $\mathrm{Im}(\zeta) > 0$, when $\zeta \in \mathrm{Im}(\zeta) > 0$, $\zeta \to \xi$, and from $\mathrm{Im}(\zeta) < 0$, when $\zeta \in \mathrm{Im}(\zeta) < 0$, $\zeta \to \xi$.

Below, we will determine the matrix $\chi^+(\xi)$ and take into account that $\overline{\chi^+(\xi)} = \chi^-(\xi)$. The use will be made of the notation $\chi^+(\xi) = \chi(\xi)$, $\vartheta_j^+ = \vartheta_j$, $j = \overline{1, m}$,

$$\begin{aligned}
&\chi(\xi) = T\vartheta_m\theta_m^*(\xi), \quad a_{m-1} < \xi < a_m,\\
&\chi(\xi) = T\vartheta_m T_{m-1}\theta_{m-1}(\xi), \;\; \theta_m^* = T_{m-1}\theta_{m-1}(\xi), \;\; a_{m-1} < \xi < a_m,\\
&\chi(\xi) = T\vartheta_m T_{m-1}\vartheta_{m-1}\theta_m^*(\xi), \quad a_{m-2} < \xi < a_{m-1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\chi(\xi) = T\vartheta_m T_{m-1}\vartheta_{m-1}T_{m-2}\vartheta_{m-2}\cdots T_1\theta_1(\xi), \quad a_1 < \xi < a_2,\\
&\chi(\xi) = T\vartheta_m T_{m-1}\vartheta_{m-1}\cdots T_1\vartheta_1\theta_1^*(\xi), \quad -\infty < \xi < a_1,\\
&\chi(\xi) = T\vartheta_m T_{m-1}\vartheta_{m-1}\cdots T_1\vartheta_1 T_{-\infty}\theta_\infty(\xi), \quad -\infty < \xi < a_1,\\
&\chi(\xi) = TT_{+\infty}\theta_\infty^*(\xi), \quad a_m < \xi < +\infty.
\end{aligned} \tag{4.20}$$

Note that

$$
\begin{aligned}
&\theta_m^*(\xi) = T_{m-1}\theta_{m-1}(\xi), \quad a_{m-1} < \xi < a_m, \\
&\theta_{m-1}^*(\xi) = T_{m-2}\theta_{m-2}(\xi), \quad a_{m-2} < \xi < a_{m-1}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\theta_2^*(\xi) = T_1\theta_1(\xi), \quad a_1 < \xi < a_2, \\
&\theta_1^*(\xi) = T_{-\infty}\theta_\infty(\xi), \quad -\infty < \xi < a_1, \\
&\theta_m(\xi) = T_m\theta_\infty(\xi), \quad a_m < \xi < +\infty.
\end{aligned}
\tag{4.21}
$$

From the system (4.21) we define the matrices T_j, $j = \overline{1, m-1}$, $T_{-\infty}$, T_∞ by means of the matrices $\theta_j(\xi)$, $\theta_j^*(\xi)$, $j = \overline{1, m}$, $\theta_\infty(\xi)$ which depend on a_j, c_j, $j = \overline{1, m}$.

Substituting the matrices $\chi(\xi)$, $\overline{\chi}(\xi)$ defined in the intervals (a_{j-1}, a_j) successively as $j = m, m-1, \dots, 1$ into the boundary condition (3.10) and then multiplying from the right each of the equalities by $[\theta_j^*(\xi)]^{-1}$, $j = m, m-1, \dots, 1$, we obtain the system of matrix equations

$$T\vartheta_m = g_{m-1}T\overline{\vartheta}_m, \quad \xi = a_m, \tag{4.22}$$

$$T\vartheta_m T_{m-1}\vartheta_{m-1} = g_{m-2}T\overline{\vartheta}_m T_{m-1}\overline{\vartheta}_{m-1}, \quad \xi = a_{m-1}, \tag{4.23}$$

$$
\begin{aligned}
&T\vartheta_m T_{m-1}\vartheta_{m-1}T_{m-2}\vartheta_{m-2} = \\
&\qquad = g_{m-3}T\overline{\vartheta}_m T_{m-1}\overline{\vartheta}_{m-1}T_{m-2}\overline{\vartheta}_{m-2}, \quad \xi = a_{m-2},
\end{aligned}
\tag{4.24}
$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$T\vartheta_m T_{m-1}\vartheta_{m-1}\cdots T_1\vartheta_1 = T\overline{\vartheta}_m T_{m-1}\overline{\vartheta}_{m-1}\cdots T_1\overline{\vartheta}_1, \quad \xi = a_1. \tag{4.25}$$

We can immediately verify that when passing in (4.22)–(4.25) to the conjugate matrix equations, the matrix equations (4.22)–(4.25) remain unchanged, since $\overline{g(\xi)} = g^{-1}(\xi) = \overline{G}^{-1}(\xi)G(\xi)$.

From every matrix equation (4.22)–(4.25), we obtain two scalar equations with respect to the elements of the matrices $T, T_m, T_{m-1}, \dots, T_1$ ([32]–[37]). Indeed, we rewrite the matrix equation (4.22) in the form

$$\vartheta_m\overline{\vartheta}_m^{-1} = T^{-1}g_m^{-1}g_{m-1}T, \quad g_m = E. \tag{4.26}$$

It follows from (4.26) that the matrices $\vartheta_m\overline{\vartheta}_m^{-1}$, $T^{-1}g_m^{-1}g_{m-1}T$ are similar.

For the sake of brevity, if we assume that the matrix ϑ_m is diagonal, then from (4.22) we will obtain the system of equations

$$p\exp(i\pi\alpha_{1m}) = g_{m-1}^{11}p\exp(-i\pi\alpha_{1m}) + g_{m-1}^{12}r\exp(-i\pi\alpha_{1m}), \tag{4.27}$$

$$q\exp(i\pi\alpha_{2m}) = g_{m-1}^{11}q\exp(-i\pi\alpha_{2m}) + g_{m-1}^{12}s\exp(-i\pi\alpha_{2m}), \tag{4.28}$$

$$r\exp(i\pi\alpha_{1m}) = g_{m-1}^{21}p\exp(-i\pi\alpha_{1m}) + g_{m-1}^{22}r\exp(-i\pi\alpha_{1m}), \tag{4.29}$$

$$s\exp(i\pi\alpha_{2m}) = g_{m-1}^{21}q\exp(-i\pi\alpha_{2m}) + g_{m-1}^{22}s\exp(-i\pi\alpha_{2m}), \tag{4.30}$$

where g_{m-1}^{ij}, $i, j = 1, 2$, are the elements of the matrix g_{m-1}, $\alpha_{km} = \frac{1}{2\pi i}\ln\lambda_{km}$, $k, m = 1, 2$, are the characteristic numbers.

It is not difficult to verify that the equations (4.27) and (4.28) identically coincide respectively with the equations (4.29) and (4.30).

Indeed, solving (4.27) and (4.29) with respect to p/r and then equating the obtained expressions to each other, and just in the same manner, solving (4.28) and (4.30) with respect to s/q and equating the obtained expressions to each other, we arrive to the following identities:

$$(\lambda_{1m} - g_{m-1}^{22})(g_{m-1}^{21})^{-1} = g_{m-1}^{12}(\lambda_{1m} - g_{m-1}^{11})^{-1}, \tag{4.31}$$

$$g_{m-1}^{11}(\lambda_{2m} - g_{m-1}^{11})^{-1} = (\lambda_{1m} - g_{m-1}^{22})^{-1}(g_{m-1}^{21})^{-1}, \tag{4.32}$$

where

$$g_{m-1}^{11} + g_{m-1}^{22} = \lambda_{1m} + \lambda_{2m}, \quad g_{m-1}^{11}g_{m-1}^{22} - g_{m-1}^{12}g_{m-1}^{21} = \lambda_{1m}\lambda_{2m}. \tag{4.33}$$

The equation (4.23) in view of (4.22) can be written in the form

$$T_{m-1}\vartheta_{m-1}\overline{\vartheta}_{m-1}^{-1}T_{m-1}^{-1} = \overline{\vartheta}_{m-1}^{-1}T_m^{-1}g_{m-1}^{-1}g_{m-2}T\overline{\vartheta}_m. \tag{4.34}$$

The matrices appearing in (4.24) in the left and in the right side are similar, therefore we can calculate (4.24) analogously to (4.29)–(4.34) and prove that the matrix equation (4.34) results in two scalar equations.

For the point $\xi = a_{m-2}$ we have

$$\begin{aligned} T_{m-2}\vartheta_{m-2}\overline{\vartheta}_{m-2}^{-1}T_{m-2}^{-1} = \\ = \overline{\vartheta}_{m-1}^{-1}T_{m-1}^{-1}\overline{\vartheta}_m^{-1}T^{-1}g_{m-2}^{-1}g_{m-3}T\overline{\vartheta}_mT_{m-1}\overline{\vartheta}_{m-1}. \end{aligned} \tag{4.35}$$

Next, for all the points $\xi = a_{m-2}, \dots, a_1$ we can write out similar matrices, and this proves our statement above.

From the system (4.21), depending on the parameters a_j, c_j, $j = \overline{1,m}$, we find the elements of the matrices T_j, $j = \overline{1,m}$, and substitute them in (4.22)–(4.25). Thus we obtain a system of matrix equations with respect to p, q, r, s, a_j, c_j, $j = \overline{1,m}$, from which, as is said above, for the points $\xi = a_k$, $k = \overline{1,m}$, we obtain two scalar equations with respect to the parameters a_j, c_j, p/s, q/s, r/s, $j = \overline{1,m}$. Hence we obtain a system consisting of $2m$ equations.

According to the Riemann theorem, we can choose and fix three of the parameters $\xi = a_j$, $j = \overline{1,m}$. From the system (3.14),(3.51) and (3.52) we define three parameters, for example, c_1, c_2 and c_3, and the number of unknown essential parameters a_j, c_j, $j = \overline{1,m}$, will be $2(m-3)$ ([23], [24], [32]–[37]]). In the general case, to the above-mentioned parameters a_j, c_j, $j = \overline{1,m}$, we have to add three complex parameters of integration of the Schwartz equation. Thus the number of unknown parameters will be $2m$. In our case, for $g_m(\xi) = E$ we have three parameters of integration: p/s, q/s, r/s. Consequently, the number of unknown parameters is $2(m-3)+3 = 2m-3$, and that of equations is $2m$. Then the difference $2m-(2m-3) = 3$.

The contour $\ell(w_0)$ of the circular polygon may have vertices with angles 2π made by the cuts (circular or linear). For every such vertex $\ell(w_0)$ we obtain one equation of type (3.37) due to the fact that in the theory of filtration the coordinates of vertices on the contour $\ell(w_0)$ are not given a priori. If we assume that a number of such vertices is two, then the number of equations with respect to unknown parameters will be equal to $2(m-1)$, and the difference $2m - 2(m-1) = 2$. For example, in the theory of filtration there exist circular pentagons with only one cut. Then a number of essentially unknown parameters is reduced to three. To these parameters we add p/s, q/s and r/s. Consequently, the number of unknown parameters is equal to six, and the number of equations is equal to nine of which for finding p/s, q/s and r/s we have three equations. Hence there remains a system of six equations with three unknowns. The difference between the number of equations and unknowns is equal to three, which is considered to be normal.

As is known, in the case of linear polygons, the number of equations exceeds that of unknown parameters by two. It is difficult to obtain a solution of a system of three higher transcendent equations with respect to three essential parameters, but, in principle, it is quite possible. Having found the above-mentioned parameters, we have to substitute them in the remaining equations and thus find the parameters c_j, $j = \overline{1,s}$, p/s, q/s and r/s ([32]–[37]]).

If we denote by $u_1(\xi)$ and $u_2(\xi)$ the elements of the matrix $\Phi_1(\xi)$, or, which is the same thing, the elements of the first column of the matrix $\chi(\xi)$, then by the formula

$$w(\xi) = u_1(\xi)/u_2(\xi), \quad -\infty < \xi < +\infty, \tag{4.36}$$

we obtain the general solution (3.18).

The vector elements $\Phi'(\xi)$, $\omega'(\xi)$ and $\sigma'(\xi)$ are defined by the equalities

$$d\omega_0(\xi) = u_1(\xi)\chi_{01}(\xi)\, d\xi, \quad -\infty < \xi < +\infty, \tag{4.37}$$

$$d\sigma_0(\xi) = u_2(\xi)\chi_{01}(\xi)\, d\xi, \quad -\infty < \xi < +\infty, \tag{4.38}$$

where $\omega_0'(\xi) = u_1(\xi)\chi_{01}(\xi)$ and $\sigma'(\xi) = u_2(\xi)\chi_{01}(\xi)$ satisfy the boundary conditions (3.10) and also the conditions at the singular points $\xi = e_j$, $j = \overline{1,n}$, $t = \infty$.

The integration of (4.37) and (4.38) in the intervals $(-\infty, \xi)$, (e_j, ξ), $j = \overline{1,n}$, results in

$$\omega_0(\xi) = \int\limits_{-\infty}^{\xi} u_1(\xi)\chi_{01}(\xi)\, d\xi + \omega_0(-\infty), \tag{4.39}$$

$$\sigma_0(\xi) = \int\limits_{-\infty}^{\xi} u_2(\xi)\chi_{01}(\xi)\, d\xi + \sigma_0(-\infty), \tag{4.40}$$

$$\omega_0(\xi) = \int\limits_{e_j}^{\xi} u_1(\xi)\chi_{01}(\xi)\, d\xi + \omega_0(e_j + 0), \tag{4.41}$$

$$\sigma_0(\xi) = \int_{e_j}^{\xi} u_2(\xi)\chi_{01}(\xi)\, d\xi + \sigma_0(e_j + 0). \tag{4.42}$$

Considering (4.41) and (4.42) for $\xi = e_{j+1}$, we obtain the third system of equations with respect to both removable singular points $\xi = e_j$, $\xi = e_{j+k}$, the parameter Q, where Q is the liquid discharge, and another unknown parameters, for example s, where we substitute numerical values of the parameters e_j, c_j, $j = \overline{1,n}$, p/s, q/s and r/s.

Having found all unknown parameters on which the functions $u_1(\xi)$, $u_2(\xi)$ and $\chi_{01}(\xi)$ depend, by the formulas (4.41) and (4.42) we can determine the equations of unknown parts of boundaries of the domains $S(\sigma_0)$, $S(\omega_0)$, $S(w_0)$ and also another geometric and mechanical parameters of the filtrational liquid flow ([32]–[37]).

5. Some Applications to the Problems of Filtration

Sometimes it is convenient to make use of the symbols of the complex differentiation

$$\frac{\partial}{\partial\overline{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial\xi} + i\,\frac{\partial}{\partial\eta}\right), \tag{5.1}$$

$$\frac{\partial}{\partial\zeta} = \frac{1}{2}\left(\frac{\partial}{\partial\xi} - i\,\frac{\partial}{\partial\eta}\right), \quad \zeta = \xi + i\eta, \quad \overline{\zeta} = \xi - i\eta. \tag{5.2}$$

As an example, we apply the symbol (5.1) to the complex function

$$w(\xi) = \frac{1}{2}\left(\frac{\partial u}{\partial\xi} - \frac{\partial v}{\partial\eta}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\zeta}\right). \tag{5.3}$$

In particular, by means of the symbol (5.1) we write the Cauchy–Riemann condition as follows:

$$\frac{\partial w}{\partial\overline{\zeta}} = 0. \tag{5.4}$$

We will need the Riemann–Green formula ([5]–[6]),

$$\frac{1}{2i}\int_C w(\zeta)\, d\zeta = \iint_D \frac{\partial w}{\partial\overline{\zeta}}\, d\xi\, d\eta, \tag{5.5}$$

where $w(\zeta) = u(\xi,\eta) + iv(\xi,\eta)$, C is the boundary of the domain D, and $w(\zeta)$ possesses in D continuous partial derivatives.

The formula (5.5) expresses the Cauchy theorem:

$$w(\zeta) = \frac{1}{2\pi i}\int_C \frac{w(z)\, dz}{z-\zeta} - \frac{1}{\pi}\iint_D \frac{\partial w}{\partial\overline{\zeta}}\,\frac{dx\, dy}{z-\zeta}\,. \tag{5.6}$$

The theory of partial differential equations employs the methods of representation of solutions in the complex form ([3], [5], [6]). As an example, we consider the Carleman system

$$\frac{\partial u}{\partial \zeta} = \frac{\partial v}{\partial \eta} + au + bv, \quad \frac{\partial u}{\partial \eta} = -\frac{\partial v}{\partial \zeta} + cu + dv, \tag{5.7}$$

where a, b, c and d are continuous functions of variables ξ and η in the domain D. For the solution of the system (5.7) we apply the formula (5.6) and obtain

$$\frac{\partial w}{\partial \overline{\zeta}} = Aw + B\overline{w}, \tag{5.8}$$

where

$$w = u + iv, \quad A = \frac{1}{4}[a + d + i(c - b)], \quad B = \frac{1}{4}[a - d + i(c + b)]. \tag{5.9}$$

The formula (5.6) provides us with a solution of the system (5.8) in the form

$$w(\zeta) = \frac{1}{2\pi i}\int\limits_C \frac{w(z)\,dz}{z - \zeta} - \frac{1}{\pi}\iint\limits_D \frac{A(z)w(z) + B(z)\overline{w(z)}}{z - \zeta}\,dx\,dy\,. \tag{5.10}$$

Let us consider first the system of equations (1.21), (1.22). Introduce the notation ([5])

$$u(\xi, \eta) = \rho(\xi, \eta)\varphi(\xi, \eta), \quad v(\xi, \eta) = \psi(\xi, \eta). \tag{5.11}$$

Differentiating $\varphi(\xi, \eta) = u(\xi, \eta)/\rho(\xi, \eta)$, we obtain

$$\varphi'_\xi = -\frac{1}{\rho^2}\,\frac{\partial \rho}{\partial \xi}\,u + \frac{1}{\rho}\,\frac{\partial u}{\partial \xi}\,, \tag{5.12}$$

$$\varphi'_\eta = -\frac{1}{\rho^2}\,\frac{\partial \rho}{\partial \eta}\,u + \frac{1}{\rho}\,\frac{\partial u}{\partial \eta}\,. \tag{5.13}$$

Substituting now (5.12) and (5.13) in (1.21) and (1.22), we get

$$u'_\xi - v'_\eta = \frac{1}{\rho}\,\frac{\partial \rho}{\partial \xi}\,u, \tag{5.14}$$

$$u'_\eta + v'_\xi = \frac{1}{\rho}\,\frac{\partial \rho}{\partial \eta}\,u. \tag{5.15}$$

The system (5.14), (5.15) can be written in the complex form,

$$\frac{\partial w}{\partial \overline{\zeta}} - \frac{1}{\rho}\left(\frac{\partial \rho}{\partial \xi} + i\,\frac{\partial \rho}{\partial \eta}\right)u = 0, \tag{5.16}$$

where

$$w = u + iv, \quad \frac{\partial w}{\partial \overline{\zeta}} = \frac{1}{2}\,(w_\xi + iw_\eta), \tag{5.17}$$

and solution of (5.16) can be reduced to solution of the system of integral equations

$$u(\xi, \eta) + \frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} \frac{\frac{1}{\rho}\,u\{\frac{\partial \rho}{\partial x}\,(x - \xi) + \frac{\partial \rho}{\partial y}\,(y - \eta)\}\,dx\,dy}{(x - \xi)^2 + (y - \eta)^2} = \varphi_0(\xi, \eta), \tag{5.18}$$

$$v(\xi,\eta)+\frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0}\frac{\frac{1}{\rho}\,u\{\frac{\partial\rho}{\partial y}\,(x-\xi)+\frac{\partial\rho}{\partial x}\,(y-\eta)\}\,dx\,dy}{(\xi-x)^2+(\eta-y)^2}=\psi_0(\xi,\eta), \quad (5.19)$$

where

$$\Delta\varphi_0(\xi,\eta)=0,\ \ \Delta\psi_0(\xi,\eta)=0,\ \ \frac{\partial\varphi}{\partial\xi}=\frac{\partial\psi_0}{\partial\eta}\,,\ \ \frac{\partial\varphi_0}{\partial\eta}=-\frac{\partial\psi_0}{\partial\zeta}\,. \quad (5.20)$$

The functions $\omega_0(\zeta)=\varphi_0(\xi,\eta)+i\psi_0(\xi,\eta)$, $\sigma_0(\zeta)=z(\xi,\eta)+i\rho(\xi,\eta)$, $w(\zeta)=\omega'(\zeta)/\sigma'(\zeta)$ are considered as defined already (see Sections 3–4). The same is true for the formulas (5.25)–(5.30).

Getting back to the notation (5.11), we obtain

$$\rho(\xi,\eta)\varphi(\xi,\eta)+\frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0}\frac{\varphi(x,y)\{\frac{\partial\rho}{\partial x}\,(x-\xi)+\frac{\partial\rho}{\partial y}\,(y-\eta)\}\,dx\,dy}{(x-\xi)^2+(y-\eta)^2}=$$
$$=\varphi_0(\xi,\eta), \quad (5.21)$$

$$\psi(\xi,\eta)-\frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0}\frac{\varphi(x,y)\{\frac{\partial\rho}{\partial x}\,(y-\eta)-\frac{\partial\rho}{\partial y}\,(x-\xi)\}\,dx\,dy}{(x-\xi)^2+(y-\eta)^2}=$$
$$=\psi_0(\xi,\eta). \quad (5.22)$$

The system (5.21), (5.22) can be solved by means of the method of successive approximations according to the following scheme ([4], [5], [9]):

$$\rho(\xi,\eta)\varphi_{n+1}(\xi,\eta)+\frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0}\varphi_n(x,y)\frac{\{\frac{\partial\rho}{\partial x}\,(x-\xi)+\frac{\partial\rho}{\partial y}\,(y-\eta)\}\,dx\,dy}{(x-\xi)^2+(y-\eta)^2}=$$
$$=\varphi_0(\xi,\eta),\quad n=0,1,2,\dots\ , \quad (5.23)$$

$$\psi_{n+1}(\xi,\eta)-\frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0}\varphi_n(x,y)\frac{\{\frac{\partial\rho}{\partial x}\,(y-\eta)-\frac{\partial\rho}{\partial y}\,(x-\xi)\}\,dx\,dy}{(x-\xi)^2+(y-\eta)^2}=$$
$$=\psi_0(\xi,\eta),\quad n=0,1,2,\dots\ . \quad (5.24)$$

We can write the system (1.25) and (1.26) in the complex form

$$\frac{\partial w}{\partial\overline{\zeta}}+\frac{1}{\rho}\left(\frac{\partial\rho}{\partial\xi}\,\frac{\partial u}{\partial\zeta}-\frac{\partial\rho}{\partial\eta}\,\frac{\partial v}{\partial\eta}\right),\quad u=\frac{\partial\varphi}{\partial\xi}\,,\quad v=-\frac{\partial\varphi}{\partial\eta}\,. \quad (5.25)$$

The differential equation (5.25) can be reduced to the integro-differential equation

$$u+iv=-\frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0}\frac{1}{\rho(x,y)}\,\frac{(\frac{\partial\rho}{\partial x}\,u-\frac{\partial\rho}{\partial y}\,v)dx\,dy}{z-\zeta}+u_0+iv_0. \quad (5.26)$$

Separating in the equation (5.26) the real and imaginary parts, we obtain

$$\frac{\varphi(\xi,\eta)}{\partial\xi}=-\frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0}\frac{\frac{1}{\rho(x,y)}(\frac{\partial\rho}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{\partial\rho}{\partial y}\frac{\partial\varphi}{\partial y})(x-\xi)dx\,dy}{(x-\xi)^2+(y-\eta)^2}+\frac{\partial\varphi_0(\xi,\eta)}{\partial\xi}, \quad (5.27)$$

$$\frac{\varphi(\xi,\eta)}{\partial\eta}=\frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0}\frac{\frac{1}{\rho(x,y)}(\frac{\partial\rho}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{\partial\rho}{\partial y}\frac{\partial\varphi}{\partial y})(y-\eta)dx\,dy}{(x-\xi)^2+(y-\eta)^2}+\frac{\partial\varphi_0(\xi,\eta)}{\partial\eta}. \quad (5.28)$$

After integration of (5.27) and (5.28) and taking into account (1.21) and (1.22), we obtain

$$\varphi(\xi,\eta)=-\frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0}\frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{\partial\rho}{\partial y}\frac{\partial\varphi}{\partial y}\right)dx\,dy\times$$
$$\times\int\limits_{(\xi_0,\eta_0)}^{(\xi,\eta)}\frac{(x-\xi)\,d\xi-(y-\eta)\,d\eta}{(x-\xi)^2+(y-\eta)^2}+\varphi_0(\xi,\eta)+c, \quad (5.29)$$

where c is a constant,

$$\psi(\xi,\eta)=-\frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)\geq 0}\frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{\partial\rho}{\partial y}\frac{\partial\varphi}{\partial y}\right)dx\,dy\times$$
$$\times\int\limits_{(\xi_0,\eta_0)}^{(\xi,\eta)}\frac{(y-\eta)\rho(\xi,\eta)d\xi-(x-\xi)\rho(\xi,\eta)\,d\eta}{(x-\xi)^2+(y-\eta)^2}+$$
$$+\int\limits_{(\xi_0,\eta_0)}^{(\xi,\eta)}\rho(\xi,\eta)\,d\psi_0(\xi,\eta). \quad (5.30)$$

For solving the system (5.29), (5.30), the use can be made of the method of successive approximations we applied above.

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(Received 12.10.2005)

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