Memoirs on Differential Equations and Mathematical Physics
Volume 4. 1995

S. Kharibegashvili

GOURSAT AND DARBOUX TYPE PROBLEMS FOR LINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS AND SYSTEMS

Abstract. In the present paper, for hyperbolic equations and systems in angular domains, we consider the formulations of problems representing natural continuation and further development of the well-known classical formulations of Goursat and Darboux type problems. For a wide class of linear normally hyperbolic equations and systems of second order, the dependence of unique solvability of the problems under consideration on the structure of an angular domain as well as on the weighted space in which the solution is sought, is established. Some correct multidimensional analogues of Goursat and Darboux type problems for hyperbolic equations are also considered.

1991 Mathematics Subject Classification. 35L20.

Key words and Phrases. Hyperbolic equations and systems, problems of Goursat and Darboux type, characteristic problem, multidimensional analogues of Goursat and Darboux type problems, angular domain, weighted space.

რეზიუმე. ნაშრომში ჰიპერბოლური განტოლებებისა და სისტემებისათვის კუთხიან არეებში განხილულია ამოცანები, რომლებიც წარმოადგენენ ცნობილი გურსას და დარბუს ამოცანების ბუნებრივ გაგრძელებას და განვითარებას. ორგანზომილებიან შემთხვევაში მეორე რიგის წრფივი ნორმალურად ჰიპერბოლური განტოლებებისა და სისტემების ფართო კლასისათვის დადგენილია განხილული ამოცანების ცალსახად ამოხსნადობის დამოკიდებულება კუთხიანი არის სტრუქტურისაგან და, აგრეთვე, იმ წონიანი სივრცისაგან, რომელშიც იძებნება ამონახსნი. გახილულია, აგრეთვე, გურსას და დარბუს ტიპის ამოცანების ზოგიერთი მრავალგანზომილებიანი ანალოგი ჰიპერბოლური ტიპის განტოლებებისათვის.

The questions of searching for and investigation of correctly posed boundary value problems are of great interest in the theory of equations and systems of hyperbolic type. Among these problems boundary value problems for hyperbolic equations and systems representing natural continuation and further development of the well-known classical formulations of the Goursat and Darboux problems are especially interesting.

Unlike the multidimensional case, more simple structure of characteristic manifolds in the two-dimensional case allows one to obtain most complete results on the solvability of these problems for hyperbolic equations.

In the two-dimensional case for the equation of string oscillation written in terms of the characteristic variables

$$u_{xy} = 0, \tag{1}$$

the Goursat problem is formulated as follows [19, 24, 64, 75]: in a rectangular domain $D_0 : 0 < x < a$, $0 < y < b$ find a regular solution $u(x, y)$ of equation (1) of the class $C(\overline{D}_0)$, satisfying on the segments of characteristics $\gamma_1 : y = 0$, $0 \leq x \leq a$ and $\gamma_2 : x = 0$, $0 \leq y \leq b$ the following boundary conditions

$$u\big|_{\gamma_i} = f_i, \quad i = 1, 2, \tag{2}$$

where f_i, $i = 1, 2$, are given real functions satisfying the agreement condition $f_1(O) = f_2(O)$ at the origin $O(0, 0)$.

The solution $u(x, y)$, continuous together with its partial derivatives u_x, u_y and u_{xy}, is called regular in the domain D_0 solution of equation (1).

To solve the problem (1), (2) the use can be made of the well-known Asgeirsson's mean value theorem [17] which in the case of equation (1) is formulated as follows: if $\Omega : a_1 \leq x \leq a_2$, $b_1 \leq y \leq b_2$ is a characteristic rectangle wholly contained in $\overline{D}_0$, then for any regular solution $u(x, y)$ of equation (1) of the class $C(\overline{D}_0)$, the equality

$$u(A) + u(C) = u(B) + u(K), \tag{3}$$

is valid, where $A(a_1, b_1)$, $B(a_1, b_2)$, $C(a_2, b_2)$, $K(a_2, b_1)$ are vertices of the rectangle Ω.

Let $M(x, y)$ be an arbitrary point of the domain D_0, and let $P_1(x, 0) \in \gamma_1$ and $Q_1(0, y) \in \gamma_2$ be the points of intersection with γ_1 and γ_2 of the characteristics of equation (1) coming out of $M(x, y)$. Then by virtue of (3) applied to the characteristic rectangle OP_1MQ_1, the regular solution $u(x, y)$ of the Goursat problem (1), (2) of the class $C(\overline{D}_0)$, for $f_1 \in C^1(0, a] \cap C[0, a]$, $f_2 \in C^1(0, b] \cap C[0, b]$ is given by the formula

$$u(M) = f_1(P_1) + f_2(Q_1) - f_1(O). \tag{4}$$

Let us now consider the Darboux problems [6, 19] for equation (1). Denote by D_1 the domain lying at the angle $x > 0$, $y > 0$ and bounded by the characteristics $\mu_1 : y = 0,\ 0 \le x \le a,\ l_1 : y = b,\ kb \le x \le a,\ l_2 : x = a,$ $0 \le y \le b$ of equation (1) and by a non-characteristic curve $\mu_2 : x = ky$, $0 \le y \le b$, where a, b and k are positive constants with $kb < a$.

The first Darboux problem: find in D_1 a regular solution $u(x,y)$ of equation (1) of the class $C(\overline{D}_1)$ satisfying the boundary conditions

$$u\big|_{\mu_i} = f_i, \quad i = 1, 2, \tag{5}$$

where f_1 and f_2 are given real functions belonging respectively to the classes $C^1(0,a) \cap C[0,a]$ and $C^1(0,b) \cap C[0,b]$, and satisfying $f_1(O) = f_2(O)$.

If $M(x,y)$ is an arbitrary point of D_1, then by $P_1(x,0) \in \mu_1$ and $Q_1(ky,y)$ $\in \mu_2$ we denote the points of intersection with the curves μ_1 and μ_2 of characteristics of equation (1) coming out of $M(x,y)$. Let $P_2(ky,0) \in \mu_1$ be the point of intersection with μ_1 of the characteristic coming out of Q_1.

Applying equality (3) to the characteristic rectangle $P_2Q_1MP_1$, we obtain for the regular solution $u(x,y)$ of the first Darboux problem (1), (5) the following formula

$$u(M) = f_1(P_1) + f_2(Q_1) - f_1(P_2). \tag{6}$$

Denote now by D_2 the domain lying at the angle $x > 0$, $y > 0$ and bounded by the characteristics $l_3 : y = b,\ k_2b \le x \le a,\ l_4 : x = a,\ k_1a \le y \le\ b$ of equation (1) and by non-characteristic curves $\sigma_1 : y = k_1x$, $0 \le x \le a$, $\sigma_2 : x = k_2y$, $0 \le y \le b$, where a, b and k_i, $i = 1, 2$, are positive constants satisfying $k_1a < b$ and $k_2b < a$.

The second Darboux problem: find in D_2 a regular solution $u(x,y)$ of equation (1) of the class $C(\overline{D}_2)$ satisfying on the curves σ_1 and σ_2 the boundary conditions

$$u\big|_{\sigma_i} = f_i, \quad i = 1, 2, \tag{7}$$

where f_i, $i = 1, 2$, are given real functions belonging to the same classes as in the case of the problem (1), (5), and $f_1(O) = f_2(O)$.

Remark. It is seen from the formulas (4) and (6) that the value of the solution $u(x,y)$ of both the Goursat problem (1), (2) and the first Darboux problem (1), (5) at a point $M(x,y)$ depends on the values of functions f_1, f_2 at a finite number of points. At the same time, as it will be seen below, the value of the solution $u(x,y)$ of the second Darboux problem (1), (7), if it exists, will depend on the values of functions f_1, f_2 at an infinite number of points convergent to zero.

Let $M_0(x_0, y_0)$ be an arbitrary point of D_2. By $L_1(M_0)$ and $L_2(M_0)$ we denote, respectively, the characteristics $x = x_0$ and $y = y_0$ of equation (1) passing through M_0. Let $P_1 \in \sigma_1$ and $Q_1 \in \sigma_2$ be the points of intersection of the characteristics $L_1(M_0)$ and $L_2(M_0)$ of equation (1) with the curves σ_1 and σ_2. If the points $P_{n-1} \in \sigma_1$ and $Q_{n-1} \in \sigma_2$ are well determined, then by $P_n \in \sigma_1$ and $Q_n \in \sigma_2$ we denote the points of intersection of the characteristics $L_1(Q_{n-1})$ and $L_2(P_{n-1})$ with σ_1 and σ_2, respectively. Continuing this process, we shall get the sequences $P_1, P_2, \dots, P_n, \dots$ and $Q_1, Q_2, \dots, Q_n, \dots$ of points lying respectively on σ_1 and σ_2 and tending for $n \to \infty$ to the origin O.

Denote by $M_n \in D_2$ the point of intersection of the characteristics $L_2(P_n)$ and $L_1(Q_n)$. Obviously, the sequence of points M_n also tends to the origin O for $n \to \infty$. Without restriction of generality we can assume $u(O) = f_1(O) = f_2(O) = 0$, since otherwise the function $v = u - f_1(O)$ is considered as a new unknown function.

Applying (3) to the rectangle $M_{n-1}P_nM_nQ_n$, we obtain

$$u(M_{n-1}) = f_1(P_n) + f_2(Q_n) - u(M_n), \quad n = 1, 2, \dots . \tag{8}$$

From (8) we have

$$u(M_0) = \sum_{i=1}^{n} (-1)^{i+1} \big[f_1(P_i) + f_2(Q_i) \big] + (-1)^n u(M_n). \tag{9}$$

If the problem (1), (7) is solvable, then passing in (9) to the limit for $n \to \infty$ and taking into account that $\lim\limits_{n\to\infty} u(M_n) = u(O) = f_1(O)$, we get that the series

$$I = \sum_{i=1}^{\infty} (-1)^{i+1} \big[f_1(P_i) + f_2(Q_i) \big] \tag{10}$$

converges. Thus the convergence of (10) is necessary and sufficient for the problem (1), (7) to be solvable in the class of regular solutions introduced above.

Passage in (9) to the limit for $n \to \infty$ when $f_1 = f_2 = 0$ also shows that in the class of regular solutions the second Darboux problem cannot have more than one solution.

Now let us show that the series (10) converges not for all functions f_1 and f_2 from the above mentioned classes. For the sake of simplicity let $a = b = 1$, $0 < k_1 = k_2 = k < 1$, $f_2 \equiv 0$, and let $x_0 = y_0 = 1$ be the coordinates of M_0. As a function $f_1 = f_1(x)$ of the class $C^1(0,1) \cap C[0,1]$, we take

$$f_1(x) = \frac{\cos(\pi \frac{\ln x}{\ln k})}{\ln \frac{1}{2} x} .$$

In this case (10) takes the form

$$I = \sum_{i=1}^{\infty} \frac{1}{\ln \frac{1}{2} k^{i-1}} = \sum_{i=1}^{\infty} \frac{1}{(i-1)\ln k + \ln \frac{1}{2}} ,$$

and, obviously, diverges.

Since in (10)

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} Q_n = O,$$

to ensure convergence of this series we additionally require of the functions f_1 and f_2 to be regular in a neighborhood of O. For example, it suffices to require that

$$f_1 \in C^1(0,a] \cap C[0,a], \quad f_2 \in C^1(0,b] \cap C[0,b]$$

and for some α, $0 < \alpha = \text{const} < 1$, the first order derivatives of these functions have integrable at O singularities of the type

$$\big|f_1^{(1)}(x)\big| \le \frac{C}{x^\alpha}\,, \quad \big|f_2^{(1)}(y)\big| \le \frac{C}{y^\alpha}\,, \quad C = \text{const} > 0. \tag{11}$$

In this case, the series (10) and that obtained from (10) by termwise differentiation with respect to x or y converge uniformly in D_2 and the regular solution of the problem (1), (7) is given by the formula

$$u(M_0) = \sum_{i=1}^{\infty} (-1)^{i+1} \big[f_1(P_i) + f_2(Q_i)\big].$$

The solution and its partial derivatives with respect to x and y satisfy in a neighborhood of O the estimates

$$\begin{gathered} |u(x,y)| \le C_1 \big(|x| + |y|\big)^{1-\alpha}, \quad |u_x(x,y)| \le \frac{C_1}{(|x|+|y|)^\alpha}\,, \\ |u_y(x,y)| \le \frac{C_1}{(|x|+|y|)^\alpha}\,, \quad C_1 = \text{const} > 0. \end{gathered} \tag{12}$$

Thus, to ensure the solvability of the second Darboux problem (1), (7), we have naturally come to the consideration of weighted spaces defined by inequalities (11) for the functions f_1, f_2 and by inequalities (12) for the regular solutions of equation (1).

Chapter I of the present paper deals with the boundary value problems for equation (1) which are formulated more generally than the above-mentioned Goursat and Darboux type problems.

The results obtained for equation (1) are in a definite sense complete and simple by form and serve as a visual model for investigation of boundary value problems for second order hyperbolic systems with two independent variables.

Let $\gamma_1 : y = \gamma_1(x)$, $0 \le x \le x_0$, and $\gamma_2 : x = \gamma_2(y)$, $0 \le y \le y_0$, be the two simple smooth curves coming out of the origin O and lying wholly at the angle $x \ge 0$, $y \ge 0$. Below it is assumed that the functions $\gamma_1(x)$ and $\gamma_2(y)$ are monotonically non-decreasing, i.e., $\gamma_1^{(1)}(x) \ge 0$, $\gamma_2^{(1)}(y) \ge 0$, and $\gamma_1(\gamma_2(y)) < y$ for $0 < y \le y_0$. Denote by D the domain lying at the angle $x > 0$, $y > 0$ bounded by the curves γ_1, γ_2 and the characteristics $L_1(P_0) : x = x_0$ and $L_2(P_0) : y = y_0$ coming out of the point $P_0(x_0, y_0)$.

Consider the boundary value problem formulated as follows [29]: find in the domain D a regular solution $u(x,y)$ of equation (1) satisfying on the curves γ_1 and γ_2 the following conditions

$$\big(M_i u_x + N_i u_y\big)\big|_{\gamma_i} = f_i, \quad i = 1,2, \tag{13}$$

where M_i, N_i, f_i, $i = 1,2$, are given real functions.

Remark. Note that the Goursat and Darboux type problems considered above are reduced to a problem of the type (1), (13) by differentiating the corresponding boundary conditions along the tangents of data carriers of these problems.

The solution of the problem (1),(13) is sought in the following weight space

$$C^{1,1}_{\alpha}(\overline{D}) = \Big\{ u \in C(\overline{D}) : \ u_x, u_y, u_{xy} \in C(\overline{D}\backslash O), \ u(0,0) = 0,$$
$$\sup_{z\in\overline{D}\backslash O} |z|^{-\alpha}|u_x(z)| < \infty, \ \sup_{z\in\overline{D}\backslash O} |z|^{-\alpha}|u_y(z)| < \infty,$$
$$\sup_{z\in\overline{D}\backslash O} |z|^{-(\alpha-1)}|u_{xy}(z)| < \infty \Big\},$$

where $z = x + iy$, $i = \sqrt{-1}$, and $\alpha > -1$ is a real parameter.

Obviously, if $u \in C^{1,1}_{\alpha}(\overline{D})$, then $\sup_{z\in\overline{D}\backslash O} |z|^{-(1+\alpha)}|u(z)| < \infty$.

If the solution $u(x,y)$ of the problem (1), (13) is sought in the space $C^{1,1}_{\alpha}(\overline{D})$, then we require of the boundary functions f_1, f_2 that

$$f_1(x) \in C_{\alpha}(\gamma_1) = \Big\{ f_1 \in C(0,x_0] : \sup_{0<x\le x_0} \big|x^{-\alpha} f_1(x)\big| < \infty \Big\},$$
$$f_2(y) \in C_{\alpha}(\gamma_2) = \Big\{ f_2 \in C(0,y_0] : \sup_{0<y\le y_0} \big|y^{-\alpha} f_2(y)\big| < \infty \Big\}.$$

It is shown that the correctness of the problem (1), (13) in the class $C^{1,1}_{\alpha}(\overline{D})$ depends essentially on the parameter α, as well as on the angle between the supports of boundary data γ_1 and γ_2 at the common point O and their configuration [29]. For example, if the curves γ_1, γ_2 are not characteristics of equation (1), do not have a common tangent line at O, and $M_i|_{\gamma_i} \neq 0$, $N_i|_{\gamma_i} \neq 0$, $i = 1,2$, then for $\alpha > -\frac{\ln|\sigma|}{\ln \tau_0}$ the problem (1), (13) is uniquely solvable in the class $C^{1,1}_{\alpha}(\overline{D})$, while for $\alpha < -\frac{\ln|\sigma|}{\ln \tau_0}$ the homogeneous problem corresponding to (1), (13) has an infinite number of linearly independent solutions, where $\sigma = (M_1^{-1}N_1M_2N_2^{-1})(O)$, $0 < \tau_0 = \gamma_1^{(1)}(0)\gamma_2^{(1)}(0) < 1$.

In the case where the curves γ_1, γ_2 have the same tangent line at O, i.e. $\tau_0 = \gamma_1^{(1)}(0)\gamma_2^{(1)}(0) = 1$ and $M_i|_{\gamma_i} \neq 0$, $N_i|_{\gamma_i} \neq 0$, $i = 1,2$, then for $|\sigma| < 1$ the problem (1), (13) is uniquely solvable in the class $C^{1,1}_{\alpha}(\overline{D})$, while for $|\sigma| > 1$ the homogeneous problem corresponding to (1), (13) has an infinite number of linear independent solutions [29]. We should also note the work [53] in which sufficient conditions for unique solvability of the problem (1), (13) in the class $C^2(\overline{D})$ are obtained in the case where γ_1 and γ_2 are segments of non-characteristic straight lines coming out of the common point O. The case $|\sigma| = 1$ which corresponds to the case where the directions of differentiation operators $\frac{\partial}{\partial l_i} = M_i \frac{\partial}{\partial x} + N_i \frac{\partial}{\partial y}$, $i = 1, 2$, appearing in the boundary conditions (13) coincide at the point O, turned out to be more complicated. More interesting results in this direction are obtained by T. M. Makharadze [51, 52]. He has established that the correctness of formulation of the problem under consideration depends on the parameter α, the order of tangency of the curves γ_1, γ_2 and the directions of differentiation operators $\frac{\partial}{\partial l_1}$, $\frac{\partial}{\partial l_2}$ at O. The results of Firmani concerning the second Darboux problem in the case where the curves σ_1 and σ_2 have a common tangent line at O are also worth mentioning [20–22].

In the same chapter it is shown that when condition $M_1(x,y) \neq 0$ or $N_2(x,y) \neq 0$ is violated on the whole curve γ_1 or γ_2, the existence of the lowest terms in this problem may affect the correctness of formulation of the problem (1), (13). The case where condition $M_1(x,y) \neq 0$ or $N_2(x,y) \neq 0$ violates at one point O only, is also considered. In this case, in the class $C^{1,1}_{\alpha}(\overline{D})$ the homogeneous problem corresponding to (1), (13) has an infinite number of linearly independent solutions. At the same time, the functional space $C_{\alpha,\chi}(\overline{D})$ is determined such that the problem (1), (13) is uniquely solvable.

Additional difficulties arise when we pass to second order hyperbolic systems. This has been first shown by A. V. Bitsadze [7] who constructed examples of second order hyperbolic systems for which the corresponding homogeneous characteristic problem (the Goursat problem with data on the characteristics) has a finite or even an infinite number of linearly independent solutions. Characteristic problem for second order hyperbolic systems with two independent variables and constant leading coefficients has been investigated in the works of the author [30–32]. In particular, these works reveal new effects connected with the problems of smoothness of solutions and the possibility for the characteristic problem to have a non-zero finite index. Simple examples of second order hyperbolic systems in A. V. Bitsadze's work [8] illustrate how the lowest terms affect the correctness of formulation of the characteristic problems.

S. L. Sobolev [68], V. P. Mikhaĭlov [58, 59] and L. A. Mel'tser [55] investigated some analogues of the Goursat type problem in the case of first order hyperbolic systems with two independent variables.

Chapter II deals with the boundary value problems for second order linear normal hyperbolic systems with variable coefficients of the type

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + A_1 u_x + B_1 u_y + C_1 u = F$$

in the weighted spaces $\overset{\circ}{C}{}^k_{\alpha}(\overline{D})$ [33–37, 54]. Boundary conditions in these problems are determined by a first order differential operator, while the carrier of these conditions are the two arcs γ_1 and γ_2 with a common point at the origin. The sufficient conditions imposed both on the coefficients of the system and on the curves γ_1, γ_2 ensuring correctness of the problems in the spaces $\overset{\circ}{C}{}^k_\alpha(\overline{D})$ are also given in the same chapter. The structure of the domain of definition of the solution is determined depending on the location of data carriers with respect to the characteristics of the system.

Characteristic problems for second order linear hyperbolic systems of the types

$$y^m A u_{xx} + 2y^{\frac{m}{2}} B u_{xy} + C u_{yy} + a u_x + b u_y + c u = F \qquad (14)$$

and

$$A u_{xx} + 2y^{\frac{m}{2}} B u_{xy} + y^m C u_{yy} + a u_x + b u_y + c u = F \qquad (15)$$

with parabolic degeneration along the straight line $y = 0$ are studied in Chapter III. Boundary conditions in these problems are determined by means of Goursat type data, while the carrier of these conditions are the two arcs of adjoint characteristics of the system coming out of the point of parabolic degeneration. Under certain conditions imposed on the coefficients of the system and boundary operator, we prove theorems on the unique solvability of these problems in special weighted spaces determined with regard to the character of parabolic degeneration [38–40]. The condition obtained in this case and imposed on the lowest terms of the system is an exact analogue of the well-known Gellerstedt's condition for one equation.

It should be noted that the characteristic problem with boundary conditions $u|_{\gamma_i} = f_i$, $i = 1, 2$, on segments of characteristics γ_1 and γ_2 coming out of the origin $O(0,0)$, has been investigated by L. Sh. Agababyan and A. B. Nersesyan [1–3] for one second order hyperbolic equation with parabolic degeneration of the type

$$y^m u_{xx} - u_{yy} + a u_x + b u_y + c u = F$$

in a rectangle bounded by characteristics of that equation coming out of the points $O(0,0)$ and $P(0,1)$. The characteristic problem for the equation

$$y^2 u_{xx} - u_{yy} + a u_x = 0$$

has been studied by T. Sh. Kalmenov [27] in a triangular domain bounded by the segment $[0,1]$ of the axis x and by pieces of characteristics coming out of the points $O(0,0)$ and $Q_1(1,0)$. Note also the works of V. N. Vragov [76] and B. A. Bubnov [15] where, in particular, the characteristic problem in domains containing a segment of a line of degeneration is studied. The case when OP_1 is a segment of a axis x and OP_2 is that of a characteristic of one hyperbolic equation with parabolic degeneration for $y = 0$, has been studied in the works of V. N. Vragov [76] and A. M. Nakhushev [60–62], while for the systems of the type

$$K(y)u_{xx} - Eu_{yy} + au_x + bu_y + cu = F,$$

this case has been studied by M. Meredov [56, 57].

In this chapter, the class of hyperbolic systems of the type (14) and (15) for which characteristic problems are investigated, contains the systems with non-split principal parts and the higher term $2y^{\frac{m}{2}}Bu_{xy}$ different from zero.

The last Chapter IV concerns with certain multidimensional variants of Goursat and Darboux type problems for linear hyperbolic differential equations.

If in the two-dimensional case the problems of the Goursat and Darboux type for hyperbolic equations and systems are investigated with sufficient completeness, in the multidimensional case we have in this direction only individual results. One of the main reasons is probably the existence of a continual bundle of bicharacteristics of a hyperbolic equation, owing to which, in particular, to ensure the correctness of this or that problem, one should require definite orientation of data supports.

A multidimensional analogue of the Goursat problem (the Cauchy characteristic problem) when the solution of a second order hyperbolic equation is sought inside a characteristic conoid, has been studied by D'Adhemar [18], Hadamard [25], S. L. Sobolev [69], Riesz [67], Lundberg [50], A. A. Borgardt and D. A. Karnenko [14]. In the case when a second order hyperbolic system is split in its principal part, the same problem has been investigated by Cagnac [16] in the four-dimensional space.

It should be noted that the Cauchy characteristic problem for a non-split in the principal part second order hyperbolic system has not been studied so far. Here, alongside with technical difficulties, there arise principal algebraic difficulties connected with determination of geometric structure of a characteristic conoid in a vicinity of the vertex.

Certain multidimensional analogues of the first or the second Darboux problems are treated by C. L. Sobolev [70], Gårding [23], A. V. Bitsadze [9], V. N. Vragov [76], T. Sh. Kalmenov [28] and Rassias [65, 66] for the case where the solution of a second order hyperbolic equation is sought in a conic domain, one part of whose boundary is of time-type and the other is either characteristic or wholly of time-type. One variant of the second Darboux problem in a conic domain of time-type is studied by S. S. Kharibegashvili in the case where a second order hyperbolic system is non-split in its principal part and for one hyperbolic equation of higher order with constant coefficients at higher derivatives [41–43]. Note that for general hyperbolic equations and systems both variants of the Darboux problems in conic domains are not treated.

Other multidimensional analogues of the Goursat and Darboux problems for one second order hyperbolic equation in a bihedral angle when either both sides are characteristic or one side is characteristic and the other is a hypersurface of time-type, have been considered in the works of Beudon [5], Hadamard [25], Tolen [71] and S. S. Kharibegashvili [44–46]. The second Darboux problem when both sides are hypersurfaces of time-type is more complicated. This case is considered by S. S. Kharibegashvili in [47].

In Chapter IV we shall restrict ourselves to the statement of the results obtained in the course of investigation of multidimensional analogues of the Goursat and Darboux problems for the second order hyperbolic equation with the wave operator in its principal part in a bihedral angle of a quite definite orientation [44–47]. The final paragraph of this chapter concerns with a multidimensional variant of the second Darboux problem for a higher order hyperbolic equation with constant coefficients at higher derivatives in a conic domain located fully in the interior cone of rays [43].

§

In the plane of variables x, y let us consider a second order hyperbolic equation of the type

$$u_{xy} + a_1 u_x + b_1 u_y + c_1 u = F, \tag{1.1}$$

where a_1, b_1, c_1, F are given real functions and u is an unknown one.

Let $\gamma_1 : y = \gamma_1(x)$, $0 \le x \le x_0$, and $\gamma_2 : x = \gamma_2(y)$, $0 \le y \le y_0$, be two simple curves of the class C^1 coming out of the origin $O(0,0)$ of the plane of variables x, y and located completely in the angle $x \ge 0$, $y \ge 0$.

Below we shall assume that $\gamma_1(\gamma_2(y)) < y$, $0 < y \le y_0$, and each of the curves γ_i, $i = 1,2$, either is a characteristic of equation (1.1) or it has characteristic direction at none of its points, except maybe $O(0,0)$. This implies that if γ_1 (γ_2) is not a characteristic, then the function $y = \gamma_1(x)$ $(x = \gamma_2(y))$ is strictly monotonically increasing. Denote by D the domain lying at the angle $x > 0$, $y > 0$, bounded by the curves γ_1, γ_2 and the characteristics $L_1(P_0) : x = x_0$ and $L_2(P_0) : y = y_0$ of equation (1.1), coming out of the point $P_0(x_0, y_0)$.

Consider the boundary value problem formulated as follows: in the domain D find a regular solution $u(x,y)$ of (1.1) satisfying on γ_1 and γ_2

$$\left(M_i u_x + N_i u_y + S_i u\right)\Big|_{\gamma_i} = f_i, \quad i = 1,2, \tag{1.2}$$

where M_i, N_i, S_i, f_i, $i = 1,2$, are given real functions.

The solution of the problem (1.1), (1.2) is sought in the weighted space

$$\begin{gathered} C^{1,1}_\alpha(\overline{D}) = \Big\{ u \in C(\overline{D}) : \ u_x, u_y, u_{xy} \in C(\overline{D} \setminus O), \ u(0,0) = 0, \\ \sup_{z \in \overline{D} \setminus O} |z|^{-\alpha} |u_x(z)| < \infty, \ \sup_{z \in \overline{D} \setminus O} |z|^{-\alpha} |u_y(z)| < \infty, \\ \sup_{z \in \overline{D} \setminus O} |z|^{-(\alpha-1)} |u_{xy}(z)| < \infty \Big\}, \end{gathered}$$

where $z = x + iy$, $i = \sqrt{-1}$, $\alpha > -1$ is a real parameter.

Obviously, if $u \in C^{1,1}_\alpha(\overline{D})$, then $\sup_{z \in \overline{D} \setminus O} |z|^{-(1+\alpha)} |u(z)| < \infty$.

When considering the problems (1.1), (1.2) in the space $u \in C^{1,1}_\alpha(\overline{D})$, we require that $a_1, b_1, c_1 \in C(\overline{D})$, $M_i, N_i, S_i \in C(\gamma_i)$, $i = 1,2$,

$$f_1(x) \in C_\alpha(\gamma_1) = \Big\{ f_1 \in C(0, x_0] : \sup_{0 < x \le x_0} \left| x^{-\alpha} f_1(x) \right| < \infty \Big\},$$

$$f_2(x) \in C_\alpha(\gamma_2) = \Big\{ f_2 \in C(0,y_0] : \sup_{0<y\le y_0} \big| y^{-\alpha} f_2(y) \big| < \infty \Big\},$$
$$F(z) \in C_{\alpha-1}(\overline{D}) = \Big\{ F \in C(\overline{D}\backslash O) : \sup_{z\in \overline{D}\backslash O} |x^{-(\alpha-1)}| F(z)| < \infty \Big\}.$$

For the sake of simplicity, we shall restrict ourselves to the consideration of the equation of the string oscillation

$$u_{xy} = 0, \tag{1.3}$$

and in the boundary conditions (1.2) we shall assume $S_i = 0$, $i = 1, 2$, i.e.,

$$\big(M_i u_x + N_i u_y \big)\big|_{\gamma_i} = f_i, \quad i = 1, 2. \tag{1.4}$$

Denoting $v = u_x$ and $w = u_y$, we can rewrite the problem (1.3), (1.4) equivalently in the form

$$v_y = 0, \tag{1.5}$$
$$w_x = 0, \tag{1.6}$$
$$u_y = w \tag{1.7}$$

with boundary conditions

$$\big(M_i v + N_i w \big)\big|_{\gamma_i} = f_i, \quad i = 1, 2, \tag{1.8}$$
$$\big(u_x + \gamma_1^{(1)} u_y \big)\big|_{\gamma_1} = \big(v + \gamma_1^{(1)} w \big)\big|_{\gamma_1}. \tag{1.9}$$

Indeed, if $u(x, y)$ is a solution of the problem (1.3), (1.4), then it is clear that the system of functions u, v and w satisfies (1.5)–(1.9). Conversely, let u, v, w be a solution of the problem (1.5)–(1.9). Then, obviously, equalities $u_x = v$, $w = u_y$ imply that $u(x, y)$ is a solution of the problem (1.3), (1.4). Therefore, by virtue of (1.7) it suffices to prove that $u_x = v$.

Let $g = v - u_x$. Then owing to (1.5)–(1.7), we have

$$g_y = v_y - u_{xy} = 0 - (u_y)_x = 0 - w_x = 0.$$

Hence $g(x, y) = g(x)$, i.e.,

$$g(P) = g(P^*) = (v - u_x)\big|_{\gamma_1}, \tag{1.10}$$

where P^* is the projection of an arbitrarily taken point $P(x, y) \in D$ on the curve γ_1, parallel to the axis Oy.

By (1.7) and (1.9) we have

$$\big(u_x + \gamma_1^{(1)} u_y \big)\big|_{\gamma_1} = \big(u_x + \gamma_1^{(1)} w \big)\big|_{\gamma_1} = \big(v + \gamma_1^{(1)} w \big)\big|_{\gamma_1},$$

whence $u_x|_{\gamma_1} = v|_{\gamma_1}$ and, according to (1.10), we get $g \equiv 0$ which means that $u_x = v$ in D.

Denoting $v|_{\gamma_1} = \varphi(x)$ and $w|_{\gamma_2} = \psi(y)$, we rewrite the boundary conditions (1.8) as a system of two functional equations

$$M_1\varphi(x) + N_1\psi\big(\gamma_1(x)\big) = f_1(x), \quad 0 < x \le x_0, \tag{1.11}$$

$$M_2\varphi\big(\gamma_2(y)\big) + N_2\psi(y) = f_2(y), \quad 0 < y \le y_0, \tag{1.12}$$

with respect to the unknown functions $(\varphi, \psi) \in C_\alpha(\gamma_1) \times C_\alpha(\gamma_2)$.

Evidently, if $\varphi(x)$ and $\psi(y)$ are a solution of the system (1.11), (1.12), then the functions u, v and w of the problem (1.5)–(1.9) can be uniquely defined by the formulas

$$v(x,y) = \varphi(x), \quad w(x,y) = \psi(y), \quad u(x,y) = \int\limits_{OP} vdx + wdy,$$

where $OP \subset D$ is a curve connecting the point $P(x,y) \in D$ with the origin $O(0,0)$.

Below we shall assume that

$$M_1\Big|_{\gamma_1} \ne 0, \quad N_2\Big|_{\gamma_2} \ne 0. \tag{1.13}$$

Excluding in the system (1.11), (1.12) the unknown function $\psi(y)$, for $\varphi(x)$ we obtain the functional equation

$$T\varphi \equiv \varphi(x) - a(x)\varphi\big(\tau(x)\big) = f(x), \quad 0 < x \le x_0. \tag{1.14}$$

Here

$$a(x) = M_1^{-1}(x)N_1(x)N_2^{-1}\big(\gamma_1(x)\big)M_2\big(\gamma_1(x)\big), \tag{1.15}$$

$$\tau(x) = \gamma_2\big(\gamma_1(x)\big), \tag{1.16}$$

$$f(x) = M_1^{-1}(x)f_1(x) - M_1^{-1}(x)N_1(x)N_2^{-1}\big(\gamma_1(x)\big)f_2\big(\gamma_1(x)\big).$$

Remark. It is obvious that when the conditions (1.13) are fulfilled, the problem (1.3), (1.4) in the class $C_\alpha^{1,1}(\overline{D})$ is equivalently reduced to one functional equation (1.14) with respect to the unknown function $\varphi(x)$ of the class $C_\alpha(0,x_0]$.

§ γ_1 γ_2 $O(0,0)$

Let $\tau_0 = \tau^{(1)}(0)$. If the curves γ_1 and γ_2 do not have a common tangent line at $O(0,0)$, then due to the requirements imposed on γ_1 and γ_2 we have $0 \le \tau_0 < 1$, where $\tau_0 = 0$ if and only if one of the curves γ_1 or γ_2 has a characteristic direction at this point.

If at least one of the curves γ_1 or γ_2 is a characteristic of equation (1.3), *then equation* (1.14) *is uniquely solvable in the class $C_\alpha(0,x_0]$ for $\alpha > 0$.*

Proof. Obviously, in this case $\tau(x) \equiv 0$. Moreover, since $\alpha > 0$, from $\varphi(x)$ $(f(x)) \in C_\alpha(0, x_0]$ we have $\varphi(x)$ $(f(x)) \in C[0, x_0]$ and $\varphi(0) = 0$ $(f(0) = 0)$. Therefore in this case equation (1.14) takes the trivial form

$$\varphi(x) = f(x). \quad \blacksquare$$

Let now γ_1 and γ_2 not be characteristics of (1.3). Then according to the requirements imposed on γ_1 and γ_2, the continuously differentiable function $\tau(x)$ defined by (1.16) is strictly monotonically increasing on $[0, x_0]$ and

$$\tau(0) = 0, \quad 0 < \tau(x) < x \quad \text{for} \quad 0 < x \leq x_0. \tag{1.17}$$

Therefore if $\tau_k(x) = \tau(\tau_{k-1}(x))$, $\tau_1(x) = \tau(x)$, $0 \leq x \leq x_0$, then according to (1.17) the sequence $\{\tau_k(x)\}_{k=1}^{\infty}$ on the interval $[0, x_0]$ tends uniformly to zero, as $k \to \infty$. Hence there exists a natural number n such that

$$\tau_k(x) \leq \varepsilon, \quad 0 \leq x \leq x_0, \quad k \geq n. \tag{1.18}$$

Let equation (1.14) *be uniquely solvable on the interval* $(0, \varepsilon]$, $0 < \varepsilon = \text{const} < x_0$, *in the class* $C_\alpha(0, \varepsilon]$. *Then equation* (1.14) *is likewise uniquely solvable on the whole interval* $(0, x_0]$ *in the class* $C_\alpha(0, x_0]$, *and its solution* $\varphi(x)$ *can be represented in the form*

$$\varphi(x) = \begin{cases} \varphi_0(x), & 0 < x \leq \varepsilon, \\ (\Lambda^n \varphi_0)(x) + f(x) + \sum\limits_{i=1}^{n-1} (\Lambda^i f)(x), & x > \varepsilon, \end{cases} \tag{1.19}$$

where $\varphi_0(x)$ *is the solution of equation* (1.14) *on* $(0, \varepsilon]$ *of the class* $C_\alpha(0, \varepsilon]$, $(\Lambda^n \varphi)(x) = a(x) a(\tau(x)) \cdots a(\tau_{n-1}(x)) \varphi(\tau_n(x))$, *and the number* n *is chosen by inequality* (1.18).

Proof of Lemma 1.2 is trivial.

The following lemma is obvious.

In the class $C_\alpha(0, x_0]$, (1.14) *is equivalent to the equation*

$$\psi(x) - a(x) \Big(\frac{\tau(x)}{x} \Big)^\alpha \psi\big(\tau(x)\big) = g(x), \quad 0 < x \leq x_0, \tag{1.20}$$

in the class $C_0(0, x_0]$, *where* $\psi(x) = x^{-\alpha} \varphi(x) \in C_0(0, x_0]$, $g(x) = x^{-\alpha} f(x) \in C_0(0, x_0]$.

Lemmas 1.2 and 1.3 immediately yield

Equation (1.14) *is uniquely solvable in the class* $C_\alpha(0, x_0]$ *if and only if equation* (1.20) *is uniquely solvable for some* ε, $0 < \varepsilon < x_0$, *in the class* $C_0(0, \varepsilon]$.

Let the curves $\gamma_1 \backslash O$ and $\gamma_2 \backslash O$ not be characteristics of equation (1.3) *and at least one of them have characteristic direction at the point O. Then equation* (1.14) *is uniquely solvable in the class $C_\alpha(0, x_0]$ for $\alpha > 0$. If, however, $-1 < \alpha \leq 0$, then* (1.14) *is uniquely solvable in the class $C_\alpha(0, x_0]$ when the condition*

$$\overline{\lim_{x \to +0}} \left| a(x) \left(\frac{\tau(x)}{x} \right)^\alpha \right| < 1 \tag{1.21}$$

is fulfilled.

Proof. By virtue of Lemma 1.4, it suffices to prove that for sufficiently small $\varepsilon > 0$ the operator

$$(T_0 \psi)(x) = a(x) \left(\frac{\tau(x)}{x} \right)^\alpha \psi\big(\tau(x)\big), \tag{1.22}$$

appearing in (1.20) has in the space $C_0(0, \varepsilon]$ the norm which is less than unity, i.e.,

$$\|T_0\|_{C_0(0,\varepsilon] \to C_0(0,\varepsilon]} < 1. \tag{1.23}$$

Really, in this case the Neumann series

$$(I - T_0)^{-1} = I + T_0 + \cdots + T_0^n + \cdots$$

for the operator T_0 converges in the space $C_0(0, \varepsilon]$ and the unique solution $\psi(x)$ of (1.20) can be represented in the form

$$\psi = f + T_0 f + \cdots + T_0^n f + \cdots,$$

where I is an identical operator.

In the first case, when $\alpha > 0$ and at least one of the curves γ_1 or γ_2 has the characteristic direction at O, we have $\tau_0 = \tau^{(1)}(0) = 0$, and

$$\lim_{x \to +0} \left| a(x) \left(\frac{\tau(x)}{x} \right)^\alpha \right| = \lim_{x \to +0} |a(x)| \lim_{x \to +0} \left(\frac{\tau(x)}{x} \right)^\alpha = \\ = |a(0)| \big(\tau^{(1)}(0)\big)^\alpha = 0.$$

Therefore, since the function $a(x)(\frac{\tau(x)}{x})^\alpha$ is continuous in a vicinity of zero, there exists a sufficiently small number $\varepsilon > 0$ such that for $0 < x \leq \varepsilon$ we have

$$\max_{0 < x \leq \varepsilon} \left| a(x) \left(\frac{\tau(x)}{x} \right)^\alpha \right| \leq q = \text{const} < 1,$$

whence we get

$$\|T_0 \psi\|_{C_0(0,\varepsilon]} = \sup_{0 < x \leq \varepsilon} \left| a(x) \left(\frac{\tau(x)}{x} \right)^\alpha \psi\big(\tau(x)\big) \right| \leq \\ \leq q \sup_{0 < x \leq \varepsilon} \left| \psi\big(\tau(x)\big) \right| \leq q \sup_{0 < x \leq \varepsilon} |\psi(x)| = q \|\psi\|_{C_0(0,\varepsilon]},$$

i.e.,

$$\|T_0\|_{C_0(0,\varepsilon]\to C_0(0,\varepsilon]} \le q < 1.$$

In the second case, when $-1 < \alpha \le 0$ and (1.21) is fulfilled, the estimate (1.23) for the norm of the operator T_0 defined by (1.22) can be proved analogously. ■

Let now the curves γ_1 and γ_2 not be characteristics of equation (1.3) and have no characteristic direction at O. In this case $0 < \tau_0 < 1$. Put $\sigma = a(0)$.

Let the curves $\gamma_1 \backslash O$ and $\gamma_2 \backslash O$ not be the characteristics of equation (1.3) *and have no characteristic direction at O. Then for $\alpha > -\frac{\log|\sigma|}{\log\tau_0}$, equation* (1.14) *is uniquely solvable in the class $C_\alpha(0, x_0]$.*

Remark. In Lemma 1.6 for $\sigma = 0$, that is for $N_1(0)M_2(0) = 0$, one should assume $-\frac{\log|\sigma|}{\log\tau_0} = -\infty$, and in this case equation (1.14) is uniquely solvable for any $\alpha > -1$.

Proof. It follows from the condition $\alpha > -\frac{\log|\sigma|}{\log\tau_0}$ that

$$|\sigma|\tau_0^\alpha < 1,$$

whence we directly obtain (1.21)

$$\overline{\lim_{x\to+0}}\Big|a(x)\Big(\frac{\tau(x)}{x}\Big)^\alpha\Big| = |\sigma|\tau_0^\alpha < 1$$

which, as is shown in Lemma 1.5, ensures the unique solvability of equation (1.14) in the class $C_\alpha(0,\varepsilon]$. ■

Let the curves $\gamma_1 \backslash O$ and $\gamma_2 \backslash O$ not be characteristics of equation (1.3) *and have no characteristic direction at O. If $N_1(0)M_2(0) \ne 0$, then for $\alpha < -\frac{\log|\sigma|}{\log\tau_0}$ equation* (1.14) *is solvable in the class $C_\alpha(0, x_0]$, and the homogeneous equation corresponding to* (1.14) *has an infinite number of linearly independent solutions in this class.*

Proof. Since $N_1(0)M_2(0) \ne 0$, i.e. $\sigma \ne 0$ and $\alpha < -\frac{\log|\sigma|}{\log\tau_0}$, there exists a positive number ε, $\varepsilon < x_0$, such that for $0 < x \le \varepsilon$ we have $N_1(x) \ne 0$, $M_2(x) \ne 0$, and

$$\Big|a(x)\Big(\frac{\tau(x)}{x}\Big)^\alpha\Big| \ge \frac{1}{q} = \text{const} > 1, \quad 0 < x \le \varepsilon. \tag{1.24}$$

Since the function $\tau(x)$ is strictly monotone, for any x from the interval $0 < x < \tau(\varepsilon)$ there exists a unique natural number $n_1 = n_1(x)$ satisfying

$$\tau(\varepsilon) < \tau^{-n_1}(x) \le \varepsilon.$$

Analogously, for any x satisfying $\varepsilon_1 < x \le x_0$ there exists a unique natural number $n_2 = n_2(x)$ such that

$$\tau(\varepsilon) \le \tau^{n_2}(x) < \varepsilon.$$

By virtue of Lemma 1.3, it suffices to prove that equation (1.20) is solvable in the class $C_0(0,\varepsilon]$, and for the homogeneous equation corresponding to (1.20) there exists an infinite number of linearly independent solutions of this class.

Since the function $\tau(x)$ is strictly monotonically increasing, there exists a function inverse to $\tau(x)$ which we denote by $\tau^{-1}(x)$. It is easily seen that the operator T_0 defined by (1.22) is invertible, and

$$\big(T_0^{-1}\psi\big)(x) = a^{-1}\big(\tau^{-1}(x)\big)\Big(\frac{x}{\tau^{-1}(x)}\Big)^{-\alpha}\psi\big(\tau^{-1}(x)\big). \tag{1.25}$$

It can be easily verified that every solution of (1.20) which is continuous in a half-interval $0 < x \le x_0$ is given by

$$\psi(x) = \begin{cases} \psi_0(x), & \tau(\varepsilon) \le x \le \varepsilon, \\ \big(T_0^{-n_1(x)}\psi_0\big)(x) - \sum\limits_{i=1}^{n_1(x)} \big(T_0^{-i}g\big)(x), & 0 < x < \tau(\varepsilon), \\ \big(T_0^{n_2(x)}\psi_0\big)(x) + \sum\limits_{i=0}^{n_2(x)-1} \big(T_0^{i}g\big)(x), & \varepsilon < x \le x_0, \end{cases} \tag{1.26}$$

where $\psi_0(x)$ is an arbitrary function of the class $C[\tau(\varepsilon),\varepsilon]$ satisfying the condition $\psi_0(\varepsilon) - a(\varepsilon)(\frac{\tau(\varepsilon)}{\varepsilon})^\alpha\psi_0(\tau(\varepsilon)) = g(\varepsilon)$.

Let us show that if $g \in C_0(0,\varepsilon]$, the function $\psi(x)$ given by (1.26) belongs to the class $C_0(0,x_0]$ for any $\psi_0 \in C[\tau(\varepsilon),\varepsilon]$, $\psi_0(\varepsilon) - a(\varepsilon)(\frac{\tau(\varepsilon)}{\varepsilon})^\alpha\psi_0(\tau(\varepsilon)) = g(\varepsilon)$. From this and owing to the the arbitrariness of the function ψ_0, we obtain the assertion of Lemma 1.7.

Obviously, in order to prove that $\psi \in C_0(0,x_0]$, it suffices to show that the functions

$$\big(T_0^{-n_1(x)}\psi_0\big)(x) \quad\text{and}\quad \sum_{i=1}^{n_1(x)} \big(T_0^{-i}g\big)(x)$$

are bounded in the interval $0 < x < \tau(\varepsilon)$.

(1.24) and (1.25) yield

$$\big|\big(T_0^{-n_1(x)}\psi_0\big)(x)\big| \le q^{n_1(x)} \max_{\tau(\varepsilon)\le x\le\varepsilon} |\psi_0(x)| < \max_{\tau(\varepsilon)\le x\le\varepsilon} |\psi_0(x)|,$$

$$\Big|\sum_{i=1}^{n_1(x)} \big(T_0^{-i}g\big)(x)\Big| \le \sum_{i=1}^{n_1(x)} q^i \sup_{0<x\le x_0} |g(x)| < \frac{1}{1-q} \sup_{0<x\le x_0} |g(x)|. \ \blacksquare$$

Remark. One can prove that in the critical case where $\alpha = -\frac{\log|\sigma|}{\log\tau_0}$, equation (1.14) in the class $C_\alpha(0,\varepsilon]$ is not Hausdorff normally solvable, that is, the set of all right-hand sides $f \in C_\alpha(0,\varepsilon]$ for which (1.14) is solvable, is everywhere dense in $C_\alpha(0,\varepsilon]$ but not coinciding with it.

From the above proven lemmas it follows that the following theorems are valid.

Let the conditions (1.13) *be fulfilled and at least one of the curves* γ_1 *or* γ_2 *be characteristics of equation* (1.3). *Then the problem* (1.3), (1.4) *is uniquely solvable in the class* $C^{1,1}_\alpha(\overline{D})$ *for* $\alpha > 0$.

Let the conditions (1.13) *be fulfilled, the curves* $\gamma_1 \backslash O$ *and* $\gamma_2 \backslash O$ *not be characteristics of equation* (1.3) *and at least one of them have characteristic direction at* O. *Then the problem* (1.3), (1.4) *is uniquely solvable in the class* $C^{1,1}_\alpha(\overline{D})$ *for* $\alpha > 0$. *If, however,* $-1 < \alpha \le 0$, *then the problem* (1.3), (1.4) *is uniquely solvable in the class* $C^{1,1}_\alpha(\overline{D})$, *when the condition*

$$\overline{\lim_{x\to+0}} \Big| a(x)\Big(\frac{\tau(x)}{x}\Big)^\alpha \Big| < 1$$

is fulfilled.

Let the conditions (1.13) *be fulfilled, the curves* $\gamma_1 \backslash O$ *and* $\gamma_2 \backslash O$ *not be characteristics of equation* (1.3) *and have no characteristic direction at* O. *If* $N_1(0)M_2(0) = 0$, *then the problem* (1.3), (1.4) *is uniquely solvable in the class* $C^{1,1}_\alpha(\overline{D})$ *for* $\alpha > -1$.

Let the conditions (1.13) *be fulfilled, the curves* $\gamma_1 \backslash O$ *and* $\gamma_2 \backslash O$ *not be characteristics of equation* (1.3) *and have no characteristic direction at* O. *If* $N_1(0)M_2(0) \neq 0$, *then for* $\alpha > -\frac{\log|\sigma|}{\log \tau_0}$ *the problem* (1.3), (1.4) *is uniquely solvable in the class* $C^{1,1}_\alpha(\overline{D})$, *while for* $\alpha < -\frac{\log|\sigma|}{\log \tau_0}$ *it is solvable in the class* $C^{1,1}_\alpha(\overline{D})$, *and the homogeneous problem corresponding to* (1.3), (1.4) *has an infinite number of linearly independent solutions in this class.*

Remark. Using Picard's method of successive approximations, one can prove that the assertions of Theorems 1.1–1.3 and those of the first part of Theorem 1.4 are also valid for the problem (1.1), (1.2) in the class $C^{1,1}_\alpha(\overline{D})$; moreover, the estimate

$$\|u\|_{C^{1,1}_\alpha(\overline{D})} \le C\Big(\sum_{i=1}^{2} \|f_i\|_{C_\alpha(\gamma_i)} + \|F\|_{C_{\alpha-1}(\overline{D})}\Big)$$

with a positive constant C not depending on f_i and F, is valid for the solution $u(x,y)$.

Here

$$\|u\|_{C^{1,1}_\alpha(\overline{D})} = \sup_{z\in\overline{D}\backslash O} |z|^{-\alpha}|u_x(z)| + \sup_{z\in\overline{D}\backslash O} |z|^{-\alpha}|u_y(z)| + \\ + \sup_{z\in\overline{D}\backslash O} |z|^{-(\alpha-1)}|u_{xy}(z)|,$$

$$\|f_i\|_{C_\alpha(\gamma_i)} = \sup_{z\in\gamma_i\backslash O} |z|^{-\alpha}|f_i(z)|, \quad \|F\|_{C_{\alpha-1}(\overline{D})} = \sup_{z\in\overline{D}\backslash O} |z|^{-(\alpha-1)}|F(z)|.$$

The assertion of the second part of Theorem 1.4 is likewise valid, but in this case instead of the solvability of the problem (1.1), (1.2) in the class $C_\alpha^{1,1}(\overline{D})$ there takes place the Hausdorff normal solvability [6]. Note also that in the critical case where $\alpha = -\frac{\log|\sigma|}{\log \tau_0}$, the Hausdorff normal solvability of the problem (1.1), (1.2) in the class $C_\alpha^{1,1}(\overline{D})$ will, generally speaking, be violated.

§ γ_1 γ_2 $O(0,0)$

By virtue of the requirements imposed on the curves γ_1 and γ_2 in the case where they have a common tangent line at $O(0,0)$, we have $\tau_0 = \tau^{(1)}(0) = 1$. The fact that $|\sigma| = |a(0)| = |N_1 M_2 M_1^{-1} N_2^{-1}(0)| \neq 1$ means that the directions of differentiation in the boundary conditions (1.4) do not coincide at $O(0,0)$.

Repeating the same arguments as in §2, we can prove the validity of the following

Let the conditions (1.13) *be fulfilled, the curves* γ_1 *and* γ_2 *have a common tangent line at the point* $O(0,0)$*, but the directions of differentiation in the boundary conditions* (1.4) *not coincide at this point, i.e.,* $|\sigma| \neq 1$*. If* $N_1(0)M_2(0) = 0$*, then the problem* (1.3), (1.4) *is uniquely solvable in the class* $C_\alpha^{1,1}(\overline{D})$ *for* $\alpha > -1$*. If, however,* $N_1(0)M_2(0) \neq 0$*, then in the case* $|\sigma| < 1$ *the problem* (1.3), (1.4) *is uniquely solvable in the class* $C_\alpha^{1,1}(\overline{D})$ *for* $\alpha > -1$*, while in the case* $|\sigma| > 1$ *the problem* (1.3), (1.4) *is solvable in the class* $C_\alpha^{1,1}(\overline{D})$ *for* $\alpha > -1$*; moreover, the homogeneous problem corresponding to* (1.3), (1.4) *has an infinite number of linearly independent solutions.*

Note that in this case, the remark following after Theorem 1.4 of the previous paragraph is also valid.

§ γ_1 γ_2 $O(0,0)$

For the sake of simplicity we shall assume below that the curves γ_1, γ_2 and the coefficients M_i, N_i, $i = 1,2$, in the boundary conditions (1.4) belong to the class C^∞. In this case it is obvious that $\tau(x) \in C^\infty[0, x_0]$, and the coefficient $a(x) \in C^\infty[0, x_0]$ in the functional equation (1.14).

Let γ_1 and γ_2 have a common tangent line at $O(0,0)$ and the order of tangency be equal to k. This, obviously, is equivalent to the conditions

$$\tau_0 = \tau^{(1)}(0) = 1, \quad \tau^{(i)}(0) = 0, \quad 1 < i \leq k, \quad \tau^{(k+1)}(0) \neq 0. \quad (1.27)$$

Therefore the function $\tau(x) \in C^\infty[0, x_0]$ can be represented in the form

$$\tau(x) = x + \frac{\tau^{(k+1)}(0)}{(k+1)!} x^{k+1} + \lambda(x) x^{k+1}, \tag{1.28}$$

where $\lambda(x) = o(x)$ for $x \to 0$, i.e., $\lim\limits_{x \to 0} \lambda(x) = 0$.

Since $\tau(x) < x$ for $0 < x \le x_0$, (1.27) and (1.28) imply

$$c = -\frac{\tau^{(k+1)}(0)}{(k+1)!} > 0. \tag{1.29}$$

Taking into account (1.29), we rewrite (1.28) as

$$\tau(x) = x - cx^{k+1} + \lambda(x) x^{k+1}. \tag{1.30}$$

Assume $\tau_n(x) = \tau(\tau_{n-1}(x))$, $\tau_1(x) = \tau(x)$, $0 < x \le x_0$. As it is noted above, the monotonicity of the function $\tau(x)$ and the validity of the conditions (1.17) imply that the sequence of the functions $\{\tau_n(x)\}_{n=1}^\infty$ vanishes uniformly on $[0, x_0]$ for $n \to \infty$, i.e., $\tau_n(x) \rightrightarrows 0$, $n \to \infty$.

Below we shall concern ourselves with the asymptotics when the sequence $x_n = \tau(x_{n-1})$, $x_1 = x \in (0, x_0]$ tends to zero with respect to n.

The following lemma holds.

The behavior of the sequence $x_n = \tau_n(x)$ for $n \to \infty$ can be written by the formula

$$x_n = \frac{\xi_n}{\sqrt[k]{ckn}}\ , \tag{1.31}$$

where the function $\xi_n = \xi_n(x)$ tends uniformly on the segment $0 \le x \le x_0$ to unity as $n \to \infty$, i.e., $\xi_n(x) \rightrightarrows 1$, $n \to \infty$.

Proof. Because of (1.30) and the well-known equality $(1+\eta)^p = 1 + p\eta + \lambda_1(\eta)\eta$ for $p \ge 0$, where $\lim\limits_{\eta \to 0} \lambda_1(\eta) = 0$, we have

$$\begin{aligned}
\frac{1}{x_n^p} = \frac{1}{[\tau(x_{n-1})]^p} &= \frac{1}{(x_{n-1} - cx_{n-1}^{k+1} + \lambda(x_{n-1}) x_{n-1}^{k+1})^p} = \\
&= \frac{1}{x_{n-1}^p} \frac{1}{(1 - cx_{n-1}^k + \lambda(x_{n-1}) x_{n-1}^k)^p} = \\
&= \frac{1}{x_{n-1}^p} \frac{1}{(1 - pcx_{n-1}^k + \lambda_2(x_{n-1}) x_{n-1}^k)} = \\
&= \frac{1}{x_{n-1}^p} \big(1 + pcx_{n-1}^k + \lambda_3(x_{n-1}) x_{n-1}^k\big) = \\
&= \frac{1}{x_{n-1}^p} + pcx_{n-1}^{k-p} + \lambda_3(x_{n-1}) x_{n-1}^{k-p},
\end{aligned} \tag{1.32}$$

where $\lim\limits_{\eta \to 0} \lambda_i(\eta) = 0$, $i = 2, 3$.

Assuming $p = k$ and $n = i$ in (1.32), we find that

$$\frac{1}{x_i^k} = \frac{1}{x_{i-1}^k} + ck + \lambda_3(x_{i-1}). \tag{1.33}$$

Adding equalities (1.33) for $i = 2, 3, \ldots, n$, we get

$$\frac{1}{x_n^k} = \frac{1}{x_1^k} + ck(n-1) + \sum_{i=2}^{n} y_i,$$

i.e.,

$$\frac{1}{cknx_n^k} = \frac{1}{cknx_1^k} + \frac{n-1}{n} + \frac{1}{ck}\frac{\sum_{i=2}^{n} y_i}{n}, \tag{1.34}$$

where the sequence $y_i = y_i(x) = \lambda_3(x_{i-1}) = \lambda_3(\tau_{i-1}(x))$ tends uniformly on the segment $0 \le x \le x_0$ to zero, i.e., $y_i(x) \rightrightarrows 0$, $n \to \infty$.

Since

$$\lim_{n\to\infty} \frac{1}{cknx_1^k} = 0, \quad \lim_{n\to\infty} \frac{n-1}{n} = 1, \quad \lim_{n\to\infty} y_n = 0$$

and hence

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=2}^{n} y_i = 0,$$

we obtain finally from (1.34) that the sequence $\frac{1}{\xi_n^k} = \frac{1}{cknx_n^k}$ tends uniformly on $[0, x_0]$ to unity. ■

As already noted, coincidence of the directions of differentiation in the boundary conditions (1.4) means that $|\sigma| = |a(0)| = 1$. Let first $a(0) = \sigma = 1$. Then since $a(x) \in C^\infty[0, x_0]$, the representation

$$a(x) = 1 + dx^m + \mu(x)x^m, \tag{1.35}$$

where $\lim\limits_{x\to 0} \mu(x) = 0$ and

$$a^{(i)}(0) = 0, \quad 1 \le i \le m-1, \quad a^{(m)}(0) \ne 0, \quad d = \frac{a^{(m)}(0)}{m!}.$$

is valid.

Because of the fact that $a(x) = \frac{K_1(x)}{K_2(x)}$, where $K_1(x) = \frac{N_1(x)}{M_1(x)}$, $K_2(x) = \frac{N_2(x)(\gamma_1(x))}{M_2(\gamma_1(x))}$ we have $K_1(x) - K_2(x) = O(x^m)$ for $x \to 0$. Therefore, geometrically the value $m-1$ can be interpreted as the order of tangency of the directions of differentiation at $O(0,0)$ in the boundary conditions (1.4).

We rewrite equation (1.4) in the form

$$(T\varphi)(x) \equiv \varphi(x) - (\Lambda\varphi)(x) = f(x), \quad 0 < x \le x_0, \tag{1.36}$$

where $(\Lambda\varphi)(x) = a(x)\varphi(\tau(x))$.

From (1.36) we have

$$\varphi(x) = (\Lambda^n \varphi)(x) + \sum_{i=0}^{n-1} (\Lambda^i f)(x), \tag{1.37}$$

where $\Lambda^0 = I$ is the unit operator.

For $m > k$, equation (1.36) *cannot have more than one solution in the class $C_\alpha(0, x_0]$, $\alpha > 0$.*

Proof. Let $\varphi(x)$ be a solution of the homogeneous equation corresponding to (1.36) in the class $C_\alpha(0, x_0]$, $\alpha > 0$. Then because of (1.37) the equality $\varphi(x) = (\Lambda^n \varphi)(x)$ holds.

Equality (1.35) yields

$$|a(x)| \leq 1 + d_1 x^m, \quad 0 \leq x \leq x_0, \tag{1.38}$$

for some $d_1 = \text{const} > 0$. Therefore

$$\begin{gathered} |(\Lambda^n \varphi)(x)| = \big|a(x) a(\tau(x)) \cdots a(\tau_{n-1}(x)) \varphi(\tau_n(x))\big| \leq \\ \leq (1 + d_1 x^m)\big(1 + d_1 \tau^m(x)\big) \cdots \big(1 + d_1 \tau_{n-1}^m(x)\big) \big|\varphi(\tau_n(x))\big|. \end{gathered} \tag{1.39}$$

As is known, the convergence of an infinite product $\prod\limits_{i=1}^{\infty} (1 + \eta_i)$ is equivalent to that of the series $\sum\limits_{i=1}^{\infty} \eta_i$ if the values η_i have the same sign. Therefore the convergence of the product

$$\prod_{i=1}^{\infty} \big(1 + d_1 \tau_i^m(x)\big)$$

is equivalent to that of the series

$$\sum_{i=1}^{\infty} \tau_i^m(x),$$

which, in its turn, is equivalent to the convergence of the series $\sum\limits_{i=1}^{\infty} \frac{1}{i^{m/k}}$ in virtue of (1.31). The series $\sum\limits_{i=1}^{\infty} \frac{1}{i^{m/k}}$ converges for $m > k$. Therefore there exists a number $M = \text{const} > 0$ such that for $n \geq 1$ the equality

$$\prod_{i=1}^{n} \big(1 + d_1 \tau_i^m(x)\big) \leq M \tag{1.40}$$

is valid.

Inequalities (1.39) and (1.40) imply

$$|(\Lambda^n \varphi)(x)| \leq M \big|\varphi(\tau_n(x))\big|. \tag{1.41}$$

Since $\varphi \in C_\alpha(0, x_0]$ and $\alpha > 0$, it is obvious that $\varphi \in C[0, x_0]$ and $\varphi(0) = 0$. Therefore, since the sequence $\{\tau_n(x)\}_{n=1}^{\infty}$ converges uniformly on the segment $0 \le x \le x_0$ to zero, we have

$$\lim_{n\to\infty} \varphi\big(\tau_n(x)\big) = 0, \quad 0 \le x \le x_0. \tag{1.42}$$

By virtue of (1.41) and (1.42), passing in the equality $\varphi(x) = (\Lambda^n \varphi)(x)$ to the limit for $n \to \infty$, we finally obtain that $\varphi(x) \equiv 0$. ■

If $f(x) \in C_\beta(0, x_0]$, $\beta > k$, then for $m > k$ equation (1.36) *has the solution in the class $C_\alpha(0, x_0]$, $0 < \alpha < \beta - k$.*

Proof. It can be easily verified that the functional series

$$T^{-1} f = \sum_{i=0}^{\infty} \Lambda^i f \tag{1.43}$$

is formally a solution of equation (1.36). Therefore, to prove that equation (1.36) is solvable, it is sufficient to show that the series (1.43) converges in the class $C_\alpha(0, x_0]$, $\alpha < \beta - k$.

Since $f \in C_\beta(0, x_0]$, $\beta > k$, the equality

$$|f(x)| \le M_1 x^\alpha x^{\beta_1}, \quad 0 \le x \le x_0, \tag{1.44}$$

where $M_1 = \text{const} > 0$, is valid for $\beta_1 = k + \varepsilon$, $\varepsilon = \beta - k - \alpha > 0$.

From (1.31), (1.40), (1.44) and because $\tau_i(x) \le x$, we have

$$\begin{aligned} |(\Lambda^i f)(x)| &= |a(x)|\big|a(\tau(x))\big| \cdots \big|a(\tau_{i-1}(x))\big|\big|f(\tau_i(x))\big| \le \\ &\le M M_1 [\tau_i(x)]^\alpha \left[\frac{\xi_i}{\sqrt[k]{cki}}\right]^{\beta_1} \le M M_1 \xi_i^{\beta_1} (ck)^{-\frac{\beta_1}{k}} x^\alpha \frac{1}{i^{\beta_1/k}} \,. \end{aligned} \tag{1.45}$$

Since $\beta_1 > k$, (1.45) implies the convergence of (1.43) in the class $C_\alpha(0, x_0]$. ■

Remark. The fact that the solution $\varphi(x)$ of equation (1.36) for $f \in C_\beta(0, x_0]$, $\beta > k$, does not, in general, belong to the class $C_\alpha(0, x_0]$ for $\alpha > \beta - k$, is seen from the following example. It is not difficult to see that the function $\varphi_0(x) = x^{\beta-k} \in C_{\beta-k}(0, x_0]$. By (1.30) and (1.35), we have

$$\begin{gathered} (T\varphi_0)(x) = x^{\beta-k} - a(x)(\tau(x))^{\beta-k} = \\ = x^{\beta-k} - (1 + dx^m + \mu(x)x^m)(x - cx^{k+1} + \lambda(x)x^{k+1})^{\beta-k} = \\ = x^{\beta-k} - (1 + dx^m + \mu(x)x^m)x^{\beta-k}(1 - c(\beta-k)x^k + \widetilde{\lambda}(x)x^k) = \\ = c(\beta-k)x^\beta + \widetilde{\mu}(x)x^\beta, \end{gathered}$$

where $\lim\limits_{x\to 0} \widetilde{\lambda}(x) = \lim\limits_{x\to 0} \widetilde{\mu}(x) = 0$. Hence the function $f_0(x) = (T\varphi_0)(x) \in C_\beta(0, x_0]$, and the function $\varphi_0(x) = x^{\beta-k}$ itself which does not belong to the class $C_\alpha(0,x_0]$ for any $\alpha > \beta - k$, is the unique solution of equation (1.36) for $f(x) = f_0(x)$.

Note that the above proven lemmas are also valid in the case $a(0) = \sigma = -1$.

Owing to Lemmas 1.9 and 1.10, the following theorem is valid.

Let $\tau_0 = 1$, $|\sigma| = 1$ and $m > k$. Then the problem (1.3), (1.4) *cannot have more than one solution in the class $C_\alpha(0,x_0]$, $\alpha > 0$. If $f_i \in C_\beta(\gamma_i)$, $i = 1,2$, where $\beta > k$, then the problem* (1.3), (1.4) *has a unique solution in the class $C_\alpha^{1,1}(\overline{D})$, $0 < \alpha < \beta - k$.*

We shall give the following results from [52] without proofs.

Let $\tau_0 = 1$, $|\sigma| = 1$ and $m = k$, $\sigma d > 0$. Then for any $f_i \in C_\beta(0,x_0]$, $i = 1,2$, $\beta > k + 2\frac{|d|}{c}$, the problem (1.3), (1.4) *is uniquely solvable in the class $C_\alpha^{1,1}(\overline{D})$, where $\frac{|d|}{c} < \alpha < \beta - k - \frac{|d|}{c}$.*

Theorem below does not involve the dependence between m and k.

Let $\tau_0 = 1$, $|\sigma| = 1$ and $\sigma d < 0$. Then for any $f_i \in C_\beta(0,x_0]$, $i = 1,2$, $\beta > k$, the problem (1.3), (1.4) *is uniquely solvable in the class $C_\alpha^{1,1}(\overline{D})$, where $0 < \alpha < \beta - k$.*

§

γ_1 γ_2

As the example of the equation $u_{xy} = 0$ shows, the problem (1.1), (1.2) may appear to be ill-posed when the conditions (1.13) are violated. Below we shall show that the existence of lower terms in equation (1.1) and in the boundary conditions (1.2) may affect the correctness of the statement of the problem (1.1), (1.2).

For simplicity let $M_i = \text{const}$, $N_i = \text{const}$, $S_i = \text{const}$ and $|M_i| + |N_i| + |S_i| \neq 0$, $i = 1,2$. Without loss of generality we may assume $|M_i| + |N_i| \neq 0$, $i = 1,2$, since, otherwise, this can be achieved by differentiating the corresponding boundary condition with respect to a tangent curve γ_i.

As γ_1 and γ_2 let us take the characteristic segments $\gamma_1 : y = 0$, $0 \leq x \leq x_0$, $\gamma_2 : x = 0$, $0 \leq y \leq y_0$.

Let the second condition in (1.13) be fulfilled, while the first one be violated on the whole segment γ_1, i.e.,

$$M_1\big|_{\gamma_1} = 0. \tag{1.46}$$

Below we shall restrict ourselves to consideration of the problem (1.1), (1.2) in the class

$$\overset{\circ}{C}{}^2(\overline{D}) = \left\{ u \in C^2(\overline{D}) : \ \frac{\partial^{i+j} u(0,0)}{\partial x^i \partial y^j} = 0, \ 0 \leq i + j \leq 2 \right\}$$

and assume that $a, b, c \in C^2(\overline{D})$, $F \in C^1(\overline{D})$, $F(0,0) = 0$, $f_i = \overset{\circ}{C}{}^1(\gamma_i) = \{f_i \in C^1(\gamma_i) : f_i(0) = f_i^{(1)}(0) = 0\}$, $i = 1, 2$.

Denote by $R(x, y; x_1, y_1)$ the Riemann function which, by definition, is the solution of the so-called conjugate equation [10]

$$R_{xy} - (aR)_x - (bR)_y + cR = 0 \tag{1.47}$$

which on the characteristics $x = x_1$, $y = y_1$ takes the values

$$\begin{aligned} R(x_1, y; x_1, y_1) &= \exp\Big(\int\limits_{y_1}^{y} a(x_1, \eta) d\eta\Big), \\ R(x, y_1; x_1, y_1) &= \exp\Big(\int\limits_{x_1}^{x} b(\xi, y_1) d\xi\Big), \end{aligned} \tag{1.48}$$

where (x_1, y_1) is an arbitrarily fixed point in the domain D_1.

Due to (1.47) and (1.48), the function $R(x, y; x_1, y_1)$ satisfies the integral equation

$$\begin{aligned} R(x, y; x_1, y_1) - \int\limits_{x_1}^{x} b(\xi, y) R(\xi, y; x_1, y_1) d\xi - \\ - \int\limits_{y_1}^{y} a(x, \eta) R(x, \eta; x_1, y_1) d\eta + \\ + \int\limits_{x_1}^{x} d\xi \int\limits_{y_1}^{y} c(\xi, \eta) R(\xi, \eta; x_1, y_1) d\eta = 1. \end{aligned} \tag{1.49}$$

It is known that equation (1.49) has the unique solution $R(x, y; x_1, y_1)$ which, as it can be easily verified, possesses the following continuous derivatives

$$\begin{gathered} \partial_{x,y}^{i,j} \partial_{x_1,y_1}^{i_1,j_1} R(x, y; x_1, y_1) \in C(\overline{D} \times \overline{D}), \\ 0 \le i + j \le 1, \quad 0 \le i_1 + j_1 \le 2, \end{gathered} \tag{1.50}$$

where $\partial_{x,y}^{i,j} = \frac{\partial^{i+j}}{\partial x^i \partial y^j}$, $\partial_{x_1,y_1}^{i_1,j_1} = \frac{\partial^{i_1+j_1}}{\partial x_1^{i_1} \partial y_1^{j_1}}$.

From (1.48) we have

$$\left.\begin{gathered} R_y(x_1, y; x_1, y_1) - a(x_1, y) R(x_1, y; x_1, y_1) = 0, \\ R_x(x, y_1; x_1, y_1) - b(x, y_1) R(x, y_1; x_1, y_1) = 0, \\ R(x_1, y_1; x_1, y_1) = 1, \\ R_{y_1}(x, y; x, y_1) + a(x, y_1) R(x, y; x, y_1) = 0, \\ R_{x_1}(x, y; x_1, y) + b(x_1, y) R(x, y; x_1, y) = 0, \\ R(x, y; x, y) = 1. \end{gathered}\right\} \tag{1.51}$$

On account of (1.50), every solution $u(x,y)$ of equation (1.1) of the class $C^2(\overline{D})$ can be represented in the form [10]

$$
\begin{aligned}
u(x,y) = R(x,0;x,y)\varphi(x) + R(0,y;x,y)\psi(y) - R(0,0;x,y)\varphi(0) + \\
+ \int\limits_0^y \big[a(0,\eta)R(0,\eta;x,y) - R_y(0,\eta;x,y)\big]\psi(\eta)d\eta + \\
+ \int\limits_0^x \big[b(\xi,0)R(\xi,0;x,y) - R_x(\xi,0;x,y)\big]\varphi(\xi)d\xi + \\
+ \int\limits_0^x d\xi \int\limits_0^y R(\xi,\eta;x,y)F(\xi,\eta)d\eta,
\end{aligned}
\tag{1.52}
$$

as the solution of the Goursat problem

$$
u(x,0) = \varphi(x), \quad u(0,y) = \psi(y), \quad \varphi(0) = \psi(0),
$$

where φ and ψ are given functions of the class C^2.

When considering the problem (1.1), (1.2) in the class $\overset{\circ}{C}{}^2(\overline{D})$, one should assume that

$$
\varphi^{(i)}(0) = \psi^{(i)}(0) = 0, \quad i = 0,1,2. \tag{1.53}
$$

From (1.52) and because of (1.53) we have

$$
\begin{aligned}
u_x(x,y) = \big(R_x(x,0;x,y) + R_{x_1}(x,0;x,y)\big)\varphi(x) + \\
+R(x,0;x,y)\varphi^{(1)}(x) + R_{x_1}(0,y;x,y)\psi(y) + \\
+ \int\limits_0^y \big[a(0,\eta)R_{x_1}(0,\eta;x,y) - R_{yx_1}(0,\eta;x,y)\big]\psi(\eta)d\eta + \\
+\big[b(x,0)R(x,0;x,y) - R_x(x,0;x,y)\big]\varphi(x) + \\
+ \int\limits_0^x \big[b(\xi,0)R_{x_1}(\xi,0;x,y) - R_{xx_1}(\xi,0;x,y)\big]\varphi(\xi)d\xi + \\
+ \int\limits_0^y R(x,\eta;x,y)F(x,\eta)d\eta + \int\limits_0^x d\xi \int\limits_0^y R_{x_1}(\xi,\eta;x,y)F(\xi,\eta)d\eta,
\end{aligned}
\tag{1.54}
$$

$$
\begin{aligned}
u_y(x,y) = R_{y_1}(x,0;x,y)\varphi(x) + \big(R_y(0,y;x,y) + R_{y_1}(0,y;x,y)\big)\psi(y) + \\
+R(0,y;x,y)\psi^{(1)}(y) + \big[a(0,y)R(0,y;x,y) - R_y(0,y;x,y)\big]\psi(y) + \\
+ \int\limits_0^y \big[a(0,\eta)R_{y_1}(0,\eta;x,y) - R_{yy_1}(0,\eta;x,y)\big]\psi(\eta)d\eta +
\end{aligned}
$$

$$+\int\limits_0^x \big[b(\xi,0)R_{y_1}(\xi,0;x,y)-R_{xy_1}(\xi,0;x,y)\big]\varphi(\xi)d\xi+$$

$$+\int\limits_0^x R(\xi,y;x,y)F(\xi,y)d\xi+\int\limits_0^x d\xi\int\limits_0^y R_{y_1}(\xi,\eta;x,y)F(\xi,\eta)d\eta. \quad (1.55)$$

Assuming in equalities (1.54), (1.55) $x=0$, $y=0$ and taking into account (1.53), we obtain

$$u_x(0,y)=R_{x_1}(0,y;0,y)\psi(y)+\int\limits_0^y \big[a(0,\eta)R_{x_1}(0,\eta;0,y)-$$

$$-R_{yx_1}(0,\eta;0,y)\big]\psi(\eta)d\eta+\int\limits_0^y R(0,\eta;0,y)F(0,\eta)d\eta, \quad (1.56)$$

$$u_y(x,0)=R_{y_1}(x,0;x,0)\varphi(x)+\int\limits_0^x \big[b(\xi,0)R_{y_1}(\xi,0;x,0)-$$

$$-R_{xy_1}(\xi,0;x,0)\big]\varphi(\xi)d\xi+\int\limits_0^x R(\xi,0;x,0)F(\xi,0)d\xi. \quad (1.57)$$

It easily follows from (1.51) that

$$R_{x_1}(0,y;0,y)=-b(0,y), \quad R_{y_1}(x,0;x,0)=-a(x,0). \quad (1.58)$$

Substituting the expressions obtained in (1.56), (1.57) for u_x and u_y into the boundary conditions (1.2) and taking into consideration the equalities $u(x,0)=\varphi(x)$, $u(0,y)=\psi(y)$ and (1.58), (1.46), we find that

$$-N_1a(x,0)\varphi(x)+N_1\int\limits_0^x \big[b(\xi,0)R_{y_1}(\xi,0;x,0)-$$

$$-R_{xy_1}(\xi,0;x,0)\big]\varphi(\xi)d\xi+S_1\varphi(x)=f_3(x), \quad 0\le x\le x_0, \quad (1.59)$$

$$-M_2b(0,y)\psi(y)+$$

$$+M_2\int\limits_0^y \big[a(0,\eta)R_{x_1}(0,\eta;0,y)-R_{yx_1}(0,\eta;0,y)\big]\psi(\eta)d\eta+$$

$$+N_2\psi^{(1)}(y)+S_2\psi(y)=f_4(y), \quad 0\le y\le y_0, \quad (1.60)$$

where

$$f_3(x)=f_1(x)-N_1\int\limits_0^x R(\xi,0;x,0)F(\xi,0)d\xi,$$

$$f_4(y) = f_2(y) - M_2 \int\limits_0^y R(0,\eta;0,y)F(0,\eta)d\eta.$$

Obviously, the problem (1.1), (1.2) in the class $\overset{\circ}{C}{}^2(\overline{D})$ is equivalent to the system of equations (1.59), (1.60) with respect to unknown functions $\varphi \in \overset{\circ}{C}{}^2[0,x_0]$, $\psi \in \overset{\circ}{C}{}^2[0,y_0]$.

Let the condition

$$(S_1 - aN_1)\big|_{\gamma_1} \neq 0. \tag{1.61}$$

be fulfilled. From (1.48), (1.49) we have

$$K_1(\xi,x) = b(\xi,0)R_{y_1}(\xi,0;x,0) - R_{xy_1}(\xi,0;x,0) =$$

$$= \big(a_x(\xi,0) + a(\xi,0)b(\xi,0) - c(\xi,0)\big)\exp\Big(\int\limits_x^\xi b(\tau,0)d\tau\Big), \tag{1.62}$$

$$K_2(\eta,y) = a(0,\eta)R_{x_1}(0,\eta;0,y) - R_{yx_1}(0,\eta;0,y) =$$

$$= \big(b_y(0,\eta) + a(0,\eta)b(0,\eta) - c(0,\eta)\big)\exp\Big(\int\limits_y^\eta a(0,\tau)d\tau\Big). \tag{1.63}$$

Let

$$\psi_0(y) = \psi^{(1)}(y), \quad \psi(y) = \int\limits_0^y \psi_0(\tau)d\tau. \tag{1.64}$$

By virtue of (1.61)–(1.64) and owing to the condition $N_2|_{\gamma_2} \neq 0$, the system of equations (1.59), (1.60) can be rewritten in the form

$$\varphi(x) + N_1\lambda(x)\int\limits_0^x K_1(\xi,x)\varphi(\xi)d\xi = f_5(x), \quad 0 \le x \le x_0, \tag{1.65}$$

$$\psi_0(y) + \mu(y)\int\limits_0^y \psi_0(\tau)d\tau +$$

$$+M_2N_2^{-1}\int\limits_0^y K_3(\tau,y)\psi_0(\tau)d\tau = f_6(y), \quad 0 \le y \le y_0, \tag{1.66}$$

where $\lambda(x) = (S_1 - aN_1)^{-1}(x,0)$, $\mu(y) = N_2^{-1}(S_2 - bM_2)(0,y)$, $K_3(\tau,y) = \int_\tau^y K_2(\eta,y)d\eta$, $f_5(x) = \lambda(x)f_3(x)$, $f_6(y) = N_2^{-1}f_4(y)$.

Since equations (1.65) and (1.66) are second order Volterra type integral equations, for equations (1.65) and (1.66) to be solvable, respectively, in the classes $\overset{\circ}{C}{}^2[0,x_0]$ and $\overset{\circ}{C}{}^1[0,y_0]$, it is sufficient to require that

$$\begin{gathered} K_1(\xi,x)\in C^1\Big(0\le {\xi \atop x}\le x_0\Big),\\ \frac{\partial^2 K_1(\xi,x)}{\partial x^2}\in C\Big(0\le {\xi \atop x}\le x_0\Big),\quad f_5\in \overset{\circ}{C}{}^2[0,x_0], \end{gathered} \tag{1.67}$$

$$\begin{gathered} K_2(\eta,y)\in C\Big(0\le {\eta \atop y}\le y_0\Big),\\ \frac{\partial K_2(\eta,y)}{\partial y}\in C\Big(0\le {\eta \atop y}\le y_0\Big),\quad f_6\in \overset{\circ}{C}{}^1[0,y_0]. \end{gathered} \tag{1.68}$$

Due to the requirements imposed on the coefficients a, b, c of equation (1.1) and the functions F, f_1, f_2, the condition (1.68) will obviously be fulfilled. However, for the condition (1.67) to be valid, one should additionally require that

$$f_1\in C^2(OP_1),\quad f_1^{(2)}(0)-N_1F_x(0,0)=0.$$

Consider now the case where the condition (1.61) is violated, i.e.,

$$(S_1-aN_1)\big|_{\gamma_1}=0. \tag{1.69}$$

Since, by the assumption, $|M_1|+|N_1|\ne 0$, $M_1,N_1,S_1=\text{const}$, we have on account of (1.46) and (1.69)

$$a\big|_{OP_1}=\text{const}\,. \tag{1.70}$$

When the condition (1.69) is fulfilled, equation (1.59) with respect to the unknown function $\varphi(x)$ is an integral Volterra type equation of the first kind

$$\int\limits_0^x K_1(\xi,x)\varphi(\xi)d\xi=N_1^{-1}f_3(x),\quad 0\le x\le x_0. \tag{1.71}$$

Differentiating both parts of equation (1.71) with respect to x and taking into account (1.70), we get

$$(ab-c)(x,0)\varphi(x)-b(x,0)\int\limits_0^x K_1(\xi,x)\varphi(\xi)d\xi=N_1^{-1}f_3^{(1)}(x). \tag{1.72}$$

Similarly, when the condition

$$(ab-c)\big|_{\gamma_1}\ne 0 \tag{1.73}$$

is fulfilled, in order that equation (1.72) to be solvable in the class $\overset{\circ}{C}{}^2[0,x_0]$, we should require that

$$f_1 \in C^3(OP_1), \quad F \in C^2(OP_1), \quad f_1^{(2)}(0) - N_1 F_x(0,0) = 0,$$
$$f_1^{(3)}(0) - N_1 F_{xx}(0,0) + N_1 b(0,0) F_x(0,0) = 0.$$

If, however, the condition (1.73) is violated, i.e.,

$$(ab-c)\big|_{\gamma_1} = 0,$$

then, according to (1.62), (1.70), we have

$$K_1(\xi,x) \equiv 0.$$

In this case the left-hand side of equation (1.71) is equal identically to zero and the equality

$$f_3(x) = f_1(x) - N_1 \int\limits_0^x \Big(\exp \int\limits_x^\xi b(\tau,0)d\tau\Big) F(\xi,0)d\xi \equiv 0, \quad 0 \le x \le x_0,$$

is a necessary and sufficient condition for the problem (1.1), (1.2) to be solvable in the class $\overset{\circ}{C}{}^2(\overline{D})$; moreover, the homogeneous problem corresponding to (1.1), (1.2) has an infinite number of linearly independent solutions which are given by

$$u(x,y) = R(x,0;x,y)\varphi(x) + \\ + \int\limits_0^x \big[b(\xi,0)R(\xi,0;x,y) - R_x(\xi,0;x,y)\big]\varphi(\xi)d\xi,$$

where $\varphi(x)$ is an arbitrary function of the class $\overset{\circ}{C}{}^2[0,x_0]$.

Thus the following theorem is valid.

Let the conditions $M_1 = 0$, $N_2 \ne 0$ be fulfilled. Then for $(S_1 - aN_1)|_{\gamma_1} \ne 0$, the problem (1.1), (1.2) *is uniquely solvable in the class $\overset{\circ}{C}{}^2(\overline{D})$ if $f_1 \in C^2(\gamma_1)$, $f_1^{(2)}(0) - N_1 F_x(0,0) = 0$. If, however, $(S_1 - aN_1)|_{\gamma_1} = 0$, then for $(ab-c)|_{\gamma_1} \ne 0$ the problem* (1.1), (1.2) *is uniquely solvable in the class $\overset{\circ}{C}{}^2(\overline{D})$ if $f_1 \in C^3(\gamma_1)$, $F \in C^2(\gamma_1)$, $f_1^{(2)}(0) - N_1 F_x(0,0) = 0$, $f_1^{(3)}(0) - N_1 F_{xx}(0,0) + N_1 b(0,0) F_x(0,0) = 0$. In the case where $(S_1 - aN_1)|_{\gamma_1} = 0$ and $(ab-c)|_{\gamma_1} = 0$, for the problem* (1.1), (1.2) *to be solvable in the class $\overset{\circ}{C}{}^2(\overline{D})$, it is necessary and sufficient that*

$$f_1(x) - N_1 \int\limits_0^x \exp\Big(\int\limits_x^\xi b(\tau,0)d\tau\Big) F(\xi,0)d\xi \equiv 0, \quad 0 \le x \le x_0;$$

moreover, the homogeneous problem corresponding to (1.1), (1.2) *has an infinite number of linearly independent solutions which are given by*

$$u(x,y) = R(x,0;x,y)\varphi(x) + \\ + \int\limits_0^x \big[b(\xi,0)R(\xi,0;x,y) - R_x(\xi,0;x,y)\big]\varphi(\xi)d\xi,$$

where $\varphi(x)$ *is an arbitrary function of the class* $\overset{\circ}{C}{}^2[0,x_0]$, *and* $R(x,y;x_1,y_1)$ *is a Riemann function for equation* (1.1).

The cases $M_1|_{\gamma_1} \neq 0$, $N_2|_{\gamma_2} = 0$ and $M_1|_{\gamma_1} = N_2|_{\gamma_2} = 0$ can be considered in a similar manner.

§

$O(0,0)$

For simplicity, below we shall assume that in the problem (1.3), (1.4)

$$\gamma_1 : y = \rho_1 x,\ \ 0 \leq x \leq x_0, \quad \gamma_2 : x = \rho_2 y,\ \ 0 \leq y \leq y_0,$$
$$\rho_i = \text{const} > 0,\ \ i = 1,2, \quad \rho_1 x_0 < y_0, \quad \rho_2 y_0 < x_0.$$

Let $N_1|_{\gamma_1} \neq 0$, $M_2|_{\gamma_2} \neq 0$, and let the second condition of (1.13) be fulfilled, while the first one be violated only at one point $O(0,0)$ in the form

$$M_1(x) = x^p \omega(x),$$

where $\omega(x) \neq 0$, $0 \leq x \leq x_0$, $p > 0$ and $\omega(x) \in C[0,x_0]$.

It is known that every solution $u(x,y)$ of equation (1.3) of the class $C_\alpha^{1,1}(\overline{D})$, $\alpha > -1$, can be represented uniquely as [6]

$$u(x,y) = \widetilde{\varphi}(x) + \widetilde{\psi}(y),$$

where $\widetilde{\varphi}(x) \in C[0,x_0]$, $\widetilde{\varphi}^{(1)}(x) \in C_\alpha(0,x_0]$, $\widetilde{\psi}(y) \in C[0,y_0]$, $\widetilde{\psi}^{(1)}(y) \in C_\alpha(0,y_0]$, $\widetilde{\varphi}(0) = \widetilde{\psi}(0) = 0$.

In the notations $\varphi(x) = u_x(x,y) = \widetilde{\varphi}^{(1)}(x)$, $\psi(y) = u_y(x,y) = \widetilde{\psi}^{(1)}(y)$, we rewrite the boundary conditions (1.13) in the form of a system of equations

$$x^p\omega(x)\varphi(x) + N_1(x)\psi(\rho_1 x) = f_1(x), \quad 0 < x \leq x_0, \tag{1.74}$$
$$M_2(y)\varphi(\rho_2 y) + N_2(y)\psi(y) = f_2(y), \quad 0 < y \leq y_0, \tag{1.75}$$

with respect to unknown functions $\varphi(x) \in C_\alpha(0,x_0]$, $\psi(y) \in C_\alpha(0,y_0]$.

It is easily seen that the system of equations (1.74), (1.75) is equivalent to the system

$$x^p\varphi(x) - b_1(x)\varphi(\tau_0 x) = f_3(x), \quad 0 < x \leq x_0, \tag{1.76}$$
$$\psi(y) = -(N_2^{-1}M_2)(y)\varphi(\rho_2 y) + (N_2^{-1}f_2)(y), \quad 0 < y \leq y_0, \tag{1.77}$$

where $\tau_0=\rho_1\rho_2 < 1$, $b_1(x)=(\omega^{-1}N_1)(x)(N_2^{-1}M_2)(\rho_1 x)$, $f_3(x)=(\omega^{-1}f_1)(x)-(\omega^{-1}N_1)(x)(N_2^{-1}f_2)(\rho_1 x)$.

The following lemma holds.

The homogeneous equation corresponding to (1.76) *has an infinite number of linearly independent solutions in the class* $C_\alpha(0,x_0]$ *for all* α.

Proof. It can be easily verified that the function

$$\chi(t)=t^{\frac{p}{2}\left(\frac{\log t}{\log\tau_0}-1\right)}$$

belongs to the class $C^\infty[0,\infty)$, tends to zero as $t\to+0$ more rapidly than any power t^m, $m>0$, $\chi(t)>0$ for $t>0$ and strictly monotonically increases on the segment $0\le t\le\tau_0^{1/2}$; moreover,

$$\chi(\tau_0 t)=t^p\chi(t). \tag{1.78}$$

Bearing in mind (1.78), after substitution $\varphi(x)=\chi(x)\varphi_1(x)$, the homogeneous equation corresponding to (1.76) takes with respect to the unknown function φ_1 the form

$$\varphi_1(x)-b_1(x)\varphi_1(\tau_0 x)=0,\quad 0<x\le x_0. \tag{1.79}$$

For simplicity, let $b_1(x)=\text{const}\neq 0$. According to (1.26), every solution of (1.79), continuous in the half-interval $0<x\le x_0$, can be represented in the form

$$\varphi_1(x)=\begin{cases}\varphi_1^0(x), & \tau_0x_0\le x\le x_0,\\ b_1^{-n_1(x)}\varphi_1^0\big(\tau_0^{-n_1(x)}x\big), & 0<x<\tau_0x_0,\\ & \quad n_1(x)=\left[\frac{\log x}{\log\tau_0}\right],\end{cases} \tag{1.80}$$

where $[\frac{\log x}{\log\tau_0}]$ is an integer part of the number $[\frac{\log x}{\log\tau_0}]$, while φ_1^0 is an arbitrary function of the class $C[\tau_0x_0,x_0]$ satisfying $\varphi_1^0(x_0)-b_1\varphi_1^0(\tau_0x_0)=0$.

If $|b_1|<1$, then

$$|b_1|^{-n_1(x)}\le|b_1|^{-\frac{\log x}{\log\tau_0}}=x^{-\frac{\log|b_1|}{\log\tau_0}}, \tag{1.81}$$

and for $|b_1|\ge1$ we have

$$|b_1|^{-n_1(x)}\le|b_1|^{-\frac{\log x}{\log\tau_0}+1}=|b_1|x^{-\frac{\log|b_1|}{\log\tau_0}}. \tag{1.82}$$

From (1.80)–(1.82) we have

$$|\varphi_1(x)|\le\widetilde{c}x^{-\frac{\log|b_1|}{\log\tau_0}}\|\varphi_1^0\|_{C[\tau_0x_0,x_0]}, \tag{1.83}$$

where $\widetilde{c}=\max(1,|b_1|)$.

Since the function $\chi(x)$ along with all its derivatives vanishes for $x \to +0$ more rapidly than any power x^m, $m > 0$, owing to (1.83) we have

$$\lim_{x \to +0} \left| x^{-\alpha} \chi(x) \varphi_1(x) \right| = 0$$

for any α. Therefore the function $\varphi(x) = \chi(x)\varphi_1(x)$, being the solution of equation (1.76), belongs to the class $C_\alpha(0, x_0]$.

Because the function $\varphi_1^0(x)$ in (1.80) is arbitrary, equation (1.76) has in fact an infinite number of linearly independent solutions of the class $C_\alpha(0, x_0]$. ■

By Lemma 1.11, when condition $M_1|_{\gamma_1} \neq 0$ is violated at one point $O(0,0)$ only, the homogeneous problem corresponding to (1.3), (1.4) has an infinite number of linearly independent solutions in the class $C_\alpha^{1,1}(\overline{D})$ for all $\alpha > -1$. At the same time, we can find a functional space $C_{\alpha,\chi}^{1,1}(\overline{D})$ in which the problem (1.3), (1.4) is uniquely solvable.

Introduce into consideration the space

$$C_{\alpha,\chi}^{1,1}(\overline{D}) = \Big\{ u \in C(\overline{D}) \cap C^1(\overline{D} \backslash O) : \ u(0,0) = 0, \ \chi^{-1}(x) u_x \in C_\alpha(\overline{D} \backslash O),$$
$$y^{-p\frac{\log \rho_2}{\log \tau_0}} \chi^{-1}(y) u_y \in C_\alpha(\overline{D} \backslash O), \ u_{xy} \in C(\overline{D} \backslash O) \Big\},$$

where

$$C_\alpha(\overline{D} \backslash O) = \Big\{ u \in C(\overline{D} \backslash O) : \ \sup_{z \in \overline{D} \backslash O} |z|^{-\alpha} |u(z)| < +\infty \Big\}.$$

As it is shown above, the problem (1.3), (1.4) in the class $C_{\alpha,\chi}^{1,1}(\overline{D})$ is equivalently reduced to the system of equations (1.76), (1.77) with respect to the unknown functions $\varphi(x)$ and $\psi(y)$, where

$$\chi^{-1}(x)\varphi(x) \in C_\alpha(0, x_0], \quad y^{-p\frac{\log \rho_2}{\log \tau_0}} \chi^{-1}(y) \psi(y) \in C_\alpha(0, y_0].$$

The spaces consisting of the functions $\varphi(x)$ and $\psi(y)$ and satisfying these conditions we denote, respectively, by $C_{\alpha,\chi}(0, x_0]$ and $C_{\alpha, y^{q_1}\chi}(0, y_0]$, where $q_1 = p\frac{\log \rho_2}{\log \tau_0}$.

If $\varphi(x) \in C_{\alpha,\chi}(0, x_0]$, then it is obvious that $\varphi(\rho_2 y) \in C_{\alpha, y^{q_1}\chi}(0, y_0]$. Therefore by virtue of (1.77) we require that $f_2 \in C_{\alpha, y^{q_1}\chi}(0, y_0]$.

Since

$$x^p \varphi(x), \ \ \varphi(\tau_0 x) \in C_{\alpha, x^p \chi}(0, x_0]$$

and

$$f_2(\rho_1 x) \in C_{\alpha, y^{q_2} y^{q_1}}(0, y_0] = C_{\alpha, y^p \chi}(0, y_0],$$

where $q_2 = p\frac{\log \rho_1}{\log \tau_0}$ and $q_1 + q_2 = p$ owing to $\rho_1 \rho_2 = \tau_0$, in equation (1.76) in order to $f_3 \in C_{\alpha, x^p\chi}(0, x_0]$ we require of the boundary function f_1 that $f_1 \in C_{\alpha, x^p\chi}(0, x_0]$. Therefore if we consider the problem (1.3), (1.4) in the class $C_{\alpha,\chi}^{1,1}(\overline{D})$, then we shall assume that in the boundary conditions (1.3)

$$f_1 \in C_{\alpha, x^p\chi}(0, x_0], \quad f_2 \in C_{\alpha, y^{q_1}\chi}(0, y_0].$$

Let $\sigma_1 = b_1(0) = (\omega^{-1} N_2^{-1} N_1 M_2)(0)$.

For $\alpha > -\frac{\log|\sigma_1|}{\log \tau_0}$ *equation* (1.76) *is uniquely solvable in the class* $C_{\alpha,\chi}(0,x_0]$, *while for* $\alpha < -\frac{\log|\sigma_1|}{\log \tau_0}$ *equation* (1.76) *is solvable in the class* $C_{\alpha,\chi}(0,x_0]$*; moreover, the homogeneous equation corresponding to* (1.76) *has an infinite number of linearly independent solutions in this class.*

Proof. Because of (1.78), substituting in equation (1.76) $\varphi(x) = \chi(x)\varphi_1(x)$ for the unknown function $\varphi_1(x)$, we obtain the equation

$$\varphi_1(x) - b_1(x)\varphi_1(\tau_0 x) = f(x), \tag{1.84}$$

where $\varphi_1(x) \in C_\alpha(0,x_0]$, if $\varphi(x) \in C_{\alpha,\chi}(0,x_0]$ and $f(x) = x^{-p}\chi^{-1}(x)f_3(x) \in C_\alpha(0,x_0]$.

It is now evident that Lemma 1.12 is a direct consequence of Lemma 1.7 applied to equation (1.84). ■

Thus the following theorem is valid.

Let $N_i|_{\gamma_i} \neq 0$, $i = 1,2$, $M_2|_{\gamma_2} \neq 0$ *and* $M_1(x) = x^p\omega(x)$, $p > 0$, $\omega(x) \in C[0,x_0]$, $\omega(x) \neq 0$, $x \in [0,x_0]$. *Then for* $\alpha > -\frac{\log|\sigma_1|}{\log \tau_0}$ *the problem* (1.3), (1.4) *is uniquely solvable in the class* $C^{1,1}_{\alpha,\chi}(\overline{D})$, *while for* $\alpha < -\frac{\log|\sigma_1|}{\log \tau_0}$ *the problem* (1.3), (1.4) *is solvable in the class* $C^{1,1}_{\alpha,\chi}(\overline{D})$*; moreover, the homogeneous problem corresponding to* (1.3), (1.4) *has an infinite number of linearly independent solutions in this class.*

§

In the plane of variables x, y let us consider a system of linear differential equations of the type

$$Au_{xx}+2Bu_{xy}+Cu_{yy}+A_1u_x+B_1u_y+C_1u=F, \tag{2.1}$$

where A, B, C, A_1, B_1, C_1 are given real $n\times n$-matrices, F is a given and u is an unknown n-dimensional real vector, respectively, and it is assumed that $\det C\neq 0$, $n>1$.

Denote by $p(x,y;\xi,\eta)$ the characteristic determinant of the system (2.1), that is,

$$p(x,y;\xi,\eta)=\det Q(x,y;\xi,\eta),$$

where $Q(x,y;\xi,\eta)=A(x,y)\xi^2+2B(x,y)\xi\eta+C(x,y)\eta^2$.

Since $\det C\neq 0$, we have the representation

$$p(x,y;1,\lambda)=\det C\prod_{i=1}^{l}\big(\lambda-\lambda_i(x,y)\big)^{k_i},\quad \sum_{i=1}^{l}k_i=2n,$$
$$l=l(x,y),\quad k_i=k_i(x,y),\quad i=1,\dots,l.$$

Obviously, the system (2.1) degenerates parabolically only at the point (x,y) in the case $l=1$. The system (2.1) is said to be hyperbolic at (x,y) if $l>1$ and all the roots $\lambda_1(x,y),\dots,\lambda_l(x,y)$ of the polynomial $p(x,y;1,\lambda)$ are real numbers.

It can be easily verified that [6]

$$k_i(x,y)\geq n-\operatorname{rank}Q\big(x,y;1,\lambda_i(x,y)\big),\quad i=1,\dots,l.$$

The hyperbolic system (2.1) is said to be normally hyperbolic at the point (x,y) if the equalities [6]

$$k_i(x,y)=n-\operatorname{rank}Q\big(x,y;1,\lambda_i(x,y)\big),\quad i=1,\dots,l,$$

are fulfilled.

Below we shall assume that at every point (x,y) of the domain of definition of the coefficients A, B, C the system (2.1) is normally hyperbolic, and the multiplicities $k_1(x,y),\dots,k_l(x,y)$ of the roots $\lambda_1(x,y),\dots,\lambda_l(x,y)$ of the characteristic polynomial $p(x,y;1,\lambda)$ do not depend on the variables x, y, i.e., $k_i=\text{const}$, $i=1,\dots,l$.

Note that strictly hyperbolic systems, i.e. when $l=2n$, $k_i=1$, $i=1,\dots,2n$, form a subclass of normally hyperbolic systems.

Let $\gamma_i: x=x_i(t)$, $y=y_i(t)$, $0\leq t<\infty$, $i=1,2$, be simple curves of the class C^k, $k\geq 2$, coming out of the origin $O(0,0)$, having no common point at

$t > 0$ and dividing the plane into two simply connected unbounded angles. Denote by D the angle between γ_1 and γ_2 whose size at the point $O(0,0)$ is less than π. In §2 under certain restrictions imposed on the curves γ_1, γ_2 and characteristics of the system (2.1), we construct the domain $D_1 \subset D$ representing either a curvilinear quadrangle or a triangle (depending on the location of certain points P_1, P_2 on γ_1, γ_2) with a vertex at $O(0,0)$ which is bounded by γ_1, γ_2 and well-defined characteristics of the system (2.1), coming out of the points P_1, P_2. D_1 is assumed to be a subdomain of the domain of definition of the system (2.1).

Consider the boundary value problem formulated as follows [6]: find in the domain D_1 a regular solution $u(x,y)$ of the system (2.1), satisfying on the segments OP_1 and OP_2 of the curves γ_1 and γ_2 the conditions

$$\big(M_1 u_x + N_1 u_y + S_1 u\big)\big|_{OP_1} = f_1, \tag{2.2}$$

$$\big(M_2 u_x + N_2 u_y + S_2 u\big)\big|_{OP_2} = f_2, \tag{2.3}$$

where M_i, N_i, S_i, $i = 1,2$, are given real $m_i \times n$-matrices, f_i, $i = 1,2$, are given m_i-dimensional vectors and m_1 and m_2 are non-negative integers which will be defined below.

Introduce the functional spaces

$$\overset{\circ}{C}{}^k(\overline{D}_1) = \Big\{u \in C^k(\overline{D}_1) :\ \partial^{i,j} u(0,0) = 0,\ 0 \le i+j \le k\Big\},$$

$$\partial^{i,j} = \frac{\partial^{i+j}}{\partial x^i \partial y^j}\ ,$$

$$\overset{\circ}{C}{}^k_\alpha(\overline{D}_1) = \Big\{u \in \overset{\circ}{C}{}^k(\overline{D}_1) :\ \max_{i+j=k}\ \sup_{z\in\overline{D}_1\setminus O} |z|^{-\alpha} |\partial^{i,j} u(z)| < \infty\Big\},$$

$$-\infty < \alpha < \infty.$$

Obviously, $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1) = \overset{\circ}{C}{}^k(\overline{D}_1)$ for $\alpha \le 0$. Analogously we introduce the weighted spaces $\overset{\circ}{C}{}^k_\alpha(OP_i)$, $i = 1,2$.

When considering problems (2.1)–(2.3) in the class $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$, $k \ge 2$, $\alpha \ge 0$, we shall require that $A, B, C \in C^k(R^2)$, where R^2 is the plane of variables x, y; $A_1, B_1, C_1 \in C^{k-1}(\overline{D}_1)$; $M_i, N_i, S_i \in C^{k-1}(OP_i)$, $i = 1,2$; $f_i \in \overset{\circ}{C}{}^{k-1}_\alpha(OP_i)$, $i = 1,2$; $F \in \overset{\circ}{C}{}^{k-1}_{\alpha-1}(\overline{D}_1)$.

§ γ_1 γ_2 m_1

m_2 D_1 D_P

In §3 it will be shown that under certain assumptions made with respect to the coefficients A, B, C of the system (2.1), the roots of the characteristic polynomial $p(x,y;1,\lambda)$ at every point $(x,y) \in R^2$ can be renumerated so that $\lambda_i(x,y) \in C^k(R^2)$, $i = 1,\dots,l$.

Through every point $(x, y) \in R^2$ there pass l characteristic curves $L_i(x, y)$, $i = 1, \dots, l$, of the system (2.1), satisfying the ordinary differential equations

$$dx + \lambda_i(x, y)dy = 0, \quad i = 1, \dots, l.$$

Let the location of the curves γ_1, γ_2 on the plane be such that while moving towards $O(0,0)$ along γ_2 and then along γ_1, the domain D bounded by γ_1, γ_2 remains to the left. Renumerate the roots of the polynomial $p(x, y; 1, \lambda)$ in such a way that the characteristic curves $L_1(P_1), \dots, L_l(P_1)$ corresponding to the roots $\lambda_1, \dots, \lambda_l$ and coming out of the point P_1 into the domain $\{P \in D : |P - P_1| < \varepsilon\}$ would turn out to be renumerated counter-clockwise if we count from $L_1(P_1)$, where ε is a sufficiently small positive number.

If the curves γ_1 and γ_2 do not have a common tangent line at $O(0,0)$, then we denote by l_0, $0 \le l_0 \le l$, the number of different characteristics issued from $O(0,0)$ into the domain $\{(x, y) \in D : x^2 + y^2 < \varepsilon^2\}$. In the case where γ_1, γ_2 have a common tangent line at $O(0,0)$, we assume $l_0 = 0$.

Below we impose on the curves γ_1, γ_2 and the characteristics $L_i(P)$, $P \in \overline{D}$, $i = 1, \dots, l$, the following restrictions.

1. Each of the curves γ_1, γ_2 either is a characteristic of the system (2.1) or it has characteristic direction at none of its point.

2. For $i > l_0$ every characteristic $L_i(P)$, $P \in \overline{D} \backslash O$, extended maximally to either side in $\overline{D}$ possesses one of the following properties:

a) it entirely coincides with one of the curves γ_1 or γ_2;

b) it intersects γ_1 (γ_2) only at one point, when γ_1 (γ_2) is a non-characteristic curve or γ_1 (γ_2) is a characteristics of the system (2.1), not belonging to the family L_i.

If, however, $1 \le i \le l_0$, then the characteristics $L_i(O)$ divide D into two simply-connected unbounded angles and the characteristics $L_i(P)$ intersect the curve γ_1 or γ_2 at one point only, depending on the location of the point P in $\overline{D} \backslash L_i(O)$.

3. The family of characteristics L_i is described in $\overline{D}$ by the equation $L_i : \Omega_i(x, y) = \text{const}$, $1 \le i \le l$, where $\Omega_i \in C^k(\overline{D})$ and $|\operatorname{grad} \Omega_i||_{\overline{D}} \ne 0$.

For the sake of simplicity, let the characteristics $L_i(P_1)$, $i = 1, \dots, l$, issued from the point P_1 into D not intersect the curve γ_2 at the point P_2. We take the number m_1 of boundary conditions in (2.2) to be equal to the number of characteristics, with regard for their multiplicities, issued from the point P_1 into D and not intersecting with the closed segment $OP_2 \subset \gamma_2$. Substituting the point P_1 by P_2 and the segment OP_2 by $OP_1 \subset \gamma_1$, we can determine analogously the value m_2. In particular, if $m_i = 0$, then the segment $OP_i \subset \gamma_i$, $i = 1, 2$, becomes completely free from the boundary conditions. It is clear that under such a choice the numbers m_1 and m_2 depend on the location of the points P_1 and P_2 on the curves γ_1 and γ_2; moreover, $0 \le m_i \le 2n$, $i = 1, 2$.

Let us introduce into the consideration the domains D_1 and D_P, $P \in \overline{D} \backslash O$. If $m_i > 0$, $i = 1, 2$, then let D_1 be a curvilinear quadrangle with a vertex at the point $O(0,0)$, bounded by the curves γ_1, γ_2, $L_{s_0}(P_1)$, $L_{s_1}(P_2)$, where $L_{s_0}(P_1)$ is the last (moving counter-clockwise) characteristic, coming out of the point P_1 into the domain D and not intersecting the closed segment OP_2, while $L_{s_1}(P_2)$ is the last (moving clockwise) characteristic coming out of the point P_2 into the angle D and not intersecting with the closed segment OP_1. In this case D_P is a curvilinear quadrangle with a vertex at the point $O(0,0)$, bounded by the curves γ_1, γ_2, $L_{s_0}(P)$ and $L_{s_1}(P)$. Clearly, $s_1 = s_0 + 1$ for $0 < m_1 < 2n$ and $s_1 = 1$, $s_0 = l$ for $m_1 = 2n$, $l_0 > 0$, but in the case $m_1 = 2n$, $l_0 = 0$ the number $m_2 = 0$. If, however, $m_1 = 0$, then D_1 and D_P are curvilinear triangles bounded, respectively, by the curves γ_1, γ_2, $L_1(P_2)$ and γ_2, $L_1(P)$, $L_l(P)$. Similarly, for $m_2 = 0$ the domains D_1 and D_P are bounded, respectively, by the curves γ_1, γ_2, $L_l(P_1)$ and γ_1, $L_1(P)$, $L_l(P)$.

§

Owing to normal hyperbolicity of the system (2.1), at every point (x,y) we have $\operatorname{rank} Q(x,y;1,\lambda_i(x,y)) = n-k_i$, $1 \le i \le l$. Hence $\dim \operatorname{Ker} Q(x,y;1,\lambda_i(x,y)) = k_i$, where $\operatorname{Ker} Q(x,y;1,\lambda_i(x,y))$ is a kernel of the matrix operator $Q(x,y;1,\lambda_i(x,y))$ acting in R^n. Let $\{\nu_{ij}\}_{j=1}^{k_i}$ be a basis chosen arbitrarily in $\operatorname{Ker} Q(x,y;1,\lambda_i(x,y))$. It can be easily verified that at every point (x,y), the value $\lambda_i(x,y)$, $1 \le i \le l$, is an eigen-value, while the $2n$-dimensional vectors $(\nu_{ij}, \lambda_i\nu_{ij})(x,y)$ corresponding to $\lambda_i(x,y)$ are eigen-vectors of the matrix operator

$$A_0(x,y) = \begin{Vmatrix} 0 & -E \\ C^{-1}A & 2C^{-1}B \end{Vmatrix} (x,y).$$

Note that if the $2n$-dimensional vector $(\nu^1_{ij}, \nu^2_{ij})(x,y)$ is a an eigen-vector of the operator A_0 corresponding to the eigen-value $\lambda_i(x,y)$, then $\nu^2_{ij}(x,y) = \lambda_i(x,y)\nu^1_{ij}(x,y)$, and $\nu^1_{ij}(x,y) \in \operatorname{Ker} Q(x,y;1,\lambda_i(x,y))$. Since the system (2.1) is normally hyperbolic, the vectors $(\nu_{ij}, \lambda_i\nu_{ij})$, $i = 1,\dots,l$, $j = 1,\dots,k_i$, form a complete system of eigen-vectors of the operator $A_0(x,y)$, and hence diagonalizing the operator, A_0 we obtain the equality

$$K^{-1}A_0K = D_0 \tag{2.4}$$

at the point (x,y), where

$$K = \begin{pmatrix} \nu_{11} & \cdots & \nu_{1k_1} & \nu_{21} & \cdots & \nu_{lk_l} \\ \lambda_1\nu_{11} & \cdots & \lambda_1\nu_{1k_1} & \lambda_2\nu_{21} & \cdots & \lambda_l\nu_{lk_l} \end{pmatrix},$$
$$D_0 = \operatorname{diag}\big[-\lambda_1,\dots,-\lambda_1,-\lambda_2,\dots,-\lambda_2,\dots,-\lambda_l\big].$$

Denote by Δ_r the square $\{(x,y) \in R^2 : |x| < r, |y| < r\}$. Since the matrix operator A_0 is diagonalizable, belongs to the class $C^k(R^2)$ and the multiplicities k_i of the eigen-values λ_i, $i = 1,\dots,l$, do not depend on the variables x, y, owing to the results of [72], for any fixed $r > 0$ at every point

$(x,y) \in \Delta_r$ we can renumerate the numbers $\lambda_i(x,y)$, $i = 1,\dots,l$, and choose the basis vectors $\nu_{ij}(x,y)$, $j = 1,\dots,k_i$, in the space $\operatorname{Ker} Q(x,y;1,\lambda_i(x,y))$ such that $\lambda_i(x,y) \in C^k(\overline{\Delta}_r)$, $i = 1,\dots,l$, and $\nu_{ij}(x,y) \in C^k(\overline{\Delta}_r)$, $i = 1,\dots,l$; $j = 1,\dots,k_i$. From this it is not difficult to see that we can choose the numbering of $\lambda_1,\dots,\lambda_l$ such that $\lambda_i(x,y) \in C^k(R^2)$, $i = 1,\dots,l$. Indeed, performing additional renumeration, we may assume that for any $r > 0$

$$\lambda_1^r(0,0) < \lambda_2^r(0,0) < \cdots < \lambda_l^r(0,0) \tag{2.5}$$

and $\lambda_i^r(x,y) \in C^k(\overline{\Delta}_r)$, $i = 1,\dots,l$. Now let us show that (2.5) implies the validity of the same inequalities at any other point $(x,y) \in \overline{\Delta}_r$, i.e.,

$$\lambda_1^r(x,y) < \lambda_2^r(x,y) < \cdots < \lambda_l^r(x,y). \tag{2.6}$$

If at a point $(x_0,y_0) \in \overline{\Delta}_r$ the inverse inequality $\lambda_i^r(x_0,y_0) > \lambda_j^r(x_0,y_0)$ took place for $i < j$, then due to the continuity of the function $g_{ij}(x,y) = \lambda_i^r(x,y) - \lambda_j^r(x,y)$ and because of the inequalities $g_{ij}(0,0) < 0$, $g_{ij}(x_0,y_0) > 0$, one could find on the portion of the straight line connecting the points $(0,0)$ and (x_0,y_0), a point $(x_1,y_1) \in \overline{\Delta}_r$ such that $g_{ij}(x_1,y_1) = 0$, i.e., $\lambda_i^r(x_1,y_1) = \lambda_j^r(x_1,y_1)$, but this equality contradicts the fact that at every point (x,y) all the numbers $\lambda_1^r(x,y),\dots,\lambda_l^r(x,y)$ differ. Since inequalities (2.6) are valid for any r and for $0 < r_1 < r_2$ the sets $\{\lambda_1^{r_1}(x,y),\dots,\lambda_l^{r_1}(x,y)\}$ and $\{\lambda_1^{r_2}(x,y),\dots,\lambda_l^{r_2}(x,y)\}$ coincide at every point $(x,y) \in \overline{\Delta}_{r_1}$, we get

$$\lambda_i^{r_1}(x,y) = \lambda_i^{r_2}(x,y) \quad \text{for} \quad (x,y) \in \overline{\Delta}_{r_1}, \;\; i = 1,\dots,l. \tag{2.7}$$

It follows from (2.7) that the functions

$$\lambda_i(x,y) = \lambda_i^r(x,y) \quad \text{for} \quad (x,y) \in \Delta_r, \;\; i = 1,\dots,l,$$

belong to the class $C^k(R^2)$.

Since the domain D_1 constructed in §2 is bounded, $D_1 \subset \Delta_r$ for some $r > 0$. Therefore, owing to the above arguments, the basis vectors $\nu_{ij}(x,y)$ will be assumed to be chosen in the space $\operatorname{Ker} Q(x,y;1,\lambda_i(x,y))$ such that $\nu_{ij}(x,y) \in C^k(\overline{D}_1)$, $i = 1,\dots,l$, $j = 1,\dots,k_i$.

Without loss of generality we may assume that the domain D_P, $P(x_0,y_0) \in \overline{D}_1$, constructed in §2 is located entirely in the half-plane $y \le y_0$; moreover, every characteristic $L_i(x_0,y_0)$, $1 \le i \le l$, of the system (2.1) issued from the point $P(x_0,y_0)$ into the closed domain $\overline{D}_P$ to the intersection with one of the curves γ_1 or γ_2 admits parametrization of the type $L_i(x_0,y_0) : x = z_i(x_0,y_0;y) \in C^k$, $y = t$. Otherwise, as it can be easily verified, because of the requirement 3 imposed on the characteristics L_i, this can be achieved by means of a non-degenerate transform of variables $\widetilde{x} = J_1(x,y)$, $\widetilde{y} = J_2(x,y)$, $J_1(0,0) = J_2(0,0) = 0$ which translates the families of characteristics $L_{s_0}(x,y)$ and $L_{s_1}(x,y)$ to those of straight lines $\widetilde{y} + \widetilde{x} = \text{const}$ and $\widetilde{y} - \widetilde{x} = \text{const}$, respectively, while the domain D_1 to a subdomain $\widetilde{D}_1$ of the half-plane $\widetilde{y} \ge 0$. In the plane of variables $\widetilde{x}$, $\widetilde{y}$, every characteristic $\widetilde{L}_i(\widetilde{x}_0,\widetilde{y}_0)$, $1 \le i \le l$, issued from the point $\widetilde{P}(\widetilde{x}_0,\widetilde{y}_0) \in \overline{\widetilde{D}}_1$ into the the domain $\overline{\widetilde{D}}_{\widetilde{P}}$ to the intersection with the curve $\widetilde{\gamma}_1$ or $\widetilde{\gamma}_2$ will entirely lie in the quarter-plane $\widetilde{y}+\widetilde{x} \le \widetilde{y}_0+\widetilde{x}_0$, $\widetilde{y}-\widetilde{x} \le \widetilde{y}_0-\widetilde{x}_0$, and hence at every point $\widetilde{P}(\widetilde{x}_0,\widetilde{y}_0) \in \overline{\widetilde{D}}_1$ the tangent to the characteristic $\widetilde{L}_i(\widetilde{x}_0,\widetilde{y}_0)$ is not parallel to the axis $o\widetilde{x}$. This, in its turn, implies that the portion of the characteristic $\widetilde{L}_i(\widetilde{x}_0,\widetilde{y}_0)$ which is located in the domain $\overline{\widetilde{D}}_1$ admits a parametrization of the form $\widetilde{x} = \widetilde{z}_i(\widetilde{x}_0,\widetilde{y}_0;t) \in C^k$, $\widetilde{y} = t$.

Denote by $\omega_i(x_0,y_0)$ the ordinate of the point of intersection of the characteristic $L_i(x_0,y_0)$, issued from the point $P(x_0,y_0) \in \overline{D}_1$ into the domain $\overline{D}_P$, with one of the curves γ_1 or γ_2. This curve depends both on the index i of the characteristic L_i and on the location of the point P in $\overline{D}_1$ and we denote it by $\gamma_{i(P)}$. According to the requirements imposed on the characteristics L_i and the curves γ_1, γ_2 we have $\omega_i \in C^k(\overline{D}_1)$, $\omega_i(x_0,y_0) \le y_0$, $(x_0,y_0) \in \overline{D}_1$; moreover, $L_i(P) \cap \overline{D}_P \,:\, x = z_i(x_0,y_0;t) \in C^k$, $y = t$, $\omega_i(x_0,y_0) \le t \le y_0$.

Below we shall assume that a portion OP_i of the curve γ_i is described in terms of the equation $x = \gamma_i(y)$, $0 \le y \le d_i$, $i = 1,2$. One can easily verify that the problem (2.1)–(2.3) in the class $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$ can be equivalently rewritten in the form

$$v_y + A_0 v_x + B_0 v + C_0 u^0 = F^0, \tag{2.8}$$

$$\Big(-\lambda_1\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)u = -\lambda_1 v_1 + v_2, \tag{2.9}$$

$$\big(M_1 v_1 + N_1 v_2 + S_1 u\big)\big|_{OP_1} = f_1, \tag{2.10}$$

$$\big(M_2 v_1 + N_2 v_2 + S_2 u\big)\big|_{OP_2} = f_2, \tag{2.11}$$

$$\Big(\frac{d\gamma_i}{dy}\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)u\Big|_{OP_i} = \Big(\frac{d\gamma_i}{dy}v_1 + v_2\Big)\Big|_{OP_i}, \quad i = 1,2, \tag{2.12}$$

where

$$A_0 = \begin{Vmatrix} 0 & -E \\ C^{-1}A & 2C^{-1}B \end{Vmatrix}, \quad B_0 = \begin{Vmatrix} 0 & 0 \\ C^{-1}A_1 & C^{-1}B_1 \end{Vmatrix},$$

$$C_0 = \begin{Vmatrix} 0 & 0 \\ C^{-1}C_1 & 0 \end{Vmatrix},$$

$$v_1 = u_x, \ \ v_2 = u_y, \ \ u \in \overset{\circ}{C}{}^k_\alpha(\overline{D}_1), \ \ v = (v_1,v_2) \in \overset{\circ}{C}{}^{k-1}_\alpha(\overline{D}_1),$$

$u^0 = (u,0)$, $F^0 = (0, C^{-1}F)$ and E is the unit $n \times n$-matrix.

In the case $l_0 = 0$, one should write instead of (2.12) the equality

$$\Big(\frac{d\gamma_1}{dy}\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)u\Big|_{OP_1} = \Big(\frac{d\gamma_1}{dy}v_1 + v_2\Big)\Big|_{OP_1}.$$

If $u \in \overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$ is a solution of the problem (2.1)–(2.3), then the system of vectors u, $v_1 = u_x$, $v_2 = u_y$ will, obviously, give the solution of the problem

(2.8)–(2.12). Conversely, let $u \in \overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$, $v = (v_1, v_2) \in \overset{\circ}{C}{}^{k-1}_\alpha(\overline{D}_1)$ be a solution of the problem (2.8)–(2.12). Let us show that u is a solution of the problem (2.1)–(2.3), and $v_1 = u_x$, $v_2 = u_y$. For simplicity, let us assume that $\lambda_1 = \text{const}$. It follows from the first n equations of the system (2.8) that $v_{1y} = v_{2x}$. Next, because of (2.9) we have

$$\Big(-\lambda_1\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)(u_x - v_1) =$$
$$= \frac{\partial}{\partial x}\Big(-\lambda_1\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)u - \Big(-\lambda_1\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)v_1 =$$
$$= \frac{\partial}{\partial x}(-\lambda_1 v_1 + v_2) + \lambda_1 v_{1x} - v_{1y} =$$
$$= -\lambda_1 v_{1x} + v_{2x} + \lambda_1 v_{1x} - v_{1y} = v_{2x} - v_{1y} = 0.$$

Thus $u_x - v_1 \equiv 0$, since, by requirements imposed both on the characteristics L_i and on the curves γ_1, γ_2, the system of equations (2.9), (2.12) is uniquely solvable with respect to u_x and u_y on the segments $OP_1 \subset \gamma_1$, $OP_2 \subset \gamma_2$. Moreover, $(u_x - v_1)|_{OP_1} = (u_x - v_1)|_{OP_2} = 0$, if $l_0 > 1$ and $(u_x - v_1)|_{OP_1} = 0$ for $l_0 = 0$. Because of $u_x = v_1$, it follows from (2.9) that $u_y = v_2$ and by (2.8), (2.10), (2.11) we easily obtain that u is a solution of the problem (2.1)–(2.3). In the case $\lambda_1(x,y) \not\equiv \text{const}$, we shall act as follows. Denote by $\widetilde{\lambda}(x,y)$ a function of the class $C^1(\overline{D}_1)$ such that $\nabla_1\widetilde{\lambda} = \nabla\lambda_1$ and $\widetilde{\lambda} - \lambda \neq 0$ in $\overline{D}_1$, where $\nabla_1 = -\lambda_1\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, $\nabla_2 = -\widetilde{\lambda}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. By equalities $v_{1y} = v_{2x}$ and $\nabla_1\widetilde{\lambda} = \nabla_2\lambda_1$, we can easily verify that

$$\nabla_1\nabla_2 = \nabla_2\nabla_1, \quad \nabla_2(-\lambda_1 v_1 + v_2) = \nabla_1(-\widetilde{\lambda}_1 v_1 + v_2),$$

whence, taking into account (2.9), we get

$$\nabla_1\big(\nabla_2 u - (-\widetilde{\lambda} v_1 + v_2)\big) = \nabla_2\nabla_1 u - \nabla_2(-\lambda_1 v_1 + v_2) =$$
$$= \nabla_2\big(\nabla_1 u - (-\lambda_1 v_1 + v_2)\big) = 0.$$

From this, due to the unique solvability of the system of equations (2.9), (2.12) with respect to u_x and u_y on OP_1 and OP_2 and, as a consequence, of the equalities $(\nabla_2 u - (-\widetilde{\lambda} v_1 + v_2))|_{OP_1\cup OP_2} = 0$ or $(\nabla_2 u - (-\widetilde{\lambda} v_1 + v_2))|_{OP_1} = 0$ for $l_0 > 0$ and $l_0 = 0$, respectively, we find that $\nabla_2 u - (-\widetilde{\lambda} v_1 + v_2) = 0$ in $\overline{D}_1$. Since $\widetilde{\lambda} - \lambda \neq 0$ in $\overline{D}_1$, it follows from (2.9) and the obtained equality $\nabla_2 u - (-\widetilde{\lambda} v_1 + v_2) = 0$ that $u_x = v_1$, $u_y = v_2$ which, in its turn, implies that u is a solution of the problem (2.1)–(2.3). To construct the function $\widetilde{\lambda}(x,y)$ with properties indicated above, we rewrite the equality $\nabla_1\widetilde{\lambda} = \nabla_2\lambda_1$ in terms of the linear first order differential equation

$$\Big(-\lambda_1\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)\widetilde{\lambda} + \lambda_{1x}\widetilde{\lambda} = \lambda_{1y}.$$

Integrating this equation as an ordinary differential equation along the first characteristic L_1 of the system (2.1) and taking as the initial Cauchy data sufficiently large absolute values $\widetilde{\lambda}$ on $OP_1 \cup OP_2$ for $l_0 > 0$ or on OP_1 for $l_0 = 0$, we get the function $\widetilde{\lambda}$ satisfying the conditions $\nabla_1 \widetilde{\lambda} = \nabla_2 \lambda_1$ and $\widetilde{\lambda} - \lambda_1 \neq 0$ in $\overline{D}_1$.

Substitution of the unknown function $v = Kw$ by (2.8)–(2.12) results in

$$w_y + D_0 w_x = B_2 w + C_2 u^0 + F^1, \tag{2.13}$$

$$\Big(-\lambda_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \Big) u = (-\lambda_1 K_1 + K_2) w, \tag{2.14}$$

$$\big((M_1 K_1 + N_1 K_2) w + S_1 u \big) \big|_{OP_1} = f_1, \tag{2.15}$$

$$\big((M_2 K_2 + N_2 K_2) w + S_2 u \big) \big|_{OP_2} = f_2, \tag{2.16}$$

$$\Big(\frac{d\gamma_i}{dy} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \Big) u \Big|_{OP_i} = \Big(\frac{d\gamma_i}{dy} K_1 + K_2 \Big) w \Big|_{OP_i}, \quad i = 1, 2, \tag{2.17}$$

where $B_2 = -K^{-1}(K_y + A_0 K_x + B_0 K)$, $C_2 = -K^{-1} C_0$, $F^1 = K^{-1} F^0$, and K_1 and K_2 are the matrices of order $n \times 2n$ composed, respectively, of the first and the last n rows of the matrix K.

Integrate the $(q_i + j)$-th equation of the system (2.13), where $q_1 = 0$, $q_i = k_1 + \cdots + k_{i-1}$, $j = 1, \dots, k_i$, along the i-th characteristic $L_i(x,y)$ coming out of $P(x,y) \in \overline{D}_1$ into the domain $\overline{D}_P$, from $P(x,y)$ to the point of intersection of $L_i(x,y)$ with the curve γ_1 or γ_2, depending both on the index i of the characteristic L_i and on the location of P in $\overline{D}_1$, and integrate equation (2.14) with respect to the first characteristic. We obtain

$$\begin{aligned} w_{q_i+j}(x,y) &= w_{q_i+j}\big(\gamma_{i(P)}(\omega_i(x,y), \omega_i(x,y)\big) + \\ &+ \int\limits_{\omega_i(x,y)}^{y} \Big(\sum_{p=1}^{2n} a^1_{ijp} w_p + \sum_{p=1}^{n} b^1_{ijp} u_p \Big) \big(z_i(x,y;t), t \big)\, dt + F^2_{ij}(x,y), \end{aligned} \tag{2.18}$$

$$1 \leq i \leq l, \quad 1 \leq j \leq k_i,$$

$$\begin{aligned} u(x,y) &= g(\omega_1(x,y)) + \\ &+ \int\limits_{\omega_1(x,y)}^{y} \big((-\lambda_1 K_1 + K_2) w \big) \big(z_1(x,y;t), t \big)\, dt, \end{aligned} \tag{2.19}$$

where a^1_{ijp}, b^1_{ijp}, F^2_{ij} are well-defined functions depending only on the coefficients and the right-hand side of the system (2.1); moreover, by (2.17) we have

$$\begin{aligned} g(\omega_1(x,y)) &= u\big(\gamma_{1(P)}(\omega_1(x,y), \omega_1(x,y)\big) = \\ &= \int\limits_{0}^{\omega_1(x,y)} \Big(\frac{d\gamma_{1(P)}}{dy} K_1 + K_2 \Big) w \big(\gamma_{1(P)}(t), t \big)\, dt. \end{aligned}$$

Let

$$\varphi^1_{q_i+j}(y) = w_{q_i+j}(\gamma_1(y), y), \quad 0 \le y \le d_1,$$
$$i = 1, \dots, s_0; \;\; j = 1, \dots, k_i,$$
$$\varphi^2_{q_i+j}(y) = w_{q_i+j}(\gamma_2(y), y), \quad 0 \le y \le d_2,$$
$$i = 1, \dots, l_0; \;\; j = 1, \dots, k_i,$$
$$\varphi^2_{q_i+j-k_0}(y) = w_{q_i+j}(\gamma_2(y), y), \quad 0 \le y \le d_2,$$
$$i = s_0 + 1, \dots, l; \;\; j = 1, \dots, k_i,$$

where $k_0 = \sum\limits_{i=l_0+1}^{s_0} k_i$, the numbers l_0 and s_0 are determined in §2, and the number of components of the vector $\varphi^i(y)$ is obviously equal to m_i, $i = 1, 2$.

Due to the requirements imposed on the curves γ_1, γ_2 and L_i, we can see that

$$\omega_i(\gamma_1(y), y) = \begin{cases} y & \text{for} \quad i = 1, \dots, s_0, \\ \tau^1_i(y) & \text{for} \quad i = s_0 + 1, \dots, l, \end{cases}$$

$$\omega_i(\gamma_2(y), y) = \begin{cases} y & \text{for} \quad i = 1, \dots, l_0, \\ \tau^2_i(y) & \text{for} \quad i = l_0 + 1, \dots, s_0, \\ y & \text{for} \quad i = s_0 + 1, \dots, l, \end{cases}$$

where $\omega_i(x, y) \in C^k(\overline{D}_1)$, $\tau^1_i(y) \in C^k[0, d_1]$, $i = s_0 + 1, \dots, l$, $\tau^2_j(y) \in C^k[0, d_2]$, $j = l_0 + 1, \dots, s_0$, and $\tau^1_l(y) \equiv 0$, if γ_1 is a characteristic of the system (2.1). Analogously, $\tau^2_{l_0+1}(y) \equiv 0$, if γ_2 is a characteristic, and the remaining functions $\tau^p_i(y)$ satisfy the inequality $\tau^p_i(y) < y$ for $0 < y \le d_p$, $p = 1, 2$.

Substituting the expressions for $w(x, y)$ and $u(x, y)$ from (2.18) and (2.19) into the boundary conditions (2.15) and (2.16), we get

$$\begin{gathered} G^1_0(y)\varphi^1(y) + \sum_{i=s_0+1}^{l} G^1_i(y)\varphi^2\big(\tau^1_i(y)\big) + \\ +(T_1 w)(y) + (T_2 u)(y) = f_3(y), \quad 0 \le y \le d_1, \\ G^2_0(y)\varphi^2(y) + \sum_{j=l_0+1}^{s_0} G^2_j(y)\varphi^1\big(\tau^2_j(y)\big) + \\ +(T_3 w)(y) + (T_4 u)(y) = f_4(y), \quad 0 \le y \le d_2, \end{gathered} \tag{2.20}$$

where G^1_i, G^2_j, $i = s_0 + 1, \dots, l$; $j = l_0 + 1, \dots, s_0$ are well-defined matrices of the class C^{k-1}, and T_i, $i = 1, \dots, 4$, are linear integral operators.

Obviously, G^i_0, $i = 1, 2$, from (2.20) are the matrices of order $m_i \times m_i$ which can be represented as the product

$$G^i_0 = \Gamma_i \times V_i, \quad i = 1, 2,$$

where $\Gamma_i = (M_i, N_i)$, $i = 1, 2$, are rectangular $m_i \times 2n$-matrices and V_i, $i = 1, 2$, are matrices of order $2n \times m_i$ written in the form

$$V_1 = \begin{pmatrix} \nu_{11} & \cdots & \nu_{1k_1} & \cdots & \nu_{s_0 1} & \cdots & \nu_{s_0 k_{s_0}} \\ \lambda_1 \nu_{11} & \cdots & \lambda_1 \nu_{1k_1} & \cdots & \lambda_{s_0} \nu_{s_0 1} & \cdots & \lambda_{s_0} \nu_{s_0 k_{s_0}} \end{pmatrix},$$

$$V_2 = \begin{pmatrix} \nu_{11} & \cdots & \nu_{l_0 k_{l_0}} & \nu_{s_0+1,1} & \cdots & \nu_{l k_l} \\ \lambda_1 \nu_{11} & \cdots & \lambda_{l_0} \nu_{l_0 k_{l_0}} & \lambda_{s_0+1} \nu_{s_0+1,1} & \cdots & \lambda_l \nu_{l k_l} \end{pmatrix}.$$

Under the assumption that

$$\det \left(\Gamma_i \times V_i\right)\big|_{OP_i} \neq 0, \quad i = 1, 2, \tag{2.21}$$

we can rewrite equation (2.20) in the form

$$\begin{gathered} \varphi^1(y) - \sum_{i=s_0+1}^{l} \sum_{j=l_0+1}^{s_0} G^3_{ij}(y) \varphi^1\left(\tau^1_{ij}(y)\right) + \\ +(T_5 w)(y) + (T_6 u)(y) = f_5(y), \quad 0 \leq y \leq d_1, \\ \varphi^2(y) - \sum_{i=l_0+1}^{s_0} \sum_{j=s_0+1}^{l} G^4_{ij}(y) \varphi^2\left(\tau^2_{ij}(y)\right) + \\ +(T_7 w)(y) + (T_8 u)(y) = f_6(y), \quad 0 \leq y \leq d_2, \end{gathered} \tag{2.22}$$

where $\tau^1_{ij}(y) = \tau^2_j(\tau^1_i(y))$, $\tau^2_{ij}(y) = \tau^1_j(\tau^2_i(y))$, G^3_{ij} and G^4_{ij} are matrices of orders $m_1 \times m_1$ and $m_2 \times m_2$, and T_5, T_6, T_7, T_8 are linear integral operators.

If γ_1 or γ_2 is a characteristic of the system (2.1), then we will have respectively $\tau^1_{lj}(y) = \tau^2_{il}(y) \equiv 0$, $i, j = l_0 + 1, \dots, s_0$, and $\tau^1_{il_0+1}(y) = \tau^2_{l_0+1j}(y) \equiv 0$, $i, j = s_0 + 1, \dots, l$. Therefore our discussion below will concern the remaining functions τ^1_{ij} and τ^2_{ij} which, as is easily verified, possess the following properties:

1) $\tau^p_{ij} \in C^k[0, d_p]$, $\tau^p_{ij}(0) = 0$, $p = 1, 2$;
2) τ^p_{ij}, $p = 1, 2$, are strictly monotonically increasing functions;
3) $\tau^p_{ij}(y) < y$, $0 < y \leq d_p$, $p = 1, 2$;
4) if the curves γ_1 and γ_2 do not have a common tangent line at the point $O(0, 0)$, then

$$0 \leq \sigma^p_{ij} = \frac{d\tau^p_{ij}}{dy}(0) < 1, \quad p = 1, 2, \tag{2.23}$$

or

$$\sigma^p_{ij} = \frac{d\tau^p_{ij}}{dy}(0) = 1, \quad p = 1, 2$$

otherwise.

The validity of property 1) is obvious. To prove the validity of the other properties, we shall give geometric interpretation of the functions τ^p_{ij}. Let a characteristic $L_i(Q_1)$ be issued from $Q_1(y, \gamma_1(y)) \in OP_1 \subset \gamma_1$ to the intersection with γ_2 at the point Q_2, and let a characteristic $L_j(Q_2)$ be issued from Q_2 to the intersection with γ_1 at Q_3. It is easily seen that the ordinate of Q_2 is equal to $\tau_i^1(y)$, while that of Q_3 is equal to $\tau_{ij}^1(y) = \tau_j^2(\tau_i^1(y))$. In a similar manner we can determine $\tau_{ij}^2(y)$ by interchanging the curves γ_1 and γ_2. The validity of properties 2) and 3) follows directly from the geometrical meaning of the functions τ_{ij}^p if we take into account the requirements which have been imposed on the curves γ_1, γ_2 and characteristics L_i.

Let us now prove the validity of property 4).

In a neighborhood V of $O(0,0)$ one can specify a family of characteristics L_i in the form of the equality $L_i : \mu_i(x,y) = \text{const}$, where $\mu_i \in C^k(V)$, $|\nabla\mu_i||_V \neq 0$, $i = 1,\dots,l$. Since $\nabla\mu_i(0,0) = (\frac{\partial\mu_i}{\partial x}, \frac{\partial\mu_i}{\partial y})(0,0) = c_i(1,\lambda_i(0,0))$, $c_i = \text{const} \neq 0$, the Jacobian of transformation of the independent variables $\widetilde{y} = \mu_i(x,y)$, $\widetilde{x} = \mu_j(x,y)$ at the point $O(0,0)$ is different from zero for the fixed i and j, $i \neq j$. Therefore, in a sufficiently small neighborhood V of the point $O(0,0)$ this mapping will be a diffeomorphism. In the plane of variables $\widetilde{x}$, $\widetilde{y}$ let us denote by $\widetilde{\gamma}_i$ the image of the curve $\gamma_i \cap V$, $i = 1,2$, under this mapping. By the assumptions on the curves γ_1, γ_2 and characteristics L_i, L_j, the curves $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$ are located in the angle $\widetilde{x} \geq 0$, $\widetilde{y} \geq 0$ and described by the equations $\widetilde{\gamma}_1 : \widetilde{y} = \widetilde{\gamma}_1(\widetilde{x})$, $\widetilde{\gamma}_2 : \widetilde{y} = \widetilde{\gamma}_2(\widetilde{x})$, $0 \leq \widetilde{x} \leq \varepsilon$, $\varepsilon > 0$, where $\widetilde{\gamma}_1, \widetilde{\gamma}_2 \in C^k$, $0 < \widetilde{\gamma}_1(\widetilde{x}) < \widetilde{\gamma}_2(\widetilde{x})$ for $0 < \widetilde{x} \leq \varepsilon$ and $\widetilde{\gamma}_1(0) = \widetilde{\gamma}_2(0) = 0$. Introduce into the consideration the function $\widetilde{\tau}_{ij}^1(\widetilde{x})$, $0 \leq \widetilde{x} \leq \varepsilon$, which under the above-mentioned transform corresponds to the function $\tau_{ij}^1(y)$. Let us draw the straight line parallel to the axis $o\widetilde{x}$ from $\widetilde{Q}_1(\widetilde{x}, \widetilde{\gamma}_1(\widetilde{x})) \in \widetilde{\gamma}_1$ to the intersection with the curve $\widetilde{\gamma}_2$ at $\widetilde{Q}_2$ and the straight line parallel to the axis $o\widetilde{y}$ from $\widetilde{Q}_2$ to the intersection with $\widetilde{\gamma}_1$ at $\widetilde{Q}_3$. The value $\widetilde{\tau}_{ij}^1(\widetilde{x})$ is equal to the abscissa of $\widetilde{Q}_3$, and hence,

$$\widetilde{\tau}_{ij}^1(\widetilde{x}) = \widetilde{\gamma}_2^{-1}\big(\widetilde{\gamma}_1(\widetilde{x})\big), \quad 0 \leq \widetilde{x} \leq \varepsilon.$$

If γ_1 and γ_2 have a common tangent line at $O(0,0)$, then it is evident that $\widetilde{\gamma}_1^{(1)}(0) = \widetilde{\gamma}_2^{(1)}(0)$, otherwise $0 \leq \widetilde{\gamma}_1^{(1)}(0) < \widetilde{\gamma}_2^{(1)}(0)$. Consequently, $\frac{d\widetilde{\tau}_{ij}^1}{d\widetilde{x}}(0) = \frac{\widetilde{\gamma}_1^{(1)}(0)}{\widetilde{\gamma}_2^{(1)}(0)} = 1$, if γ_1 and γ_2 have a common tangent line at $O(0,0)$, and $0 \leq \frac{d\widetilde{\tau}_{ij}^1}{d\widetilde{x}}(0) < 1$, otherwise.

Let us now show that

$$\frac{d\tau_{ij}^1}{dy}(0) = \frac{d\widetilde{\tau}_{ij}^1}{d\widetilde{x}}(0)\ ,$$

which will imply the validity of property 4). As is easily seen, the functions $\tau_{ij}^1(y)$ and $\widetilde{\tau}_{ij}^1(\widetilde{x})$ are connected by the relation

$$\tau_{ij}^1(y) = \chi_2\Big(\widetilde{\tau}_{ij}^1(\mu_j(\gamma_1(y),y)), \widetilde{\gamma}_1\big(\widetilde{\tau}_{ij}^1(\mu_j(\gamma_1(y),y))\big)\Big)$$

for sufficiently small y, where $x = \chi_1(\widetilde{x},\widetilde{y})$, $y = \chi_2(\widetilde{x},\widetilde{y})$ realize the mapping inverse to the given one, $\widetilde{x} = \mu_j(x,y)$, $\widetilde{y} = \mu_i(x,y)$.

Since

$$\widetilde{\gamma}_1^{(1)}(0) = \frac{d\mu_i(\gamma_1(y),y)}{dy}\Big(\frac{d\mu_j(\gamma_1(y),y)}{dy}\Big)^{-1}\Big|_{y=0} =$$

$$= \frac{\frac{\partial\mu_i}{\partial x}(0,0)\gamma_1^{(1)}(0) + \frac{\partial\mu_i}{\partial y}(0,0)}{\frac{\partial\mu_j}{\partial x}(0,0)\gamma_1^{(1)}(0) + \frac{\partial\mu_j}{\partial y}(0,0)}\ ,$$

$$\frac{\partial\chi_2}{\partial\widetilde{x}} = \frac{-\frac{\partial\mu_i}{\partial x}}{\Delta}\ , \quad \frac{\partial\chi_2}{\partial\widetilde{y}} = \frac{\frac{\partial\mu_j}{\partial x}}{\Delta}\ ,$$

$$\Delta = \frac{\partial\mu_j}{\partial x}\frac{\partial\mu_i}{\partial y} - \frac{\partial\mu_i}{\partial x}\frac{\partial\mu_j}{\partial y}\ ,$$

we have

$$\frac{d\tau_{ij}^1}{dy}(0) = \frac{\partial\chi_2}{\partial\widetilde{x}}(0,0)\frac{d\widetilde{\tau}_{ij}^1}{d\widetilde{x}}(0)\Big(\frac{\partial\mu_j}{\partial x}(0,0)\gamma_1^{(1)}(0) + \frac{\partial\mu_j}{\partial y}(0,0)\Big) +$$

$$+ \frac{\partial\chi_2}{\partial\widetilde{y}}(0,0)\widetilde{\gamma}_1^{(1)}(0)\frac{d\widetilde{\tau}_{ij}^1}{d\widetilde{x}}(0)\Big(\frac{\partial\mu_j}{\partial x}(0,0)\gamma_1^{(1)}(0) + \frac{\partial\mu_j}{\partial y}(0,0)\Big) =$$

$$= \frac{d\widetilde{\tau}_{ij}^1}{d\widetilde{x}}(0)\Big(\frac{\partial\mu_j}{\partial x}(0,0)\gamma_1^{(1)}(0) + \frac{\partial\mu_j}{\partial y}(0,0)\Big) \times$$

$$\times \Big(- \frac{\partial\mu_i}{\partial x}(0,0) + \frac{\partial\mu_j}{\partial x}\frac{\frac{\partial\mu_i}{\partial x}(0,0)\gamma_1^{(1)}(0) + \frac{\partial\mu_i}{\partial y}(0,0)}{\frac{\partial\mu_j}{\partial x}(0,0)\gamma_1^{(1)}(0) + \frac{\partial\mu_j}{\partial y}(0,0)}\Big)\Delta^{-1} =$$

$$= \frac{d\widetilde{\tau}_{ij}^1}{d\widetilde{x}}(0).$$

Now we can easily calculate the value

$$\sigma_{ij}^1 = \frac{d\tau_{ij}^1}{dy}(0) = \frac{d\widetilde{\tau}_{ij}^1}{d\widetilde{x}}(0) = \frac{\widetilde{\gamma}_1^{(1)}(0)}{\widetilde{\gamma}_2^{(1)}(0)} =$$

$$= \frac{\frac{\partial\mu_i}{\partial x}(0,0)\gamma_1^{(1)}(0) + \frac{\partial\mu_i}{\partial y}(0,0)}{\frac{\partial\mu_j}{\partial x}(0,0)\gamma_1^{(1)}(0) + \frac{\partial\mu_j}{\partial y}(0,0)}\Big(\frac{\frac{\partial\mu_i}{\partial x}(0,0)\gamma_2^{(1)}(0) + \frac{\partial\mu_i}{\partial y}(0,0)}{\frac{\partial\mu_j}{\partial x}(0,0)\gamma_2^{(1)}(0) + \frac{\partial\mu_j}{\partial y}(0,0)}\Big)^{-1} =$$

$$= \frac{(\gamma_1^{(1)}(0) + \lambda_i(0,0))(\gamma_2^{(1)}(0) + \lambda_j(0,0))}{(\gamma_1^{(1)}(0) + \lambda_j(0,0))(\gamma_2^{(1)}(0) + \lambda_i(0,0))}\ ,$$

since $\nabla\mu_i(0,0) = c_i(1,\lambda_i(0,0))$, $c_i = \text{const}$, $i = 1,\dots,l$. The case of the function τ_{ij}^2 is considered in a similar way.

Remark. It is obvious that when conditions (2.21) are fulfilled, the problem (2.1)–(2.3) in the class $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$ is equivalent to the system of integro-functional equations (2.18), (2.19), (2.22) with respect to unknown functions $u \in \overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$, $w \in \overset{\circ}{C}{}^{k-1}_\alpha(\overline{D}_1)$ and $\varphi^i \in \overset{\circ}{C}{}^{k-1}_\alpha[0,d_i]$, $i = 1,2$.

§

Let us consider functional equations of the type

$$(K_{1p}\varphi)(y)=\varphi(y)-\sum_{i=s_0+1}^{l}\sum_{j=l_0+1}^{s_0}G^3_{ijp}(y)\varphi\big(\tau^1_{ij}(y)\big)=g_1(y),\quad (2.24)$$
$$0\le y\le d_1,\quad p=0,1,\dots,k-1,$$
$$(K_{2p}\psi)(y)=\psi(y)-\sum_{i=l_0+1}^{s_0}\sum_{j=s_0+1}^{l}G^4_{ijp}(y)\psi\big(\tau^2_{ij}(y)\big)=g_2(y),\quad (2.25)$$
$$0\le y\le d_2,\quad p=0,1,\dots,k-1,$$

where

$$G^3_{ijp}(y)=G^3_{ij}(y)\Big(\frac{d\tau^1_{ij}}{dy}(y)\Big)^p,\quad G^4_{ijp}(y)=G^4_{ij}(y)\Big(\frac{d\tau^2_{ij}}{dy}(y)\Big)^p,$$

and the values G^3_{ij}, G^4_{ij}, τ^1_{ij}, τ^2_{ij} are determined in equations (2.22).

Remark. As is easily seen, the expressions $K_{1p}\varphi^1$ and $K_{2p}\varphi^2$ for $p=0$ coincide with the functional parts of equations (2.22). Moreover, if we differentiate p times the expression $(K_{10}\varphi)(y)$ with respect to y, then in the expression obtained after differentiation the sum of those summands which involve the function $\varphi(y)$ with the derivative $\varphi^{(p)}(y)$, yields $(K_{1p}\varphi^{(p)})(y)$. Similar remark holds also for the operator K_{2p}.

We shall consider equations (2.24) and (2.25) in the spaces $\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]$ and $\overset{\circ}{C}_{k-1+\alpha-p}[0,d_2]$.

Denote by $\widetilde{m}_1$ the number of characteristics taking into account their multiplicities, issued from the point P_1 into the domain D_1 and intersecting an open segment OP_2. The number $\widetilde{m}_2$ can be defined in a similar manner by substituting the point P_1 by P_2 and OP_2 by an open segment OP_1. It is easily seen that $\widetilde{m}_1\widetilde{m}_2=0$ if, for example, either $l_0=2n$ or $m_1m_2=0$.

Obviously, the columns $2n\times m_i$ of the matrix V_i, $i=1,2$, are composed of the well-defined columns of the matrix K, where the matrices K, V_1, V_2 have been introduced in §3. Denote by $\widetilde{V}_i$, $i=1,2$, the matrix of order $2n\times(2n-m_i)$, composed of the remaining columns of the matrix K, i.e., of the columns not belonging to the matrix V_i.

We have the following

Let either $\widetilde{m}_1\widetilde{m}_2=0$ or at least one of the equalities $(\Gamma_1\times\widetilde{V}_1)|_{OP_1}=0$ or $(\Gamma_2\times\widetilde{V}_2)|_{OP_2}=0$ hold. Then equations (2.24) and (2.25) are uniquely solvable in the spaces $\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]$ and $\overset{\circ}{C}_{k-1+\alpha-p}[0,d_2]$ for all $k\ge 2$, $\alpha\ge 0$.

The proof follows from the fact that under the conditions of Lemma 2.1 either all values $\tau^1_{ij} \equiv \tau^2_{ij} \equiv 0$ or all matrices $G^3_{ij} \equiv G^4_{ij} \equiv 0$. In both cases the operators K_{1p} and K_{2p} are identical in the spaces $\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]$ and $\overset{\circ}{C}_{k-1+\alpha-p}[0,d_2]$, i.e., $K_{1p}\varphi = \varphi$, $K_{2p}\psi = \psi$.

Consider the functions

$$h_1(\rho) = \sum_{i=s_0+1}^{l} \sum_{j=l_0+1}^{s_0} (\sigma^1_{ij})^{\rho-1} \|G^3_{ij}(0)\|, \quad -\infty < \rho < \infty,$$

$$h_2(\rho) = \sum_{i=l_0+1}^{s_0} \sum_{j=s_0+1}^{l} (\sigma^2_{ij})^{\rho-1} \|G^4_{ij}(0)\|, \quad -\infty < \rho < \infty,$$

where $\|\cdot\|$ is the norm of the matrix operator, acting from one Euclidean space of the other.

Assume that the curves γ_1 and γ_2 do not have a common tangent line at the point $O(0,0)$. If for some values of the indices i, j, $\|\sigma^1_{ij} G^3_{ij}(0)\|$ and $\|\sigma^2_{ij} G^4_{ij}(0)\|$ are different from zero, then by (2.23) the functions h_1 and h_2 are continuous and strictly monotonically decreasing on $(-\infty,\infty)$; moreover, $\lim\limits_{\rho\to-\infty} h_i(\rho) = +\infty$ and $\lim\limits_{\rho\to+\infty} h_i(\rho) = 0$, $i = 1,2$. Therefore there exist unique real numbers ρ_1 and ρ_2 such that $h_1(\rho_1) = 1$ and $h_2(\rho_2) = 1$. If, however, all the values $\|\sigma^1_{ij} G^3_{ij}(0)\| = 0$, then we assume $\rho_1 = -\infty$. Similarly, assume $\rho_2 = -\infty$ if all the values $\|\sigma^2_{ij} G^4_{ij}(0)\| = 0$. It is evident that all these cases are realizable if either $\widetilde{m}_1 \widetilde{m}_2 = 0$ or at least one of the equalities $(\Gamma_i \times \widetilde{V}_i)(0) = 0$, $i = 1,2$, holds.

Assume that the curves γ_1, γ_2 do not have a common tangent line at the point $O(0,0)$, and $\widetilde{m}_1 \widetilde{m}_2 \neq 0$, $(\Gamma_i \times \widetilde{V}_i)|_{OP_i} \neq 0$, $i = 1,2$. Then for $k + \alpha > \rho_0$ the equations (2.24) *and* (2.25) *are uniquely solvable in the spaces $\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]$ and $\overset{\circ}{C}_{k-1+\alpha-p}[0,d_2]$, and the estimates*

$$\begin{aligned}\left\|(K^{-1}_{1p} g_1)(y)\right\|_{R^{m_1}} &= \|\varphi(y)\|_{R^{m_1}} \leq \\ &\leq c_1 y^{k-1+\alpha-p} \|g_1\|_{\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]},\end{aligned} \tag{2.26}$$

$$\begin{aligned}\left\|(K^{-1}_{2p} g_2)(y)\right\|_{R^{m_2}} &= \|\psi(y)\|_{R^{m_2}} \leq \\ &\leq c_2 y^{k-1+\alpha-p} \|g_2\|_{\overset{\circ}{C}_{k-1+\alpha-p}[0,d_2]},\end{aligned} \tag{2.27}$$

hold, where c_1, c_2 are positive constants not depending on g_1, g_2.

Proof. Condition $k + \alpha > \rho_0$ imples

$$h_1(k+\alpha) = \sum_{i=s_0+1}^{l} \sum_{j=l_0+1}^{s_0} (\sigma^1_{ij})^{k-1+\alpha} \|G^3_{ij}(0)\| < 1.$$

Therefore, owing to the continuity of τ^1_{ij}, $\frac{d\tau^1_{ij}}{dy}$, G^3_{ij} and equalities $\frac{d\tau^1_{ij}}{dy}(0) = \sigma^1_{ij} < 1$, there exist positive numbers ε ($\varepsilon < d_1$), δ and β such that for $0 \le y \le \varepsilon$ the inequalities

$$\|G^3_{ij}(y)\| \le \|G^3_{ij}(0)\| + \delta, \tag{2.28}$$

$$\frac{d\tau^1_{ij}}{dy}(y) \le \sigma^1_{ij} + \delta, \quad \|G^3_{ijp}(y)\| =$$

$$= \Big\| \Big(\frac{d\tau^1_{ij}}{dy}(y)\Big)^p G^3_{ij}(y) \Big\| \le (\sigma^1_{ij} + \delta)^p \big(\|G^3_{ij}(0)\| + \delta\big), \tag{2.29}$$

$$\tau^1_{ij}(y) \le (\sigma^1_{ij} + \delta) y, \tag{2.30}$$

$$\sum_{i=s_0+1}^{l} \sum_{j=l_0+1}^{s_0} (\sigma^1_{ij} + \delta)^{k-1+\alpha} \big(\|G^3_{ij}(0)\| + \delta\big) = \beta > 1. \tag{2.31}$$

are valid.

Since the functions τ^1_{ij} possess properties 1)–3) cited in §3, there exists a natural number q_0 such that for $q \ge q_0$

$$\tau^1_{i_q j_q}\big(\tau^1_{i_{q-1} j_{q-1}}(\cdots(\tau^1_{i_1 j_1}(y))\cdots)\big) \le \varepsilon, \quad 0 \le y \le d_1, \tag{2.32}$$

where $s_0 + 1 \le i_s \le l$, $l_0 + 1 \le j_s \le s_0$, $s = 1, \dots, q$.

Because of the property 3), for the functions τ^1_{ij} and the inequalities (2.30) and (2.32) we have

$$\begin{aligned}
&\tau^1_{i_q j_q}\big(\tau^1_{i_{q-1} j_{q-1}}(\cdots(\tau^1_{i_1 j_1}(y))\cdots)\big) = \\
&= \tau^1_{i_q j_q}\big(\tau^1_{i_{q-1} j_{q-1}}(\cdots(\tau^1_{i_{q_0} j_{q_0}}(\tau^1_{i_{q_0-1} j_{q_0-1}}(\cdots(\tau^1_{i_1 j_1}(y))\cdots)))\cdots)\big) \le \\
&\le (\sigma^1_{i_q j_q} + \delta)\tau^1_{i_{q-1} j_{q-1}}\big(\cdots(\tau^1_{i_{q_0} j_{q_0}}(\tau^1_{i_{q_0-1} j_{q_0-1}}(\cdots(\tau^1_{i_1 j_1}(y))\cdots)))\cdots\big) \le \\
&\le \cdots \le (\sigma^1_{i_q j_q} + \delta)(\sigma^1_{i_{q-1} j_{q-1}} + \delta)\cdots(\sigma^1_{i_{q_0+1} j_{q_0+1}} + \delta) \times \\
&\times \tau^1_{i_{q_0} j_{q_0}}\big(\tau^1_{i_{q_0-1} j_{q_0-1}}(\cdots(\tau^1_{i_1 j_1}(y))\cdots)\big) \le \\
&\le \Big[\prod_{s=q_0+1}^{q} (\sigma^1_{i_s j_s} + \delta)\Big] y, \quad 0 \le y \le d_1, \quad q > q_0.
\end{aligned} \tag{2.33}$$

Introduce into the consideration the operators Λ_{1p}, K^{-1}_{1p} acting by the formulas

$$(\Lambda_{1p}\varphi)(y) = \sum_{i=s_0+1}^{l} \sum_{j=l_0+1}^{s_0} G^3_{ijp}(y)\varphi\big(\tau^1_{ij}(y)\big),$$

$$K^{-1}_{1p} = I + \sum_{q=1}^{\infty} \Lambda^q_{1p},$$

where I is the identical operator. Obviously, K_{1p}^{-1} is formally inverse to K_{1p}, i.e., $K_{1p}K_{1p}^{-1} = K_{1p}^{-1}K_{1p} = I$. Therefore it suffices for us to show that K_{1p}^{-1} is continuous in the space $\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]$.

It can be easily seen that the expression $\Lambda_{1p}^q g_1$ is a sum consisting of the summands of the form

$$\begin{gathered}
J_{i_1j_1\cdots i_qj_q}(y) = \\
= G^3_{i_1j_1p}(y)G^3_{i_2j_2p}\big(\tau^1_{i_1j_1}(y)\big)G^3_{i_3j_3p}\big(\tau^1_{i_2j_2}(\tau^1_{i_1j_1}(y))\big)\cdots \\
\cdots G^3_{i_qj_qp}\big(\tau^1_{i_{q-1}j_{q-1}}(\tau^1_{i_{q-2}j_{q-2}}(\cdots(\tau^1_{i_1j_1}(y))\cdots))\big)\times \\
\times g_1\big(\tau^1_{i_qj_q}(\tau^1_{i_{q-1}j_{q-1}}(\cdots(\tau^1_{i_1j_1}(y))\cdots))\big),
\end{gathered}$$

where $s_0+1 \le i_s \le l$, $l_0+1 \le j_s \le s_0$, $s = 1,\dots,q$.

Let

$$\max_{s_0+1\le i\le l}\ \max_{l_0+1\le j\le s_0}\ \max_{0\le y\le d_i} \big\|G^3_{ijp}(y)\big\|_{R^{m_1}} = \eta_p.$$

By virtue of (2.28)–(2.33) we have

$$\begin{gathered}
\big\|J_{i_1j_1\cdots i_qj_q}(y)\big\|_{R^{m_1}} \le \big\|G^3_{i_1j_1p}(y)\big\|\cdots \\
\cdots\big\|G^3_{i_{q_0}j_{q_0}p}\big(\tau^1_{i_{q_0-1}j_{q_0-1}}(\tau^1_{i_{q_0-2}j_{q_0-2}}(\cdots(\tau^1_{i_1j_1}(y))\cdots))\big)\big\|\times \\
\times\big\|G^3_{i_{q_0+1}j_{q_0+1}p}\big(\tau^1_{i_{q_0}j_{q_0}}(\tau^1_{i_{q_0-1}j_{q_0-1}}(\cdots(\tau^1_{i_1j_1}(y))\cdots))\big)\big\|\cdots \\
\cdots\big\|G^3_{i_qj_qp}\big(\tau^1_{i_{q-1}j_{q-1}}(\cdots(\tau^1_{i_1j_1}(y))\cdots))\big)\big\|\times \\
\times\big\|g_1\big(\tau^1_{i_qj_q}(\cdots\tau^1_{i_1j_1}(y))\cdots)\big)\big\|_{R^{m_1}} \le \\
\le \eta_p^{q_0}(\sigma_{i_{q_0+1}j_{q_0+1}}+\delta)^p\big(\big\|G^3_{i_{q_0+1}j_{q_0+1}}(0)\big\|+\delta\big)\cdots \\
\cdots(\sigma^1_{i_qj_q}+\delta)^p\big(\big\|G^3_{i_qj_q}(0)\big\|+\delta\big)\times \\
\times\big|\tau^1_{i_qj_q}\big(\cdots(\tau^1_{i_1j_1}(y))\cdots\big)\big|^{k-1+\alpha-p}\|g_1\|_{\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]} \le \\
\le \eta_p^{q_0}\Big[\prod_{s=q_0+1}^{q}(\sigma^1_{i_sj_s}+\delta)^p\big(\big\|G^3_{i_sj_s}(0)\big\|+\delta\big)\Big]\times \\
\times\Big[\prod_{s=q_0+1}^{q}(\sigma^1_{i_sj_s}+\delta)^{k-1+\alpha-p}\Big]y^{k-1+\alpha-p}\|g_1\|_{\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]} = \\
= \eta_p^{q_0}\Big[\prod_{s=q_0+1}^{q}(\sigma^1_{i_sj_s}+\delta)^{k-1+\alpha}\big(\big\|G^3_{i_sj_s}(0)\big\|+\delta\big)\Big] \\
\times y^{k-1+\alpha-p}\|g_1\|_{\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]}
\end{gathered}\tag{2.34}$$

for $q > q_0$, $g_1 \in \overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]$, and

$$\begin{gathered}
\big\|J_{i_1j_1\cdots i_qj_q}(y)\big\|_{R^{m_1}} \le \\
\le \eta_p^{q_0}\big|\tau^1_{i_qj_q}\big(\cdots(\tau^1_{i_1j_1}(y))\cdots\big)\big|^{k-1+\alpha-p}\|g_1\|_{\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]} \le
\end{gathered}$$

$$\leq \eta_p^q y^{k-1+\alpha-p} \|g_1\|_{\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]} \tag{2.35}$$

for $0 < q \leq q_0$.

Because of (2.34), (2.35) and (2.31) we have

$$\big\|(\Lambda_{1p}^q g_1)(y)\big\|_{R^{m_1}} = \Big\| \sum_{i_1,j_1,\dots,i_q,j_q} J_{i_1 j_1 \cdots i_q j_q}(y) \Big\|_{R^{m_1}} \leq$$

$$\leq \Big(\sum_{i_1,j_1,\dots,i_{q_0},j_{q_0}} 1 \Big)^{q_0} \eta_p^{q_0} \Big[\sum_{i=s_0+1}^{l} \sum_{j=l_0+1}^{s_0} (\sigma_{ij}^1 + \delta)^{k-1+\alpha} \times$$

$$\times \big(\|G_{ij}^3(0)\| + \delta \big) \Big]^{q-q_0} y^{k-1+\alpha-p} \|g_1\|_{\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]} =$$

$$= c_3 \beta^q y^{k-1+\alpha-p} \|g_1\|_{\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]} \tag{2.36}$$

for $q > q_0$, and

$$\big\|(\Lambda_{1p}^q g_1)(y)\big\|_{R^{m_1}} \leq c_4 y^{k-1+\alpha-p} \|g_1\|_{\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]} \tag{2.37}$$

for $0 < q \leq q_0$, where

$$c_3 = \eta_p^{q_0} \beta^{-q_0} \Big(\sum_{i_1,j_1,\dots,i_{q_0},j_{q_0}} 1 \Big)^{q_0}, \quad c_4 = \eta_p^q \Big(\sum_{i_1,j_1,\dots,i_q,j_q} 1 \Big).$$

From (2.36) and (2.37) we finally obtain

$$\big\|(K_{1p}^{-1} g_1)(y)\big\|_{R^{m_1}} = \|\varphi(y)\|_{R^{m_1}} \leq$$

$$\leq \big\|g_1(y)\big\|_{R^{m_1}} + \sum_{q=1}^{q_0} \big\|(\Lambda_{1p}^q g_1)(y)\big\|_{R^{m_1}} + \sum_{q=q_0+1}^{\infty} \big\|(\Lambda_{1p}^q g_1)(y)\big\|_{R^{m_1}} \leq$$

$$\leq \Big(1 + c_4 q_0 + c_3 \frac{\beta^{q_0+1}}{1-\beta} \Big) y^{k-1+\alpha-p} \|g_1\|_{\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]},$$

whence it follows that the operator K_{1p}^{-1} is continuous in the space $\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]$, and the estimate (2.26) is valid. The operator K_{2p}^{-1} is considered in a similar manner. ■

Let the curves γ_1, γ_2 have a common tangent line at the point $O(0,0)$, and $\widetilde{m}_1 \widetilde{m}_2 \neq 0$, $(\Gamma_i \times \widetilde{V}_i)|_{OP_i} \not\equiv 0$, $i = 1,2$. Then for $h_i(1) < 1$, $i = 1,2$, the equations (2.24) *and* (2.25) *are uniquely solvable in the spaces $\overset{\circ}{C}_{k-1+\alpha-p}[0,d_1]$ and $\overset{\circ}{C}_{k-1+\alpha-p}[0,d_2]$ for all $k \geq 2$, $\alpha \geq 0$, and the estimates* (2.26) *and* (2.27) *take place.*

The proof of Lemma 2.3 does not differ from that of Lemma 2.2 if in inequalities (2.28)–(2.31) we substitute the different from zero numbers σ_{ij}^1 by unity.

It easily follows from Lemmas 2.1–2.3 that if either $\widetilde{m}_1\widetilde{m}_2 = 0$ or at least one of the equalities $(\Gamma_1 \times \widetilde{V}_1)(O) = 0$ or $(\Gamma_2 \times \widetilde{V}_2)(O) = 0$ holds, then the assertion of Lemma 2.1 is valid for all $k \geq 2$, $\alpha \geq 0$.

Let the conditions (2.21) *be fulfilled. If either* $\widetilde{m}_1\widetilde{m}_2 = 0$ *or at least one of the equalities* $(\Gamma_1 \times \widetilde{V}_1)(O) = 0$ *or* $(\Gamma_2 \times \widetilde{V}_2)(O) = 0$ *holds, then the problem* (2.1)–(2.3) *is uniquely solvable in the class* $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$ *for all* $k \geq 2$, $\alpha \geq 0$.

Let the conditions (2.21) *be fulfilled, and* $\widetilde{m}_1\widetilde{m}_2 \neq 0$, $(\Gamma_i \times \widetilde{V}_i)(O) \neq 0$, $i = 1, 2$. *If the curves* γ_1, γ_2 *do not have a common tangent line at the point* $O(0,0)$, *then for* $k + \alpha > \rho_0$ *the problem* (2.1)–(2.3) *is uniquely solvable in the class* $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$.

Let the conditions (2.21) *be fulfilled, and* $\widetilde{m}_1\widetilde{m}_2 \neq 0$, $(\Gamma_i \times \widetilde{V}_i)(O) \neq 0$, $i = 1, 2$. *If the curves* γ_1, γ_2 *have a common tangent line at the point* $O(0,0)$, *then for* $h_i(1) < 1$, $i = 1, 2$, *the problem* (2.1)–(2.3) *is uniquely solvable in the class* $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$ *for all* $k \geq 2$, $\alpha \geq 0$.

Before passing to the proof of Theorems 2.1–2.3, let us make some remarks.

1. Since the $2n \times m_i$-matrix V_i, $i = 1, 2$, has a maximal rank equal to m_i, for any normally hyperbolic system (2.1) one can always indicate boundary conditions (2.2), (2.3) such that the conditions (2.21) are fulfilled when conditions cited in §2 hold.
2. The values ρ_0 and $h_i(1)$, $i = 1, 2$, in Theorems 2.2 and 2.3 depend only on the coefficients A, B, C, M_i, N_i, S_i, $i = 1, 2$, of the problem (2.1)–(2.3) and the direction of the tangents to γ_1 and γ_2 at the point $O(0,0)$.
3. When conditions of Theorems 2.1–2.3 are violated, as it has been shown in Chapter I for one equation of hyperbolic type, the problem (2.1)–(2.3) may turn out to be ill-posed. In particular, the homogeneous problem corresponding to (2.1)–(2.3) may have an infinite number of linearly independent solutions.

Proof of Theorems 2.1–2.3. Using the method of successive approximations we solve the system of equations (2.18), (2.19) and (2.2) with respect to unknown functions $u \in \overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$, $w \in \overset{\circ}{C}{}^{k-1}_\alpha(\overline{D}_1)$ and $\varphi^i \in \overset{\circ}{C}{}^{k-1}_\alpha[0, d_i]$, $i = 1, 2$.

Assume

$$u_0(x,y) \equiv 0, \quad w_0(x,y) \equiv 0, \quad \varphi^i_0(y) \equiv 0, \ \ i = 1, 2,$$

$$w_{q_i+j,m}(x,y) = \widetilde{\varphi}^{i(P)}_{q_i+j,m}(\omega_i(x,y)) + \int\limits_{\omega_i(x,y)}^{y} \Big(\sum_{p=1}^{2n} a^1_{ijp} w_{p,m-1} +$$

$$+\sum_{p=1}^{n} b^1_{ijp} u_{p,m-1}\Big)\big(z_i(x,y;t),t\big)dt + F^2_{ij}(x,y), \tag{2.38}$$

$$1 \le i \le l, \quad 1 \le j \le k_i,$$

$$u_m(x,y) = \int\limits_0^{\omega_1(x,y)} \Big(\frac{d\gamma_{1(P)}}{dy}K_1 + K_2\Big) w_{m-1}\big(\gamma_{1(P)}(t),t\big)dt +$$

$$+ \int\limits_{\omega_1(x,y)}^{y} \big((-\lambda_1 K_1 + K_2)w_{m-1}\big)\big(z_1(x,y;t),t\big)dt, \tag{2.39}$$

where

$$\widetilde{\varphi}^{i(P)}_{q_i+j,m}(\omega_i(x,y)) = \begin{cases} \varphi^{i(P)}_{q_i+j,m}(\omega_i(x,y)), & 1 \le i \le l_0, \ \ 1 \le j \le k_i, \\ \varphi^{1}_{q_i+j,m}(\omega_i(x,y)), & l_0+1 \le i \le s_0, \ \ 1 \le j \le k_i, \\ \varphi^{2}_{q_i+j-k_0,m}(\omega_i(x,y)), & s_0+1 \le i \le l, \ \ 1 \le j \le k_i, \\ & \qquad k_0 = \sum\limits_{i=l_0+1}^{s_0} k_i, \end{cases}$$

The values $\varphi^1_m(y)$ and $\varphi^2_m(y)$ are determined from the equations

$$(K_{10}\varphi^1_m)(y) + (T_5 w_{m-1})(y) + (T_6 u_{m-1})(y) = f_5(y) \tag{2.40}$$

and

$$(K_{20}\varphi^2_m)(y) + (T_7 w_{m-1})(y) + (T_8 u_{m-1})(y) = f_6(y). \tag{2.41}$$

The operators K_{10} and K_{20} here act by formulas (2.24), (2.25) for $p = 0$.

We rewrite the system of equations (2.18), (2.19) in a more convenient form

$$w_m(x,y) = \widetilde{\varphi}_{P,m}(x,y) +$$

$$+\sum_{i=1}^{l} \int\limits_{\omega_i(x,y)}^{y} \big(\Omega_{1i} w_{m-1} + \Omega_{2i} u_{m-1}\big)\big(z_i(x,y;t),t\big)dt + F^2(x,y), \tag{2.42}$$

$$u_m(x,y) = \int\limits_0^{\omega_1(x,y)} \widetilde{\Omega}_3 w_{m-1}\big(\gamma_{1(P)}(t),t\big)dt +$$

$$+ \int\limits_{\omega_1(x,y)}^{y} \widetilde{\Omega}_4 w_{m-1}\big(z_1(x,y;t),t\big)dt, \tag{2.43}$$

where the (q_i+j)-th component of the vector $\widetilde{\varphi}_{P,m}(x,y)$ is equal to $\widetilde{\varphi}^{i(P)}_{q_i+j,m}(\omega_i(x,y))$, $1 \le i \le l$, $1 \le j \le k_i$, and Ω_{1i}, Ω_{2i}, $\widetilde{\Omega}_3$, $\widetilde{\Omega}_4$ are well-defined matrices.

It is easily seen that the operators $T_5 w_{m-1} + T_6 u_{m-1}$ and $T_7 w_{m-1} + T_8 u_{m-1}$ from (2.40) and (2.41) can be represented in the form

$$T_9(w_{m-1}, u_{m-1})(y) = (T_5 w_{m-1} + T_6 u_{m-1})(y) =$$
$$= \sum_{i=s_0+1}^{l} \int\limits_{\tau_i^1(y)}^{y} \big(E_{1i}^1 w_{m-1} + E_{2i}^1 u_{m-1}\big)\big(z_i(\gamma_1(y), y; t), t\big)\,dt +$$
$$+ \sum_{j=s_0+1}^{l} \sum_{i=l_0+1}^{s_0} \int\limits_{\tau_{ji}^1(y)}^{\tau_j^1(y)} \big(E_{3ij}^1 w_{m-1} + E_{4ij}^1 u_{m-1}\big)\big(z_i(\gamma_2(\tau_j^1(y)), \tau_j^1(y); t), t\big)\,dt,$$
$$T_{10}(w_{m-1}, u_{m-1})(y) = (T_7 w_{m-1} + T_8 u_{m-1})(y) =$$
$$= \sum_{i=l_0+1}^{s_0} \int\limits_{\tau_i^2(y)}^{y} \big(E_{1i}^2 w_{m-1} + E_{2i}^2 u_{m-1}\big)\big(z_i(\gamma_2(y), y; t), t\big)\,dt +$$
$$+ \sum_{j=l_0+1}^{s_0} \sum_{i=s_0+1}^{l} \int\limits_{\tau_{ji}^2(y)}^{\tau_j^2(y)} \big(E_{3ij}^2 w_{m-1} + E_{4ij}^2 u_{m-1}\big)\big(z_i(\gamma_1(\tau_j^2(y)), \tau_j^2(y); t), t\big)\,dt,$$

where E_{1i}^p,E_{2i}^p, E_{3ij}^p, E_{4ij}^p, $p = 1, 2$, are well-defined matrices.

The following estimates hold:

$$\big\|u_{m+1}(x, y) - u_m(x, y)\big\| \le M^* \frac{M_*^m}{m!} y^{m+k+\alpha-1}, \tag{2.44}$$
$$\big\|w_{m+1}(x, y) - w_m(x, y)\big\| \le M^* \frac{M_*^m}{m!} y^{m+k+\alpha-1}, \tag{2.45}$$
$$\big\|\varphi_{m+1}^1(y) - \varphi_m^1(y)\big\| \le M^* \frac{M_*^m}{m!} y^{m+k+\alpha-1}, \tag{2.46}$$
$$\big\|\varphi_{m+1}^2(y) - \varphi_m^2(y)\big\| \le M^* \frac{M_*^m}{m!} y^{m+k+\alpha-1}, \tag{2.47}$$

where M_* and M^* are sufficiently large positive numbers not depending on m.

Due to the requirements imposed on f_1, f_2 and F, we have $f_5 \in \overset{\circ}{C}{}_\alpha^{k-1}[0, d_1]$, $f_6 \in \overset{\circ}{C}{}_\alpha^{k-1}[0, d_2]$, $F \in \overset{\circ}{C}{}_\alpha^{k-1}(\overline{D}_1)$. Therefore, it is obvious that the estimates

$$\big\|\partial^{i,j} F^2(x, y)\big\| \le \Theta_1 y^{k-1+\alpha-(i+j)}, \tag{2.48}$$
$$(x, y) \in \overline{D}_1, \quad 0 \le i + j \le k - 1,$$
$$\big\|\partial^i f_{4+j}(y)\big\| \le \Theta_{1+j} y^{k-1+\alpha-i}, \tag{2.49}$$
$$0 \le y \le d_j, \ \ j = 1, 2, \quad 0 \le i \le k - 1,$$
$$\Theta_i = \text{const} > 0, \quad i = 1, 2, 3,$$

are valid since, by the assumption, D_1 is such that for any point $z = x + \sqrt{-1}y \in \overline{D}_1$ the two-sided estimate $y \le |z| = \sqrt{x^2+y^2} \le (\max\limits_{i=1,2} \max\limits_{0 \le y \le d_i} |\gamma_i^{(1)}(y)|)y$ is valid.

Since $u_0 \equiv w_0 \equiv 0$, $\varphi_0^1 \equiv \varphi_0^2 \equiv 0$ and under the conditions of Theorems 2.1–2.3 the estimates (2.26), (2.27) are valid for $p = 0$, we have from (2.40), (2.41) and (2.49) that

$$\big\|\varphi_1^i(y) - \varphi_0^i(y)\big\| = \big\|\varphi_1^i(y)\big\| \le c_3\Theta_4 y^{k-1+\alpha}, \ \ i = 1,2 \qquad (2.50)$$
$$c_3 = \max(c_1, c_2), \quad \Theta_4 = \max(\Theta_2, \Theta_3).$$

In its turn, it follows from (2.50) that

$$\begin{aligned} &\big\|\widetilde{\varphi}_{P,1}(x,y) - \widetilde{\varphi}_{P,0}(x,y)\big\| = \big\|\widetilde{\varphi}_{P,1}(x,y)\big\| = \\ &= \sum_{1\le i\le l}\sum_{1\le j\le k_i} \big|\widetilde{\varphi}^{i(P)}_{q_i+j,1}(\omega_i(x,y))\big| \le \\ &\le \sum_{1\le i\le l}\sum_{1\le j\le k_i} c_3\Theta_4(\omega_i(x,y))^{k-1+\alpha} \le 2nc_3\Theta_4 y^{k-1+\alpha}, \end{aligned} \qquad (2.51)$$

since $\sum\limits_{1\le i\le l}\sum\limits_{1\le j\le k_i} 1 = 2n$, and as noted in §3, $0 \le \omega_i(x,y) \le y$, $i = 1,\dots,l$.

Now, by virtue of (2.48) and (2.51), from (2.42) and (2.43) we have

$$\begin{aligned} &\big\|w_1(x,y) - w_0(x,y)\big\| = \big\|w_1(x,y)\big\| \le \\ &\le \big\|\widetilde{\varphi}_{P,1}(x,y)\big\| + \big\|F^2(x,y)\big\| \le \\ &\le 2nc_3\Theta_4 y^{k-1+\alpha} + \Theta_1 y^{k-1+\alpha} = (2nc_3\Theta_4 + \Theta_1)y^{k-1+\alpha}, \end{aligned} \qquad (2.52)$$

$$\big\|u_1(x,y) - u_0(x,y)\big\| = \big\|u_1(x,y)\big\| = 0. \qquad (2.53)$$

Under the assumption that the estimates (2.44)–(2.47) are valid for m, $m > 0$, let us prove their validity for $m+1$ for sufficiently large M_* and M^*.

From (2.40) we have

$$\big(K_{10}(\varphi^1_{m+2} - \varphi^1_{m+1})\big)(y) = -T_9(w_{m+1} - w_m, u_{m+1} - u_m)(y). \quad (2.54)$$

Furthermore, for the right-hand side of equation (2.54) the estimate

$$\begin{aligned} &\big\|T_9(w_{m+1} - w_m, u_{m+1} - u_m)(y)\big\| \le \\ &\le \sum_{i=s_0+1}^{l} \int\limits_{\tau_i^1(y)}^{y} \big(\|E^1_{1i}\|\|w_{m+1} - w_m\| + \\ &+\|E^1_{2i}\|\|u_{m+1} - u_m\|\big)\big(z_i(\gamma_1(y), y; t), t\big)\,dt + \end{aligned}$$

$$+\sum_{j=s_0+1}^{l}\sum_{i=l_0+1}^{s_0}\int\limits_{\tau_{ij}^1(y)}^{\tau_j^1(y)}\big(\|E_{3ij}^1\|\|w_{m+1}-w_m\|+$$

$$+\|E_{4ij}^1\|\|u_{m+1}-u_m\|\big)\big(z_i(\gamma_2(\tau_j^1(y)),\tau_j^1(y);t),t\big)dt. \qquad (2.55)$$

is valid.

The largest of the numbers $\max\limits_{y,t}\|E_{1j}^p(y,t)\|$, $\max\limits_{y,t}\|E_{2j}^p(y,t)\|$, $\max\limits_{y,t}\|E_{3ij}^p(y,t)\|$, $\max\limits_{y,t}\|E_{4ij}^p(y,t)\|$, we denote by ξ_p, $p=1,2$. Since $0\le\tau_{ji}^1(y)\le\tau_j^1(y)\le y$ and owing to (2.44) and (2.45), we have from (2.55) that

$$\big\|T_9(w_{m+1}-w_m,u_{m+1}-u_m)(y)\big\|\le$$

$$\le\xi_1M^*\frac{M_*^m}{m!}\Big(\sum_{i=s_0+1}^{l}\int\limits_{\tau_i^1(y)}^{y}2t^{m+k+\alpha-1}dt+$$

$$+\sum_{j=s_0+1}^{l}\sum_{i=l_0+1}^{s_0}\int\limits_{\tau_{ij}^1(y)}^{\tau_j^1(y)}2t^{m+k+\alpha-1}dt\Big)\le$$

$$\le2\xi_1M^*\frac{M_*^m}{m!}\Big(\sum_{i=s_0+1}^{l}1+\sum_{j=s_0+1}^{l}\sum_{i=l_0+1}^{s_0}1\Big)\int\limits_0^y t^{m+k+\alpha-1}dt\le$$

$$\le2\xi_1M^*\frac{M_*^m}{m!}(l+l^2)\frac{1}{m+k+\alpha}y^{m+k+\alpha}\le$$

$$\le2(l+l^2)\xi_1M^*\frac{M_*^m}{(m+1)!}y^{m+1+k+\alpha-1}. \qquad (2.56)$$

Now (2.54), (2.56) and (2.26) imply

$$\big\|\varphi_{m+2}^1(y)-\varphi_{m+1}^1(y)\big\|\le$$

$$\le2(l+l^2)c_1\xi_1M^*\frac{M_*^m}{(m+1)!}y^{m+1+k+\alpha-1}. \qquad (2.57)$$

for $p=0$.

Similarly, from (2.41), (2.44), (2.45) and (2.27) we find

$$\big\|\varphi_{m+2}^2(y)-\varphi_{m+1}^2(y)\big\|\le$$

$$\le2(l+l^2)c_2\xi_2M^*\frac{M_*^m}{(m+1)!}y^{m+1+k+\alpha-1}. \qquad (2.58)$$

Proceeding similarly as in deducing the estimate (2.51), we obtain

$$\big\|\widetilde{\varphi}_{P,m+2}(x,y)-\widetilde{\varphi}_{P,m+1}(x,y)\big\|\le\xi_3M^*\frac{M_*^m}{(m+1)!}y^{m+1+k+\alpha-1}, \quad (2.59)$$

where $\xi_3 = 4n(l+l^2)c_3\widetilde{\xi}_2$, $\widetilde{\xi}_2 = \max(\xi_1, \xi_2)$.

Denote by η the largest of the numbers $\max\limits_{\overline{D}_1}\|\Omega_{1i}\|$, $\max\limits_{\overline{D}_1}\|\Omega_{2i}\|$, $\max\limits_{\overline{D}_1}\|\widetilde{\Omega}_3\|$, $\max\limits_{\overline{D}_1}\|\widetilde{\Omega}_4\|$, where the matrices Ω_{1i}, Ω_{2i}, $i = 1,\dots,l$, $\widetilde{\Omega}_3$, $\widetilde{\Omega}_4$ are determined in (2.42), (2.43). By virtue of (2.59) we have from (2.42) and (2.43)

$$\big\|w_{m+2}(x,y) - w_{m+1}(x,y)\big\| \le \big\|\widetilde{\varphi}_{P,m+2}(x,y) - \widetilde{\varphi}_{P,m+1}(x,y)\big\| +$$

$$+\sum_{i=1}^{l}\int\limits_{\omega_i(x,y)}^{y}\big(\|\Omega_{1i}\|\|w_{m+1} - w_m\| + \|\Omega_{2i}\|\|u_{m+1} - u_m\|\big)\big(z_i(x,y;t),t\big)\,dt \le$$

$$\le \xi_3 M^*\frac{M_*^m}{(m+1)!}y^{m+1+k+\alpha-1} + 2l\eta\int\limits_0^y M^*\frac{M_*^m}{m!}t^{m+k+\alpha-1}\,dt \le$$

$$\le (\xi_3 + 2l\eta)M^*\frac{M_*^m}{(m+1)!}y^{m+1+k+\alpha-1}, \tag{2.60}$$

$$\big\|u_{m+2}(x,y) - u_{m+1}(x,y)\big\| \le 2\eta M^*\frac{M_*^m}{(m+1)!}y^{m+1+k+\alpha-1}, \tag{2.61}$$

since $0 \le \omega_i(x,y) \le y$, $i = 1,\dots,l$.

It immediately follows from (2.50), (2.52), (2.53), (2.57), (2.58), (2.60) and (2.61) that if we put

$$M^* = 2nc_3\Theta_4 + \Theta_1, \quad M_* = \max\big(2(l+l^2)c_1\xi_1,\, 2(l+l^2)c_2\xi_2,\, \xi_3 + 2l\eta\big),$$

the estimates (2.44)–(2.47) will be valid for any integer $m \ge 0$.

Differentiating the equalities (2.40)–(2.47) with respect to x and y and using the obtained estimates (2.44)–(2.47) as well as the solvability of equations (2.24) and (2.25) and the estimates (2.26) and (2.27) for $p = 1$, we analogously obtain

$$\Big\|\frac{\partial}{\partial x}(u_{m+1} - u_m)(x,y)\Big\| \le M_1^*\frac{M_{*1}^{m-1}}{(m-1)!}y^{m+k+\alpha-2},$$

$$\Big\|\frac{\partial}{\partial y}(u_{m+1} - u_m)(x,y)\Big\| \le M_1^*\frac{M_{*1}^{m-1}}{(m-1)!}y^{m+k+\alpha-2},$$

$$\Big\|\frac{\partial}{\partial x}(w_{m+1} - w_m)(x,y)\Big\| \le M_1^*\frac{M_{*1}^{m-1}}{(m-1)!}y^{m+k+\alpha-2},$$

$$\Big\|\frac{\partial}{\partial y}(w_{m+1} - w_m)(x,y)\Big\| \le M_1^*\frac{M_{*1}^{m-1}}{(m-1)!}y^{m+k+\alpha-2},$$

$$\Big\|\frac{\partial}{\partial y}(\varphi_{m+1}^1 - \varphi_m^1)(y)\Big\| \le M_1^*\frac{M_{*1}^{m-1}}{(m-1)!}y^{m+k+\alpha-2},$$

$$\Big\|\frac{\partial}{\partial x}(\varphi_{m+1}^2 - \varphi_m^2)(y)\Big\| \le M_1^*\frac{M_{*1}^{m-1}}{(m-1)!}y^{m+k+\alpha-2}.$$

Continuing this process, we find that for $m \geq i+j$, $0 \leq i+j \leq k-1$

$$\left.\begin{aligned} &\left\|\partial^{i,j}(u_{m+1}-u_m)(x,y)\right\| \leq M^*_{i+j}\frac{M^{m-i-j}_{*i+j}}{(m-i-j)!}y^{m+k+\alpha-i-j-1},\\ &\left\|\partial^{i,j}(w_{m+1}-w_m)(x,y)\right\| \leq M^*_{i+j}\frac{M^{m-i-j}_{*i+j}}{(m-i-j)!}y^{m+k+\alpha-i-j-1},\\ &\left\|\tfrac{\partial^{i+j}}{\partial y^{i+j}}(\varphi^p_{m+1}-\varphi^p_m)(y)\right\| \leq M^*_{i+j}\frac{M^{m-i-j}_{*i+j}}{(m-i-j)!}y^{m+k+\alpha-i-j-1},\\ &\qquad\qquad p=1,2, \end{aligned}\right\} \tag{2.62}$$

where M^*_i, M_{*i}, $i=1,\dots,k-1$, are sufficiently large positive numbers not depending on m.

It follows from (2.62) that the series

$$u(x,y)=\lim_{m\to\infty}u_m(x,y)=\sum_{m=1}^{\infty}\big(u_m(x,y)-u_{m-1}(x,y)\big),$$

$$w(x,y)=\lim_{m\to\infty}w_m(x,y)=\sum_{m=1}^{\infty}\big(w_m(x,y)-w_{m-1}(x,y)\big),$$

$$\varphi^p(y)=\lim_{m\to\infty}\varphi^p_m(y)=\sum_{m=1}^{\infty}\big(\varphi^p_m(y)-\varphi^p_{m-1}(y)\big),\ \ p=1,2,$$

converge in the spaces $\overset{\circ}{C}{}^{k-1}_{\alpha}(\overline{D}_1)$, $\overset{\circ}{C}{}^{k-1}_{\alpha}[0,d_p]$, $p=1,2$, and on account of (2.40)–(2.43) the limit functions u, w, φ^1, φ^2 satisfy the system of equations (2.18), (2.19), (2.22). Hence it follows that $u_x=K_1w$, $u_y=K_2w$, where $K=\binom{K_1}{K_2}$ is the $2n\times 2n$-matrix from (2.4). Consequently, $u_x,u_y\in \overset{\circ}{C}{}^{k-1}_{\alpha}(\overline{D}_1)$ since $w\in \overset{\circ}{C}{}^{k-1}_{\alpha}(\overline{D}_1)$, $K\in C^k(\overline{D}_1)$, and therefore $u\in \overset{\circ}{C}{}^{k}_{\alpha}(\overline{D}_1)$. Thus we have shown that the obtained function $u(x,y)$ is a solution of the problem (2.1)–(2.3) in the class $\overset{\circ}{C}{}^{k}_{\alpha}(\overline{D}_1)$.

Let us now show that under the conditions of Theorem 2.1–2.3 the problem (2.1)–(2.3) has no other solution in the class $\overset{\circ}{C}{}^{k}_{\alpha}(\overline{D}_1)$. Indeed, if $u(x,y)\in \overset{\circ}{C}{}^{k}_{\alpha}(\overline{D}_1)$ is a solution of the homogeneous problem corresponding to (2.1)–(2.3), then the corresponding functions u, w, φ^1, φ^2 satisfy the homogeneous system of equations

$$
\left.\begin{aligned}
&w_{q_i+j}(x,y)=\widetilde{\varphi}^{\,i(P)}_{q_i+j}(\omega_i(x,y))+\\
&+\int\limits_{\omega_i(x,y)}^{y}\Big(\sum_{p=1}^{2n}a^1_{ijp}w_p+\sum_{p=1}^{n}b^1_{ijp}u_p\Big)\big(z_i(x,y;t),t\big)\,dt,\\
&\qquad 1\le i\le l,\quad 1\le j\le k_i,\\
&u(x,y)=\int\limits_{0}^{\omega_1(x,y)}\Big(\frac{d\gamma_{1(P)}}{dy}K_1+K_2\Big)w\big(\gamma_{1(P)}(t),t\big)\,dt+\\
&+\int\limits_{\omega_1(x,y)}^{y}\big((-\lambda_1K_1+K_2)w\big)\big(z_1(x,y;t),t\big)\,dt,\\
&(K_{10}\varphi^1)(y)+(T_5w)(y)+(T_6u)(y)=0,\\
&(K_{20}\varphi^2)(y)+(T_7w)(y)+(T_8u)(y)=0.
\end{aligned}\right\}\qquad(2.63)
$$

To the system of equations (2.63), let us apply the method of successive approximations taking u, w, φ^1, φ^2 as zero approximations. Since these functions satisfy the system of equations (2.63), every next approximation will coincide with it, that is,

$$
u_m(x,y)\equiv u(x,y),\quad w_m(x,y)\equiv w(x,y),
$$
$$
\varphi^p_m(y)\equiv\varphi^p(y),\quad p=1,2.
$$

Taking into consideration that these functions satisfy the estimates of the type (2.48), (2.49), and arguing as in deducing the estimates (2.44)–(2.47), we obtain

$$
\begin{aligned}
&\|u(x,y)\|=\big\|u_{m+1}(x,y)\big\|\le\widetilde{M}^*\frac{\widetilde{M}_*^m}{m!}y^{m+k+\alpha-1},\\
&\|w(x,y)\|=\big\|w_{m+1}(x,y)\big\|\le\widetilde{M}^*\frac{\widetilde{M}_*^m}{m!}y^{m+k+\alpha-1},\\
&\|\varphi^1(y)\|=\big\|\varphi^1_{m+1}(y)\big\|\le\widetilde{M}^*\frac{\widetilde{M}_*^m}{m!}y^{m+k+\alpha-1},\\
&\|\varphi^2(y)\|=\big\|\varphi^2_{m+1}(y)\big\|\le\widetilde{M}^*\frac{\widetilde{M}_*^m}{m!}y^{m+k+\alpha-1},
\end{aligned}
$$

whence in the limit as $m\to\infty$, we find that

$$
u\equiv w\equiv\varphi^1\equiv\varphi^2\equiv 0. \ \blacksquare
$$

The particular case of the boundary value problem (2.1)–(2.3) is the problem of Goursat type, when the boundary conditions (2.2), (2.3) have the form

$$
u\big|_{OP_1}=f_1, \qquad (2.64)
$$

$$u\big|_{OP_2} = f_2. \tag{2.65}$$

Differentiating the equalities (2.64) and (2.65) with respect to the tangent to the curves γ_1 and γ_2, we have

$$\Big(\frac{d\gamma_1}{dy}u_x + u_y\Big)\Big|_{OP_1} = f_1^{(1)}, \tag{2.66}$$

$$\Big(\frac{d\gamma_2}{dy}u_x + u_y\Big)\Big|_{OP_2} = f_2^{(1)}. \tag{2.67}$$

Below we shall assume that all the requirements imposed on the curves γ_1,γ_2 and the characteristics of the system (2.1) quoted in §2, are fulfilled; moreover, the number $l_0 = 0$ and the points P_1 and P_2 are located on the curves γ_1 and γ_2 such that $m_1 = m_2 = n$.

It is easily seen that in the class $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$, $k \geq 2$, $\alpha \geq 0$, the problem (2.1), (2.64), (2.65) is equivalent to the problem (2.1), (2.66), (2.67).

Since the matrix coefficients for the problem (2.1), (2.66), (2.67) have in the boundary conditions (2.66), (2.67) the form

$$M_i = \frac{d\gamma_i}{dy}E, \quad N_i = E, \quad S_i = 0, \ \ i = 1, 2,$$

where E is the unit $n \times n$-matrix, it is obvious that the conditions (2.21) are equivalent to the following ones

$$\operatorname{rank}\big\{\nu_{ij},\ 1 \leq i \leq s_0,\ 1 \leq j \leq k_i\big\}\big|_{OP_1} = n, \tag{2.68}$$

$$\operatorname{rank}\big\{\nu_{ij},\ s_0 < i \leq l,\ 1 \leq j \leq k_i\big\}\big|_{OP_2} = n. \tag{2.69}$$

In this case the equalities $\widetilde{U}_1 = U_2$ and $\widetilde{U}_2 = U_1$ are valid, the condition $(\Gamma_i \times \widetilde{U}_i)(O) = 0$ being fulfilled if and only if $\gamma_i = L_{i_0}(O)$, $1 \leq i_0 \leq l$, and $k_{i_0} = n$.

From Theorems 2.1–2.3 we have the following assertions:

1. Let the conditions (2.68), (2.69) be fulfilled. If either $\widetilde{m}_1\widetilde{m}_2 = 0$ or at least one of the equalities $(\Gamma_1 \times U_2)(O) = 0$ or $(\Gamma_2 \times U_1)(O) = 0$ holds, then the problem (2.1), (2.64), (2.65) is uniquely solvable in the class $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$ for all $k \geq 2$, $\alpha \geq 0$.

2. Let the conditions (2.68), (2.69) be fulfilled, and $\widetilde{m}_1\widetilde{m}_2 \neq 0$, $(\Gamma_1 \times U_2)(O) \neq 0$, $(\Gamma_2 \times U_1)(O) \neq 0$. If the curves γ_1, γ_2 do not have a common tangent line at the point $O(0,0)$, then for $k + \alpha > \rho_0$ the problem (2.1), (2.64), (2.65) is uniquely solvable in the class $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$.

3. Let the conditions (2.68), (2.69) be fulfilled, and $\widetilde{m}_1\widetilde{m}_2 \neq 0$, $(\Gamma_1 \times U_2)(O) \neq 0$, $(\Gamma_2 \times U_1)(O) \neq 0$. If the curves γ_1, γ_2 do not have a common tangent line at the point $O(0,0)$, then for $h_i(1) < 1$, $i = 1, 2$, the problem (2.1), (2.64), (2.65) is uniquely solvable in the class $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$ for all $k \geq 2$, $\alpha \geq 0$.

Remark. Let D_P, $P \in \overline{D}_1$, be the domain constructed in §2 of the present chapter, and let $\gamma_{iP} = \gamma_i \cap \partial D_P$, $i = 1, 2$. As is seen from the proofs of Theorems 2.1–2.3, when conditions of these theorems are fulfilled, the domain of dependence of the solution $u(x, y)$ of the problem (2.1)–(2.3) for the point $P \in \overline{D}_1$ is contained in the domain D_P, and for the solution $u(x, y)$ the estimate

$$\|u\|_{\overset{\circ}{C}{}^k_\alpha(\overline{D}_P)} \leq c\Big(\sum_{i=1}^{2} \|f_i\|_{\overset{\circ}{C}{}^{k-1}_\alpha(\gamma_{iP})} + \|F\|_{\overset{\circ}{C}{}^{k-1}_{\alpha-1}(\overline{D}_P)}\Big),$$

is valid, where $c = \text{const} > 0$ does not depend on F and f_i, $i = 1, 2$,

$$\|u\|_{\overset{\circ}{C}{}^k_\alpha(\overline{D}_P)} = \max_{i+j=k} \sup_{z \in \overline{D}_P \setminus O} |z|^{-\alpha} \big|\partial^{i,j} u(z)\big|, \quad \partial^{i,j} = \frac{\partial^{i+j}}{\partial x^i \partial y^j}\ .$$

The norms in the spaces $\overset{\circ}{C}{}^{k-1}_\alpha(\gamma_{i_P})$ and $\overset{\circ}{C}{}^{k-1}_{\alpha-1}(\overline{D}_P)$ are defined analogously.

§

Let us consider a normally hyperbolic system with constant coefficients of the type

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0. \tag{2.70}$$

As the curves γ_1 and γ_2 let us take straight beams $\gamma_i : x = \gamma_i^0 y$, $y \geq 0$, $\gamma_i^0 = \text{const}$, $i = 1, 2$, $\gamma_1^0 > \gamma_2^0$. Denote by D the angle contained between the beams γ_1 and γ_2 and located in a half-plane $y \geq 0$. On the beams γ_1 and γ_2 let us take arbitrarily the points P_1 and P_2 different from $O(0, 0)$ and assume that the straight line passing through P_1 and P_2 is not a characteristic of the system (2.70). Because of the fact that γ_1 and γ_2 are the straight beams, and the characteristics $L_i : x + \lambda_i y = \text{const}$, $\lambda_i = \text{const}$, $i = 1, \dots, l$, of the system (2.70) are the straight lines, all the requirements of §2, imposed both on γ_1, γ_2 and L_i, $i = 1, \dots, l$, will be fulfilled. In a similar way as in §2, we construct the domain D_1 and determine the numbers m_1 and m_2.

Introduce into the consideration the following spaces

$$\overset{\circ}{C}{}^k_{\alpha,\beta}(\overline{D}) = \Big\{u \in \overset{\circ}{C}{}^k(\overline{D}) : \max_{i+j=k} \sup_{0<|z|\leq 1, z \in \overline{D}} |z|^{-\alpha} \big|\partial^{i,j} u(z)\big| < \infty,$$
$$\max_{i+j=k} \sup_{|z|\geq 1, z \in \overline{D}} |z|^{-\beta} \big|\partial^{i,j} u(z)\big| < \infty\Big\}, \quad k \geq 2, \ \ \alpha \geq 0, \ \ \beta \geq 0,$$
$$\overset{\circ}{C}{}^k_\alpha(\overline{D}) = \Big\{u \in \overset{\circ}{C}{}^k(\overline{D}) : \max_{i+j=k} \sup_{0<|z|\leq 1, z \in \overline{D}} |z|^{-\alpha} \big|\partial^{i,j} u(z)\big| < \infty\Big\},$$
$$k \geq 2, \quad \alpha \geq 0.$$

The space $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$ has been introduced at the end of §1 in Chapter II.

Below the $m_i \times n$-matrices M_i, N_i, $i = 1, 2$, appearing in the boundary conditions (2.2), (2.3) are assumed to be constant, and $S_i = 0$, $i = 1, 2$.

When considering the problem (2.70), (2.2), (2.3) in the spaces $\overset{\circ}{C}{}^k_\alpha(\overline{D})$ and $\overset{\circ}{C}{}^k_{\alpha,\beta}(\overline{D})$, we assume that equalities (2.2) and (2.3) take place respectively on the beams γ_1 and γ_2.

When investigating the same problem in the above-mentioned spaces, the use will be made of the Bochner method of solution of functional equations which will be cited below.

When considering the problem (2.70), (2.2), (2.3) in the classes $\overset{\circ}{C}{}^k_\alpha(\overline{D})$, $\overset{\circ}{C}{}^k_{\alpha,\beta}(\overline{D})$, $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$ it is required that $f_i \in \overset{\circ}{C}{}^{k-1}_\alpha(\gamma_i)$, $f_i \in \overset{\circ}{C}{}^{k-1}_{\alpha,\beta}(\gamma_i)$, $f_i \in \overset{\circ}{C}{}^{k-1}_\alpha(OP_i)$, $i = 1, 2$, respectively, where f_1 and f_2 are the right-hand sides of equalities (2.2), (2.3).

Similarly, as in §3, the problem (2.70), (2.2), (2.3) in the class $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$ is reduced equivalently to the system of equations (2.20) in which $G_0^i = \Gamma_i \times V_i$, $\varphi^i \in \overset{\circ}{C}{}^{k-1}_\alpha[0, d_i]$, $i = 1, 2$, $\tau_j^1(y) = \sigma_j^1 y$, $0 \le \sigma_j^1 = \text{const}$, $j = s_0 + 1, \dots, l$, $\tau_i^2(y) = \sigma_i^2 y$, $0 \le \sigma_i^2 = \text{const}$, $i = l_0 + 1, \dots, s_0$, $T_j = 0$, $j = 1, \dots, 4$, $f_3 = f_1$, $f_4 = f_2$.

After substitution $\widetilde{\varphi}(t) = (\varphi^1(d_1 t), \varphi^2(d_2 t))$ we rewrite the obtained system of equations in the form of one equation

$$G_0 \widetilde{\varphi}(t) + \sum_{i=1}^{r} G_i \widetilde{\varphi}(\tau_i t) = \widetilde{f}(t), \tag{2.71}$$

where $\widetilde{\varphi} \in \overset{\circ}{C}{}^{k-1}_\alpha[0, 1]$, G_i, $i = 0, \dots, r$, are well-defined real constant $(m_1 + m_2) \times (m_1 + m_2)$-matrices; moreover, $G_0 = \begin{pmatrix} G_0^1 & 0 \\ 0 & G_0^2 \end{pmatrix}$, $0 < \tau_i = \text{const} < 1$, $i = 1, \dots, r$, and $\widetilde{f}(t) \in \overset{\circ}{C}{}^{k-1}_\alpha[0, 1]$.

Analogously one can show that the problem (2.70), (2.2), (2.3) in the classes $\overset{\circ}{C}{}^k_\alpha(\overline{D})$, $\overset{\circ}{C}{}^k_{\alpha,\beta}(\overline{D})$ is equivalent to the system of equations (2.71) with respect to an unknown function $\widetilde{\varphi}$ belonging, respectively, to the spaces $\overset{\circ}{C}{}^{k-1}_\alpha[0, \infty)$ and $\overset{\circ}{C}{}^{k-1}_{\alpha,\beta}[0, \infty)$.

Differentiating equation (2.71) $(k-1)$ times with respect to t, we get

$$(G\psi)(t) = G_0 \psi(t) + \sum_{i=1}^{r} \tau_i^{k-1} G_i \psi(\tau_i t) = f(t), \tag{2.72}$$

where $\psi(t) = \widetilde{\varphi}^{(k-1)}(t)$, $f(t) = \widetilde{f}^{(k-1)}(t)$.

Obviously, equation (2.71) with respect to $\widetilde{\varphi} \in \overset{\circ}{C}{}^{k-1}_{\alpha,\beta}[0, \infty)$ ($\overset{\circ}{C}{}^{k-1}_\alpha[0, \infty)$, $\overset{\circ}{C}{}^{k-1}_\alpha[0, 1]$) is equivalent to equation (2.72) with respect to $\psi \in \overset{\circ}{C}_{\alpha,\beta}[0, \infty)$ ($\overset{\circ}{C}_\alpha[0, \infty)$, $\overset{\circ}{C}_\alpha[0, 1]$).

Denote by σ the set of all real numbers $\{\sigma_0, \sigma_1, \dots, \sigma_i, \dots\}$ representable in the form $\sum_{i=1}^{r} n_i \log \tau_i$, where n_i are arbitrary integers, and $\sigma_0 = 0$, $\sigma_i \neq \sigma_j$ for $i \neq j$.

Let

$$\Delta(s) = \det\Big(G_0 + \sum_{i=1}^{r} \tau_i^{k-1} G_i e^{s \log \tau_i}\Big).$$

It is obvious that $\Delta(s)$ is an entire function represented as

$$\Delta(s) = \sum_{i=0}^{m_0} \eta_i e^{\widetilde{\sigma}_i s}, \quad \widetilde{\sigma}_i \in \sigma, \tag{2.73}$$

where η_i, $\widetilde{\sigma}_i$ are certain real numbers, and $\widetilde{\sigma}_{m_0} < \widetilde{\sigma}_{m_0-1} < \cdots < \widetilde{\sigma}_0 \leq 0$.

We can easily see that in the case $\Delta(s) \equiv 0$, the homogeneous problem corresponding to (2.72) has for any s a non-trivial solution of the type $\psi(t) = c(s)t^s$, $\|c(s)\| \neq 0$. Evidently, if $\det G_0 \neq 0$, then $\Delta(s) \not\equiv 0$.

Below we shall assume that $\Delta(s) \not\equiv 0$, and in this case one can suppose that $\eta_i \neq 0$, $i = 0, \dots, m_0$ in equality (2.73).

The set $\mathfrak{M}$ of real parts of all zeros of the entire function $\Delta(s)$ is a finite or countable bounded closed set; moreover, this set is empty if and only if $\Delta(s) = \eta_0 e^{\widetilde{\sigma}_0 s}$ [11]. The set $\mathfrak{M}$ divides the real axis of the plane of the variable $s = \operatorname{Re} s + i \operatorname{Im} s$ into not more than a countable set of intervals $\widetilde{\Gamma}_i$, $i = 0, 1, 2, \dots$, among which there are the half-lines $(-\infty < \operatorname{Re} s < b_0) = \widetilde{\Gamma}_0$, $(a_0 < \operatorname{Re} s < \infty) = \widetilde{\Gamma}_1$.

It is shown in [12], [13] that the analytic almost-periodic function $\frac{1}{\Delta(s)}$ expands in the strip $\Pi_i = \{s : \operatorname{Re} s \in \widetilde{\Gamma}_i\}$ into an absolutely convergent series of the type

$$\frac{1}{\Delta(s)} = \sum_{j=0}^{\infty} \gamma_{ij} e^{\sigma_j s}, \quad \sigma_j \in \sigma, \tag{2.74}$$

whose coefficients can be uniquely determined.

Since $\widetilde{\sigma}_0 > \widetilde{\sigma}_j$, $j = 1, \dots, m_0$, we have

$$\Big| \sum_{j=1}^{m_0} \eta_0^{-1} \eta_j e^{(\widetilde{\sigma}_j - \widetilde{\sigma}_0)s} \Big| < 1$$

for $\operatorname{Re} s > c_0$, where c_0 is a sufficiently large real number. Therefore for $\operatorname{Re} s > c_0$ there takes place an expansion

$$\frac{1}{\Delta(s)} = \Big[\eta_0 e^{\widetilde{\sigma}_0 s}\Big(1 + \sum_{j=1}^{m_0} \eta_0^{-1} \eta_j e^{(\widetilde{\sigma}_j - \widetilde{\sigma}_0)s}\Big)\Big]^{-1} =$$

$$= \eta_0^{-1} e^{-\widetilde{\sigma}_0 s}\Big(1 + \sum_{i=1}^{\infty} (-1)^i \Big(\sum_{j=1}^{m_0} \eta_0^{-1} \eta_j e^{(\widetilde{\sigma}_j - \widetilde{\sigma}_0)s}\Big)^i\Big). \tag{2.75}$$

Due to the uniqueness theorem for analytic almost-periodic functions [49], the coefficients γ_{1j} of the series (2.74) in the strip Π_1 can be defined from the expansion (2.75), and hence

$$\gamma_{1j} = 0 \quad \text{for} \quad \sigma_j > -\widetilde{\sigma}_0 \geq 0. \tag{2.76}$$

Denote by Δ_{ij} the algebraic supplement of the element with the indices j, i of the determinant $\Delta(s)$,

$$\Delta_{ij}(s) = \sum_{p=0}^{N_0} \xi_{ijp} e^{\sigma_p s}, \quad i,j = 1,\ldots,m_1+m_2,$$

where N_0 is a natural number and ξ_{ijp} are definite real numbers.

Denote by g_p^{ij} the element of the matrix $\tau_p^{k-1} G_p$ with indices i, j, where $\tau_0 = 1$, $p = 0,\ldots,r$, $i,j = 1,\ldots,m_1+m_2$.

Because of determinant properties, we can easily see that for $\operatorname{Re} s \in \Pi_{i_0}$, $i_0 \geq 0$,

$$\begin{aligned}
&\frac{1}{\Delta(s)} \sum_{j=1}^{m_1+m_2} \Big(\sum_{\rho'=0}^{r} g_{\rho'}^{ij} e^{s \log \tau_{\rho'}} \Big) \Delta_{j\rho}(s) = \\
&= \sum_{p=0}^{\infty} \sum_{j=1}^{m_1+m_2} \sum_{\rho'=0}^{r} \sum_{q=0}^{N_0} g_{\rho'}^{ij} \xi_{j\rho q} \gamma_{i_0 p} e^{(\log \tau_{\rho'} + \sigma_p + \sigma_q)s} = \\
&= \sum_{\nu=0}^{\infty} \Big(\sum_{(p,j,\rho',q)\in J_\nu} g_{\rho'}^{ij} \xi_{j\rho q} \gamma_{i_0 p} \Big) e^{\sigma_\nu s} = \begin{cases} 1 & \text{for} \quad i = \rho, \\ 0 & \text{for} \quad i \neq \rho, \end{cases}
\end{aligned} \tag{2.77}$$

where J_ν is the set of all collections (p,j,ρ',q) of numbers p, j, ρ', q for which $\log \tau_{\rho'} + \sigma_p + \sigma_q = \sigma_\nu$.

From (2.77), due to the absolute convergence of the series (2.74) in the strip Π_{i_0}, and because of the uniqueness theorem for analytic almost-periodic functions, we obtain

$$\sum_{(p,j,\rho',q)\in J_\nu} g_{\rho'}^{ij} \xi_{j\rho q} \gamma_{i_0 p} = \begin{cases} 1 & \text{for} \quad i = \rho, \quad \nu = 0, \\ 0 & \text{for} \quad i = \rho, \quad \nu \geq 1, \\ & \text{or} \quad i \neq \rho, \quad \nu \geq 0. \end{cases} \tag{2.78}$$

Analogous reasonings as in the case of the expression

$$\frac{1}{\Delta(s)} \sum_{\rho=1}^{m_1+m_2} \Big(\sum_{\rho'=0}^{r} g_{\rho'}^{\rho j} e^{s \log \tau_{\rho'}} \Big) \Delta_{i\rho}(s)$$

result in the equalities

$$\sum_{(p,\rho,\rho',q)\in J_\nu} g_{\rho'}^{\rho j} \xi_{i\rho q} \gamma_{i_0 p} = \begin{cases} 1 & \text{for} \quad i = j, \quad \nu = 0, \\ 0 & \text{for} \quad i = j, \quad \nu \geq 1, \\ & \text{or} \quad i \neq j, \quad \nu \geq 0. \end{cases} \tag{2.79}$$

Let now $\widetilde{G}_{i_0} = (\widetilde{G}_{i_0 1}, \ldots, \widetilde{G}_{i_0 m_1+m_2})$ be the operator acting by the formula

$$(\widetilde{G}_{i_0 i} f)(t) = \sum_{p=0}^{\infty} \sum_{\rho=1}^{m_1+m_2} \sum_{q=0}^{N_0} \xi_{i\rho q} \gamma_{i_0 p} f_\rho(e^{\sigma_p+\sigma_q} t), \tag{2.80}$$
$$i = 1, \ldots, m_1 + m_2.$$

The lemma below is due to Bochner [11].

The operator G defined by the formula (2.72) *is invertible in the space $\overset{\circ}{C}_{\alpha,\beta}[0,\infty)$ and $G^{-1} = \widetilde{G}_{i_0}$ if*

$$\mathfrak{M} \cap I_{\alpha,\beta} = \varnothing, \quad I_{\alpha,\beta} = \big[\min(\alpha,\beta), \max(\alpha,\beta)\big] \subset \Pi_{i_0}.$$

In the spaces $\overset{\circ}{C}_\alpha[0,\infty)$ and $\overset{\circ}{C}_\alpha[0,1]$ the following lemma takes place.

The assertion of Lemma 2.4 *is valid in the space $\overset{\circ}{C}_\alpha[0,\infty)$ for $\alpha > \sup \mathfrak{M} = \sup_{x\in\mathfrak{M}} x$ and in $\overset{\circ}{C}_\alpha[0,1]$ if $\det G_0 \neq 0$ and $\alpha > \sup \mathfrak{M}$, in both cases G^{-1} and $\widetilde{G}_1$ being equal.*

To prove Lemma 2.5 we shall use the Bochner method [11]. $\alpha > \sup \mathfrak{M}$ implies that $\alpha \in \Pi_1$, and hence, since the series (2.74) is absolutely convergent in Π_1, we have

$$c_1 = \sum_{j=0}^{\infty} |\gamma_{1j}| e^{\sigma_j \alpha} < \infty. \tag{2.81}$$

Suppose $p_t(f) = \sup_{\tau\in(0,t]} \|\tau^{-\alpha} f(\tau)\|_{R^{m_1+m_2}}$.

By (2.76) the function $\widetilde{G}_{1i} f$ at the point $t > 0$ depends only on those values of f which it takes on the segment $[0, t_0 t]$, where $t_0 = e^{(-\widetilde{\sigma}_0 + \max_{0\le q\le N_0} \sigma_q)}$. Therefore we have

$$p_t(\widetilde{G}_1 f) \le \max_{1\le i\le m_1+m_2} p_t(\widetilde{G}_{1i} f) \le$$
$$\le \sum_{p=0}^{\infty} \sum_{\rho=1}^{m_1+m_2} \sum_{q=0}^{N_0} \max_{1\le i\le m_1+m_2} \big|\xi_{i\rho q} e^{\sigma_q \alpha}\big| |\gamma_{1p}| e^{\sigma_q \alpha} p_{t_0 t}(f) \le$$
$$\le (m_1+m_2)(N_0+1)\big(\max_{i,\rho,q} \big|\xi_{i\rho q} e^{\sigma_q \alpha}\big|\big) c_1 p_{t_0 t}(f). \tag{2.82}$$

When deducing (2.82), the use has been made of (2.81) and the fact that $p_t(\widetilde{f}_\rho) \le e^{(\sigma_p+\sigma_q)\alpha} p_{t_0 t}(f_\rho)$, where $\widetilde{f}_\rho(t) = f_\rho(e^{\sigma_p+\sigma_q} t)$. From (2.82) it follows that the operator $\widetilde{G}_1$ is continuous in the space $\overset{\circ}{C}_\alpha[0,\infty)$.

Let us check that $G\widetilde{G}_1 = I$, where I is the identity operator. If $G\widetilde{G}_1 = ((G\widetilde{G}_1)_1, \dots, (G\widetilde{G}_1)_{m_1+m_2})$, then by (2.78) and (2.80) we have

$$\begin{aligned}\big((G\widetilde{G}_1)_i f\big)(t) &= \sum_{\rho'=0}^{r} \sum_{j=1}^{m_1+m_2} g_{\rho'}^{ij} (\widetilde{G}_{1j} f)(\tau_{\rho'} t) = \\ &= \sum_{\rho'=0}^{r} \sum_{j=1}^{m_1+m_2} \sum_{p=0}^{\infty} \sum_{\rho=1}^{m_1+m_2} \sum_{q=0}^{N_0} g_{\rho'}^{ij} \xi_{j\rho q} \gamma_{1p} f_\rho \big(e^{\log \tau_{\rho'} + \sigma_p + \sigma_q} t\big) = \\ &= \sum_{\rho=1}^{m_1+m_2} \sum_{\nu=0}^{\infty} \Big(\sum_{(p,j,\rho',q) \in J_\nu} g_{\rho'}^{ij} \xi_{j\rho q} \gamma_{1p} \Big) f_\rho (e^{\sigma_\nu} t) = f_i(t),\end{aligned}$$

which proves the equality $G\widetilde{G}_1 = I$. In a similar way, using equality (2.79), we can easily check that $\widetilde{G}_1 G = I$. Thus $G^{-1} = \widetilde{G}_1$, and Lemma 2.5 is proved in the space $\overset{\circ}{C}_\alpha[0,\infty)$.

Let now $\det G_0 \neq 0$ and $\alpha > \sup \mathfrak{M}$. From (2.73) it follows that $\widetilde{\sigma}_0 = 0$ for $\det G_0 \neq 0$. Therefore by (2.76) we have $\gamma_{1j} = 0$ for $\sigma_j > 0$. Since $\log \tau_i < 0$, $i = 1, \dots, r$, in the expansion

$$\Delta_{ij}(s) = \sum_{p=0}^{N_0} \xi_{ijp} e^{\sigma_p s}$$

we have $\xi_{ijp} = 0$ for $\sigma_p > 0$, and thus $\xi_{i\rho q} \gamma_{1p} = 0$ or $\sigma_p + \sigma_q > 0$. Hence the operator $\widetilde{G}_1$ defined by (2.80) acts from the space $\overset{\circ}{C}_\alpha[0,1]$ into itself. It remains for us to note that the operator G in the space $\overset{\circ}{C}_\alpha[0,\infty)$ is invertible for $\alpha > \sup \mathfrak{M}$, and $G^{-1} = \widetilde{G}_1$. ■

For $\alpha > \sup \mathfrak{M}$ *and* $\det G_0 = 0$ *the equation* (2.72) *is solvable in the space* $\overset{\circ}{C}_\alpha[0,1]$, *and the homogeneous equation corresponding to* (2.72) *has in the space* $\overset{\circ}{C}_\alpha[0,1]$ *an infinite number of linearly independent solutions.*

Proof. If $f \in \overset{\circ}{C}_\alpha[0,1]$, then let $\widetilde{f}$ be an arbitrary continuous extension of f from the segment $[0,1]$ to $[0,\infty)$. Clearly, $\widetilde{f} \in \overset{\circ}{C}_\alpha[0,\infty)$, since $\widetilde{f}(t) = f(t)$ for $0 \le t \le 1$. By Lemma 2.5 the equation $G\psi = \widetilde{f}$ is uniquely solvable in the space $\overset{\circ}{C}_\alpha[0,\infty)$ for $\alpha > \sup \mathfrak{M}$. It is also clear that the vector function $\widetilde{\psi}(t) = \psi(t) = (G^{-1}\widetilde{f})(t)$ defined on the segment $0 \le t \le 1$ belongs to the space $\overset{\circ}{C}_\alpha[0,1]$ and is the solution of (2.72).

Let us show that $\dim \operatorname{Ker} G = \infty$. Since $\det G_0 = 0$, there exists a non-degenerate $(m_1 + m_2) \times (m_1 + m_2)$-matrix Ω such that the last q_0 rows of the matrix $G_0\Omega$ are zero, where $q_0 = (m_1 + m_2) - \operatorname{rank} G_0 > 0$. Consider the operator G^* defined by

$$(G^*\psi)(t) = G_0\Omega\psi(t) + \sum_{i=1}^{r} G_i\Omega\psi(\tau_i t), \quad \psi \in \overset{\circ}{C}_\alpha[0,1].$$

Assume $\widetilde{\tau}_0 = \max_{1\le i\le r} \tau_i$, $0 < \widetilde{\tau}_0 < 1$. Let $\widetilde{\psi} = (\widetilde{\psi}_1, \dots, \widetilde{\psi}_{m_1+m_2})$ be an arbitrary vector function of the class $\overset{\circ}{C}_\alpha[0,1]$ such that $\widetilde{\psi}_i \equiv 0$ when $i = 1, \dots, m_1 + m_2 - q_0$ and $\widetilde{\psi}_i(t) \not\equiv 0$, $\widetilde{\psi}_i(t) = 0$ for $t \in [0, \widetilde{\tau}_0]$ when $i = m_1+m_2-q_0+1, \dots, m_1+m_2$. It can be easily seen that $G^*\widetilde{\psi} = 0$, and hence $G\psi = 0$, where $\psi = \Omega\widetilde{\psi}$. Therefore $\dim \operatorname{Ker} G = \infty$, since $\det \Omega \ne 0$. ■

Remark. As the example of equation (2.72) with $r = 1$ shows, in the case where conditions of Lemmas 2.4–2.6 are violated, the unique solvability of the equation (2.72) may not hold. For $r = 1$ the following assertion is valid: a) equation (2.72) in the space $\overset{\circ}{C}_{\alpha,\beta}[0,\infty)$ for $\mathfrak{M} \cap \{\alpha, \beta\} \ne \varnothing$ as well as in the spaces $\overset{\circ}{C}_\alpha[0,\infty)$ and $\overset{\circ}{C}_\alpha[0,1]$ for $\alpha \in \mathfrak{M}$ is not normally Hausdorff solvable; b) equation (2.72) is normally Hausdorff solvable in the space $\overset{\circ}{C}_{\alpha,\beta}[0,\infty)$ for $\mathfrak{M} \cap I_{\alpha,\beta} \ne \varnothing$, $\mathfrak{M} \cap \{\alpha,\beta\} = \varnothing$, and for $\alpha < \beta$ we have $\varkappa = d_0 - d_0^* = +\infty$, $d_0 = \dim \operatorname{Ker} G = \infty$, $d_0^* = \dim \operatorname{Ker} G^* = 0$, while for $\alpha > \beta$ we have conversely $\varkappa = -\infty$, $d_0 = 0$, $d_0^* = \infty$; c) equation (2.72) is normally Hausdorff solvable in the spaces $\overset{\circ}{C}_\alpha[0,\infty)$ and $\overset{\circ}{C}_\alpha[0,1]$ for $\alpha \le \sup \mathfrak{M}$, $\alpha \notin \mathfrak{M}$; moreover, in both cases $\varkappa = +\infty$, $d_0 = \infty$, $d_0^* = 0$. This assertion can be proved by using the same method as we have used in proving Lemma 1.7 in §2 of Chapter I.

Recall that the condition $\det G_0 \ne 0$ is equivalent to the fulfilment of the conditions (2.21); moreover, if $\det G_0 \ne 0$, then the entire function $\Delta(s) \not\equiv 0$. Denote by $\mathfrak{M}_0$ the set of real parts of zeros of the entire function

$$\Delta_0(s) = \det\Big(G_0 + \sum_{i=1}^{r} G_i e^{(s-1)\log \tau_i}\Big).$$

Since $\Delta(s) = \Delta_0(s+k)$, we have $\mathfrak{M} = \mathfrak{M}_0 - k = \{x - k : x \in \mathfrak{M}_0\}$.

From Lemmas 2.4–2.6 we have the following

The problem (2.70), (2.2), (2.3) *is uniquely solvable in: a) the class* $\overset{\circ}{C}{}^k_{\alpha,\beta}(\overline{D})$ *for* $\Delta_0(s) \not\equiv 0$ *and* $(\mathfrak{M}_0 - k) \cap I_{\alpha,\beta} = \varnothing$*; b) the class* $\overset{\circ}{C}{}^k_\alpha(\overline{D})$ *for* $\Delta_0(s) \not\equiv 0$ *and* $k + \alpha > \sup \mathfrak{M}_0$*; c) the class* $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$ *if the conditions* (2.21) *are fulfilled and* $k + \alpha > \sup \mathfrak{M}_0$*. In the case* $\Delta_0(s) \not\equiv 0$, $k + \alpha > \sup \mathfrak{M}_0$*, if at least one of the conditions* (2.21) *is violated, then the problem* (2.70), (2.2), (2.3) *is solvable in the class* $\overset{\circ}{C}{}^k_\alpha(\overline{D}_1)$*, and the* *homogeneous problem corresponding to* (2.70), (2.2), (2.3) *has an infinite number of linearly independent solutions.*

Remarks.

1. As noted above, when the conditions of Theorem 2.4 are violated in the classes $\overset{\circ}{C}{}^k_{\alpha,\beta}(\overline{D})$, $\overset{\circ}{C}{}^k_{\alpha}(\overline{D})$, $\overset{\circ}{C}{}^k_{\alpha}(\overline{D}_1)$ the problem (2.70), (2.2), (2.3) may turn out to be ill-posed.

2. If the set $\mathfrak{M}_0$ is empty, then $\Delta_0(s) \not\equiv 0$, and owing to Theorem 2.4, the problem (2.70), (2.2), (2.3) is uniquely solvable in the classes $\overset{\circ}{C}{}^k_{\alpha,\beta}(\overline{D})$, $\overset{\circ}{C}{}^k_{\alpha}(\overline{D})$ for all $k \geq 2$, $\alpha \geq 0$, $\beta \geq 0$, as well as in the class $\overset{\circ}{C}{}^k_{\alpha}(\overline{D}_1)$ when the conditions (2.21) are fulfilled for all $k \geq 2$, $\alpha \geq 0$. When the conditions of Theorem 2.1 are fulfilled, it is obvious that $\Delta_0(s) \equiv \det G_0 \neq 0$, and hence the set $\mathfrak{M} = \varnothing$. Therefore, in the case of the problem (2.70), (2.2), (2.3) Theorem 2.1 is a direct consequence of the assertion b) of Theorem 2.4.

3. It can be easily verified that $\rho_0 \geq \sup \mathfrak{M}_0$, where ρ_0 is a number occurring in the condition $k + \alpha > \rho_0$ of Theorem 2.2 in the case of the problem (2.70), (2.2), (2.3). Therefore, the condition $\alpha > \sup(\mathfrak{M}_0 - k)$ or, what is the same, the condition $k + \alpha > \sup \mathfrak{M}_0$ in Theorem 2.4 is more exact than the condition $k + \alpha > \rho_0$ in Theorem 2.2.

4. In the case $\Delta_0(s) \equiv 0$, one can easily verify that in the classes $\overset{\circ}{C}{}^k_{\alpha,\alpha}(\overline{D})$, $\overset{\circ}{C}{}^k_{\alpha}(\overline{D})$, $\overset{\circ}{C}{}^k_{\alpha}(\overline{D}_1)$ for all $k \geq 2$, $\alpha \geq 0$ the homogeneous problem corresponding to (2.70), (2.2), (2.3) has an infinite number of linearly independent solutions.

CHAPTER III

§

In the plane of variables x, y let us consider a system of linear differential equations of the type

$$y^m A u_{xx} + 2y^{\frac{m}{2}} B u_{xy} + C u_{yy} + a u_x + b u_y + c u = F, \tag{3.1}$$

where A, B, C, a, b, c are given real $n \times n$-matrices, F and u are, respectively, given and unknown n-dimensional vectors, $m = \text{const} > 0$, $n > 1$.

Below A, B, C are assumed to be constant matrices, $\det C \neq 0$, and the polynomial $p_0(\lambda) = \det(A + 2B\lambda + C\lambda^2)$ is assumed to have only simple real roots $\lambda_1, \dots, \lambda_{2n}$. In this case the system (3.1) is strictly hyperbolic for $y > 0$, and the line of parabolic degeneration $y = 0$ is not a characteristic of the system (3.1). Under these conditions the numbers $y^{\frac{m}{2}}\lambda_1, \dots, y^{\frac{m}{2}}\lambda_{2n}$ are the roots of the characteristic polynomial $p(y;\lambda) = \det(y^m A + 2y^{\frac{m}{2}} B\lambda + C\lambda^2)$ of the system (3.1), and the curves determined by the equations

$$L_i(P) : x + \frac{2\lambda_i}{m+2} y^{\frac{m+2}{2}} = x_0 + \frac{2\lambda_i}{m+2} y_0^{\frac{m+2}{2}}, \quad i = 1, \dots, 2n, \ \ y_0 > 0,$$

and passing through the point $P(x_0, y_0)$ are characteristics of the system (3.1).

Denote by D a domain lying in the half-plane $y > 0$ and bounded by two adjoint characteristics

$$\gamma_1 : x + \frac{2\lambda_{i_1}}{m+2} y^{\frac{m+2}{2}} = 0, \quad \gamma_2 : x + \frac{2\lambda_{i_2}}{m+2} y^{\frac{m+2}{2}} = 0, \ \ \lambda_{i_1} < \lambda_{i_2},$$

of the system (3.1), coming out of the origin $O(0,0)$. Take arbitrarily on γ_1 a point P_1 different from $O(0,0)$ and choose the numbering of characteristic curves $L_i(P_1)$, $i = 1, \dots, 2n$, coming out of P_1 into the angle D such that starting from $L_1(P_1)$, they follow each other counter-clockwise. On the curve γ_2 let us fix the point P_2 lying strictly between the two points of intersection of characteristics $L_n(P_1)$ and $L_{n+1}(P_1)$ with the curve γ_2. Denote by $D_1 \subset D$ the characteristic quadrangle with a vertex at $O(0,0)$, bounded by the characteristics γ_1, γ_2, $L_n(P_1)$ and $L_{n+1}(P_2)$. Under these assumptions it is evident that

$$\gamma_1 = L_{2n}(O) : x + \frac{2\lambda_{2n}}{m+2} y^{\frac{m+2}{2}} = 0, \quad \gamma_2 = L_1(O) : x + \frac{2\lambda_1}{m+2} y^{\frac{m+2}{2}} = 0.$$

For convenience we shall assume below that $\lambda_n > 0$ and $\lambda_{n+1} < 0$.

Consider the characteristic problem formulated as follows [38]: find in the domain D_1 a regular solution $u(x,y)$ of the system (3.1), satisfying on the segments OP_i of the characteristics γ_i the following conditions

$$u|_{OP_i} = f_i, \quad i = 1, 2, \tag{3.2}$$

where f_1, f_2 are given n-dimensional real vectors, $f_1(O) = f_2(O)$.

Below we asume that a, b, c, $F \in C^1(\overline{D}_1)$, $f_i \in C^2(OP_i)$, $i = 1, 2$, and moreover, in the domain D_1

$$\sup_{\overline{D}_1 \setminus O} \|y^{(1-\frac{m}{2})} a\| < \infty, \quad \sup_{\overline{D}_1 \setminus O} \|y^{(1-\frac{m}{2})} a_x\| < \infty,$$

$$\sup_{\overline{D}_1 \setminus O} \|y^{-(\alpha+\frac{m}{2}-1)} F\| < \infty, \quad \sup_{\overline{D}_1 \setminus O} \|y^{-(\alpha-2)} F_x\| < \infty, \quad \alpha = \text{const} > 0,$$

$$f_i(O) = 0, \quad \sup_{OP_i \setminus O} \|y^{-(\alpha+\frac{m}{2}+1-j)} f_i^{(j)}\| < \infty, \quad i = 1, 2; \; j = 1, 2,$$

where $\|\cdot\|$ denotes the norm in R^n.

Since the roots $\lambda_1, \dots, \lambda_{2n}$ of the polynomial $p_0(\lambda)$ are simple, we can easily verify that $\dim \operatorname{Ker}(A + 2B\lambda_i + C\lambda_i^2) = 1$, $i = 1, \dots, 2n$. Let the vectors $\nu_i \in \operatorname{Ker}(A + 2B\lambda_i + C\lambda_i^2)$ and $\|\nu_i\| \neq 0$, $i = 1, \dots, 2n$.

In §4 we shall prove the following

Let the condition

$$\operatorname{rank}\{\nu_1, \dots, \nu_n\} = \operatorname{rank}\{\nu_{n+1}, \dots, \nu_{2n}\} = n \tag{3.3}$$

be fulfilled. Then there exists a positive integer α_0 depending only on the coefficients A, B, C, α of the system (3.1) *such that for $\alpha > \alpha_0$ the problem* (3.1), (3.2) *is uniquely solvable in the class*

$$\Big\{ u \in C^2(\overline{D}_1) : \partial^{i,j} u(0,0) = 0, \\ \sup_{\overline{D}_1 \setminus O} \|y^{-(\alpha+\frac{m}{2}+1-(\frac{m}{2}+1)i-j)} \partial^{i,j} u\| < \infty, \; 0 \le i + j \le 2 \Big\}, \tag{3.4}$$

$$\partial^{i,j} = \frac{\partial^{i+j}}{\partial x^i \partial y^j} \,.$$

It should be noted that the condition

$$\sup_{\overline{D}_1 \setminus O} \|y^{(1-\frac{m}{2})} a\| < \infty$$

for the lower coefficient a of the system (3.1) is a generalization of the well-known Gellerstedt's condition when (3.1) is a scalar equation ($n = 1, A = -C = 1, B = 0$).

§

Consider the following $2n \times 2n$-matrices:

$$A_0 = \begin{Vmatrix} 0 & -E \\ C^{-1}A & 2C^{-1}B \end{Vmatrix}, \quad \widetilde{A}_0 = \begin{Vmatrix} 0 & -E \\ y^m C^{-1}A & 2y^{\frac{m}{2}} C^{-1}B \end{Vmatrix},$$

$$K = \begin{pmatrix} \nu_1 & \dots & \nu_{2n} \\ \lambda_1\nu_1 & \dots & \lambda_{2n}\nu_{2n} \end{pmatrix}, \quad \widetilde{K} = \begin{pmatrix} y^{-\frac{m}{2}}\nu_1 & \dots & y^{-\frac{m}{2}}\nu_{2n} \\ \lambda_1\nu_1 & \dots & \lambda_{2n}\nu_{2n} \end{pmatrix},$$

where E is the unit $n \times n$-matrix.

It can be easily verified that

$$K^{-1}A_0K = D_0, \quad \widetilde{K}^{-1}\widetilde{A}_0\widetilde{K} = \widetilde{D}_0. \tag{3.5}$$

Here $D_0 = \operatorname{diag}[-\lambda_1, \dots, -\lambda_{2n}]$, $\widetilde{D}_0 = \operatorname{diag}[-y^{\frac{m}{2}}\lambda_1, \dots, -y^{\frac{m}{2}}\lambda_{2n}]$.

Assume $K = \operatorname{colon}(K_1, K_2)$, $K^{-1} = (K_1^0, K_2^0)$, where K_1, K_2 and K_1^0, K_2^0 are matrices of orders $n \times 2n$ and $2n \times n$, respectively.

Obviously,

$$\widetilde{K} = \operatorname{colon}\left(y^{-\frac{m}{2}}K_1, K_2\right), \quad \widetilde{K}^{-1} = \left(y^{\frac{m}{2}}K_1^0, K_2^0\right). \tag{3.6}$$

Owing to (3.6) we have

$$\widetilde{K}_y = -\frac{m}{2}\operatorname{colon}\left(y^{-\frac{m}{2}-1}K_1, 0\right), \quad \widetilde{K}^{-1}\widetilde{K}_y = -\frac{m}{2y}K_1^0 \times K_1. \tag{3.7}$$

If

$$B_0 = \begin{Vmatrix} 0 & 0 \\ C^{-1}a & C^{-1}b \end{Vmatrix}, \tag{3.8}$$

then

$$\widetilde{K}^{-1}B_0\widetilde{K} = \frac{1}{y}\widetilde{B}_0 + \widetilde{B}_1, \tag{3.9}$$

where $\widetilde{B}_0 = y^{1-\frac{m}{2}}K_2^0C^{-1}aK_1$, $\widetilde{B}_1 = K_2^0C^{-1}bK_2$. Since by the assumption

$$\sup_{\overline{D}_1\setminus O}\|y^{(1-\frac{m}{2})}a\| < \infty, \quad \sup_{\overline{D}_1\setminus O}\|y^{(1-\frac{m}{2})}a_x\| < \infty,$$

we have

$$\begin{aligned} \sup_{\overline{D}_1\setminus O}\|\widetilde{B}_0\| &= \sup_{\overline{D}_1\setminus O}\|y^{(1-\frac{m}{2})}K_2^0C^{-1}aK_1\| < \infty, \\ \sup_{\overline{D}_1\setminus O}\|\widetilde{B}_{0x}\| &= \sup_{\overline{D}_1\setminus O}\|y^{(1-\frac{m}{2})}K_2^0C^{-1}a_xK_1\| < \infty. \end{aligned} \tag{3.10}$$

§

In the class (3.4) we can rewrite equivalently the problem (3.1), (3.2) in the form

$$v_y + \widetilde{A}_0 v_x + B_0 v + C_0 u^0 = F^0, \tag{3.11}$$

$$\Big(- y^{\frac{m}{2}} \lambda_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big) u = - y^{\frac{m}{2}} \lambda_1 v_1 + v_2, \tag{3.12}$$

$$\left.\begin{aligned} \Big(- y^{\frac{m}{2}} \lambda_{2n} v_1 + v_2\Big)\Big(- \tfrac{2\lambda_{2n}}{m+2} y^{\frac{m+2}{2}}, y\Big) = f_1^{(1)}(y), \quad 0 \le y \le d_1, \\ \Big(- y^{\frac{m}{2}} \lambda_1 v_1 + v_2\Big)\Big(- \tfrac{2\lambda_1}{m+2} y^{\frac{m+2}{2}}, y\Big) = f_2^{(1)}(y), \quad 0 \le y \le d_2, \end{aligned}\right\} \tag{3.13}$$

$$u\Big(- \frac{2\lambda_{2n}}{m+2} y^{\frac{m+2}{2}}, y\Big) = f_1(y), \quad 0 \le y \le d_1, \tag{3.14}$$

where d_i is the ordinate of the point $P_i \in \gamma_i$, $i = 1, 2$, and the $2n \times 2n$-matrices $\widetilde{A}_0$, B_0 have been introduced in §2,

$$\begin{gathered} C_0 = \operatorname{diag}(0, C^{-1} c), \quad u^0 = (0, u), \quad F^0 = (0, F), \\ v_1 = u_x, \quad v_2 = u_y, \\ v = (v_1, v_2), \quad v_1 \in C^1_{\alpha, \frac{m}{2}+1, 1}, \quad v_2 \in C^1_{\alpha + \frac{m}{2}, \frac{m}{2}+1, 1}. \end{gathered} \tag{3.15}$$

Here

$$C^k_{\alpha, p_1, p_2} = \Big\{ u \in C^k(\overline{D}_1) : \partial^{i,j} u(0,0) = 0, \\ \sup_{\overline{D}_1 \setminus O} \big\| y^{-(\alpha - p_1 i - p_2 j)} \partial^{i,j} u \big\| < \infty, 0 \le i + j \le k \Big\}.$$

In fact, if u is a solution of the problem (3.1), (3.2) from the above-mentioned class, then it is obvious that u, $v_1 = u_x$, $v_2 = u_y$ satisfy the problem (3.11)–(3.14), and (v_1, v_2) belongs to the class (3.15). Conversely, let u, v_1, v_2 be solutions of the problem (3.11)–(3.14) for which (3.4), (3.15) hold. Let us show that u is a solution of the problem (3.1), (3.2) and $v_1 = u_x$, $v_2 = u_y$. From the first n equations of the system (3.11) we have that $v_{1y} = v_{2x}$. Furthermore, equation (3.12) yields

$$\Big(- y^{\frac{m}{2}} \lambda_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big)(u_x - v_1) = \\ = \frac{\partial}{\partial x}\Big(- y^{\frac{m}{2}} \lambda_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big) u - \Big(- y^{\frac{m}{2}} \lambda_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big) v_1 = \\ = \frac{\partial}{\partial x}\big(- y^{\frac{m}{2}} \lambda_1 v_1 + v_2\big) + y^{\frac{m}{2}} \lambda_1 v_{1x} - v_{1y} = v_{2x} - v_{1y} = 0.$$

which in its turn implies that $u_x - v_1 = 0$, since because of (3.12)–(3.14) and the inequality $\lambda_1 \ne \lambda_{2n}$ we have $(u_x - v_1)|_{OP_1 \setminus O} = 0$, while at the point $O(0,0)$ the function $(u_x - v_1)$ vanishes by the assumption u_x, $v_1 \in C^1(\overline{D}_1)$,

$\sup\limits_{\overline{D}_1\setminus O} \|y^{-\alpha}u_x\| < \infty$, $\sup\limits_{\overline{D}_1\setminus O} \|y^{-\alpha}v_1\| < \infty$ and $\alpha > 0$. Since $u_x = v_1$, (3.12) implies $u_y = v_2$ and by (3.11)-(3.14) we can easily get that u is a solution of the problem (3.1)–(3.2).

Note that for the above converse assertion to be valid, it suffices to require of the unknown function u that $u \in C^1_{\alpha+\frac{m}{2},\frac{m}{2},0}$. In this case one should consider the differential expression $(-y^{\frac{m}{2}}\lambda_1\frac{\partial}{\partial x}+\frac{\partial}{\partial y})(u_x-v_1)$ in a generalized sense. By virtue of (3.12) and the equality $v_{1y} - v_{2x} = 0$, for any function $\varphi \in C_0^\infty(D_1)$ we have

$$\int\limits_{D_1}(u_x - v_1)\Big(- y^{\frac{m}{2}}\lambda_1\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big)\varphi dxdy =$$

$$=-\int\limits_{D_1} u\frac{\partial}{\partial x}\Big(- y^{\frac{m}{2}}\lambda_1\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)\varphi dxdy+\int\limits_{D_1}\Big[\Big(- y^{\frac{m}{2}}\lambda_1\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)v_1\Big]\varphi dxdy =$$

$$=\int\limits_{D_1}\Big[\Big(- y^{\frac{m}{2}}\lambda_1\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)u\Big]\frac{\partial}{\partial x}\varphi dxdy + \int\limits_{D_1}\Big[\Big(- y^{\frac{m}{2}}\lambda_1\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)v_1\Big]\varphi dxdy =$$

$$= \int\limits_{D_1}\Big(- y^{\frac{m}{2}}\lambda_1 v_1 + v_2\Big)\frac{\partial}{\partial x}\varphi dxdy + \int\limits_{D_1}\Big[\Big(- y^{\frac{m}{2}}\lambda_1\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big)v_1\Big]\varphi dxdy =$$

$$= -\int\limits_{D_1}\Big[\frac{\partial}{\partial x}\Big(- y^{\frac{m}{2}}\lambda_1 v_1 + v_2\Big)\Big]\varphi dxdy +$$

$$+\int\limits_{D_1}\Big[\Big(- y^{\frac{m}{2}}\lambda_1\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big)v_1\Big]\varphi dxdy = \int\limits_{D_1}(v_{1y} - v_{2x})\varphi dxdy = 0,$$

whence by Theorem 1.4.2 of [26, p. 19] we can conclude that the continuous function $u_x - v_1$ is constant along the characteristics $L_1 : x + \frac{2\lambda_1}{m+2}y^{\frac{m+2}{2}} =$ const, and since $(u_x-v_1)|_{OP_1} = 0$, we have $u_x-v_1 = 0$ in $\overline{D}_1$. The remaining part of our discussion is similar.

As a result of the substitution $v = \widetilde{K}w$ of the unknown function, and owing to (3.15), instead of (3.11)–(3.14) we shall have

$$\left.\begin{array}{c} w_y + \widetilde{D}_0 w_x = B_2 w + C_2 u^0 + F^1, \\ \Big(- y^{\frac{m}{2}}\lambda_1\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big)u = \big(- y^{\frac{m}{2}}\lambda_1\widetilde{K}_1 + \widetilde{K}_2\big)w, \\ \big(- y^{\frac{m}{2}}\lambda_{2n}\widetilde{K}_1 + \widetilde{K}_2\big)w\Big(- \frac{2\lambda_{2n}}{m+2}y^{\frac{m+2}{2}}, y\Big) = f_1^{(1)}(y), \ \ 0 \le y \le d_1, \\ \big(- y^{\frac{m}{2}}\lambda_1\widetilde{K}_1 + \widetilde{K}_2\big)w\Big(- \frac{2\lambda_1}{m+2}y^{\frac{m+2}{2}}, y\Big) = f_2^{(1)}(y), \ \ 0 \le y \le d_2, \\ u\Big(- \frac{2\lambda_{2n}}{m+2}y^{\frac{m+2}{2}}, y\Big) = f_1(y), \ \ 0 \le y \le d_1, \end{array}\right\} \quad (3.16)$$

where $B_2 = -\widetilde{K}^{-1}\widetilde{K}y - \widetilde{K}^{-1}B_0\widetilde{K}$, $C_2 = -\widetilde{K}^{-1}C_0$, $F^1 = \widetilde{K}^{-1}F^0$, and $\widetilde{K}_1$ and $\widetilde{K}_2$ are the matrices of order $n \times 2n$ composed, respectively, of the first and the last n rows of the matrix $\widetilde{K}$.

By (3.6)–(3.9) we have

$$\begin{aligned} \widetilde{K}_1 &= y^{-\frac{m}{2}} K_1, \quad \widetilde{K}_2 = K_2, \\ B_2 &= \frac{m}{2y} K_1^0 \times K_1 - \frac{1}{y} \widetilde{B}_0 - \widetilde{B}_1, \end{aligned} \tag{3.17}$$

$$\begin{aligned} -y^{\frac{m}{2}} \lambda_1 \widetilde{K}_1 + \widetilde{K}_2 &= -\lambda_1 K_1 + K_2 = \\ &= \big(0, (\lambda_2 - \lambda_1)\nu_2, \dots, (\lambda_{2n} - \lambda_1)\nu_{2n}\big), \\ -y^{\frac{m}{2}} \lambda_{2n} \widetilde{K}_1 + \widetilde{K}_2 &= -\lambda_{2n} K_1 + K_2 = \\ &= \big((\lambda_1 - \lambda_{2n})\nu_1, \dots, (\lambda_{2n-1} - \lambda_{2n})\nu_{2n-1}, 0\big). \end{aligned}$$

Taking into account (3.17), we rewrite the problem (3.16) in the form

$$w_y + \widetilde{D}_0 w_x = \frac{1}{y}(B_3 w + y C_2 u^0) + F^1, \tag{3.18}$$

$$\Big(-y^{\frac{m}{2}} \lambda_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \Big) u = \big(-\lambda_1 K_1 + K_2 \big) w, \tag{3.19}$$

$$\left.\begin{aligned} \big(-\lambda_{2n} K_1 + K_2 \big) w \Big(-\tfrac{2\lambda_{2n}}{m+2} y^{\frac{m+2}{2}}, y \Big) &= f_1^{(1)}(y), \ \ 0 \le y \le d_1, \\ \big(-\lambda_1 K_1 + K_2 \big) w \Big(-\tfrac{2\lambda_1}{m+2} y^{\frac{m+2}{2}}, y \Big) &= f_2^{(1)}(y), \ \ 0 \le y \le d_2, \end{aligned}\right\} \tag{3.20}$$

$$u\Big(-\frac{2\lambda_{2n}}{m+2} y^{\frac{m+2}{2}}, y \Big) = f_1(y), \ \ 0 \le y \le d_1. \tag{3.21}$$

Here $B_3 = \frac{m}{2} K_1^0 K_1 - \widetilde{B}_0 - y\widetilde{B}_1$, and by (3.10) we have

$$\sup_{\overline{D}_1 \setminus O} \|B_3\| < \infty, \quad \sup_{\overline{D}_1 \setminus O} \|B_{3x}\| < \infty. \tag{3.22}$$

It follows from (3.6) that $v_1 = y^{-\frac{m}{2}} K_1 w$, $v_2 = K_2 w$, $w = y^{\frac{m}{2}} K_1^0 v_1 + K_2^0 v_2$. Therefore (v_1, v_2) belongs to the class (3.15) if and only if $w \in C^1_{\alpha+\frac{m}{2}, \frac{m}{2}+1, 1}$.

Let $L_i(x_0, y_0) : x = z_i(x_0, y_0; t) = x_0 + \frac{2\lambda_i}{m+2} y_0^{\frac{m+2}{2}} - \frac{2\lambda_i}{m+2} t^{\frac{m+2}{2}}$, $y = t$ be a parametrization of the characteristic curve $L_i(x_0, y_0)$ passing through the point $(x_0, y_0) \in \overline{D}_1$, $i = 1, \dots, 2n$. Denote by $\omega_i(x, y)$ the ordinate of the point of intersection of the characteristic $L_i(x, y)$ with the curve γ_1 for $1 \le i \le n$ and with the curve γ_2 for $n < i \le 2n$, $(x, y) \in \overline{D}_1$. It can be easily verified that

$$\omega_i(x, y) = \\ = \begin{cases} \Big[\frac{m+2}{2} (\lambda_i - \lambda_{2n})^{-1} \Big(x + \frac{2\lambda_i}{m+2} y^{\frac{m+2}{2}} \Big) \Big]^{\frac{2}{m+2}}, & i = 1, \dots, n, \\ \Big[\frac{m+2}{2} (\lambda_i - \lambda_1)^{-1} \Big(x + \frac{2\lambda_i}{m+2} y^{\frac{m+2}{2}} \Big) \Big]^{\frac{2}{m+2}}, & i = n+1, \dots, 2n. \end{cases} \tag{3.23}$$

Let D_Q, $Q \in \overline{D}_1 \setminus O$, be the curvilinear quadrangle with a vertex at $O(0,0)$, bounded by the characteristics γ_1, γ_2, $L_n(Q)$ and $L_{n+1}(Q)$. Since by the assumption $\lambda_n > 0$ and $\lambda_{n+1} < 0$, the domain D_Q, $Q(x_0, y_0) \in \overline{D}_1 \setminus O$, is located entirely in the half-plane $y \le y_0$. Therefore it follows from the construction of the function $\omega_i(x,y)$ that

$$0 \le \omega_i(x,y) \le y, \quad (x,y) \in \overline{D}_1, \ \ i = 1, \dots, 2n, \tag{3.24}$$

because the segment of the characteristic $L_i(Q)$, issued from the point $Q \in \overline{D}_1 \setminus O$ up to the intersection with the curve γ_1 for $1 \le i \le n$ and with the curve γ_2 for $n < i \le 2n$, is contained entirely in $\overline{D}_Q$.

By virtue of (3.23) we can easily see that

$$\begin{aligned} \omega_i\Big|_{OP_1} &= \begin{cases} y, & i = 1, \dots, n, \\ \tau_i y, & i = n+1, \dots, 2n-1, \\ 0, & i = 2n, \end{cases} \\ \omega_i\Big|_{OP_2} &= \begin{cases} 0, & i = 1, \\ \tau_i y, & i = 2, \dots, n, \\ y, & i = n+1, \dots, 2n. \end{cases} \end{aligned} \tag{3.25}$$

Here

$$\tau_i = \begin{cases} \big[(\lambda_i - \lambda_1)^{-1}(\lambda_i - \lambda_{2n})\big]^{\frac{2}{m+2}}, & i = n+1, \dots, 2n-1, \\ \big[(\lambda_i - \lambda_{2n})^{-1}(\lambda_i - \lambda_1)\big]^{\frac{2}{m+2}}, & i = 2, \dots, n; \end{cases}$$

moreover, by (3.24) we have

$$\tau_1 = \tau_2 = 0, \quad 0 < \tau_i < 1, \quad i = 2, \dots, 2n-1. \tag{3.26}$$

Suppose

$$\varphi_i(y) = \begin{cases} w_i|_{OP_1} = w_i\Big(-\frac{2\lambda_{2n}}{m+2} y^{\frac{m+2}{2}}, y\Big), & 0 \le y \le d_1, \quad i = 1, \dots, n, \\ w_i|_{OP_2} = w_i\Big(-\frac{2\lambda_1}{m+2} y^{\frac{m+2}{2}}, y\Big), & 0 \le y \le d_2, \quad i = n+1, \dots, 2n. \end{cases}$$

Since $\alpha > 0$, it is obvious that $\varphi_i(0) = w_i(0,0) = 0$, $i = 1, \dots, 2n$.

Integrating the i-th equation of the system (3.18) along the i-th characteristic $L_i(x,y)$ from $P(x,y) \in \overline{D}_1$ to the point of intersection of $L_i(x,y)$ with the curve γ_1 for $i \le n$ and with the curve γ_2 for $i > n$, while the equation (3.19) along the first characteristic, and taking into account (3.21), we obtain

$$w_i(x,y) = \varphi_i\big(\omega_i(x,y)\big) +$$
$$+ \int\limits_{\omega_i(x,y)}^{y} \frac{1}{t}\Big(\sum_{j=1}^{2n} B_{3ij} w_j + \sum_{j=1}^{n} t C_{2ij} u_j\Big)\big(z_i(x,y;t), t\big)\, dt + F_i^2(x,y), \tag{3.27}$$
$$i = 1, \dots, 2n,$$

$$u(x,y)=f_1\big(\omega_1(x,y)\big)+$$
$$+\int\limits_{\omega_1(x,y)}^{y}(-\lambda_1K_1+K_2)w\big(z_1(x,y;t),t\big)dt, \tag{3.28}$$

where $F_i^2(x,y)=\int_{\omega_i(x,y)}^{y}F_i^1(z_i(x,y;t),t)dt$.

We rewrite the system of equations (3.27) in the form of one equation

$$w(x,y)=\widetilde{\varphi}(x,y)+$$
$$+\sum_{i=1}^{2n}\int\limits_{\omega_i(x,y)}^{y}\frac{1}{t}(B_{4i}w+C_{3i}u)\big(z_i(x,y;t),t\big)dt+F^2(x,y), \tag{3.29}$$

where B_{4i} and C_{3i} are well-defined matrices of orders $2n\times 2n$ and $2n\times n$, respectively, and $\widetilde{\varphi}(x,y)=(\varphi_1(\omega_1(x,y)),\dots,\varphi_{2n}(\omega_{2n}(x,y)))$.

Substituting the expression for the value $w(x,y)$ from (3.29) into the boundary condition (3.20) and using the equalities (3.25), we get

$$\begin{aligned}
&G_0^1\varphi^1(y)+\sum_{i=n+1}^{2n-1}G_i^1\varphi^2(\tau_iy)+T_1(w,u)(y)=f_3(y),\\
&\qquad 0\le y\le d_1,\\
&G_0^2\varphi^2(y)+\sum_{j=2}^{n}G_j^2\varphi^1(\tau_jy)+T_2(w,u)(y)=f_4(y),\\
&\qquad 0\le y\le d_2,
\end{aligned} \tag{3.30}$$

where $\varphi^1(y)=(\varphi_1(y),\dots,\varphi_n(y))$, $\varphi^2(y)=(\varphi_{n+1}(y),\dots,\varphi_{2n}(y))$; G_i^1, G_j^2 are well-defined constant $n\times n$-matrices; $T_i(w,u)$, $i=1,2$, are linear integral operators; f_3 and f_4 are given in terms of the known functions f_1, f_2, F.

Because of (3.17), (3.20) we can easily see that

$$G_0^1=\big((\lambda_1-\lambda_{2n})\nu_1,\dots,(\lambda_n-\lambda_{2n})\nu_n\big),$$
$$G_0^2=\big((\lambda_{n+1}-\lambda_1)\nu_{n+1},\dots,(\lambda_{2n}-\lambda_1)\nu_{2n}\big).$$

Therefore, when the condition (3.3) is fulfilled, the matrices G_0^1 and G_0^2 are invertible, and we can rewrite equations (3.30) equivalently as

$$\begin{aligned}
&\varphi^1(y)-\sum_{i=n+1}^{2n-1}\sum_{j=2}^{n}G_{ij}^1\varphi^1(\tau_i\tau_jy)+T_3(w,u)(y)=f_5(y),\\
&\qquad 0\le y\le d_1,\\
&\varphi^2(y)-\sum_{i=n+1}^{2n-1}\sum_{j=2}^{n}G_{ij}^2\varphi^2(\tau_i\tau_jy)+T_4(w,u)(y)=f_6(y),\\
&\qquad 0\le y\le d_2,
\end{aligned} \tag{3.31}$$

where $G_{ij}^1 = (G_0^1)^{-1}G_i^1(G_0^2)^{-1}G_j^2$, $G_{ij}^2 = (G_0^2)^{-1}G_j^2(G_0^1)^{-1}G_i^1$, and T_3 and T_4 are linear integral operators.

It is easily seen that the operators T_3 and T_4 are given by

$$\begin{aligned}
T_3(w,u)(y) &= \sum_{i=n+1}^{2n} \int_{\tau_i y}^{y} \frac{1}{t}(B_{5i}w + C_{5i}u)\big(z_i(\gamma_1(y),y;t),t\big)dt + \\
&+ \sum_{i=n+1}^{2n}\sum_{j=1}^{n} \int_{\tau_i\tau_j y}^{\tau_i y} \frac{1}{t}(B_{6ij}w + C_{6ij}u)\big(z_j(\gamma_2(\tau_i y),\tau_i y;t),t\big)dt, \\
T_4(w,u)(y) &= \sum_{j=1}^{n} \int_{\tau_j y}^{y} \frac{1}{t}(B_{7j}w + C_{7j}u)\big(z_j(\gamma_2(y),y;t),t\big)dt + \\
&+ \sum_{i=n+1}^{2n}\sum_{j=1}^{n} \int_{\tau_i\tau_j y}^{\tau_j y} \frac{1}{t}(B_{8ij}w + C_{8ij}u)\big(z_i(\gamma_1(\tau_j y),\tau_j y;t),t\big)dt.
\end{aligned} \tag{3.32}$$

Here $x = \gamma_i(y)$ is the equation of the curve γ_i, $i = 1,2$, and B_{5i}, C_{5i}, B_{6ij}, C_{6ij}, B_{7j}, C_{7j}, B_{8ij}, C_{8ij} are well-defined matrices.

By (3.23) and the requirements imposed on f_1, f_2, and F, one can easily verify that the values F^2, f_5, f_6 from (3.29), (3.31) satisfy for $\alpha > 1$ the following conditions:

$$\begin{gathered}
F^2 \in C^1_{\alpha+\frac{m}{2},\frac{m}{2}+1,1}, \quad f_{4+i} \in C^1(OP_i), \\
\sup_{OP_i\setminus O} \|y^{-(\alpha+\frac{m}{2})} f_{4+i}\| < \infty, \quad \sup_{OP_i\setminus O} \|y^{-(\alpha+\frac{m}{2}-1)} f_{4+i}^{(1)}\| < \infty, \ \ i = 1,2.
\end{gathered}$$

Remark. Obviously, the problem (3.1), (3.2) in the class (3.4) is equivalent to the system of equations (3.28), (3.29), (3.31) with respect to unknown functions u, w, φ^1 and φ^2, where

$$\begin{gathered}
u \in C^1_{\alpha+\frac{m}{2},\frac{m}{2},0}, \quad w \in C^1_{\alpha+\frac{m}{2},\frac{m}{2}+1,1}, \\
\varphi^i \in C^1_{\alpha+\frac{m}{2},1} = \Big\{\varphi^i \in C^1[0,d_i] : \sup_{0<y\le d_i} \|y^{-(\alpha+\frac{m}{2})}\varphi^i\| < \infty, \\
\sup_{0<y\le d_i} \|y^{-(\alpha+\frac{m}{2}-1)} \frac{d}{dy}\varphi^i\| < \infty\Big\}, \ \ i = 1,2.
\end{gathered}$$

Indeed, $w \in C^1_{\alpha+\frac{m}{2},\frac{m}{2}+1,1}$ implies that $v = (v_1, v_2)$ belongs to the class (3.15), and since $u_x = v_1$ and $u_y = v_2$, the function u belonging to $C^1_{\alpha+\frac{m}{2},\frac{m}{2},0}$ will also belong to $C^2_{\alpha+\frac{m}{2}+1,\frac{m}{2}+1,1}$, i.e., to the class (3.4).

§

Introduce into consideration the functional equations

$$(\Lambda_p\varphi^p)(y)=\varphi^p(y)-\sum_{i=n+1}^{2n-1}\sum_{j=2}^{n}G_{ij}^p\varphi^p(\tau_i\tau_j y)=g_p(y), \qquad (3.33)$$
$$0\le y\le d_p,\ \ p=1,2,$$

where G_{ij}^p, τ_i, τ_j are defined in (3.31).

Assume $h_p(\rho)=\sum\limits_{i=n+1}^{2n-1}\sum\limits_{j=2}^{n}(\tau_i\tau_j)^\rho\|G_{ij}^p\|$, $p=1,2$. By (3.3), (3.17), (3.20) and (3.26) we have $0<\tau_i\tau_j<1$, $\|G_{ij}^p\|\ne 0$, $i=n+1,\dots,2n-1$; $j=2,\dots,n$; $p=1,2$. Therefore the functions $h_1(\rho)$ and $h_2(\rho)$ are continuous and strictly monotonically decreasing on $(-\infty,\infty)$; moreover, $\lim\limits_{\rho\to-\infty}h_i(\rho)=+\infty$ and $\lim\limits_{\rho\to+\infty}h_i(\rho)=0$, $i=1,2$. Hence there exist the unique real numbers ρ_1 and ρ_2 such that $h_1(\rho_1)=1$ and $h_2(\rho_2)=1$. Let $\rho_0=\max(\rho_1,\rho_2)$.

According to Lemma 2.2 of Chapter II, equations (3.33) are uniquely solvable in the spaces $\overset{\circ}{C}_\alpha[0,d_p]$, $p=1,2$, for $\alpha>\rho_0$, and we have the estimates

$$\|(\Lambda_p^{-1}g_p)(y)\|=\|\varphi^p(y)\|\le\xi_{p\alpha}y^\alpha\|g_p\|_{\overset{\circ}{C}_\alpha[0,d_p]},\ \ p=1,2, \qquad (3.34)$$

where $\xi_{p\alpha}=(1-h_p(\alpha))^{-1}>0$, $\lim\limits_{\substack{\alpha\to+\infty,\\ \alpha>\rho_0}}\xi_{p\alpha}=1$, $p=1,2$.

Equations (3.31) in terms of (3.33) take the form

$$\begin{aligned}(\Lambda_1\varphi^1)(y)+T_3(w,u)(y)&=f_5(y),\quad 0\le y\le d_1,\\ (\Lambda_2\varphi^2)(y)+T_4(w,u)(y)&=f_6(y),\quad 0\le y\le d_2.\end{aligned} \qquad (3.35)$$

We shall solve the system of equations (3.28), (3.29), (3.35) with respect to unknown functions u, w, φ^1, φ^2 by the method of successive approximations.

Assume

$$u_0(x,y)\equiv 0,\quad w_0(x,y)\equiv 0,\quad \varphi_0^i(y)\equiv 0,\ \ i=1,2,$$

$$u_k(x,y)=f_1\big(\omega_1(x,y)\big)+\int\limits_{\omega_1(x,y)}^{y}(-\lambda_1K_1+K_2)w_{k-1}\big(z_1(x,y;t),t\big)\,dt,$$

$$w_k(x,y)=\widetilde{\varphi}_k(x,y)+$$
$$+\sum_{i=1}^{2n}\int\limits_{\omega_i(x,y)}^{y}\frac{1}{t}(B_{4i}w_{k-1}+C_{3i}u_{k-1})\big(z_i(x,y;t),t\big)\,dt+F^2(x,y),$$

where $\widetilde{\varphi}_k(x,y) = (\varphi_{1,k}(\omega_1(x,y)),\dots,\varphi_{2n,k}(\omega_{2n}(x,y)))$, and the values $\varphi_k^1(y) = (\varphi_{1,k}(y),\dots,\varphi_{n,k}(y))$ and $\varphi_k^2(y) = (\varphi_{n+1,k}(y),\dots,\varphi_{2n,k}(y))$ are to be determined from the equations

$$(\Lambda_1\varphi_k^1)(y) + T_3(w_{k-1},u_{k-1})(y) = f_5(y),$$
$$(\Lambda_2\varphi_k^2)(y) + T_4(w_{k-1},u_{k-1})(y) = f_6(y).$$

Remark. By virtue of (3.22), the coefficients at the unknown functions u and w appearing in the equalities (3.29) and (3.30) along with their first derivatives with respect to x are bounded uniformly in the norm in $\overline{D}_1\backslash O$.

Owing to the estimates (3.34), equality (3.32) and the above remark, we have the following

There exists a real number $\alpha_1 \geq 1$ depending only on the coefficients of the system (3.1) *such that for $\alpha > \alpha_1$ the estimates*

$$\|u_{k+1}(x,y) - u_k(x,y)\| \leq M_{1\alpha} y^{\alpha+\frac{m}{2}} q_{1\alpha}^k,$$
$$\|w_{k+1}(x,y) - w_k(x,y)\| \leq M_{1\alpha} y^{\alpha+\frac{m}{2}} q_{1\alpha}^k,$$
$$\|\varphi_{k+1}^i(y) - \varphi_k^i(y)\| \leq M_{1\alpha} y^{\alpha+\frac{m}{2}} q_{1\alpha}^k, \quad i=1,2,$$

are valid, where positive numbers $M_{1\alpha}$ and $q_{1\alpha}$ do not depend on k, $q_{1\alpha}$ as a function of α strictly monotonically decreases for $\alpha > \alpha_1$, and $q_{1\alpha} < 1$, $\lim\limits_{\alpha\to+\infty} q_{1\alpha} = 0$.

On the basis of Lemma 1 we prove

There exists a positive number α_2, $\alpha_2 \geq \alpha_1$, depending only on the coefficients of the system (3.1) *such that for $\alpha > \alpha_2$ the estimates*

$$\left\|\frac{\partial}{\partial x}u_{k+1}(x,y) - \frac{\partial}{\partial x}u_k(x,y)\right\| \leq M_{2\alpha} y^{\alpha} q_{2\alpha}^k,$$
$$\left\|\frac{\partial}{\partial y}u_{k+1}(x,y) - \frac{\partial}{\partial y}u_k(x,y)\right\| \leq M_{2\alpha} y^{\alpha+\frac{m}{2}} q_{2\alpha}^k,$$
$$\left\|\frac{\partial}{\partial x}w_{k+1}(x,y) - \frac{\partial}{\partial x}w_k(x,y)\right\| \leq M_{2\alpha} y^{\alpha-1} q_{2\alpha}^k,$$
$$\left\|\frac{\partial}{\partial y}w_{k+1}(x,y) - \frac{\partial}{\partial y}w_k(x,y)\right\| \leq M_{2\alpha} y^{\alpha+\frac{m}{2}-1} q_{2\alpha}^k,$$
$$\left\|\frac{\partial}{\partial y}\varphi_{k+1}^i(y) - \frac{\partial}{\partial y}\varphi_k^i(y)\right\| \leq M_{2\alpha} y^{\alpha+\frac{m}{2}-1} q_{2\alpha}^k, \quad i=1,2,$$

are valid. Here positive numbers $M_{2\alpha}$ and $q_{2\alpha}$ do not depend on k, $q_{2\alpha}$ as a function of α strictly monotonically decreases for $\alpha > \alpha_2$, and $q_{2\alpha} < 1$, $\lim\limits_{\alpha\to+\infty} q_{2\alpha} = 0$.

The lemma below holds.

The homogeneous system of equations corresponding to (3.28), (3.29), (3.31) *has only the trivial solution in the class of functions*

$$u, w \in C^0_{\alpha+\frac{m}{2},0,0}, \quad \varphi^i \in C[0,d_i], \quad \sup_{0<y\le d_i} \|y^{-(\alpha+\frac{m}{2})}\varphi^i\| < \infty, \quad i = 1,2,$$

where $\alpha > \alpha_2$.

From Lemmas 3.1–3.3 we have

For $\alpha > \alpha_2$ *the system of equations* (3.28), (3.29), (3.31) *has the unique solution in the class of functions*

$$u \in C^1_{\alpha+\frac{m}{2},\frac{m}{2},0}, \quad w \in C^1_{\alpha+\frac{m}{2},\frac{m}{2}+1,1}, \quad \varphi^i \in C^1_{\alpha+\frac{m}{2},1}, \quad i = 1,2.$$

From the remark at the end of §3 and Lemma 3.4 it follows that when the conditions (3.3) are fulfilled and $\alpha > \alpha_2$, the problem (3.1), (3.2) is uniquely solvable in the class (3.4); moreover, we can choose the number α_2 to depend only on the coefficients A, B, C and a of the system (3.1). ■

§

Let us consider the system of the form

$$Au_{xx} + 2y^{\frac{m}{2}}Bu_{xy} + y^m Cu_{yy} + au_x + bu_y + cu = F, \tag{3.36}$$

where A, B, C, a, b, c are given real $n\times n$-matrices, F is a given and u is an unknown n-dimensional real vector, $0 < m = \text{const} < 2$, $n > 1$.

Below A, B, C are assumed to be constant matrices, $\det A \neq 0$, and the polynomial $p_0(\mu) = \det(A\mu^2 + 2B\mu + C)$ of degree $2n$ is assumed to have simple real roots $\mu_1,\dots,\mu_{2n}$. Under these assumptions the system (3.36) for $y > 0$ is strictly hyperbolic, and the line of parabolic degeneration $y = 0$ is a characteristic of the system (3.36). It is easily seen that the numbers $y^{\frac{m}{2}}\mu_1,\dots,y^{\frac{m}{2}}\mu_{2n}$ are the roots of the characteristic polynomial $p(y;\mu) = \det(A\mu^2 + 2y^{\frac{m}{2}}B\mu + y^m C)$ of the system (3.36), while the curves defined by the equations

$$L_i(P) : \mu_i x + \frac{2}{2-m} y^{\frac{2-m}{2}} = \mu_i x_0 + \frac{2}{2-m} y_0^{\frac{2-m}{2}}, \quad i = 1,\dots,2n, \; y_0 > 0,$$

and passing through $P(x_0,y_0)$ are characteristics of the system (3.36).

Denote by D the domain lying in the half-plane $y > 0$ and bounded by the two adjoint characteristics

$$\gamma_1 : \mu_{i_1} x + \frac{2}{2-m} y^{\frac{2-m}{2}} = 0, \quad \gamma_2 : \mu_{i_2} x + \frac{2}{2-m} y^{\frac{2-m}{2}} = 0,$$
$$\mu_{i_2} < \mu_{i_1} < 0,$$

of the system (3.36), coming out of the origin $O(0,0)$. Let us take arbitrarily on γ_1 a point P_1 different from zero and choose the numbering of the characteristic curves $L_i(P_1)$, $i = 1,\dots,2n$, coming out of P_1 into the angle

D, such that starting from $L_1(P_1)$ they follow each other counter-clockwise. Let us fix on the curve γ_2 a point P_2 lying strictly between the two points of intersection of the characteristics $L_n(P_1)$ and $L_{n+1}(P_1)$ with the curve γ_2. Let $D_0 \subset D$ be the characterisitc quadrangle with a vertex at the point O, bounded by the characteristics γ_1, γ_2, $L_n(P_1)$ and $L_{n+1}(P_2)$.

Consider the characteristic problem formulated as follows [40]: find in the domain D_0 a regular solution $u(x,y)$ of the system (3.36) satisfying on the segments OP_i of characterisitcs γ_i the following conditions

$$u|_{OP_i} = f_i, \quad i = 1, 2, \tag{3.37}$$

where f_1, f_2 are given n-dimensional real vectors, $f_1(0) = f_2(0)$.

Note that owing to the character of degeneration of the system (3.36) the condition $m < 2$ whose fulfilment is not needed when considering problem (3.1), (3.2), is of great importance. In contrast to the problem (3.1), (3.2) where a condition of Gellerstedt type is imposed on the lowest coefficient a at u_x, in considering the problem (3.36), (3.37) a condition of similar type is to be imposed on the coefficient b at u_y.

Below we assume that a, b, c, $F \in C^1(\overline{D}_0)$, $f_i \in C^2(OP_i)$, $i = 1, 2$, and moreover, in the domain D_0

$$\sup_{\overline{D}_0 \setminus O} \|y^{1-m} b\| < \infty \quad \text{for} \quad m > 1,$$

$$\sup_{\overline{D}_0 \setminus O} \|x^{-(\alpha + \frac{m}{2-m} - 1)} F\| < \infty, \quad \sup_{\overline{D}_0 \setminus O} \|x^{-(\alpha - 2)} F_y\| < \infty, \quad \alpha = \text{const} > 0,$$

$$f_i(0) = 0, \quad \sup_{OP_i \setminus O} \|x^{-(\alpha + \frac{m}{2-m})} f_i^{(1)}\| < \infty,$$

$$\sup_{OP_i \setminus O} \|x^{-(\alpha + \frac{m}{2-m} - 1)} f_i^{(2)}\| < \infty, \quad i = 1, 2.$$

Since the system (3.36) is strictly hyperbolic, we have $\dim \operatorname{Ker}(A\mu_i^2 + 2B\mu_i + C) = 1$, $i = 1, \dots, 2n$. Let $\nu_i \in \operatorname{Ker}(A\mu_i^2 + 2B\mu_i + C)$, $\|\nu_i\| \neq 0$, $i = 1, \dots, 2n$.

Under the assumption that $\mu_n < \mu_{2n} < \mu_{n+1}$, the following theorem is valid.

If

$$\operatorname{rank}\{\nu_1, \dots, \nu_n\} = \operatorname{rank}\{\nu_{n+1}, \dots, \nu_{2n}\} = n, \tag{3.38}$$

then there exists a positive number α_0 *depending only on the coefficients* A, B, C, b *of the system* (3.36) *such that for all* $\alpha > \alpha_0$ *the problem* (3.36), (3.37) *is uniquely solvable in the class of functions*

$$\Big\{ u \in C^2(\overline{D}_0) : \partial^{i,j} u(0,0) = 0,$$

$$\sup_{\overline{D}_0 \setminus O} \|x^{-(\alpha + \frac{2}{2-m} - i - \frac{2}{2-m} j)} \partial^{i,j} u\| < \infty, 0 \le i + j \le 2 \Big\},$$

$$\partial^{i,j} = \frac{\partial^{i+j}}{\partial x^i \partial y^j} \, .$$

As examples show, when either the condition (3.38) or the inequality $\alpha > \alpha_0$ is violated, the homogeneous problem corresponding to (3.36), (3.37) may have an infinite number of linearly independent solutions.

The proof of Theorem 3.2 goes by the same scheme as that of Theorem 3.1. For details the reader may refer to [40].

§

In the space of variables x_1, x_2, t let us consider the wave equation

$$\Box u \equiv \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = F, \tag{4.1}$$

where F is a given real function and u is an unknown real function.

Denote by $D : k_1 t < x_2 < k_2 t$, $0 < t < t_0$, $-1 \leq k_i = \text{const} \leq 1$, $i = 1, 2$, $k_1 < k_2$, the domain lying in a half-space $t > 0$ bounded by the plane surfaces $S_i : k_i t - x_2 = 0$, $0 \leq t \leq t_0$, $i = 1, 2$, and by the plane $t = t_0$.

For equation (4.1) let us consider the boundary value problem formulated as follows [44, 45]: find in the domain D a solution $u(x_1, x_2, t)$ of equation (4.1) satisfying the boundary conditions

$$u\big|_{S_i} = f_i, \quad i = 1, 2, \tag{4.2}$$

where f_i, $i = 1, 2$, are given real functions on S_i with $(f_1 - f_2)|_{S_1 \cap S_2} = 0$.

Note that when $|k_i| = 1$, the surface S_i is a characteristic surface for the equation (4.1), while when $|k_i| < 1$, this surface is time-type. In the case where $|k_i| = 1$, $i = 1, 2$, the problem (4.1), (4.2) represents a multidimensional analogue of the formulated in the introduction Goursat problem for the equation of string oscillation. For $|k_1| < 1$ and $|k_2| = 1$ the problem (4.1), (4.2) represents a multidimensional analogue of the first Darboux problem and for $|k_i| < 1$, $i = 1, 2$, it represents a multidimensional analogue of the second Darboux problem.

For the equation (4.1) one can also consider the boundary value problem formulated as follows: find in the domain D a solution $u(x_1, x_2, t)$ of equation (4.1) satisfying the boundary conditions

$$\frac{\partial u}{\partial n}\bigg|_{S_1} = f_1, \tag{4.3}$$

$$u\big|_{S_2} = f_2, \tag{4.4}$$

where f_i, $i = 1, 2$, are given real functions and $\frac{\partial}{\partial n}$ is the derivative along the outer normal to S_1.

Below we shall prove existence and uniqueness theorems both for regular and for strong solutions of the problems (4.1), (4.2) and (4.1), (4.3), (4.4) in the class W_2^1.

Denote by $C_*^\infty(\overline{D})$ the space of functions of the class $C^\infty(\overline{D})$, having bounded supports, i.e.

$$C_*^\infty(\overline{D}) = \big\{u \in C^\infty(\overline{D}) : \operatorname{diam} \operatorname{supp} u < \infty\big\}.$$

The spaces $C_*^\infty(S_i)$, $i = 1,2$, are defined in a similar way.

Denote by $W_2^1(D)$, $W_2^2(D)$ and $W_2^1(S_i)$, $i = 1,2$, the well-known Sobolev spaces. Note that $C_*^\infty(\overline{D})$ is an everywhere dense subspace of the spaces $W_2^1(D)$ and $W_2^2(D)$, while $C_*^\infty(S_i)$ is an everywhere dense subspace of the space $W_2^1(S_i)$, $i = 1,2$.

Let $f_i \in W_2^1(S_i)$, $i = 1,2$, $F \in L_2(D)$. A function $u \in W_2^1(D)$ is said to be a strong solution of the problem (4.1), (4.2) of the class W_2^1 if there exists a sequence $u_n \in C_*^\infty(\overline{D})$ such that $u_n \to u$, $\Box u_n \to F$ and $u_n|_{S_i} \to f_i$ $i = 1,2$, in the spaces $W_2^1(D)$, $L_2(D)$ and $W_2^1(S_i)$, $i = 1,2$, respectively, i.e., for $n \to \infty$

$$\|u_n - u\|_{W_2^1(D)} \to 0, \quad \|\Box u_n - F\|_{L_2(D)} \to 0,$$
$$\|u_n|_{S_i} - f_i\|_{W_2^1(S_i)} \to 0, \;\; i = 1,2.$$

Below we shall also introduce the notion of the strong solution of the problem (4.1), (4.3), (4.4) in the class W_2^1.

§

The following lemma holds.

When $-1 \le k_1 < 0$ and $0 < k_2 \le 1$, the estimate

$$\|u\|_{W_2^1(D)} \le C\Big(\sum_{i=1}^{2} \|f_i\|_{W_2^1(S_i)} + \|F\|_{L_2(D)}\Big), \tag{4.5}$$

is valid for any $u \in W_2^2(D)$, where $f_i = u|_{S_i}$, $i = 1,2$, $F = \Box u$, and a positive constant C does not depend on u.

Proof. Since the space $C_*^\infty(D)$ ($C_*^\infty(S_i)$) is a dense subspace of the spaces $W_2^1(D)$ and $W_2^2(D)$ ($W_2^1(S_i)$), due to the known theorems of embedding of $W_2^2(D)$ in $W_2^1(D)$ and $W_2^2(D)$ in $W_2^1(S_i)$ it suffices to prove the validity of the estimate (4.5) for the functions u of the class $C_*^\infty(\overline{D})$.

Introduce the notation:

$$D_\tau = \big\{(x,t) \in D : t < \tau\big\}, \quad D_{0\tau} = \partial D_\tau \cap \{t = \tau\}, \quad 0 < \tau \le t_0,$$
$$S_{i\tau} = \partial D_\tau \cap S_i, \quad i = 1,2,$$
$$S_\tau = S_{1\tau} \cup S_{2\tau}, \quad \alpha_1 = \cos(\widehat{n, x_1}), \quad \alpha_2 = \cos(\widehat{n, x_2}), \quad \alpha_3 = \cos(\widehat{n, t}).$$

Here $n=(\alpha_1,\alpha_2,\alpha_3)$ is the unit vector of the outer normal to ∂D_τ; moreover, as is easily seen,

$$n\big|_{S_{1\tau}}=\Big(0,\frac{-1}{\sqrt{1+k_1^2}},\frac{k_1}{\sqrt{1+k_1^2}}\Big),\quad n\big|_{S_{2\tau}}=\Big(0,\frac{1}{\sqrt{1+k_2^2}},\frac{-k_2}{\sqrt{1+k_2^2}}\Big),$$
$$n|_{D_{0\tau}}=(0,0,1).$$

Therefore for $-1\le k_1<0$ and $0<k_2\le 1$ we have

$$\alpha_3\big|_{S_{i\tau}}<0,\ \ i=1,2,\quad \alpha_3^{-1}(\alpha_3^2-\alpha_1^2-\alpha_2^2)\big|_{S_i}>0,\ \ i=1,2. \qquad (4.6)$$

Multiplying both parts of (4.1) by $2u_t$, where $u\in C_*^\infty(\overline{D})$, $F=\Box u$, integrating the obtained expression over D_τ, and taking into account (4.6), we get

$$\begin{aligned}
2\int\limits_{D_\tau}Fu_t dxdt&=\int\limits_{D_\tau}\Big(\frac{\partial u_t^2}{\partial t}+2u_{x_1}u_{tx_1}+2u_{x_2}u_{tx_2}\Big)dxdt-\\
-2\int\limits_{S_\tau}&\big(u_{x_1}u_t\alpha_1+u_{x_2}u_t\alpha_2\big)ds=\int\limits_{D_{0\tau}}\big(u_t^2+u_{x_1}^2+u_{x_2}^2\big)dx+\\
+\int\limits_{S_\tau}&\big[\big(u_t^2+u_{x_1}^2+x_{x_2}^2\big)\alpha_3-2\big(u_{x_1}u_t\alpha_1+u_{x_2}u_t\alpha_2\big)\big]ds=\\
=\int\limits_{D_{0\tau}}&\big(u_t^2+u_{x_1}^2+u_{x_2}^2\big)dx+\int\limits_{S_\tau}\alpha_3^{-1}\big[\big(\alpha_3u_{x_1}-\alpha_1u_t\big)^2+\\
&+\big(\alpha_3u_{x_2}-\alpha_2u_t\big)^2+\big(\alpha_3^2-\alpha_1^2-\alpha_2^2\big)u_t^2\big]ds\ge\\
\ge\int\limits_{D_{0\tau}}&\big(u_t^2+u_{x_1}^2+u_{x_2}^2\big)dx+\int\limits_{S_\tau}\alpha_3^{-1}\big[\big(\alpha_3u_{x_1}-\alpha_1u_t\big)^2+\\
&+\big(\alpha_3u_{x_2}-\alpha_2u_t\big)^2\big]ds. \qquad (4.7)
\end{aligned}$$

Putting

$$w(\tau)=\int\limits_{D_{0\tau}}\big(u_t^2+u_{x_1}^2+u_{x_2}^2\big)dx,\quad \widetilde{u}_i=\alpha_3u_{x_i}-\alpha_iu_t,\quad i=1,2,$$
$$C_1=\max\Big(\frac{\sqrt{1+k_1^2}}{|k_1|},\frac{\sqrt{1+k_2^2}}{|k_2|}\Big),$$

from (4.5) we have

$$w(\tau)\le C_1\int\limits_{S_\tau}\big(\widetilde{u}_1^2+\widetilde{u}_2^2\big)ds+\int\limits_{D_\tau}\big(F^2+u_t^2\big)dxdt\le$$

$$\leq C_1 \int\limits_{S_\tau} \big(\widetilde{u}_1^2 + \widetilde{u}_2^2\big) ds + \int\limits_0^\tau d\xi \int\limits_{D_{0\xi}} u_t^2 dx + \int\limits_{D_\tau} F^2 dx dt \leq$$

$$\leq C_1 \int\limits_{S_\tau} \big(\widetilde{u}_1^2 + \widetilde{u}_2^2\big) ds + \int\limits_0^\tau w(\xi) d\xi + \int\limits_{D_\tau} F^2 dx dt. \tag{4.8}$$

Let (x, τ_x) be the point of intersection of the surface $S_1 \cup S_2$ and the straight line, parallel to the axis t and passing through $(x, 0)$. We have

$$u(x,\tau) = u(x,\tau_x) + \int\limits_{\tau_x}^{\tau} u_t(x,t) dt,$$

which implies

$$\int\limits_{D_{0\tau}} u^2(x,\tau) dx \leq$$

$$\leq 2 \int\limits_{D_{0\tau}} u^2(x,\tau_x) dx + 2|\tau - \tau_x| \int\limits_{D_{0\tau}} dx \int\limits_{\tau_x}^{\tau} u_t^2(x,t) dt =$$

$$= 2 \int\limits_{S_\tau} \alpha_3^{-1} u^2 ds + 2|\tau - \tau_x| \int\limits_{D_\tau} u_t^2 dx dt \leq$$

$$\leq C_2 \Big(\int\limits_{S_\tau} u^2 ds + \int\limits_{D_\tau} u_t^2 dx dt \Big), \tag{4.9}$$

where $C_2 = 2\max(C_1, t_0)$.

Introducing the notation

$$w_0(\tau) = \int\limits_{D_{0\tau}} \big(u^2 + u_t^2 + u_{x_1}^2 + u_{x_2}^2\big) dx$$

and adding inequalities (4.8) and (4.9), we obtain

$$w_0(\tau) \leq C_2 \Big[\int\limits_{S_\tau} \big(u^2 + \widetilde{u}_1^2 + \widetilde{u}_2^2\big) ds + \int\limits_0^\tau w_0(\xi) d\xi + \int\limits_{D_\tau} F^2 dx dt \Big],$$

from which by Gronwall's lemma we find that

$$w_0(\tau) \leq C_3 \Big[\int\limits_{S_\tau} \big(u^2 + \widetilde{u}_1^2 + \widetilde{u}_2^2\big) ds + \int\limits_{D_\tau} F^2 dx dt \Big], \tag{4.10}$$

where $C_3 = \text{const} > 0$.

We can easily see that the operator $\alpha_3\frac{\partial}{\partial x_i}-\alpha_i\frac{\partial}{\partial t}$ is an interior differential operator on the surface S_τ. Therefore by virtue of (4.2) the inequality

$$\int\limits_{S_\tau}\big(u^2+\widetilde{u}_1^2+\widetilde{u}_2^2\big)ds\le C_4\sum_{i=1}^{2}\|f_i\|^2_{W_2^1(S_{i\tau})},\quad C_4=\text{const}>0,\quad(4.11)$$

is valid.

It follows from (4.10) and (4.11) that

$$w_0(\tau)\le C_5\Big(\sum_{i=1}^{2}\|f_i\|^2_{W_2^1(S_{i\tau})}+\|F\|^2_{L_2(D_\tau)}\Big),\quad C_5=\text{const}>0.\ (4.12)$$

Integrating both parts of (4.12) with respect to τ, we get (4.5). ■

Remark 1. It should be noted that the constant C in (4.5) tends to infinity for $k_1\to 0$ or $k_2\to 0$ and in the limiting case where $k_1=0$ or $k_2=0$, i.e., for $S_1:x_2=0,\ 0\le t\le t_0$ or $S_2:x_2=0,\ 0\le t\le t_0$, this estimate becomes, generally speaking, invalid. At the same time, following the proof of Lemma 4.1, we can easily see that (4.5) is also valid for $k_1=0$ or for $k_2=0$ if $f_1=u|_{S_1}=0$ or $f_2=u|_{S_2}=0$.

Remark 2. Below along with (4.1) we consider the equation

$$Lu\equiv\Box u+au_{x_1}+bu_{x_2}+cu_t+du=F,\qquad(4.13)$$

where the coefficients a, b, c and d are given bounded measurable functions in the domain D. Moreover, it will be shown that the solvability of the problem (4.13), (4.2) follows from the solvability of the problem (4.1), (4.2) and the fact that in specifically chosen equivalent norms of the spaces $L_2(D)$, $W_2^1(D)$, $W_2^1(S_i)$, $i=1,2$, the lower terms in equation (4.13) cause arbitrarily small perturbations.

In the space $W_2^1(D)$ we consider the following equivalent norm depending on a parameter γ

$$\|u\|^2_{D,1,\gamma}=\int\limits_D e^{-\gamma t}\big(u^2+u_t^2+u_{x_1}^2+u_{x_2}^2\big)dxdt,\quad\gamma>0.$$

In the same way we introduce the norms $\|F\|_{D,0,\gamma}$, $\|f_i\|_{S_i,1,\gamma}$ in the spaces $L_2(D)$, $W_2^1(S_i)$, $i=1,2$.

Using the energetic estimate (4.12), we obtain an a priori estimate for $u\in C_*^\infty(\overline{D})$ with respect to the norms $\|\cdot\|_{D,1,\gamma}$, $\|\cdot\|_{S_i,1,\gamma}$, $\|\cdot\|_{D,0,\gamma}$. Multiplying both parts of (4.12) by $e^{-\gamma\tau}$ and integrating the obtained inequality with respect to τ from 0 to t_0, we get

$$\|u\|^2_{D,1,\gamma}=\int\limits_0^{t_0}e^{-\gamma\tau}w_0(\tau)d\tau\le$$

$$\le C_5\Big(\sum_{i=1}^{2}\int_0^{t_0} e^{-\gamma\tau}\|f_i\|^2_{W_2^1(S_{i\tau})}d\tau+\int_0^{t_0} e^{-\gamma\tau}\|F\|^2_{L_2(D_\tau)}d\tau\Big). \tag{4.14}$$

We have

$$\int_0^{t_0} e^{-\gamma\tau}\|F\|^2_{L_2(D_\tau)}d\tau=\int_0^{t_0} e^{-\gamma\tau}\Big[\int_0^{\tau}\Big(\int_{D_{0\sigma}} F^2dx\Big)d\sigma\Big]d\tau=$$

$$=\int_0^{t_0}\Big[\int_{D_{0\tau}} F^2dx\int_{\sigma}^{t_0} e^{-\gamma\tau}d\tau\Big]d\sigma=\frac{1}{\gamma}\int_0^{t_0}\big(e^{-\gamma\sigma}-e^{-\gamma t_0}\big)\Big[\int_{D_{0\sigma}} F^2dx\Big]d\sigma\le$$

$$\le\frac{1}{\gamma}\int_0^{t_0} e^{-\gamma\sigma}\Big[\int_{D_{0\sigma}} F^2dx\Big]d\sigma=\frac{1}{\gamma}\|F\|^2_{D,0,\gamma}, \tag{4.15}$$

where $D_{0\tau}=\partial D_\tau\cap\{t=\tau\}$, $0<\tau\le t_0$.

Analogously we obtain

$$\int_0^{t_0} e^{-\gamma\tau}\|f_i\|^2_{W_2^1(S_{i\tau})}d\tau\le\frac{C_6}{\gamma}\|f_i\|^2_{S_i,1,\gamma},\quad i=1,2, \tag{4.16}$$

where C_6 is a positive constant independent of f_i and γ.

Under conditions of Lemma 4.1 from (4.14)–(4.16) we obtain the following a priori estimate for $u\in W_2^2(D)$

$$\|u\|_{D,1,\gamma}\le\frac{C_7}{\sqrt{\gamma}}\Big(\sum_{i=1}^{2}\|f_i\|_{S_i,1,\gamma}+\|F\|_{D,0,\gamma}\Big), \tag{4.17}$$

where $C_7=\text{const}>0$ does not depend on u and γ.

Consider now the problem (4.1), (4.3), (4.4) in the case where $k_1=0$, i.e., $S_1: x_2=0$, $0\le t\le t_0$, $0<k_2\le 1$, and in the boundary condition (4.3) the function $f_1=0$, that is,

$$\frac{\partial u}{\partial n}\Big|_{S_1}=0. \tag{4.18}$$

For any $u\in W_2^2(D)$ *satisfying the homogeneous boundary condition* (4.18), *the estimate*

$$\|u\|_{W_2^1(D)}\le C\big(\|f_2\|_{W_2^1(S_1)}+\|F\|_{L_2(D)}\big) \tag{4.19}$$

is valid, where $f_2=u|_{S_2}$, $F=\Box u$ *and a positive constant* C *does not depend on* u.

Proof. Denote by $D_- : -k_2 t < x_2 < 0$, $0 < t < t_0$ the domain which is symmetric to D with respect to the plane $x_2 = 0$ and by $D_0 : -k_2 t < x_2 < k_2 t$, $0 < t < t_0$ the domain which is the union of domains D and D_- together with a piece of a plane surface $x_2 = 0$, $0 < t < t_0$.

It can be easily verified that if we extend evenly the function $u \in W_2^2(D)$ satisfying the homogeneous boundary condition (4.18) to the domain D_-, then the obtained function u_0

$$u_0(x_1, x_2, t) = \begin{cases} u(x_1, x_2, t), & x_2 \geq 0, \\ u(x_1, -x_2, t), & x_2 < 0, \end{cases} \tag{4.20}$$

will belong to the class $W_2^2(D_0)$. By (4.5) the function $u_0 \in W_2^2(D_0)$ satisfies the estimate

$$\|u_0\|_{W_2^1(D_0)} \leq C\big(\|f_1\|_{W_2^1(S_2^-)} + \|f_2\|_{W_2^1(S_2)} + \|F_0\|_{L_2(D_0)}\big), \tag{4.21}$$

where $S_2^- : k_2 t + x_2 = 0$, $0 \leq t \leq t_0$, $f_1 = u|_{S_2^-}$, $f_2 = u|_{S_2}$, $F_0 = \Box u_0$.

Now it remains only to note that in (4.21)

$$\|u_0\|_{W_2^1(D_0)} = \sqrt{2}\|u\|_{W_2^1(D)}, \quad \|f_1\|_{W_2^1(S_2^-)} = \|f_2\|_{W_2^1(S_2)},$$
$$\|F_0\|_{L_2(D_0)} = \sqrt{2}\|F\|_{L_2(D)}$$

because of (4.20). ■

Remark 3. Arguments similar to those given in proving the estimate (4.17) enable us to prove that for any $u \in W_2^2(D)$ satisfying the homogeneous boundary condition (4.18) the estimate

$$\|u\|_{D,1,\gamma} \leq \frac{C}{\sqrt{\gamma}}\big(\|f_2\|_{S_{2,1,\gamma}} + \|F\|_{D,0,\gamma}\big), \tag{4.22}$$

is valid, where $f_2 = u|_{S_2}$, $F = \Box u$, and C is a positive constant independent of u and γ.

Remark 4. It follows from (4.5) and (4.19) that when conditions of Lemmas 4.1 and 4.2 are fulfilled, the problems (4.1), (4.2) and (4.1), (4.3), (4.4), respectively, cannot have more than one strong solution of the class W_2^1.

We can also show that for the problem (4.1), (4.2) the uniqueness theorem is likewise valid for the weak solution of the class W_2^1.

Let $k_1 = 0$ and $k_2 = 1$, i.e., $S_1 : x_2 = 0$, $0 \leq t \leq t_0$, while $S_2 : t - x_2 = 0$, $0 \leq t \leq t_0$ is a characteristic surface. Suppose $S_3 = \partial D \cap \{t = t_0\}$, $V = \{v \in W_2^1(D) : v|_{S_1 \cup S_3} = 0\}$.

Let $f_i \in W_2^1(S_i)$, $i = 1, 2$, $F \in L_2(D)$. A function $u \in W_2^1(D)$ is called a weak solution of the problem (4.1), (4.2) of the class W_2^1 if it satisfies both the boundary conditions (4.2) and the identity

$$\int\limits_D (u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2}) dx dt + \\ + \int\limits_{S_2} \frac{\partial f_2}{\partial N} v ds + \int\limits_D F v dx dt = 0 \tag{4.23}$$

for any $v \in V$, where $\frac{\partial}{\partial N}$ is a derivative with respect to a conormal to S_2, N is the unit conormal vector at the point $(x, t) \in \partial D$ with the direction cosines

$$\cos \widehat{N x_1} = \cos \widehat{n x_1}, \quad \cos \widehat{N x_2} = \cos \widehat{n x_2}, \quad \cos \widehat{N t} = -\cos \widehat{n t},$$

and n is the unit vector of the outward normal to ∂D. Since on the characteristic surface S_2 the direction of the conormal N coincides with that of a bicharacteristic lying on S_2, the value $\frac{\partial f_2}{\partial N}$ is determined correctly.

For $k_1 = 0$, $k_2 = 1$ *the problem* (4.1), (4.2) *cannot have more than one weak solution of the class* W_2^1.

Proof. Let a function $u \in W_2^1(D)$ satisfy the identity (4.23) with $u|_{S_i} = f_i = 0$, $i = 1, 2$, $F = 0$. In this identity we take as v the function

$$v(x_1, x_2, t) = \begin{cases} 0 & \text{for} \quad t \geq \tau, \\ \int_\tau^t u(x_1, x_2, \sigma) d\sigma & \text{for} \quad |x_2| \leq t \leq \tau, \end{cases} \tag{4.24}$$

where $0 < \tau \leq t_0$.

Obviously, $v \in V$ and

$$v_t = u, \quad v_{x_i} = \int\limits_\tau^t u_{x_i}(x_1, x_2, \sigma) d\sigma, \quad i = 1, 2, \\ v_{t x_i} = u_{x_i}, \quad v_{tt} = u_t. \tag{4.25}$$

By virtue of (4.24) and (4.25) the identity (4.23) for $f_2 = 0$, $F = 0$ will take the form

$$\int\limits_{D_\tau} \big(v_{tt} v_t - v_{t x_1} v_{x_1} - v_{t x_2} v_{x_2}\big) dx dt = 0,$$

i.e.,

$$\int\limits_{D_\tau} \frac{\partial}{\partial t} \big(v_t^2 - v_{x_1}^2 - v_{x_2}^2\big) dx dt = 0, \tag{4.26}$$

where $D_\tau = D \cap \{t < \tau\}$.

Applying Gauss-Ostrogradsky's formula to the left-hand side of (4.26), we obtain

$$\int\limits_{\partial D_\tau} \left(v_t^2 - v_{x_1}^2 - v_{x_2}^2\right) \cos\widehat{nt} ds = 0. \tag{4.27}$$

Since $\partial D_\tau = S_{1\tau} \cup S_{2\tau} \cup S_{3\tau}$, for $S_{i\tau} = \partial D_\tau \cap S_i$, $i = 1,2$, $S_{3\tau} = \partial D_\tau \cap \{t = \tau\}$ and

$$\cos\widehat{nt}\big|_{S_{1\tau}} = 0, \quad \cos\widehat{nt}\big|_{S_{2\tau}} = -\frac{1}{\sqrt{2}}, \quad \cos\widehat{nt}\big|_{S_{3\tau}} = 1,$$

$$u\big|_{S_{i\tau}} = f_i = 0, \quad i = 1,2, \quad v_{x_i}\big|_{S_{3\tau}} = 0, \quad i = 1,2, \quad v_t = u,$$

it follows from (4.27) that

$$\int\limits_{S_{3\tau}} u^2 dx_1 dx_2 + \frac{1}{\sqrt{2}} \int\limits_{S_{2\tau}} \left(v_{x_1}^2 + v_{x_2}^2\right) ds = 0.$$

Hence, $u|_{S_{3\tau}} = 0$ for any τ from $(0, t_0]$. Therefore, $u \equiv 0$ in D. ■

It should be noted that Lemma 4.3 is also valid for $k_1 = -1$, $k_2 = 1$.

Remark 5. Since the strong solution of the problem (4.1), (4.2) of the class W_2^1 is at the same time a weak solution of the class W_2^1, it follows from Lemma 4.3 that if the strong solution of that problem of the class W_2^1 exists, then the same solution will be the unique weak solution of the class W_2^1.

§

For a point $P_0(x_1^0, x_2^0, t^0) \in D$ the domain of dependence of the solution $u(x_1, x_2, t)$ of the problem (4.1), (4.2) *of the class $C^2(\overline{D})$ or $W_2^2(D)$ is contained inside the characteristic cone of the past*

$$\partial K_{P_0} : t = t^0 - \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2}$$

with the vertex at P_0.

Proof. Suppose

$$\Omega_{P_0} = D \cap K_{P_0}, \quad S_{iP_0} = S_i \cap \partial\Omega_{P_0}, \quad i = 1,2,$$

where $K_{P_0} : t < t^0 - \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2}$ is the interior of ∂K_{P_0}.

To prove the above lemma it suffices to show that if

$$f_i\big|_{S_{iP_0}} \equiv u\big|_{S_{iP_0}} = 0, \quad i = 1,2, \quad F\big|_{\Omega_{P_0}} \equiv \Box u\big|_{\Omega_{P_0}} = 0, \tag{4.28}$$

then $u|_{\Omega_{P_0}} = 0$.

Consider first the case $u \in C^2(\overline{D})$. Denote by S_{3P_0} the remainder part of the boundary of Ω_{P_0}, i.e., $S_{3P_0} = \partial\Omega_{P_0} \backslash (S_{1P_0} \cup S_{2P_0})$. According to our construction, the surface S_{3P_0} is a part of ∂K_{P_0}. Therefore

$$\alpha_3\big|_{S_{3P_0}} = \text{const} > 0, \quad \big(\alpha_3^2 - \alpha_1^2 - \alpha_2^2\big)\big|_{S_{3P_0}} = 0, \tag{4.29}$$

where $n = (\alpha_1, \alpha_2, \alpha_3)$ is the unit vector of outward normal to $\partial\Omega_{P_0}$.

Multiplying both parts of (4.1) by $2u_t$ and integrating the obtained expression over Ω_{P_0}, taking into account (4.6), (4.28), (4.29) and the arguments we used in obtaining inequality (4.7), we get

$$0 = 2\int\limits_{\Omega_{P_0}} Fu_t dxdt =$$

$$= \int\limits_{\partial\Omega_{P_0}} \big[\big(u_t^2 + u_{x_1}^2 + u_{x_2}^2\big)\alpha_3 - 2\big(u_{x_1}u_t\alpha_1 + u_{x_2}u_t\alpha_2\big)\big]ds =$$

$$= \int\limits_{\partial\Omega_{P_0}} \alpha_3^{-1}\big[\big(\alpha_3 u_{x_1} - \alpha_1 u_t\big)^2 + \big(\alpha_3 u_{x_2} - \alpha_2 u_t\big)^2 +$$

$$+ \big(\alpha_3^2 - \alpha_1^2 - \alpha_2^2\big)u_t^2\big]ds \geq$$

$$\geq \int\limits_{S_{3P_0}} \alpha_3^{-1}\big[\big(\alpha_3 u_{x_1} - \alpha_1 u_t\big)^2 + \big(\alpha_3 u_{x_2} - \alpha_2 u_t\big)^2\big]ds. \tag{4.30}$$

When deducing inequality (4.30), we have used the fact that the operator $\alpha_3 \frac{\partial}{\partial x_i} - \alpha_i \frac{\partial}{\partial t}$ is an inner differential operator on the surface $\partial\Omega_{P_0}$ and, in particular, by virtue of (4.28) the equalities

$$\Big(\alpha_3 \frac{\partial u}{\partial x_i} - \alpha_i \frac{\partial u}{\partial t}\Big)\Big|_{S_{1P_0} \cup S_{2P_0}} = 0, \quad i = 1, 2,$$

hold on $S_{1P_0} \cup S_{2P_0}$.

Since $\alpha_3 > 0$ on S_{3P_0}, inequality (4.30) implies

$$\big(\alpha_3 u_{x_i} - \alpha_i u_t\big)\big|_{S_{3P_0}} = 0, \quad i = 1, 2. \tag{4.31}$$

Taking into account that $u \in C^2(\overline{D})$ and the inner differential operators $\alpha_3 \frac{\partial}{\partial x_i} - \alpha_i \frac{\partial}{\partial t}$, $i = 1, 2$, are linearly independent on the two-dimensional connected surface S_{3P_0}, (4.31) immediately yields

$$u\big|_{S_{3P_0}} \equiv \text{const}\,. \tag{4.32}$$

But because of (4.28)

$$u\big|_{S_{3P_0} \cap (S_{1P_0} \cup S_{2P_0})} = 0,$$

from which due to (4.32) we conclude that

$$u\big|_{S_{3P_0}} \equiv 0. \tag{4.33}$$

In particular, (4.33) implies $u(P_0) = 0$.

If now we take an arbitrary point $Q \in \Omega_{P_0}$, then (4.28) implies the validity of the same equalities after substitution of the point P_0 by Q. Therefore, repeating the above arguments for the domain Ω_Q, we obtain $u(Q) = 0$. Hence, in the case $u \in C^2(\overline{D})$ we have $u|_{\Omega_{P_0}} = 0$.

Let now $u \in W_2^2(D)$ and equalities (4.28) be valid. It can be easily verified that for any point $Q \in \Omega_{P_0}$ the inequality (4.30) is also valid after substitution of the point P_0 by Q, that is

$$\int\limits_{S_{3Q}} \alpha_3^{-1}\big[\big(\alpha_3 u_{x_1} - \alpha_1 u_t\big)^2 + \big(\alpha_3 u_{x_2} - \alpha_2 u_t\big)^2\big]ds \le 0.$$

whence, due to the fact that $\alpha_3|_{S_{3Q}} = \text{const} > 0$, we get

$$\int\limits_{S_{3Q}} \big[\big(\alpha_3 u_{x_1} - \alpha_1 u_t\big)^2 + \big(\alpha_3 u_{x_2} - \alpha_2 u_t\big)^2\big]ds = 0. \qquad (4.34)$$

Denote by Γ_Q a piecewise smooth curve which at the same time is the boundary of a two-dimensional connected surface S_{3Q}. Obviously,

$$\Gamma_Q = S_{3Q} \cup (S_{1Q} \cup S_{2Q}). \qquad (4.35)$$

Using the fact that on S_{3Q} inner differential operators $\alpha_3 \frac{\partial}{\partial x_i} - \alpha_i \frac{\partial}{\partial t}$, $i = 1, 2$, are independent, it is not difficult to obtain for any $v \in W_2^1(S_{3Q})$ the following estimate

$$\int\limits_{S_{3Q}} v^2 ds \le C\Big(\int\limits_{\Gamma_Q} v^2 ds +$$

$$+ \int\limits_{S_{3Q}} \big[\big(\alpha_3 v_{x_1} - \alpha_1 v_t\big)^2 + \big(\alpha_3 v_{x_2} - \alpha_2 v_t\big)^2\big]ds\Big), \qquad (4.36)$$

where $C = \text{const} > 0$ does not depend on v, and the trace $v|_{\Gamma_Q} \in L_2(\Gamma_Q)$ is correctly determined in virtue of the corresponding embedding theorem.

Since $u \in W_2^2(D)$, the traces $u|_{S_{3Q}} \in W_2^1(S_{3Q})$ and $u|_{\Gamma_Q} \in L_2(\Gamma_Q)$ are correctly determined in virtue of the embedding theorems. Therefore, due to (4.25) and (4.35) we have

$$u\big|_{\Gamma_Q} = 0. \qquad (4.37)$$

From (4.34), (4.36) and (4.37) we obtain

$$\int\limits_{S_{3Q}} u^2 ds \le$$

$$\le C\Big(\int\limits_{\Gamma_Q} u^2 ds + \int\limits_{S_{3Q}} \big[\big(\alpha_3 u_{x_1} - \alpha_1 u_t\big)^2 + \big(\alpha_3 u_{x_2} - \alpha_2 u_t\big)^2\big]ds\Big) = 0,$$

from which it immediately follows that

$$\int\limits_{S_{3Q}} u^2 ds = 0, \quad u\big|_{S_{3Q}} = 0, \quad \forall Q \in \Omega_{P_0}. \tag{4.38}$$

Since $u \in W_2^2(D)$, in virtue of (4.38) and applying Fubini's theorem we can conclude that

$$u\big|_{\Omega_{P_0}} = 0. \quad \blacksquare$$

Remark 1. The assertion of Lemma 4.4 is also valid for the problem (4.1), (4.3), (4.4). Moreover, the above arguments should be modified only on the part S_{1P_0} of the boundary Ω_{P_0}. In this case for $k_1 = 0$ and due to (4.18) we have

$$\int\limits_{S_{1P_0}} \left[(u_t^2 + u_{x_1}^2 + u_{x_2}^2)\alpha_3 - 2(u_{x_1} u_t \alpha_1 + u_{x_2} u_t \alpha_2) \right] ds = 0.$$

Remark 2. It follows from Lemma 4.4 that the wave process described by the problem (4.1), (4.2) or (4.1), (4.3), (4.4) propagates with a finite speed. Therefore, if $u \in C^\infty(\overline{D})$ is a solution of the problem (4.1), (4.2) or (4.1), (4.3), (4.4) for $f_i \in C_*^\infty(S_i)$, $i = 1, 2$, $F \in C_*^\infty(\overline{D})$, then $u \in C_*^\infty(\overline{D})$.

§

In this section we intend to concern ourselves with the question of solvability of the problem (4.1), (4.2) in the case where

$$k_1 = -1, \quad k_2 = 1, \tag{4.39}$$

that is, a multidimensional analogue of the Goursat problem, and in the case where

$$-1 < k_1 < 0, \quad k_2 = 1, \tag{4.40}$$

that is, a multidimensional analogue of the first Darboux problem.

First we shall prove the existence of regular solutions of these problems of the class $C_*^\infty(\overline{D})$ and then the existence of strong solutions of the class W_2^1.

Below we shall get an integral representation of regular solutions of the problem (4.1), (4.2) by using the method suggested in [6].

Let us denote by $D_{\varepsilon\delta}$ the part of the domain D which is bounded by the surfaces S_1, S_2, a circular cone $K_\varepsilon : r^2 = (t - t^0)^2(1 - \varepsilon)$ with the vertex at $(x^0, t^0) \in D$ and by a cylinder $H_\delta : r^2 = \delta^2$, where $r^2 = (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2$, and ε and δ are sufficiently small positive numbers.

For any two twice continuously differentiable functions u and v we have the obvious identity

$$u\Box v - v\Box u = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\Big(v\frac{\partial u}{\partial x_i} - u\frac{\partial v}{\partial x_i}\Big) - \frac{\partial}{\partial t}\Big(v\frac{\partial u}{\partial t} - u\frac{\partial v}{\partial t}\Big). \tag{4.41}$$

Integrating (4.41) over the domain $D_{\varepsilon\delta}$, where $u \in C^1(\overline{D}) \cap C^2(D)$ is a regular solution of (4.1), and

$$v = E(r,t,t^0) = \frac{1}{2\pi}\log\frac{t-t^0-\sqrt{(t-t^0)^2-r^2}}{r},$$

we shall have

$$\int\limits_{\partial D_{\varepsilon\delta}} \Big[E(r,t,t^0)\frac{\partial u}{\partial N} - \frac{\partial E(r,t,t^0)}{\partial N}u\Big]ds + \\ + \int\limits_{D_{\varepsilon\delta}} FE(r,t,t^0)dxdt = 0, \tag{4.42}$$

where N is the unit conormal vector at the point $(x,t) = (x_1,x_2,t) \in \partial D_{\varepsilon\delta}$ with direction cosines

$$\cos\widehat{Nx_1} = \cos\widehat{nx_1}, \quad \cos\widehat{Nx_2} = \cos\widehat{nx_2}, \quad \cos\widehat{Nt} = -\cos\widehat{nt},$$

and n is the unit vector of the outer normal to $\partial D_{\varepsilon\delta}$.

Passing in equality (4.42) to the limit for $\varepsilon \to 0$, $\delta \to 0$, we obtain

$$\int\limits_{x_2^0}^{t^0} u(x_1^0,x_2^0,t)dt = \\ = \int\limits_{S_1^*\cup S_2^*} \Big[\frac{\partial E(r,t,t^0)}{\partial N}u - E(r,t,t^0)\frac{\partial u}{\partial N}\Big]ds - \int\limits_{D^*} FE(r,t,t^0)dxdt,$$

where D^* is the domain $D_{\varepsilon\delta}$ for $\varepsilon = \delta = 0$, and $S_i^* = S_i \cap \partial D^*$, $i = 1,2$. By differentiation we find that

$$u(x_1^0,x_2^0,t^0) = \frac{d}{dt^0}\Big[\int\limits_{S_1^*\cup S_2^*} \Big[\frac{\partial E(r,t,t^0)}{\partial N}u - E(r,t,t^0)\frac{\partial u}{\partial N}\Big]ds - \\ - \int\limits_{D^*} FE(r,t,t^0)dxdt\Big]. \tag{4.43}$$

Remark 1. Since in the case (4.39) the direction of the conormal N on the characteristic surface S_i^* coincides with that of a bicharacteristic lying on S_i^*, $i = 1,2$, we can, alongside with the value $u|_{S_i^*} = f_i$, calculate $\frac{\partial u}{\partial N}$ over the surface S_i^*. Therefore in the case (4.39) equality (4.43) gives the integral representation of a regular solution of the multidimensional analogue (4.1), (4.2) of the Goursat problem.

Remark 2. In the case (4.40) the surface S_1^* is not characteristic. Therefore to obtain an integral representation of a regular solution of the multidimensional analogue (4.1), (4.2) of the first Darboux problem one should eliminate the value $\frac{\partial u}{\partial N}|_{S_1^*}$ in the right-hand side of the representation (4.43).

In the case (4.40) without loss of generality we can assume that for the domain D the value $k_1 = 0$, i.e., $D : 0 < x_2 < t,\ 0 < t < t_0$, since the case where $k_1 \neq 0$ is reduced to the case $k_1 = 0$ by a suitable Lorentz transform under which the wave operator $\Box$ is invariant. Let us introduce the point $P'(x_1^0, -x_2^0, t^0)$ which is symmetric to $P(x_1^0, x_2^0, t^0)$ with respect to the plane $x_2 = 0$. For this aim we denote by D_ε a part of the domain D bounded by the cone $K_\varepsilon^0 : (x_1 - x_1^0)^2 + (x_2 + x_2^0)^2 = (t - t^0)^2(1 - \varepsilon)$ with the vertex at P' and the boundary ∂D. Obviously, $\partial D_\varepsilon \cap S_1 \subset S_1^*$ and $\partial D_0 \cap S_1 = S_1^*$. Assume $\partial D_0 \cap S_2 = \widetilde{S}_2$, $\widetilde{r} = \sqrt{(x_1 - x_1^0)^2 + (x_2 + x_2^0)^2}$. Integrating now (4.41) over D_ε, where $u \in C^1(\overline{D}) \cap C^2(D)$ is a regular solution of (4.1), and

$$v = E(\widetilde{r}, t, t^0) = \frac{1}{2\pi} \log \frac{t - t^0 - \sqrt{(t - t^0)^2 - \widetilde{r}^2}}{\widetilde{r}},$$

and taking into account that the function $E(\widetilde{r}, t, t^0)$ has no singularities in the domain D_0, we obtain, after passing to the limit for $\varepsilon \to 0$, the equality

$$\frac{d}{dt^0}\Bigg[\int\limits_{S_1^* \cup S_2^*} \Big[\frac{\partial E(\widetilde{r}, t, t^0)}{\partial N} u - E(\widetilde{r}, t, t^0)\frac{\partial u}{\partial N}\Big] ds - \\ - \int\limits_{D_0} F E(\widetilde{r}, t, t^0) dx dt\Bigg] = 0. \tag{4.44}$$

Since $r = \widetilde{r}$ for $x_2 = 0$, we have $E(\widetilde{r}, t, t^0) = E(r, t, t^0)$ on S_1^*. Therefore eliminating $\frac{\partial u}{\partial N}|_{S_1^*}$ from (4.43) and (4.44), we finally obtain the integral representation of a regular solution of a multidimensional analogue of the first Darboux problem (4.1), (4.2) for $k_1 = 0$, $k_2 = 1$

$$u(x_1^0, x_2^0, t^0) = \frac{d}{dt^0}\Bigg[\int\limits_{S_1^*} \Big[\frac{\partial E(r, t, t^0)}{\partial N} - \frac{\partial E(\widetilde{r}, t, t^0)}{\partial N}\Big] u ds + \\ + \int\limits_{S_2^*} \Big[\frac{\partial E(r, t, t^0)}{\partial N} - E(r, t, t^0)\frac{\partial u}{\partial N}\Big] ds - \\ - \int\limits_{\widetilde{S}_2} \Big[\frac{\partial E(\widetilde{r}, t, t^0)}{\partial N} u - E(\widetilde{r}, t, t^0)\frac{\partial u}{\partial N}\Big] ds + \\ + \int\limits_{D_0} F E(\widetilde{r}, t, t^0) dx dt - \int\limits_{D^*} F E(r, t, t^0) dx dt\Bigg]. \tag{4.45}$$

Remark 3. According to the above remarks, the formulas (4.43) and (4.45) determine uniquely regular solutions of multidimensional analogues of the Goursat and the first Darboux problems, respectively. Moreover, using the arguments of paper [25], we can show that for any $F \in C_*^\infty(\overline{D})$, $f_i \in C_*^\infty(S_i)$, $i = 1, 2$, these solutions belong to the class $C_*^\infty(\overline{D})$.

Below, using a somewhat different method, we shall show that for any $F \in C_*^\infty(\overline{D})$, $f_i \in C_*^\infty(S_i)$, $i = 1, 2$, the solution of the multidimensional analogue of the Goursat problem (4.1), (4.2) will belong to the class $C_*^\infty(\overline{D})$ in the case (4.39). This method consists in reducing the spatial-type problem (4.1), (4.2) to the plane Goursat problem with a parameter. For the solution of the problem the necessary estimates depending on the parameter will be obtained.

If u is a solution of the problem (4.1), (4.2) of the class $C_*^\infty(\overline{D})$ in the case (4.39), then after the Fourier transform with respect to the variable x_1 equation (4.1) and the boundary conditions (4.2) take the form

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x_2^2} + \lambda^2 v = \Phi, \tag{4.46}$$

$$v\big|_{l_i} = g_i, \quad i = 1, 2, \tag{4.47}$$

where

$$v(\lambda, x_2, t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u(x_1, x_2, t) e^{-ix_1\lambda} dx_1$$

is the Fourier transform of the function $u(x_1, x_2, t)$ and Φ, g_1, g_2 are the Fourier transforms respectively of the functions F, f_1, f_2 with respect to the variable x_1. Here $l_1 : t - x_2 = 0$, $0 \le t \le t_0$, $l_2 : t + x_2 = 0$, $0 \le t \le t_0$ are the segments of beams lying in the plane of variables x_2, t and coming out of the origin $O(0, 0)$.

Thus, after the Fourier transform with respect to x_1, the spatial-type problem (4.1), (4.2) is reduced to the plane Goursat problem (4.46), (4.47) with a parameter λ in the domain $D_0 : -t < x_2 < t$, $0 < t < t_0$ of the plane of variables x_2, t.

Remark 4. If $u(x_1, x_2, t)$ is a solution of the problem (4.1), (4.2) of the class $C_*^\infty(\overline{D})$, then $v(\lambda, x_2, t)$ will be a solution of the problem (4.46), (4.47) of the class $C^\infty(\overline{D}_0)$ which at the same time, according to the Paley-Wiener theorem, is an entire analytic function with respect to λ, satisfying the following growth condition: for any integer $N \ge 0$ there exists a constant K_N such that [26, 73]

$$\big|v(\lambda, x_2^0, t^0)\big| \le K_N \big(1 + |\lambda|^2\big)^{-N} e^{d|\operatorname{Im}\lambda|}, \tag{4.48}$$

where

$$d = d(x_2^0, t^0) = \max_{(x_1, x_2^0, t^0) \in \operatorname{supp} u} |x_1|;$$

moreover, as the constant K_N we can take the value [73]

$$K_N = K_N(x_2^0, t^0) = \frac{1}{\sqrt{2\pi}} \int\limits_{|x_1|<d} \Big| \Big(1 - \frac{\partial^2}{\partial x_1^2}\Big)^N u(x_1, x_2^0, t^0) \Big| dx_1.$$

According to the same theorem, if $v(\lambda, x_2, t)$ belongs to the class $C^\infty(\overline{D}_0)$ with respect to the variables x_2, t for fixed λ, and with respect to λ it is an entire analytic function satisfying the estimates (4.48) for some $d = \text{const} > 0$, then the function $u(x_1, x_2, t)$, being the inverse Fourier transform of the function $v(\lambda, x_2, t)$, belongs to the class $C_*^\infty(\overline{D})$.

According to our assumptions, the estimates similar to (4.48) are valid for the functions Φ, g_1, g_2 which belong respectively to the classes $C^\infty(\overline{D}_0)$, $C^\infty(l_1)$, $C^\infty(l_2)$ and are entire analytic functions with respect to λ.

In new variables

$$\xi = \frac{1}{2}(t + x_2), \quad \eta = \frac{1}{2}(t - x_2), \tag{4.49}$$

retaining the same notations for the functions v, Φ, g_i the problem (4.46), (4.47) will take the form

$$\frac{\partial^2 v}{\partial \xi \partial \eta} + \lambda^2 v = \Phi, \tag{4.50}$$

$$v\big|_{\gamma_i} = g_i, \quad i = 1, 2. \tag{4.51}$$

Here a solution $v = v(\lambda, \xi, \eta)$ of equation (4.50) is considered in the domain Ω_0 of the plane of variables ξ, η which is the image of the domain Ω_0 under the linear transform (4.49), γ_i being the image of l_i under the same transform. Obviously, the domain Ω_0 is the triangle OP_1P_2 with vertices $O(0,0)$, $P_1(t_0, 0)$, $P_2(0, t_0)$, and $\gamma_1 : \eta = 0$, $0 \le \xi \le t_0$ and $\gamma_2 : \xi = 0$, $0 \le \eta \le t_0$ are the sides OP_1 and OP_2 of the triangle.

As is well known, under the assumptions with respect to the functions Φ, g_i the problem (4.50), (4.51) has a unique solution v of the class $C^\infty(\overline{\Omega}_0)$ which can be represented in the form [6]

$$\begin{aligned} v(\lambda, \xi, \eta) = {} & R(\xi, 0; \xi, \eta) g_1(\lambda, \xi) + R(0, \eta; \xi, \eta) g_2(\lambda, \eta) - \\ & - R(0, 0; \xi, \eta) g_1(\lambda, 0) - \int\limits_0^\xi \frac{\partial R(\sigma, 0; \xi, \eta)}{\partial \sigma} g_1(\lambda, \sigma) d\sigma - \\ & - \int\limits_0^\eta \frac{\partial R(0, \tau; \xi, \eta)}{\partial \tau} g_2(\lambda, \tau) d\tau + \end{aligned}$$

$$+\int_0^\xi d\sigma \int_0^\eta R(\sigma,\tau;\xi,\eta)\Phi(\lambda,\sigma,\tau)d\tau, \tag{4.52}$$

where $g_1(\lambda,\xi)=v(\lambda,\xi,0)$, $0\le\xi\le t_0$, $g_2(\lambda,\eta)=v(\lambda,0,\eta)$, $0\le\eta\le t_0$, are the Goursat data for v, and $R(\xi_1,\eta_1;\xi,\eta)$ is the Riemann function for the equation (4.50).

As is known, the Riemann function $R(\xi_1,\eta_1;\xi,\eta)$ for the equation (4.50) can be expressed by the Bessel function of zero order as [17]

$$R(\xi_1,\eta_1;\xi,\eta)=J_0\Big(2\lambda\sqrt{(\xi-\xi_1)(\eta-\eta_1)}\Big). \tag{4.53}$$

Remark 5. Since the Bessel function $J_0(z)$ of a complex argument z is an entire analytic function, the formula (4.52) in virtue of (4.53) gives a solution of (4.50) satisfying the Goursat data

$$\begin{aligned} v(\lambda,\xi,0)&=g_1(\xi), \quad 0\le\xi\le t_0,\\ v(\lambda,0,\eta)&=g_2(\eta), \quad 0\le\eta\le t_0. \end{aligned} \tag{4.54}$$

The solution is the entire analytic function with respect to the complex parameter λ.

From the well-known representation of the Bessel function [63]

$$J_0(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(iz\sin\Theta)d\Theta \tag{4.55}$$

we can easily get that

$$J_0'(z)=-\frac{z}{2\pi}\int_{-\pi}^{\pi}\cos^2\Theta\exp(iz\sin\Theta)d\Theta,$$

whence

$$\frac{dJ_0(2\lambda\sqrt{\nu x})}{dx}=-\frac{\lambda^2\nu}{2\pi}\int_{-\pi}^{\pi}\cos^2\Theta\exp(i2\lambda\sqrt{\nu x}\sin\Theta)d\Theta. \tag{4.56}$$

Now (4.53), (4.55) and (4.56) yield the following equalities and estimates

$$\begin{gathered} R(\xi,0;\xi,\eta)=R(0,\eta;\xi,\eta)=1,\\ \big|R(0,0;\xi,\eta)\big|\le\exp\big(2\sqrt{\xi\eta}|\operatorname{Im}\lambda|\big)\le\exp\big(2t_0|\operatorname{Im}\lambda|\big),\\ \Big|\frac{\partial R(\sigma,0;\xi,\eta)}{\partial\sigma}\Big|\le 2|\lambda|^2\eta\exp\big(2\sqrt{\xi\eta}|\operatorname{Im}\lambda|\big)\le 2|\lambda|^2t_0\exp\big(2t_0|\operatorname{Im}\lambda|\big),\\ \Big|\frac{\partial R(0,\tau;\xi,\eta)}{\partial\tau}\Big|\le 2|\lambda|^2\xi\exp\big(2\sqrt{\xi\eta}|\operatorname{Im}\lambda|\big)\le 2|\lambda|^2t_0\exp\big(2t_0|\operatorname{Im}\lambda|\big),\\ \big|R(\sigma,\tau;\xi,\eta)\big|\le\exp\big(2\sqrt{\xi\eta}|\operatorname{Im}\lambda|\big)\le\exp\big(2t_0|\operatorname{Im}\lambda|\big). \end{gathered}$$

From this, assuming without restriction of generality that for the functions Φ, g_1, g_2 the estimates (4.48) are, owing to our assumptions, valid with respect to λ with the same constants K_N and d, we obtain for a solution $v(\lambda,\xi,\eta)$ of the problem (4.50) representable in the form (4.52), the following estimates

$$\begin{aligned}
|v(\lambda,\xi,\eta)| &\le |g_1(\lambda,\xi)| + |g_2(\lambda,\eta)| + |g_1(\lambda,0)|\exp\big(2t_0|\operatorname{Im}\lambda|\big) + \\
&+2|\lambda|^2 t_0 \exp\big(2t_0|\operatorname{Im}\lambda|\big)\int_0^\xi |g_1(\lambda,\sigma)|d\sigma + \\
&+2|\lambda|^2 t_0 \exp\big(2t_0|\operatorname{Im}\lambda|\big)\int_0^\eta |g_2(\lambda,\tau)|d\tau + \\
&+\exp\big(2t_0|\operatorname{Im}\lambda|\big)\int_0^\xi d\sigma\int_0^\eta |\Phi(\lambda,\sigma,\tau)|d\tau \le \\
&\le 2K_N(1+|\lambda|^2)^{-N}\exp\big(d|\operatorname{Im}\lambda|\big) + \\
&+\exp\big(2t_0|\operatorname{Im}\lambda|\big)K_N(1+|\lambda|^2)^{-N}\exp\big(d|\operatorname{Im}\lambda|\big) + \\
&+2|\lambda|^2 t_0\exp\big(2t_0|\operatorname{Im}\lambda|\big)\xi K_N(1+|\lambda|^2)^{-N}\exp\big(d|\operatorname{Im}\lambda|\big) + \\
&+2|\lambda|^2 t_0\exp\big(2t_0|\operatorname{Im}\lambda|\big)\eta K_N(1+|\lambda|^2)^{-N}\exp\big(d|\operatorname{Im}\lambda|\big) + \\
&+\exp\big(2t_0|\operatorname{Im}\lambda|\big)\xi\eta K_N(1+|\lambda|^2)^{-N}\exp\big(d|\operatorname{Im}\lambda|\big) \le \\
&\le \widetilde{K}_{N-1}(1+|\lambda|^2)^{N-1}\exp\big(\widetilde{d}|\operatorname{Im}\lambda|\big). \qquad (4.57)
\end{aligned}$$

Here

$$\widetilde{K}_{N-1} = (3+5t_0^2)K_N, \quad \widetilde{d} = 2t_0 + d,$$
$$d = \max_{(x_1,x_2,t)\in I}|x_1|, \quad I = \operatorname{supp} F \cup \operatorname{supp} f_1 \cup \operatorname{supp} f_2,$$
$$K_N = \frac{1}{2\pi}\int_{|x_1|<d}\max_{0\le i\le 2}\max_{(x_2^0,t)\in\overline{D}_0}\big|\varphi_i(x_1,x_2^0,t^0)\big|dx_1,$$
$$\varphi_0 = \Big(1-\frac{\partial^2}{\partial x_1^2}\Big)^N F, \quad \varphi_i = \Big(1-\frac{\partial^2}{\partial x_1^2}\Big)^N f_i, \quad i=1,2.$$

Owing to (4.57) and the Paley-Wiener theorem, the function $v(\lambda,\xi,\eta)$, after returning to the initial variables x_2, t will, by the formulas (4.49), be the Fourier transform of a function $u(x_1,x_2,t)$ of the class $C_*^\infty(\overline{D})$. Moreover, due to (4.50) and (4.51) the function $u(x_1,x_2,t)\in C_*^\infty(\overline{D})$ will be the unique solution of the problem (4.1), (4.2) of the above-mentioned class. ■

Now, using the fact that the problem (4.1), (4.2) is solvable in the class $C_*^\infty(\overline{D})$, we shall prove the existence of a strong solution of the class W_2^1 of that problem.

It is well-known that the spaces $C_*^\infty(\overline{D})$, $C_*^\infty(S_i)$, $i=1,2$, are everywhere dense in the spaces $L_2(D)$, $W_2^1(S_i)$, $i=1,2$, respectively. Therefore there exist sequences $F_n \in C_*^\infty(\overline{D})$ and $f_{in} \in C_*^\infty(S_i)$, $i=1,2$, such that

$$\lim_{n\to\infty} \|F-F_n\|_{L_2(D)} = \lim_{n\to\infty} \|f_i - f_{in}\|_{W_2^1(S_i)} = 0, \quad i=1,2. \tag{4.58}$$

Moreover, since by the condition $(f_1-f_2)|_{S_1\cap S_2}=0$, one can take the sequences f_{1n} and f_{2n} such that $(f_{1n}-f_{2n})|_{S_1\cap S_2}=0$, $n=1,2,\dots$.

As it was shown above, under the conditions (4.39) or (4.40) there exists a sequence $u_n \in C_*^\infty(\overline{D})$ of solutions of the problem (4.1), (4.2) for $F=F_n$, $f_i=f_{in}$, $i=1,2$.

By virtue of (4.5) we have

$$\|u_n-u_m\|_{W_2^1(D)} \le$$
$$\le C\Big(\sum_{i=1}^{2} \|f_{in}-f_{im}\|_{W_2^1(S_i)} + \|F_n-F_m\|_{L_2(D)}\Big). \tag{4.59}$$

It follows from (4.58) and (4.59) that the sequence of functions u_n is fundamental in the space $W_2^1(D)$. Therefore, due to the completeness of the space $W_2^1(D)$ there exists a function $u\in W_2^1(D)$ such that $u_n\to u$, $\Box u_n \to F$ and $u_n|_{S_i}\to f_i$, $i=1,2$, in $W_2^1(D)$, $L_2(D)$ and $W_2^1(S_i)$, $i=1,2$, respectively, for $n\to\infty$. Consequently, the function u is the strong solution of the problem (4.1), (4.2) of the class W_2^1. The uniqueness of the strong solution of the problem (4.1), (4.2) of the class W_2^1 follows from inequality (4.5).

Thus the following theorem is valid.

Let the condition (4.39) *or* (4.40) *be fulfilled. Then for any* $f_i\in W_2^1(S_i)$, $i=1,2$, $F\in L_2(D)$ *there exists a unique strong solution* u *of the problem* (4.1), (4.2) *of the class* W_2^1 *for which the estimate* (4.5) *is valid.*

Consider now the question of solvability of multi-dimensional analogues of the Goursat and the first Darboux problem for the hyperbolic equation (4.13) with the wave operator $\Box$ in the principal part. To prove the solvability of the problem (4.13), (4.2) under the conditions (4.39) or (4.40), we shall use the solvability of the problem (4.1), (4.2) and the a priori estimate (4.17) in specifically chosen norms of spaces $L_2(D)$, $W_2^1(D)$, $W_2^1(S_i)$, $i=1,2$, from which it follows that the lowest terms in the equation (4.13) give arbitrarily small perturbations.

Consider the space

$$V_0 = L_2(D)\times W_2^1(S_1)\times W_2^1(S_2).$$

To the problem (4.13), (4.2) there corresponds the unbounded operator

$$T: W_2^1(D)\to V_0$$

with the domain of definition $\Omega_T = C_*^\infty(\overline{D}) \subset W_2^1(D)$, acting by the formula

$$Tu = \big(Lu, u\big|_{S_1}, u\big|_{S_2}\big), \quad u \in \Omega_T.$$

It can be easily proved that the operator T admits a closure $\overline{T}$. In fact, let $u_n \in \Omega_T$, $u_n \to 0$ in $W_2^1(D)$ and let $Tu_n \to (F, f_1, f_2)$ in V_0. First we shall show that $F = 0$. For $\varphi \in C_0^\infty(D)$ we have

$$(Lu_n, \varphi) = (u_n, \Box\varphi) + (Ku_n, \varphi), \tag{4.60}$$

where $Ku = au_{x_1} + bu_{x_2} + cu_t + du$. Since in $W_2^1(D)$, $u_n \to 0$, from (4.60) we have that $(Lu_n, \varphi) \to 0$. On the other hand, by the assumption, $Lu_n \to F$ in $L_2(D)$. Therefore $(F, \varphi) = 0$ for any $\varphi \in C_0^\infty(D)$, and hence, $F = 0$. The equalities $f_1 = f_2 = 0$ follow from the facts that $u_n \to 0$ in $W_2^1(D)$, and the contraction operator $u \to (u|_{S_1}, u|_{S_2})$ acts boundedly from $W_2^1(D)$ to $L_2(S_1) \times L_2(S_2)$. ■

To the problem (4.1), (4.2) there corresponds an unbounded operator $T_0 : W_2^1(D) \to V_0$ obtained from the operator T for $a = b = c = d = 0$. As it was shown above, the operator T_0 also admits a closure $\overline{T}_0$. Obviously, the operator $K_0 : W_2^1(D) \to V_0$ acting by the formula $K_0 u = (Ku, 0, 0)$ is bounded, and

$$T = T_0 + K_0. \tag{4.61}$$

Note that the domains of definition $\Omega_{\overline{T}}$ and $\Omega_{\overline{T}_0}$ of the closed operators $\overline{T}$ and $\overline{T}_0$ coincide by virtue of (4.61) and the fact that K_0 is bounded.

It is easily seen that from the existence of the bounded operator $\overline{T}^{-1}$ right inverse to $\overline{T}$, defined on the whole space V_0 follow the existence and uniqueness of the strong solution of the problem (4.13), (4.2) of the class W_2^1, as well as the estimate (4.5) for this solution.

The fact that under the conditions (4.39) or (4.40) the operator $\overline{T}_0$ has its bounded right inverse $\overline{T}_0^{-1} : V_0 \to W_2^1(D)$, follows from the Theorem 4.1 and the estimate (4.5) which, as it is shown above, can be rewritten in equivalent norms in terms of (4.17). It is easy to see that the operator

$$K_0 \overline{T}^{-1} : V_0 \to V_0$$

is bounded, and in virtue of (4.17) its norm admits the estimate

$$\|K_0 \overline{T}_0^{-1}\| \leq \frac{C_7 C_8}{\sqrt{\gamma}}, \tag{4.62}$$

where C_8 is a positive constant depending only on the coefficients a, b, c and d of equation (4.13).

By virtue of (4.62), the operator $(I + K_0 \overline{T}_0^{-1}) : V_0 \to V_0$ has a bounded inverse $(I + K_0 \overline{T}_0^{-1})^{-1}$ for sufficiently large γ, where I is the unit operator. Now it remains for us only to note that the operator

$$\overline{T}_0^{-1}\big(I + K_0 \overline{T}_0^{-1}\big)^{-1}$$

is a bounded operator right inverse to $\overline{T}$ and defined on the whole space V_0.

Thus the following theorem is proved.

Let the condition (4.39) *or* (4.40) *be fulfilled. Then for any* $f_i \in W_2^1(S_i)$, $i = 1, 2$, *and* $F \in L_2(D)$ *there exists a unique strong solution* u *of the problem* (4.13), (4.2) *of the class* W_2^1 *for which the estimate* (4.5) *is valid.*

§

Discussion of this paragraph will be concerned with the question of solvability of the problem (4.1), (2) in the case

$$-1 < k_1 < 0, \quad 0 < k_2 < 1, \tag{4.63}$$

that is, with a multidimensional analogue of the second Darboux problem.

Unlike the cases (4.39) and (4.40) considered in the previous section, the fact that for (4.63) none of the surfaces S_1 and S_2 is characteristic, means that for regular solutions of the problem (4.1), (4.2) there is no integral representation. To a certain extent this circumstance makes investigation of this problem difficult. Below we shall prove the existence of regular and strong solutions of the problem (4.1), (4.2) of the class W_2^1 in the case (4.63) by reducing the problem to a mixed type problem for a hyperbolic equation of second order in a cylinder.

To this end we shall need the following

Let G *be a bounded subdomain of* D *with a piecewise smooth boundary, bounded from above by the plane* $t = t_0$ *and at the sides by the planes* S_1, S_2, *as well as by piecewise smooth time-type surfaces* S_3, S_4 *on which the following inequalities are valid:*

$$\alpha_3\big|_{S_3} < 0, \quad \alpha_3\big|_{S_4} < 0, \tag{4.64}$$

where $n = (\alpha_1, \alpha_2, \alpha_3)$ *is the unit vector of the outer normal to* ∂G; *moreover,* $S_3 \cap S_4 = \varnothing$. *Let* $K_{P_0}^+ : t > t^0 + \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2}$ *be the domain bounded by the characteristic cone of the future with the vertex at* $P_0(x_1^0, x_2^0, t^0)$. *Let* $u_0 \in C^\infty(\overline{G})$ *and* $g_i = u_0|_{\partial G \cap S_i}$, $i = 1, 2$, $F_0 = \Box u_0$, $X = \operatorname{supp} g_1 \cup \operatorname{supp} g_2 \cup \operatorname{supp} F_0$, $Y = \cup_{P_0 \in X} K_{P_0}^+$. *Denote by* S_3^ε, S_4^ε *the* ε*-neighbourhoods of surfaces* S_3, S_4, *where* ε *is a fixed sufficiently small positive number. Then, if*

$$u_0\big|_{S_3 \cup S_4} = 0, \tag{4.65}$$

$$Y \cap \left(S_3^\varepsilon \cup S_4^\varepsilon\right) = \varnothing, \tag{4.66}$$

then the function

$$u(P) = \begin{cases} u_0(P), & P \in G, \\ 0, & P \in D \backslash G \end{cases}$$

is a solution of the problem (4.1), (4.2) *of the class* $C_*^\infty(\overline{D})$ *with*

$$f_i(P) = \begin{cases} g_i(P), & P \in \partial G \cap S_i, \\ 0, & P \in S_i \backslash (\partial G \cap S_i), \end{cases} \quad i = 1, 2,$$

$$F(P) = \begin{cases} F_0(P), & P \in G, \\ 0, & P \in D \backslash G. \end{cases}$$

Proof. To prove the lemma it suffices to show that the function $u_0 \in C^\infty(\overline{G})$ vanishes on the set $G \cap (S_3^\varepsilon \cup S_4^\varepsilon)$.

Let $P_0 \in G \cap (S_3^\varepsilon \cup S_4^\varepsilon)$ be an arbitrary point of this set. We shall show that $u_0(P_0) = 0$.

The use will be made of the notation of Lemma 4.4 and of §3:

$$\Omega_{P_0} = G \cap K_{P_0}, \quad S_{iP_0} = S_i \cap \partial\Omega_{P_0}, \quad i = 1, 2, 3, 4,$$
$$S_{5P_0} = \partial K_{P_0} \cap \partial\Omega_{P_0}.$$

Obviously, $\partial\Omega_{P_0} = \cup_{i=1}^5 S_{iP_0}$.

According to the assumptions of Lemma 4.5, we have

$$\begin{gathered} \alpha_3\big|_{S_{iP_0}} < 0, \quad i = 1, 2, 3, 4, \\ \alpha_3^{-1}\big(\alpha_3^2 - \alpha_1^2 - \alpha_2^2\big)\big|_{S_{iP_0}} > 0, \quad i = 1, 2, 3, 4, \end{gathered} \tag{4.67}$$

$$\alpha_3\big|_{S_{5P_0}} > 0, \quad \big(\alpha_3^2 - \alpha_1^2 - \alpha_2^2\big)\big|_{S_{5P_0}} = 0, \tag{4.68}$$

where $n = (\alpha_1, \alpha_2, \alpha_3)$ is the unit vector of the outer normal to $\partial\Omega_{P_0}$.

On account of (4.65) and (4.66) and the fact that $P_0 \in G \cap (S_3^\varepsilon \cup S_4^\varepsilon)$, we have

$$u_0\big|_{S_{iP_0}} = 0, \;\; i = 1, 2, 3, 4, \quad \Box u_0\big|_{\Omega_{P_0}} = F_0\big|_{\Omega_{P_0}} = 0. \tag{4.69}$$

Multiplying both parts of the equation $\Box u_0 = F_0$ by $2\frac{\partial u_0}{\partial t}$, integrating the obtained expression over Ω_{P_0}, taking into account (4.67)–(4.69) and the arguments used when obtaining inequalities (4.7) and (4.30), we get

$$0 = 2 \int\limits_{\Omega_{P_0}} F_0 \frac{\partial u_0}{\partial t} dx dt =$$

$$= \int\limits_{\partial\Omega_{P_0}} \alpha_3^{-1} \Big[\Big(\alpha_3 \frac{\partial u_0}{\partial t} - \alpha_1 \frac{\partial u_0}{\partial t}\Big)^2 + \Big(\alpha_3 \frac{\partial u_0}{\partial x_2} - \alpha_2 \frac{\partial u_0}{\partial t}\Big)^2 +$$

$$+ \big(\alpha_3^2 - \alpha_1^2 - \alpha_2^2\big)\Big(\frac{\partial u_0}{\partial t}\Big)^2\Big] ds \geq$$

$$\geq \int\limits_{S_{5P_0}} \alpha_3^{-1} \Big[\Big(\alpha_3 \frac{\partial u_0}{\partial x_1} - \alpha_1 \frac{\partial u_0}{\partial t}\Big)^2 + \Big(\alpha_3 \frac{\partial u_0}{\partial x_2} - \alpha_2 \frac{\partial u_0}{\partial t}\Big)^2\Big] ds,$$

whence on account of $\alpha_3|_{S_{5P_0}} > 0$, we find

$$\Big(\alpha_3 \frac{\partial u_0}{\partial x_i} - \alpha_i \frac{\partial u_0}{\partial t}\Big)\Big|_{S_{5P_0}} = 0, \quad i = 1, 2.$$

The remaining reasonings repeat word by word the proof of Lemma 4.4. Consequently, $u(P_0) = 0$ and Lemma 4.5 is proved completely. ∎

Remark 1. It is easy to see that Lemma 4.5 remains also valid in the case when conditions (4.64) are violated on a set $\omega \subset S_3 \cup S_4$ of zero two-dimensional measure, i.e., $\alpha_3|_\omega = 0$. In particular, if $\omega = \overset{m}{\underset{i=1}{\cup}} \gamma_i$ is a union of a finite number of smooth curves $\gamma_i \subset S_3 \cup S_4$ and $\alpha_3|_\omega = 0$, $\alpha_3|_{(S_3 \cup S_4)\setminus\omega} < 0$, then Lemma 4.5 remains correct.

We shall need this circumstance below in proving Theorem 4.3.

Remark 2. It should be also noted that Lemmas 4.4 and 4.5, in fact, suggest us a way of constructing the solution of the problem (4.1), (4.2) in the case (4.63) which is given below and consists in reduction of the initial problem (4.1), (4.2) to a mixed-type problem for a second order hyperbolic equation in a cylinder.

Below the functions f_1 and f_2 in the boundary conditions (4.2) are assumed to vanish on the straight line $\Gamma = S_1 \cap S_2$, i.e.,

$$f_i\big|_\Gamma = 0, \quad i = 1, 2. \tag{4.70}$$

The set of functions of the class $W_2^1(S_i)$ satisfying (4.70) is denoted by $\overset{\circ}{W}{}_2^1(S_i, \Gamma)$, that is,

$$\overset{\circ}{W}{}_2^1(S_i, \Gamma) = \big\{ f \in W_2^1(S_i) : \ f\big|_\Gamma = 0 \big\}, \quad i = 1, 2.$$

We have the following

Let the condition (4.63) *be fulfilled. Then for any* $f_i \in \overset{\circ}{W}{}_2^1(S_i, \Gamma)$, $i = 1, 2$, *and* $F \in L_2(D)$ *there exists a unique strong solution* u *of the problem* (4.1), (4.2) *of the class* W_2^1 *for which the estimate* (4.5) *is valid.*

Proof. Denote by $S_i^0 : k_i t - x_2 = 0$, $0 \le t < +\infty$, $i = 1, 2$, the half-plane containing the carrier S_i in the boundary conditions (4.2) and by D_0 the dihedral angle contained between the half-planes S_1^0 and S_2^0. It is well-known that the function $f_i \in \overset{\circ}{W}{}_2^1(S_i, \Gamma)$ can be extended to the half-plane S_i^0 as a function $\widetilde{f}_i$ of the class $\overset{\circ}{W}{}_2^1(S_i)$, i.e., $(f_i - \widetilde{f}_i)|_{S_i} = 0$, $\widetilde{f}_i \in \overset{\circ}{W}{}_2^1(S_i^0)$, $i = 1, 2$. Assume

$$\widetilde{F}(P) = \begin{cases} F(P), & P \in D, \\ 0, & P \in D_0 \setminus D. \end{cases}$$

Obviously, $\widetilde{F} \in L_2(D_0)$.

If $C_0^\infty(D_0)$, $C_0^\infty(S_i^0)$, $i=1,2$, are the spaces of finite infinitely differentiable functions, then, as we know, they are everywhere dense respectively in $L_2(D_0)$, $\overset{\circ}{W}{}_2^1(S_i^0)$, $i=1,2$. Therefore there exist sequences $F_n \in C_0^\infty(D_0)$ and $f_{in} \in C_0^\infty(S_i^0)$, $i=1,2$, such that

$$\lim_{n\to\infty} \|\widetilde{F}-F_n\|_{L_2(D_0)} = \lim_{n\to\infty} \|\widetilde{f}_i - f_{in}\|_{W_2^1(S_i^0)} = 0, \quad i=1,2. \tag{4.71}$$

In the plane of variables x_2, t let us introduce the polar coordinates r, φ taking the axis t as the polar axis. We count the polar angle φ from the polar axis assuming it to be positive clockwise. Denote by φ_i the size of a bihedral angle contained between the half-planes S_i^0 and $x_2=0$, $0\le t<+\infty$, $i=1,2$. Since the half-planes S_i^0 are of time-type ($-1<k_1<0$, $0<k_2<1$), we have $0<\varphi_i<\frac{\pi}{4}$, $i=1,2$.

In passing from the rectangular coordinates x_1, x_2, t to the system of coordinates x_1, $\tau=\log r$, φ, the bihedral angle D_0 transforms to an infinite layer

$$H = \big\{ -\infty < x_1 < \infty,\ -\infty<\tau<\infty,\ -\varphi_1<\varphi<\varphi_2 \big\},$$

and the equation (4.1), written in terms of the former notation for the functions u and F, will take the form

$$e^{-2\tau} L(\tau,\varphi,\partial)u = F, \tag{4.72}$$

where $\partial = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial \tau}, \frac{\partial}{\partial \varphi})$, $L(\tau,\varphi,\partial)$ is a second order differential operator of hyperbolic type with respect to τ with infinitely differentiable coefficients depending on τ and φ.

In the plane x_1, φ let us consider a convex domain Ω of the class C^∞, bounded by the segments of straight lines $l_1:\varphi=-\varphi_1$, $l_2:\varphi=\varphi_2$ and by the curves $\gamma_1: x_1=g(\varphi)$, $-\varphi_1\le\varphi\le\varphi_2$, $\gamma_2: x_2=-g(\varphi)$, $-\varphi_1\le\varphi\le\varphi_2$. Here $g(\varphi)\in C^\infty(-\varphi_1,\varphi_2)\cap C[-\varphi_1,\varphi_2]$, $g(\varphi)>0$ for $-\varphi_1\le\varphi\le\varphi_2$, $g^{(1)}(\varphi)>0$ for $-\varphi_1<\varphi<0$, $g^{(1)}(0)=0$, $g^{(1)}(\varphi)<0$ for $0<\varphi<\varphi_2$ and $g^{(2)}(\varphi)<0$ for $-\varphi_1<\varphi<\varphi_2$; moreover,

$$\min\big(g(-\varphi_1), g(\varphi_2)\big) > 1+t_0+d, \tag{4.73}$$

where $d=\max(d_1,d_2,d_3)$,

$$d_i = \sup_{(x_1,x_2,t)\in \operatorname{supp} f_i} |x_1|, \quad i=1,2,$$
$$d_3 = \sup_{(x_1,x_2,t)\in \operatorname{supp} F} |x_1|.$$

Denote by $H_0\subset H$ a cylindrical domain $\Omega\times(-\infty,\infty)$ of the class C^∞, where $(-\infty,\infty)$ is the τ-axis, and denote by ∂H_0 its lateral surface $\partial\Omega\times(-\infty,\infty)$. Upon the inverse transform $(x_1,\tau,\varphi)\to(x_1,x_2,t)$, the cylindrical domain H_0 transforms to an unbounded domain $G_0\subset D_0$ bounded by surfaces $\widetilde{S}_i = S_i^0\cap\partial G_0$, $i=1,2$, $\widetilde{S}_3$ and $\widetilde{S}_4$.

Below we shall show that the surfaces $\widetilde{S}_3$ and $\widetilde{S}_4$ are of time-type on which the following conditions

$$\alpha_3\big|_{(\widetilde{S}_3\cup\widetilde{S}_4)\setminus\omega}<0,\quad \alpha_3\big|_{\omega}=0, \tag{4.74}$$

are fulfilled, where ω is the union of two smooth curves ω_1 and ω_2 lying on $\widetilde{S}_3\cup\widetilde{S}_4$.

Indeed, it can be easily seen that $\widetilde{S}_1$ and $\widetilde{S}_2$ are the images of cylindrical surfaces $S_1'=l_1\times(-\infty,\infty)\subset\partial H_0$ and $S_2'=l_2\times(-\infty,\infty)\subset\partial H_0$, while $\widetilde{S}_3$ and $\widetilde{S}_4$ are the images of the surfaces $S_3^0=\gamma_1\times(-\infty,\infty)\subset\partial H_0$ and $S_4^0=\gamma_2\times(-\infty,\infty)\subset\partial H_0$ when the inverse transform $(x_1,\tau,\varphi)\to(x_1,x_2,t)$ is applied. Dividing the surface S_3^0 into two parts $S_3^0=S_{3+}^0\cup S_{3-}^0$, where

$$\begin{gathered}S_{3+}^0=\gamma_{1+}\times(-\infty,\infty),\quad S_{3-}^0=\gamma_{1-}\times(-\infty,\infty),\\ \gamma_{1+}:x_1=g(\varphi),\ \ 0<\varphi<\varphi_2,\quad \gamma_{1-}:x_1=g(\varphi),\ \ -\varphi_1<\varphi<0,\end{gathered}$$

we can see that the image $\widetilde{S}_{3+}\subset\widetilde{S}_3$ of S_{3+}^0 admits upon the inverse transform $(x_1,\tau,\varphi)\to(x_1,x_2,t)$ the following parametric representation

$$\begin{gathered}\widetilde{S}_{3+}:x_1=g(\varphi),\quad x_2=\sigma\sin\varphi,\\ t=\sigma\cos\varphi,\quad 0<\varphi<\varphi_2,\quad 0<\sigma<+\infty,\end{gathered}$$

from which for the unit vector $n=(\alpha_1,\alpha_2,\alpha_3)$ of the outer normal to ∂G_0 we obtain the following on the part $\widetilde{S}_{3+}$

$$n\big|_{\widetilde{S}_{3+}}=\left(\frac{\sigma}{\sqrt{\sigma^2+g'^2(\varphi)}},\frac{-g'(\varphi)\cos\varphi}{\sqrt{\sigma^2+g'^2(\varphi)}},\frac{g'(\varphi)\sin\varphi}{\sqrt{\sigma^2+g'^2(\varphi)}}\right). \tag{4.75}$$

Taking into account the structure of the domain Ω, we can conclude from (4.75) that $\widetilde{S}_{3+}$ is a time-type surface on which $\alpha_3|_{\widetilde{S}_{3+}}<0$. Assertion similar to this one is proved for the remaining parts $\widetilde{S}_{3-}$, $\widetilde{S}_{4+}$ and $\widetilde{S}_{4-}$ of the surfaces $\widetilde{S}_3$ and $\widetilde{S}_4$. To prove finally the validity of (4.74), it suffices to note that on the curves

$$\omega_1=\partial\widetilde{S}_{3+}\cap\partial\widetilde{S}_{3-},\quad \omega_2=\partial\widetilde{S}_{4+}\cap\partial\widetilde{S}_{4-},$$

which are the images of the straight lines $\widetilde{\omega}_1:x_1=g(0)$, $\varphi=0$, $-\infty<\tau<\infty$ and $\widetilde{\omega}_2:x_1=-g(0)$, $\varphi=\pi$, $-\infty<\tau<\infty$, the third component α_3 of the unit vector of the normal n vanishes.

Let us determine on the boundary ∂G_0 of the domain G_0 the function ν_n of the class C^∞ as follows

$$\nu_n\big|_{\widetilde{S}_i}=f_{in},\ \ i=1,2,\quad \nu_n\big|_{\widetilde{S}_3}=\nu_n\big|_{\widetilde{S}_4}=0,\ \ n=1,2,\dots.$$

The fact that $\nu_n\in C_0^\infty(\partial G_0)$ follows from the structure of the domain G_0 and inequality (4.73), as well as from the smoothness and location of supports of the functions $f_{in}\in C_0^\infty(S_i^0)$, $i=1,2$.

When passing to the variables x_1, τ, φ the functions ν_n and F_n will transform to some functions for which we retain the same notation. Obviously,

$$\nu_n \in C_0^\infty(\partial H_0), \quad F_n \in C_0^\infty(H_0). \tag{4.76}$$

For hyperbolic equation (4.72) with $F = F_n$ let us consider in the cylinder H_0 the following mixed-type problem with "zero Cauchy data" for $\tau = -\infty$:

$$e^{-2\tau} L_1(\tau, \varphi, \partial) v = F_n, \tag{4.77}$$

$$v\big|_{\partial H_0} = \nu_n. \tag{4.78}$$

Taking into account (4.76), the mixed problem (4.77), (4.78), due to the results of [4], [74], has a unique solution $v = v_n$ of the class $C^\infty(\overline{H}_0)$ which turns into identical zero for $\tau < -M$, where $M = \text{const}$ is a sufficiently large positive number.

Returning to the initial variables x_1, x_2, t and retaining former notation for the functions v_n and F_n, we get that:

1) the function $u_n^0 = v_n|_{\partial G_0 \cap D}$ belongs to the class $C^\infty(\overline{G_0 \cap D})$ and satisfies the equation

$$\Box u_n^0 = F_n;$$

2) u_n^0 on the lateral part $\cup_{i=1}^4 \widetilde{S}_i^0$ of the boundary domain $G_0 \cap D$ satisfies the conditions

$$u_n\big|_{\widetilde{S}_3^0 \cup \widetilde{S}_4^0} = 0, \quad u_n\big|_{\widetilde{S}_i} = f_{in}, \quad i = 1, 2,$$

where, as is easily seen, the surface $\widetilde{S}_i^0$ is a part of S_i for $i = 1, 2$ and is a part of $\widetilde{S}_i$ for $i = 3, 4$ appearing in conditions (4.74).

Therefore, on account of (4.73), (4.74) as well as of Lemma 4.5 and Remark 1, the function

$$u_n(P) = \begin{cases} u_n^0(P), & P \in G_0, \\ 0, & P \in D \backslash G_0 \end{cases}$$

belongs to the class $C_*^\infty(\overline{D})$ and is a solution of the problem (4.1), (4.2) for $f_i = f_{in}$, $i = 1, 2$, and $F = F_n$.

By virtue of (4.5) we have

$$\|u_n - u_m\|_{W_2^1(D)} \leq$$
$$\leq C\Big(\sum_{i=1}^{2} \|f_{in} - f_{im}\|_{W_2^1(S_i)} + \|F_n - F_m\|_{L_2(D)}\Big). \tag{4.79}$$

From (4.71) and (4.79) it follows that the sequence of the functions u_n is fundamental in the space $W_2^1(D)$. Therefore, since the space $W_2^1(D)$ is complete, there exists a function $u \in W_2^1(D)$ such that $u_n \to u$, $\Box u_n \to F$ and $u_n|_{S_i} \to f_i$, $i = 1, 2$, in the spaces $W_2^1(D)$, $L_2(D)$ and $W_2^1(S_i)$, $i = 1, 2$, respectively, as $n \to \infty$. Consequently, u is a strong solution of problem (4.1), (4.2) of the class W_2^1. The uniqueness of the strong solution of the problem (4.1), (4.2) of the class W_2^1 follows from the inequality (4.5). Thus Theorem 4.3 is proved completely. ∎

Repeating word by word the same arguments connected with equivalent norms which led us to Theorem 4.2, we get that the following theorem is valid.

Let the condition (4.63) *be fulfilled. Then for any* $f_i \in \overset{\circ}{W}{}_2^1(S_i, \Gamma)$, $i = 1, 2$, *and* $F \in L_2(D)$ *there exists a unique strong solution* u *of the problem* (4.13), (4.2) *of the class* W_2^1 *for which estimate* (4.5) *is valid.*

§

Consider the problem (4.1), (4.3), (4.4) in the case where

$$k_1 = 0, \quad k_2 = 1, \tag{4.80}$$

that is, $S_1 : x_2 = 0$, $0 \le t \le t_0$ is a time-type surface, $S_2 : t - x_2 = 0$, $0 \le t \le t_0$ is a characteristic surface, and let in the boundary condition (4.3) the function $f_1 = 0$, that is,

$$\frac{\partial u}{\partial n}\Big|_{S_1} = 0. \tag{4.81}$$

We have the following

Let the condition (4.80) *be fulfilled. Then for any* $f_2 \in C_*^\infty(S_2)$ *and* $F \in C_*^\infty(\overline{D})$ *satisfying*

$$\frac{\partial^k F}{\partial n^k}\Big|_{S_1} = 0, \quad k = 1, 3, 5, \dots, \tag{4.82}$$

the problem (4.1), (4.81), (4.4) *is uniquely solvable in the class* $C_*^\infty(\overline{D})$.

Proof. Denote by $D_- : -t < x_2 < 0$, $0 < t < t_0$ the domain which is symmetric to the domain $D : 0 < x_2 < t$, $0 < t < t_0$, with respect to the plane $x_2 = 0$ and by $D_0 : -t < x_2 < t$, $0 < t < t_0$ the domain being the union of the domains D and D_- and the piece of the plane surface $x_2 = 0$, $0 < t < t_0$.

If we extend evenly the function $F \in C_*^\infty(\overline{D})$ to the domain D_-, then because of (4.82) the function F_0 obtained in the domain D_0,

$$F_0(x_1, x_2, t) = \begin{cases} F(x_1, x_2, t), & x_2 \ge 0, \\ F(x_1, -x_2, t), & x_2 < 0, \end{cases}$$

will belong to the class $C_*^\infty(\overline{D}_0)$. Denote by f_1^- the function defined on $S_1^- : t + x_2 = 0$, $0 \le t \le t_0$ by

$$f_1^-\big|_{S_1^-} = f_1^-(x_1, x_2, -x_2) = f_2(x_1, -x_2, -x_2) = f_2\big|_{S_2}. \tag{4.83}$$

Obviously, $f_1^- \in C_*^\infty(S_1^-)$.

In the domain D_0 let us now consider the problem of determination of a solution $u_0(x_1, x_2, t)$ of the equation

$$\Box u_0 = F_0 \tag{4.84}$$

belonging to the class $C_*^\infty(\overline{D}_0)$ and satisfying the boundary conditions

$$u_0\big|_{S_1^-} = f_1^-, \quad u_0\big|_{S_2} = f_2. \tag{4.85}$$

It is shown in §4 of the present chapter that a multidimensional analogue of the Goursat problem (4.84), (4.85) for $F_0 \in C_*^\infty(\overline{D}_0)$, $f_1^- \in C_*^\infty(S_1^-)$, $f_2 \in C_*^\infty(S_2)$ has a unique solution u_0 of the class $C_*^\infty(\overline{D}_0)$. Let us show now that the restriction of this function to the domain D, i.e., $u = u_0|_D$, is a solution of the problem (4.1), (4.81), (4.4) of the class $C_*^\infty(\overline{D})$. To this end it suffices to show that the function $u_0(x_1, x_2, t)$ is even with respect to the variable x_2. Because the function F_0 is even with respect to the variable x_2, and the functions f_1 and f_2 are connected by equality (4.83), we can easily verify that the function $\widetilde{u}(x_1, x_2, t) = u_0(x_1, -x_2, t)$ is also a solution of the same problem (4.84), (4.85) of the class $C_*^\infty(\overline{D}_0)$. But due to a priori estimate (4.5), the problem (4.84), (4.85) cannot have more than one solution of the above-mentioned class. Therefore, $\widetilde{u}(x_1, x_2, t) \equiv u_0(x_1, x_2, t)$, i.e., the solution $u_0(x_1, x_2, t)$ of equation (4.84) is an even function with respect to x_2. This implies $\frac{\partial u_0}{\partial n}|_{x_2=0} = 0$, i.e., the boundary condition (4.81) is fulfilled for $u = u_0|_D$. Thus, the function $u = u_0|_D \in C_*^\infty(\overline{D})$ is a solution of the problem (4.1), (4.81), (4.4). The uniqueness of this solution of the problem (4.1), (4.81), (4.4) follows from a priori estimate (4.19). ■

Let $f_2 \in W_2^1(S_2)$, $F \in L_2(D)$. The function $u \in W_2^1(D)$ is said to be a strong solution of the problem (4.1), (4.81), (4.4) of the class W_2^1 if there exists a sequence $u_n \in C_*^\infty(\overline{D})$ such that $\frac{\partial u_n}{\partial n}|_{S_1} = 0$, $u_n \to u$, $\Box u_n \to F$ and $u_n|_{S_2} \to f_2$ in the spaces $W_2^1(D)$, $L_2(D)$ and $W_2^1(S_2)$, respectively.

The following theorem holds.

Let the condition (4.80) *be fulfilled. Then for any* $f_2 \in W_2^1(S_2)$ *and* $F \in L_2(D)$ *there exists a unique strong solution* u *of the problem* (4.1), (4.81), (4.4) *of the class* W_2^1 *for which the estimate* (4.19) *is valid.*

Proof. It is known that the space $C_0^\infty(D) \subset C_*^\infty(\overline{D})$ of infinitely differentiable finite functions in the domain D is everywhere dense in $L_2(D)$, while the space $C_*^\infty(S_2)$ is everywhere dense in $W_2^1(S_2)$. Therefore there exist the sequences $F_n \in C_0^\infty(D)$ and $f_{2n} \in C_*^\infty(S_2)$ such that

$$\lim_{n\to\infty} \|F - F_n\|_{L_2(D)} = \lim_{n\to\infty} \|f_2 - f_{2n}\|_{W_2^1(S_2)} = 0. \tag{4.86}$$

Since the functions $F_n \in C_0^\infty(D)$ satisfy the conditions (4.82), according to Lemma 4.6 there exists a sequence $u_n \in C_*^\infty(\overline{D})$ of solutions of the problem (4.1), (4.81), (4.4) with $F = F_n$, $f_2 = f_{2n}$.

On account of (4.19) we have

$$\|u_n - u_m\|_{W_2^1(D)} \leq \\ \leq C\big(\|f_{2n} - f_{2m}\|_{W_2^1(S_2)} + \|F_n - F_m\|_{L_2(D)}\big). \tag{4.87}$$

It follows from (4.86), (4.87) that the sequence of functions u_n is fundamental in the space $W_2^1(D)$. Therefore, since the space $W_2^1(D)$ is complete, there exists a function $u \in W_2^1(D)$ such that $u_n \to u$, $\Box u_n \to F$ and $u_n|_{S_2} \to f_2$ respectively in the spaces $W_2^1(D)$, $L_2(D)$ and $W_2^1(S_2)$ as $n \to \infty$. Consequently, u is a strong solution of the problem (4.1), (4.81), (4.4) of the class W_2^1. The uniqueness of this solution follows from (4.19). ■

Using equivalent norms depending on a parameter and arguing as while proving Theorem 4.2 of §4, we can prove

Let the condition (4.80) *be fulfilled. Then for any* $f_2 \in W_2^1(S_2)$, *and* $F \in L_2(D)$ *there exists a unique strong solution* u *of the problem* (4.13), (4.81), (4.4) *of the class* W_2^1 *for which the estimate* (4.19) *is valid.*

§

Consider in the space R^n, $n > 2$, a strictly hyperbolic equation of the type

$$p(x, \partial)u(x) = f(x), \tag{4.88}$$

where $\partial = (\partial_1, \dots, \partial_n)$, $\partial_j = \frac{\partial}{\partial x_j}$, $p(x, \xi)$ is a real polynomial of order $2m$, $m > 1$, with respect to $\xi = (\xi_1, \dots, \xi_n)$, f is a given function, and u is an unknown real function. It is assumed that in (4.88) the coefficients at higher derivatives are constant and the other coefficients are finite and infinitely differentiable in R^n.

Let D be a conic domain in R^n, i.e., D together with a point $x \in D$ contains the entire beam tx, $0 < t < \infty$. Denote by Γ the cone ∂D. D is assumed to be homeomorphic to the conic domain $x_1^2 + \dots + x_{n-1}^2 - x_n^2 < 0$, $x_n > 0$, and $\Gamma' = \Gamma \backslash O$ is assumed to be a connected $(n-1)$-dimensional manifold of the class C^∞, where O is the vertex of Γ.

Consider the boundary value problem [43]: find in the domain D a solution $u(x)$ of the equation (4.88) satisfying the boundary conditions

$$\left.\frac{\partial^i u}{\partial \nu^i}\right|_{\Gamma'} = g_i, \quad i = 0, \dots, m-1, \tag{4.89}$$

where $\nu = \nu(x)$ is the outer normal to Γ' at the point $x \in \Gamma'$, and g_i, $i = 0, \dots, m-1$, are given real functions.

In this section we investigate the question whether the problem (4.88), (4.89) can be correctly formulated in special weighted spaces $W^k_\alpha(D)$ when the cone Γ is assumed not to be characteristic and to have a quite definite orientation.

Denote by $p_0(\xi)$ the characteristic polynomial of (4.88), i.e., the higher homogeneous part of the polynomial $p(x,\xi)$. The strict hyperbolicity of the equation (4.88) implies the existence of a vector $\zeta \in R^n$ such that the straight line $\xi = \lambda\zeta + \eta$, where $\eta \in R^n$ is an arbitrarily chosen vector not parallel to ζ and λ is a real parameter, intersects the cone of normals $K : p_0(\xi) = 0$ of the equation (4.88) at $2m$ real different points. In other words, the equation $p_0(\lambda\zeta + \eta) = 0$ has $2m$ real different roots with respect to λ. The vector ζ is called a spatial-type normal. As is known, the set of all spatial-type normals form two connected centrally symmetric convex conic domains whose boundaries K_1 and K_{2m} give the internal cavity of the cone of normals K [17]. The surface $S \subset R^n$ is called characteristic at a point $x \in S$ if the normal to S at x belongs to K.

Let the vector ζ be a spatial-type normal and the vector $\eta \neq 0$ vary in the plane orthogonal to ζ. Then the roots of the characteristic polynomial $p_0(\lambda\zeta + \eta)$ with respect to λ can be renumbered so that $\lambda_{2m}(\eta) < \lambda_{2m-1}(\eta) < \cdots < \lambda_1(\eta)$. It is obvious that the vectors $\lambda_i(\eta)\zeta + \eta$ cover the cavities K_i of K, when the η varies on the plane orthogonal to ζ. Since $\lambda_{m-j}(\eta) = -\lambda_{m+j+1}(-\eta)$, $0 \le j \le m-1$, the cones K_{m-j} and K_{m+j+1} are centrally symmetric with respect to the point $(0,\dots,0)$. It is well-known that the straight beams whose orthogonal planes are tangential planes to one of the cavities K_i at a point different from the vertex, are bicharacteristics of equation (4.88).

Assume that there exists a plane π_0 such that $\pi_0 \cap K_m = \{(0,\dots,0)\}$. This means that the cones $K_1,\dots,K_m$ are located on one side of π_0 and the cones $K_{m+1},\dots,K_{2m}$ on the other. Put $K_i^* = \cap_{\eta\in K_i}\{\xi \in R^n : \xi\cdot\eta < 0\}$, where $\xi\cdot\eta$ is the scalar product of the vectors ξ and η. Since $\pi_0 \cap K_m = \{(0,\dots,0)\}$, K_i^* is a conic domain and

$$K_m^* \subset K_{m-1}^* \subset \ldots \subset K_1^*, \quad K_{m+1}^* \subset K_{m+2}^* \subset \ldots \subset K_{2m}^*.$$

It is easy to verify that $\partial(K_i^*)$ is a convex cone whose generatrices are bicharacteristics; note that in this case none of the bicharacteristics of equation (4.88) comes from the point $(0,\dots,0)$ into the cone $\partial(K_m^*)$ or $\partial(K_{m+1}^*)$ [17].

Let us consider

The surface Γ' is characteristic at none of its points and each generatrix of the cone Γ has the direction of a spatial-type normal; moreover, $\Gamma \subset K_m^* \cup O$ or $\Gamma \subset K_{m+1}^* \cup O$.

Denote by $W^k_\alpha(D)$, $k \ge 2m$, $-\infty < \alpha < \infty$, the function space with the norm [48]

$$\|u\|^2_{W^k_\alpha(D)} = \sum_{i=0}^{k} \int\limits_D r^{-2\alpha-2(k-i)} \left| \frac{\partial^i u}{\partial x^i} \right|^2 dx,$$

where

$$r = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}, \quad \frac{\partial^i u}{\partial x^i} = \frac{\partial^i u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}, \quad i = i_1 + \cdots + i_n.$$

The space $W^k_\alpha(\Gamma)$ is defined in a similar manner.

Consider the space

$$V = W^{k+1-2m}_{\alpha-1}(D) \times \prod_{i=0}^{m-1} W^{k-i}_{\alpha-\frac{1}{2}}(\Gamma).$$

Assume that to the problem (4.88), (4.89) there corresponds an unbounded operator

$$T : W^k_\alpha(D) \to V$$

with the domain of definition $\Omega_T = W^{k+1}_{\alpha-1}(D) \subset W^k_\alpha(D)$, acting as

$$Tu = \Big(p(x,\partial)u, u\big|_{\Gamma'}, \ldots, \frac{\partial^i u}{\partial \nu^i}\Big|_{\Gamma'}, \ldots, \frac{\partial^{m-1} u}{\partial \nu^{m-1}}\Big|_{\Gamma'}\Big), \quad u \in \Omega_T.$$

It is obvious that the operator T admits the closure $\overline{T}$.

The function u is called a strong solution of the problem (4.88), (4.89) of the class $W^k_\alpha(D)$ if $u \in \Omega_{\overline{T}}$, $\overline{T}u = (f, g_0, \ldots, g_{m-1}) \in V$, which is equivalent to the existence of a sequence $u_i \in \Omega_T = W^{k+1}_{\alpha-1}(D)$ such that $u_i \to u$ in $W^k_\alpha(D)$ and

$$\Big(p(x,\partial)u_i, u_i\big|_{\Gamma'}, \ldots, \frac{\partial^{m-1} u_i}{\partial \nu^{m-1}}\Big|_{\Gamma'}\Big) \to \big(f, g_0, \ldots, g_{m-1}\big)$$

in the space V. Below, by a solution of the problem (4.88), (4.89) of the class $W^k_\alpha(D)$ will be meant a strong solution of this problem in the sense indicated above.

We shall prove

Let condition 1 *be fulfilled. Then there exists a real number* $\alpha_0 = \alpha_0(k) > 0$ *such that for* $\alpha \geq \alpha_0$ *problem* (4.88), (4.89) *is uniquely solvable in the class* $W^k_\alpha(D)$ *for any* $f \in W^{k+1-2m}_{\alpha-1}(D)$, $g_i \in W^{k-i}_{\alpha-\frac{1}{2}}(\Gamma)$, $i = 0, \ldots, m-1$, *and for the solution* u *we have the estimate*

$$\|u\|_{W^k_\alpha(D)} \leq c\Big(\sum_{i=0}^{m-1} \|g_i\|_{W^{k-i}_{\alpha-\frac{1}{2}}(\Gamma)} + \|f\|_{W^{k+1-2m}_{\alpha-1}(D)}\Big), \qquad (4.90)$$

where c *is a positive constant not depending on* f, g_i, $i = 0, \ldots, m-1$.

First we shall show that Condition 1 implies the following conditions: Take any point $P \in \Gamma'$ and choose a Cartesian system of coordinates $x_1^0, \dots, x_n^0$ having the vertex at P and such that the x_n^0-axis is directed along the generatrix of Γ passing through P, while the x_{n-1}^0-axis is directed along the inner normal to Γ at that point.

The surface Γ' is characteristic at none of its point. Each generatrix of the cone Γ has the direction of a spatial-type normal, and exactly m characteristic planes of the equation (4.88) pass through the $(n-2)$-dimensional plane $x_n^0 = x_{n-1}^0 = 0$ connected with an arbitrary point $P \in \Gamma'$ into the angle $x_n^0 > 0$, $x_{n-1}^0 > 0$.

Denote by $\widetilde{p}_0(\xi)$ the characteristic polynomial of (4.88) written in terms of the coordinate system $x_1^0, \dots, x_n^0$ connected with an arbitrarily chosen point $P \in \Gamma'$.

The surface Γ' is characteristic at none of its point. Each generatrix of Γ has the direction of a spatial-type normal and for $\operatorname{Re} s > 0$ the number of roots $\lambda_j(\xi_1, \dots, \xi_{n-2}, s)$ of the polynomial $\widetilde{p}_0(i\xi_1, \dots, i\xi_{n-2}, \lambda, s)$ with $\operatorname{Re} \lambda_j < 0$, taking into account their multiplicities, is equal to m, $i = \sqrt{-1}$.

When condition 3 is fulfilled, the polynomial $\widetilde{p}_0(i\xi_1, \dots, i\xi_{n-2}, \lambda, s)$ can be written as $\Delta_-(\lambda)\Delta_+(\lambda)$, where for $\operatorname{Re} s > 0$ the roots of the polynomials $\Delta_-(\lambda)$ and $\Delta_+(\lambda)$ lie, respectively, to the left and to the right of the imaginary axis, while the coefficients are continuous for s, $\operatorname{Re} s \geq 0$, $(\xi_1, \dots, \xi_{n-2}) \in R^{n-2}$, $\xi_1^2 + \cdots + \xi_{n-2}^2 + |s|^2 = 1$ [4]. On the left side of the boundary conditions (4.89), to the differential operator $b_j(x, \partial)$, $0 \leq j \leq m-1$, written in terms of the coordinate system $x_1^0, \dots, x_n^0$ connected with $P \in \Gamma'$, there corresponds the characteristic polynomial $b_j(\xi) = \xi_{n-1}^j$. Therefore, since the degree of $\Delta_-(\lambda)$ is equal to m, the following condition will be fulfilled.

For any point $P \in \Gamma'$ and any s, $\operatorname{Re} s \geq 0$, and $(\xi_1, \dots, \xi_{n-2}) \in R^{n-2}$, such that $\xi_1^2 + \cdots + \xi_{n-2}^2 + |s|^2 = 1$, the polynomials $b_j(i\xi_1, \dots, i\xi_{n-2}, \lambda, s) = \lambda^j$, $j = 0, \dots, m-1$, are linearly independent, as polynomials of λ, modulo $\Delta_-(\lambda)$.

We shall now show that condition 1 implies condition 2, while the latter implies condition 3. Let us consider the case $\Gamma \subset K_{m+1}^* \cup O$. The second case $\Gamma \subset K_m^* \cup O$ is treated analogously. Let $P \in \Gamma'$ and $x_1^0, \dots, x_n^0$ be the coordinate system connected with this point. Since the generatrix γ of Γ passing through P is a spatial-type normal, the plane $x_n^0 = 0$ passing through P is a spatial-type plane.

Denote by $K_j\hat{}$ the boundary of the convex shell of K_j and by $K_j^\perp$ the set which is the union of all bicharacteristics corresponding to K_j and coming out of O along the outer normal to K_j, $1 \leq j \leq 2m$. Obviously, $(K_j\hat{})^* = K_j^*$, $\partial(K_j^*) = (K_j\hat{})^\perp$.

Let us now show that the plane π_1, parallel to the plane $x_n^0 = 0$ and passing through the point $(0,\dots,0)$, is the plane of support to the cone $K_{\hat{m}}$ at the point $(0,\dots,0)$. Indeed, it is evident that the plane $N\cdot\xi = 0$, $N \in R^n\backslash(0,\dots,0)$, $\xi \in R^n$, is the plane of support to $K_{\hat{m}}$ at the point $(0,\dots,0)$ if and only if the normal vector N to this plane taken with the sign $+$ or $-$ belongs to the closure of the conic domain $(K_{\hat{m}})^* = K_m^*$. Now it remains for us to note that the conic domains K_m^* and K_{m+1}^* are centrally symmetric with respect to $(0,\dots,0)$, and the generatrix Γ passing through P is perpendicular to π_1 and, by the condition, belongs to $K_{m+1}^* \cup O$.

Since $x_n^0 = 0$ is a spatial-type plane, the two-dimensional plane $\sigma : x_1^0 = \dots = x_{n-2}^0 = 0$ which passes through the generatrix γ directed along the spatial-type normal, intersects the cone of normals K_P of the equation (4.88) with the vertex at the point P by $2m$ different real straight lines [17]. The planes orthogonal to these straight lines and passing through the $(n-2)$-dimensional plane $x_n^0 = x_{n-1}^0 = 0$, give all $2m$ characteristic planes passing through this plane.

The straight lines $x_n^0 = 0$ and $x_{n-1}^0 = 0$ divide the two-dimensional plane σ into four right angles $\sigma_1 : x_{n-1}^0 > 0$, $x_n^0 > 0$; $\sigma_2 : x_{n-1}^0 < 0$, $x_n^0 > 0$; $\sigma_3 : x_{n-1}^0 < 0$, $x_n^0 < 0$; $\sigma_4 : x_{n-1}^0 > 0$, $x_n^0 < 0$. It is easily seen that exactly m characteristic planes of equation (4.88) pass through the $(n-2)$-dimensional plane $x_n^0 = x_{n-1}^0 = 0$ into the angle $x_n^0 > 0$, $x_{n-1}^0 > 0$, if and only if exactly m straight lines from the intersection of K_P with the two-dimensional plane σ pass into the angle σ_4. The latter fact really occurs, since: 1) the plane $x_n^0 = 0$ is the plane of support to $K_{\hat{m}}$ and therefore to all $K_1,\dots,K_{2m}$; 2) the planes $x_n^0 = 0$, $x_{n-1}^0 = 0$ are not characteristic because the generatrices of Γ have a spatial-type direction and Γ is not characteristic at the point P.

Now it will be shown that condition 2 implies condition 3. By virtue of Condition 2 the plane $x_{n-1}^0 = 0$ is not characteristic and therefore the polynomial $\widetilde{p}_0(i\xi_1,\dots,i\xi_{n-2},\lambda,s)$ for λ has exactly $2m$ roots. In this case, if $\operatorname{Re} s > 0$, the number of roots $\lambda_j(\xi_1,\dots,\xi_{n-2},s)$, with the multiplicity of the polynomial $\widetilde{p}_0(i\xi_1,\dots,i\xi_{n-2},\lambda,s)$ taken into account, will be equal to m provided that $\operatorname{Re}\lambda_j < 0$.

Indeed, recalling that equation (4.88) is hyperbolic, for $\operatorname{Re} s > 0$ the equation $\widetilde{p}_0(i\xi_1,\dots,i\xi_{n-2},\lambda,s) = 0$ has no purely imaginary roots with respect to λ. Since the roots λ_j are the continuous functions of s, we can determine the number of roots λ_j with $\operatorname{Re}\lambda_j < 0$ by passing to the limit as $\operatorname{Re} s \to +\infty$.

Since the equality

$$\widetilde{p}_0(i\xi_1,\dots,i\xi_{n-2},\lambda,s) = s^{2m}\widetilde{p}_0\Big(i\frac{\xi_1}{s},\dots,i\frac{\xi_{n-2}}{s},\frac{\lambda}{s},1\Big)$$

holds, it is clear that the ratios $\frac{\lambda_j}{s}$, where λ_j are the roots of the equation $\widetilde{p}_0(i\xi_1,\dots,i\xi_{n-2},\lambda,s) = 0$, tend to the roots μ_j of the equation $\widetilde{p}_0(0,\dots,0,\mu,1) = 0$ as $\operatorname{Re} s \to +\infty$. The latter roots are real and different because equation (4.88) is hyperbolic. If s is taken positive and sufficiently large, then for $\mu_j \neq 0$ we have $\lambda_j = s\mu_j + o(s)$. But $\mu_j \neq 0$, since the plane $x_n^0 = 0$ is not characteristic. Therefore the number of roots λ_j with $\operatorname{Re}\lambda_j < 0$ coincides with the number of roots μ_j with $\mu_j < 0$. Since the characteristic planes of equation (4.88) passing through the $(n-2)$-dimensional plane $x_n^0 = x_{n-1}^0 = 0$, are determined by the equalities $\mu_j x_{n-1}^0 + x_n^0 = 0$, $j = 1, \dots, 2m$, condition 2 implies that the number of roots λ_j with $\operatorname{Re}\lambda_j < 0$ is equal to m.

We give another equivalent description of the space $W_\alpha^k(D)$. On the unit sphere $S^{n-1} : x_1^2 + \cdots + x_n^2 = 1$ let us choose a coordinate system $(\omega_1, \dots, \omega_{n-1})$ such that in the domain D the transformation

$$I : \tau = \log r, \quad \omega_j = \omega_j(x_1, \dots, x_n), \;\; j = 1, \dots, n-1,$$

is one-to-one, nondegenerate and infinitely differentiable. Since the cone $\Gamma = \partial D$ is strictly convex at the point $O(0, \dots, 0)$, such coordinates evidently exist. Under the above transformation the domain D turns to an infinite cylinder G bounded by an infinitely differentiable surface $\partial G = I(\Gamma')$.

Introduce the functional space $H_\gamma^k(G)$, $-\infty < \gamma < \infty$, with the norm

$$\|v\|^2_{H_\gamma^k(G)} = \sum_{i_1+j=0}^{k} \int\limits_G e^{-2\gamma\tau} \Big| \frac{\partial^{i_1+j} v}{\partial \tau^{i_1} \partial \omega^j} \Big|^2 d\omega d\tau,$$

where

$$\frac{\partial^{i_1+j} v}{\partial \tau^{i_1} \partial \omega^j} = \frac{\partial^{i_1+j} v}{\partial \tau^{i_1} \partial \omega_1^{j_1} \dots \partial \omega_{n-1}^{j_{n-1}}}\,, \quad j = j_1 + \cdots + j_{n-1}.$$

As it is shown in [48], the function $u(x) \in W_\alpha^k(D)$ if and only if $\widetilde{u} = u(I^{-1}(\tau, \omega)) \in H^k_{(\alpha+k)-\frac{n}{2}}(G)$, and the estimates

$$c_1 \|\widetilde{u}\|_{H^k_{(\alpha+k)-\frac{n}{2}}(G)} \leq \|u\|_{W_\alpha^k(D)} \leq c_2 \|\widetilde{u}\|_{H^k_{(\alpha+k)-\frac{n}{2}}(G)}$$

hold, where I^{-1} is the transformation inverse to I and the positive constants c_1 and c_2 do not depend on u.

It is easy to see that the condition $v \in H_\gamma^k(G)$ is equivalent to the condition $e^{-\gamma\tau} v \in W^k(G)$, where $W^k(G)$ is the Sobolev space. Denote by $H_\gamma^k(\partial G)$ the set of all ψ such that $e^{-\gamma\tau}\psi \in W^k(\partial G)$, and by $W^k_{\alpha-\frac{1}{2}}(\Gamma)$ the set of all φ for which $\widetilde{\varphi} = \varphi(I^{-1}(\tau, \omega)) \in H^k_{(\alpha+k)-\frac{n}{2}}(\partial G)$. Assume that

$$\|\varphi\|_{W^k_{\alpha-\frac{1}{2}}(\Gamma)} = \|\widetilde{\varphi}\|_{H^k_{(\alpha+k)-\frac{n}{2}}(\partial G)}.$$

Spaces $W_\alpha^k(D)$ possess the following simple properties:

1) if $u \in W_\alpha^k(D)$, then $\frac{\partial^i u}{\partial x^i} \in W_\alpha^{k-i}(D)$, $0 \leq i \leq k$;

2) $W_{\alpha-1}^{k+1}(D) \subset W_\alpha^k(D)$;

3) if $u \in W^k_{\alpha-1}(D)$, then by the well-known embedding theorems we have

$$u\big|_{\Gamma} \in W^k_{\alpha-\frac{1}{2}}(\Gamma), \quad \frac{\partial^i u}{\partial \nu^i}\Big|_{\Gamma'} \in W^{k-i}_{\alpha-\frac{1}{2}}(\Gamma), \;\; i = 1, \dots, m-1;$$

4) $f = p(x, \partial)u \in W^{k+1-2m}_{\alpha-1}(D)$ if $u \in W^{k+1}_{\alpha-1}(D)$.

In what follows we will need in spaces $W^k_\alpha(D)$, $W^k_{\alpha-\frac{1}{2}}(\Gamma)$ other norms depending on the parameter $\gamma = (\alpha + k) - \frac{n}{2}$ and equivalent to the original norms.

Put

$$R^n_{\omega,\tau} = \big\{ -\infty < \tau < \infty, \; -\infty < \omega_i < \infty, \; i = 1, \dots, n-1 \big\},$$
$$R^n_{\omega,\tau,+} = \big\{ (\omega, \tau) \in R^n_{\omega,\tau} : \; \omega_{n-1} > 0 \big\}, \quad \omega' = (\omega_1, \dots, \omega_{n-2}),$$
$$R^{n-1}_{\omega',\tau} = \big\{ -\infty < \tau < \infty, \; -\infty < \omega_i < \infty, \; i = 1, \dots, n-2 \big\}.$$

Denote by $\widetilde{v}(\xi_1, \dots, \xi_{n-2}, \xi_{n-1}, \xi_n - i\gamma)$ the Fourier transform of the function $e^{-\gamma\tau} v(\omega, \tau)$, i.e.,

$$\widetilde{v}(\xi_1, \dots, \xi_{n-1}, \xi_n - i\gamma) = (2\pi)^{-\frac{n}{2}} \int v(\omega, \tau) e^{-i\omega\xi' - i\tau\xi_n - \gamma\tau} d\omega d\tau,$$
$$i = \sqrt{-1}, \quad \xi' = (\xi_1, \dots, \xi_{n-1}),$$

and by $\widehat{v}(\xi_1, \dots, \xi_{n-2}, \omega_{n-1}, \xi_n - i\gamma)$ the partial Fourier transform of the function $e^{-\gamma\tau} v(\omega, \tau)$ with respect to ω', τ .

In the above-considered spaces $H^k_\gamma(R^n_{\omega,\tau})$ and $H^k_\gamma(R^n_{\omega,\tau,+})$ we can introduce the following equivalent norms:

$$|||v|||^2_{R^n,k,\gamma} = \int\limits_{R^n} \big(\gamma^2 + |\xi|^2\big)^k \big|\widetilde{v}(\xi_1, \dots, \xi_{n-1}, \xi_n - i\gamma)\big|^2 d\xi,$$
$$|||v|||^2_{R^n_+,k,\gamma} = \int\limits_0^\infty \int\limits_{R^{n-1}} \sum_{j=0}^k \big(\gamma^2 + |\xi'|^2\big)^{k-j} \times$$
$$\times \Big| \frac{\partial^j}{\partial \omega^j_{n-1}} \widehat{v}(\xi_1, \dots, \xi_{n-2}, \omega_{n-1}, \xi_n - i\gamma)\Big|^2 d\xi' d\omega_{n-1}.$$

Let $\varphi_1, \dots, \varphi_N$ be the partitioning of unity in $G' = G \cap \{\tau = 0\}$, where $G = I(D)$, i.e., $\sum\limits_{j=1}^N \varphi_j(\omega) \equiv 1$ in $G', \varphi_j \in C^\infty(\overline{G}')$, the supports of the functions $\varphi_1, \dots, \varphi_{N-1}$ lie in boundary half-neighborhoods, while the support of the function φ_N lies inside G'. Then for $\gamma = (\alpha + k) - \frac{n}{2}$ the equalities

$$\begin{aligned} |||u|||^2_{G,k,\gamma} &= \sum_{j=1}^{N-1} |||\varphi_j u|||^2_{R^n_+,k,\gamma} + |||\varphi_N u|||^2_{R^n,k,\gamma}, \\ |||u|||^2_{\partial G,k,\gamma} &= \sum_{j=1}^{N-1} |||\varphi_j u|||^2_{R^{n-1}_{\omega',\tau,k,\gamma}} \end{aligned} \tag{4.91}$$

define equivalent norms in the spaces $W^k_\alpha(D)$ and $W^k_{\alpha-\frac{1}{2}}(\Gamma)$, where the norms on the right side of these equalities are taken in terms of local coordinates [4].

Assume first that the equation (4.88) contains only higher terms, i.e., $p(x,\xi) \equiv p_0(\xi)$. Equation (4.88) and the boundary conditions (4.89) written in terms of the coordinates ω, τ will take the form

$$e^{-2m\tau} A(\omega,\partial)u = f,$$
$$e^{-i\tau} B_i(\omega,\partial)u\big|_{\partial G} = g_i, \quad i = 0,\dots,m-1,$$

i.e.,

$$A(\omega,\partial)u = \widetilde{f}, \tag{4.92}$$

$$B_i(\omega,\partial)u\big|_{\partial G} = \widetilde{g}_i, \quad i = 0,\dots,m-1, \tag{4.93}$$

where $A(\omega,\partial)$ and $B_i(\omega,\partial)$ are, respectively, differential operators of orders $2m$ and i with infinitely differentiable coefficients depending only on ω, while $\widetilde{f} = e^{2m\tau} f$ and $\widetilde{g}_i = e^{i\tau} g_i$, $i = 0,1,\dots,m-1$.

Thus under the transformation $I : D \to G$, the unbounded operator T of the problem (4.88), (4.89) transforms to the unbounded operator

$$\widetilde{T} : H^k_\gamma(G) \to H^{k+1-2m}_\gamma(G) \times \prod_{i=0}^{m-1} H^{k-i}_\gamma(\partial G)$$

with the domain of definition $H^{k+1}_\gamma(G)$, acting as

$$\widetilde{T}u = \big(A(\omega,\partial)u, B_0(\omega,\partial)u\big|_{\partial G}, \dots, B_{m-1}(\omega,\partial)u\big|_{\partial G}\big),$$

where $\gamma = (\alpha + k) - \frac{n}{2}$. Note that written in terms of the coordinates ω, τ, the functions $f(\omega,\tau) \in H^{k+1-2m}_{\gamma-2m}(G)$, $g_i(\omega,\tau) \in H^{k-i}_{\gamma-i}(\partial G)$, $i = 0,\dots,m-1$, if $f(x) \in W^{k+1-2m}_{\alpha-1}(D)$, $g_i(x) \in W^{k-i}_{\alpha-\frac{1}{2}}(\Gamma)$, $i = 0,1,\dots,m-1$. Therefore the functions $\widetilde{f} = e^{2m\tau} f \in H^{k+1-2m}_\gamma(G)$, $\widetilde{g}_i = e^{i\tau} g_i \in H^{k-i}_\gamma(\partial G)$, $i = 0,\dots,m-1$.

Since by Condition 1 each generatrix of the cone Γ has the direction of a spatial-type normal, due to the convexity of K_m each beam coming out of the vertex O into the conic domain D also has the direction of a spatial-type normal. Therefore equation (4.92) is strictly hyperbolic with respect to the τ-axis. It was shown above that the fulfillment of Condition

1 implies that of Condition 4. Therefore, according to the results of [4], for $\gamma \geq \gamma_0$, where γ_0 is a sufficiently large positive number, the operator $\widetilde{\widetilde{T}}$ has a bounded right inverse operator $\widetilde{\widetilde{T}}^{-1}$. Thus for any $\widetilde{f} \in H_\gamma^{k+1-2m}(G)$, $\widetilde{g}_i \in H_\gamma^{k-i}(\partial G)$, $i = 0, 1, \ldots, m-1$, $\gamma \geq \gamma_0$ the problem (4.92), (4.93) is uniquely solvable in the class $H_\gamma^k(G)$ and for the solution u we have the estimate

$$|||u|||^2_{G,k,\gamma} \leq c\Big(\sum_{i=0}^{m-1} |||\widetilde{g}_i|||_{\partial G, k-i,\gamma} + \frac{1}{\gamma}|||\widetilde{f}|||_{G,k+1-2m,\gamma}\Big) \qquad (4.94)$$

with a positive constant c not depending on γ, $\widetilde{f}$ and $\widetilde{g}_i$, $i = 0, 1, \ldots, m-1$. Hence it immediately follows that the theorem and the estimate (4.90) are valid in the case $p(x, \xi) \equiv p_0(\xi)$.

Remark. Estimate (4.94) with the coefficient $\frac{1}{\gamma}$ at $|||\widetilde{f}|||_{G,k+1-2m,\gamma}$, obtained in the appropriately chosen norms (4.91), enables one to prove Theorem 4.7 also when equation (4.88) contains lower terms, since the latter give arbitrarily small perturbations for sufficiently large γ.

1. L. Sh. Agababyan and A. B. Nersesyan, On some problems for a model weakly hyperbolic equation. (Russian) *Izv. Akad. Nauk Armyan. SSR, Ser. Mat.* (1981), No. 5, 397–407.

2. L. Sh. Agababyan and A. B. Nersesyan, On sufficient conditions of correctness of the Goursat problem for hyperbolic equation of second order with power degeneration. (Russian) *Dokl. Akad. Nauk Armyan. SSR* (1981), No. 1, 9–16.

3. L. Sh. Agababyan, On the correctness of the Goursat problem for some weakly hyperbolic equations. (Russian) *Uspekhi Mat. Nauk* (1982), 4(226), 113.

4. M. S. Agranovich, Boundary value problems for systems with a parameter. (Russian) *Mat. Sb.* (1971), No. 1, 27–65.

5. Beudon, *Bull. Soc. Math. France* (1897).

6. A. V. Bitsadze, Some classes of partial differential equations. (Russian) *Nauka, Moscow,* 1981.

7. A. V. Bitsadze, To the analysis of the formulation of characteristic problems for second order hyperbolic systems . (Russian) *Dokl. Acad. Nauk SSSR* (1975), No. 6, 1289–1292.

8. A. V. Bitsadze, Influence of lower terms on the correctness of formulation of characteristic problems for second order hyperbolic systems. (Russian) *Dokl. Akad. Nauk SSSR* (1975), No. 1, 31–34.

9. A. V. Bitsadze, On mixed type equations in three-dimensional domains. (Russian) *Dokl. Akad. Nauk SSSR* (1962), No. 5, 1017–1019.

10. A. V. Bitsadze, Mixed type equations. (Russian) *Izd. Akad. Nauk SSSR, Moscow,* 1959.

11. S. Bochner, Allgemeine lineare Differenzengleichungen mit asymptotisch konstanten Koeffizienten. *Math. Z.* (1931), 426–450.

12. S. Bochner, Beiträge zur Theorie der fastperiodishen Funktionen. *Math. Ann.* (1927), 119–147.

13. S. Bochner, Beiträg zur absoluten Konvergenz fastperiodischer Fourierreihen. *Jahresber. Deutsch. Math.-Verein.* (1930), 52–54.

14. A. A. Borgardt and D. A. Karnenko, A characteristic problem for the wave equation with mass. (Russian) *Differentsial'nye Uravneniya* (1984), No. 2, 302-308.

15. B. A. Bubnov, The Goursat and Darboux problems for a class of hyperbolic equations. (Russian) *Sibirsk. Mat. J.* (1978), No. 2, 461–465.

16. F. Cagnac, Problème de Cauchy sur un conoide caracteristique. *Ann. Mat. Pura. Appl.* (1975), 355–393.

17. R. Courant, Partial differential equations. *New York–London,* 1962.

18. D'Adhemar, Rend. *Circ. Mat. Palermo* (1909), 143.

19. G. Darboux, Leçons sur la theorie generale des surfaces. P. 4. *Gauthier–Villars, Paris,* 1896.

20. B. Firmani, Sui casi singolari della problema di Goursat. *Rend. Mat. Appl.* (1982), No. 2, 239–256.

21. B. Firmani, Su una problema singolare di Goursat per l'equazione $u_{xy} = f(x, y, u, u_x, u_y)$. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur.* (1987), No. 1, 1–6.

22. B. Firmani, Teoremi di unicità per problemi di Goursat relativi all'equazione $u_{xy} = 0$. *Rend. Accad. Naz. Sci. XL Mem. Mat.* (1988), No. 1, 167–169.

23. L. Gårding, Linear hyperbolic partial differential equations with constant coefficients. *Acta Math.* (1951), 1–62.

24. E. Goursat, Cours d'analyse Mathématique, t. III, quatr. e'd. *Gauthier–Villars, Paris,* 1927.

25. J. Hadamard, Lectures on Cauchy's problem in partial differential equations. *Yale University Press, New Haven,* 1923.

26. L. Hörmander, Linear partial differential operators. *Grundlehren. Math. Wiss., Band* 116*, Springer–Verlag, Berlin–Heidelberg–New York,* 1963.

27. T. Sh. Kalmenov, A criterion of continuity of the solution of the Goursat problem for a degenerating equation. (Russian) *Differentsial'nye Uravneniya* (1972), No. 1, 41–54.

28. T. Sh. Kalmenov, On multidimensional regular boundary value problems for the wave equation. (Russian) *Izv. Akad. Nauk Kazakh. SSR. Ser. Fiz.–Math.* (1982), No. 3, 18–25.

29. S. S. Kharibegashvili, On a boundary value problem for a second order hyperbolic equation. (Russian) *Dokl. Akad. Nauk SSSR* (1985), No. 6, 1313–1316.

30. S. S. Kharibegashvili, A characteristic problem for second order hyperbolic systems with constant coefficients. (Russian) *Differentsial'nye Uravneniya* (1978), No. 1, 123–135.

31. S. S. Kharibegashvili, A characteristic problem for second order strictly hyperbolic systems with two independent variables. (Russian) *Differentsial'nye Uravneniya* (1979), No. 1, 142–152.

32. S. S. Kharibegashvili, On a characteristic problem for linear nonstrictly hyperbolic systems. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* (1979), No. 3, 553–556.

33. S. S. Kharibegashvili, On a problem of Goursat type for a class of second order hyperbolic systems. (Russian) *Differentsial'nye Uravneniya* (1982), No. 1, 152–166.

34. S. S. Kharibegashvili, On the solvability of a problem of Goursat type for second order normal hyperbolic systems with variable coefficients. (Russian) *Differentsial'nye Uravneniya* (1983), No. 1, 134–145.

35. S. S. Kharibegashvili, On a boundary value problem for normal hyperbolic systems. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* (1984), No. 2, 269-279.

36. S. S. Kharibegashvili, On the solvability of a boundary value problem for linear second order normal systems of hyperbolic type with variable

coefficients. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* (1984), No. 3, 477–480.

37. S. S. Kharibegashvili, On a boundary value problem for second order normal hyperbolic systems with variable coefficients. (Russian) *Differentsial'nye Uravneniya* (1985), No. 1, 149–155.

38. S. S. Kharibegashvili, A characteristic problem for a class of second order hyperbolic systems with parabolic degeneration. (Russian) *Differentsial'nye Uravneniya* (1986), No. 1, 153–164.

39. S. S. Kharibegashvili, On the solvability of a characteristic problem for second order hyperbolic systems with degeneration. (Russian) *Differentsial'nye Uravneniya* (1989), No. 1, 154–162.

40. S. S. Kharibegashvili, A characteristic problem for a class of second order hyperbolic systems with the characteristic line of degeneracy. (Russian) *Differentsial'nye Uravneniya* (1990), No. 2, 305–313.

41. S. S. Kharibegashvili, The Goursat problems for some class of hyperbolic systems. (Russian) *Differentsial'nye Uravneniya* (1981), No. 1, 157–164.

42. S. S. Kharibegashvili, On a multidimensional problem of Goursat type for second order strictly hyperbolic systems. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* (1985), No. 1, 37–40.

43. S. Kharibegashvili, On the correct formulation of one multidimensional problem for strictly hyperbolic equations of higher order. *Georgian Math. J.* (1993), No. 2, 159–169.

44. S. S. Kharibegashvili, On a characteristic problem for the wave equation. *Proc. I. Vekua Inst. Appl. Math., Tbilisi State Univ.* (1992), 76–82.

45. S. Kharibegashvili, On a spatial problem of Darboux type for second order hyperbolic equation. *Georgian Math. J.* (1995), No. 3, 299–311.

46. S. Kharibegashvili, On the solvability of a spatial problem of Darboux type for the wave equation. *Georgian Math. J.* (1995), No. 4, 385–394.

47. S. Kharibegashvili, On the solvability of a non-characteristic spatial problem of Darboux type for the wave equation. *Georgian Math. J.* (1996), No. 1, 53–68, (*to appear*).

48. V. A. Kondrat'ev, Boundary value problems for elliptic equation in domain with conic or corner points. (Russian) *Trudy Moskov. Mat. Obshch.* (1967), 209–292.

49. B. M. Levitan, Almost periodic functions. (Russian) *Gostekhizdat, Moscow–Leningrad,* 1953.

50. L. Lundberg, The Klein–Gordon equation with light-cone data. *Comm. Math. Phys.* (1978), No. 2, 107-118.

51. T. M. Makharadze, On the character of dependence of the well posedness of Darboux problem for second order hyperbolic equation on the touching order of the data of the problem. (Russian) *Proc. I. Vekua Inst. Appl. Math.* (1992), 146–155.

52. T. M. Makharadze, On the solvability of a problem of Darboux type for the second order hyperbolic equation in the case of tangency of data of the problem. *Reports Enlarged sessions sem. I. N. Vekua Inst. Appl. Math.* (1993), No. 1.

53. O. Mel'nik, An example of non-classical boundary value problem for the equation of string oscillation. (Russian) *Ukrain. Mat. Zh.* (1980), No. 5, 671–674.

54. Z. O. Mel'nik, A boundary value problem free from boundary conditions for a hyperbolic system of second order. (Russian) *In: Boundary value problems of mathematical physics.* (*Russian*) *Kiev,* 1981, 81–82.

55. L. A. Mel'tser, On non-correct formulation of the Goursat problem. (Russian) *Mat. Sb.* (1946), No. 1, 59–104.

56. M. Meredov, On the Darboux problem for a degenerating hyperbolic system. (Russian) *Differentsial'nye Uravneniya* (1972), No. 1, 97–106.

57. M. Meredov, On the unique solvability of the Darboux problem for a degenerating system. (Russian) *Differentsial'nye Uravneniya* (1974), No. 1, 89–99.

58. V. P. Mikhaĭlov, On analytic solution of the Goursat problem for a system of differential equations. (Russian) *Dokl. Akad. Nauk SSSR* (1957), No. 3, 450–453.

59. V. P. Mikhaĭlov, On non-analytic solutions of the Goursat problem for a system of differential equations with two independent variables. (Russian) *Dokl. Akad. Nauk SSSR* (1957), No. 5, 759–762.

60. A. M. Nakhushev, On the Darboux problem for hyperbolic equations. (Russian) *Dokl. Akad. Nauk SSSR* (1970), No. 4, 776–779.

61. A. M. Nakhushev, On the Darboux problem for degenerating hyperbolic equations. (Russian) *Differentsial'nye Uravneniya* (1971), No. 1, 49–56.

62. A. M. Nakhushev, To a priori estimates for the Tricomi and Darboux problems. (Russian) *Differentsial'nye Uravneniya* (1972), No. 1, 107–117.

63. F. Olver, Introduction to asymptotics and special functions. *Academic Press, New York–London,* 1974.

64. E. Picard, Leçons sur quelques problèmes aux limites de la théorie des équations differentielles. *Gauthier–Villars, Paris,* 1930.

65. J. Rassias, New uniqueness theorems. *Bull. Acad. Pol. Sci., Ser. Sci. Math.* (1980), No. 11–12, 569–571.

66. J. Rassias, On a Goursat type problem. *Roy. Soc. Can. Math. Repts.* (1980), No. 1, 49–51.

67. M. Riesz, L'integrale de Riemann–Liouville et le problem de Cauchy. *Acta Math.* (1949), 107–125.

68. S. L. Sobolev, On analytic solutions of systems of partial differential equations with two independent variables. (Russian) *Mat. Sb.* (1931), No. 1-2, 107–147.

69. S. L. Sobolev, Some applications of functional analysis in mathematical physics. (Russian) *Publ. Sib. Otd. Akad. Nauk SSSR, Novosibirsk,*

1962.

70. S. L. Sobolev, Some new problems of the theory of partial differential equations of hyperbolic type. (Russian) *Mat. Sb.* (1942), No. 3, 155-200.

71. J. Tolen, Probléme de Cauchy sur les deux hypersurfaces caracteristiques sècantes. *C.R.Acad. Sci. Paris* Sér. A-B (1980), No. 1, A49–A52.

72. B. V. Verbitskiĭ, One global property of matrix-functions depending on several variables. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.,* (1978), No. 1, 8–17.

73. V. S. Vladimirov, Generalized functions in mathematical physics. (Russian) *Nauka, Moscow,* 1976.

74. L. R. Volevich and S. G. Gindikin, The method of energetic estimates in a mixed problem. (Russian) *Uspekhi Mat. Nauk* (1980), 53–120.

75. V. Volterra, Rendiconti del Circolo Matematico di Palermo, (1905).

76. V. N. Vragov, On the Goursat and Darboux problems for a class of hyperbolic equations. (Russian) *Differentsial'nye Uravneniya* (1972), No. 1, 7–16.

Goursat and Darboux TypeProblems for Linear Hyperbolic Partial Differential Equations and Systems 1

Introduction 3

Chapter I. Boundary Value Problems for A Hyperbolic Equation of Second Order

§1. Statement of the Problem and Its Reduction to the Functional Equation 12
§2. The Case Where the Curves γ_1 and γ_2 do not Have a Common Tangent Line at the Point $O(0,0)$ 14
§3. The Case Where the Curves γ_1 and γ_2 Have a Common Tangent Line at $O(0,0)$, But the Directions of Differentiation in the Boundary Conditions (1.4) do not Coincide at This Point 20
§4. The Case Where Tangents to the Curves γ_1 and γ_2 as Well as the Directions of Differentiation in the Boundary Conditions (1.4) Coincide at the Point $O(0,0)$ 20
§5. Influence of the Lower Terms on the Correctness of Statement of the Problem (1.1), (1.2) in the Case Where Conditions (1.13) are Violated on the Whole Curve γ_1 or γ_2 25
§6. The Case Where the Conditions (1.13) are Violated at One Point $O(0,0)$ Only 32

Chapter II. Boundary Value Problems for Linear Second Order Normally Hyperbolic Systems

§1. Statement of the Problem 36
§2. Requirements Imposed on the Curves γ_1, γ_2 and on the Characteristics of the System (2.1). Determination of Numbers m_1 and m_2. Construction of Domains D_1 and D_p 37
§3. Reduction of the Problem (2.1)–(2.3) to the System of Integro-Differential Equations 39
§4. Investigation of the System of Integro-Functional Equations (2.18), (2.19), (2.22) 48
§5. The Case of Hyperbolic Systems with Constant Coefficients ... 62

Chapter III. Characteristic Problems for Linear Hyperbolic Systems of Second Order with Parabolic Degeneration

§1. Characteristic Problem for Hyperbolic System of Second Order with Non-Characteristic Line of Parabolic Degeneration 70
§2. Some Structural Properties of the Hyperbolic System (3.1) 72

§3. Reduction of the Problem (3.1), (3.2) to a System of Integro-Functional Equations 73
§4. Investigation of the System of Integro-Functional Equations (3.28), (3.29), (3.31) and the Proof of Theorem 3.1 79
§5. A characteristic Problem for Hyperbolic System of Second Order with Characteristic Line of Parabolic Degeneration 81

Chapter IV. MULTIDIMENSIONAL ANALOGUES OF THE GOURSAT AND DARBOUX PROBLEMS FOR LINEAR DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

§1. Formulation of Multidimensional Analogues of the Goursat and Darboux Problems for the Wave Equation 84
§2. A Priori Estimates for Solutions of the Problems (4.1), (4.2), and (4.1), (4.3), (4.4) 85
§3. Domain of Dependence of Solutions of the Problems (4.1), (4.2), and (4.1), (4.3), (4.4) 92
§4. Solvability of Multidimensional Analogues of the Goursat and the First Darboux Problems 95
§5. Solvability of a Multidimensional Analogue of the Second Darboux Problem 104
§6. Solvability of the Problem (4.1), (4.3), (4.4) 110
§7. One Multidimensional Analogue of the Second Darboux Problem for Hyperbolic Equations of Higher Orders 112

REFERENCES 121