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A PRIORI ESTIMATES OF SOLUTIONS OF SYSTEMS OF FUNCTIONAL DIFFERENTIAL INEQUALITIES AND SOME OF THEIR APPLICATIONS

Dedicated to the blessed memory of Professor N. V. Azbelev

Abstract. A priori estimates of solutions of systems of functional differential inequalities appearing in the theory of boundary value problems, as well as in the stability theory are established. On the basis of these estimates, new sufficient conditions of boundedness, uniform stability and uniform asymptotic stability of solutions of nonlinear delay differential systems are obtained.

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[illegible]

Introduction

In the present paper, we consider systems of one-sided functional differential inequalities of types which appear in the theory of boundary value problems and in the stability theory (see, e.g., [1]–[10]). In Section 1, a priori estimates of nonnegative solutions of such systems are found. Relying on the obtained results, in Section 2 we establish new effective conditions which guarantee, respectively, the boundedness, uniform stability and uniform asymptotic stability of solutions of nonlinear differential systems with delay.

Throughout the paper, the use will be made of the following notation:

$$\mathbb{R} =]-\infty, +\infty[\,, \quad \mathbb{R}_+ = [0, +\infty[\,;$$

δ_{ik} is Kronecker's symbol, i.e.,

$$\delta_{ik} = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases};$$

$\boldsymbol{x} = (x_i)_{i=1}^n$ and $X = (x_{ik})_{i,k=1}^n$ are the n-dimensional column vector and $n \times n$-matrix with the elements x_i and $x_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$ and the norms

$$\|\boldsymbol{x}\| = \sum_{i=1}^n |x_i|, \quad \|X\| = \sum_{i,k=1}^n |x_{ik}|;$$

X^{-1} is the matrix inverse to X;

$r(X)$ is the spectral radius of X;

E is the unit matrix;

I is a compact or noncompact interval;

$C(I)$ is the space of bounded continuous functions $x : I \to \mathbb{R}$ with the norm

$$\|x\|_{C(I)} = \sup\big\{|x(t)| : \ t \in I\big\};$$

$\widetilde{C}_{loc}(I)$ is the space of functions $x : I \to \mathbb{R}$, absolutely continuous on every compact interval containing in I;

$L(I)$ is the space of Lebesgue integrable functions $x : I \to \mathbb{R}$;

$L_{loc}(I)$ is the space of functions $x : I \to \mathbb{R}$, Lebesgue integrable on every compact interval containing in I *)

1. Theorems on A Priori Estimates

On a finite or an infinite interval I, we consider the system of functional differential inequalities

$$\sigma_i u_i'(t) \le p_i(t) u_i(t) + \sum_{k=1}^n p_{ik}(t) \|u_k\|_{C(I)} + q_i(t) \ \ (i = 1, \dots, n), \tag{1.1}$$

*) $L_{loc}(I) = L(I)$, when I is a compact interval.

where

$$\sigma_i \in \{-1,1\}, \ \ p_i \in L_{loc}(I), \ \ p_{ik} \in L_{loc}(I), \ \ q_i \in L_{loc}(I) \ \ (i,k=1,\dots,n);$$
$$p_i(t) \le 0, \ \ p_{ik}(t) \ge 0, \ \ q_i(t) \ge 0 \ \text{ a.e. on } \ I \ \ (i,k=1,\dots,n).$$

Definition 1.1. The vector function $(u_i)_{i=1}^n : I \to \mathbb{R}^n$ is said to be a nonnegative solution of the system (1.1) if

$$u_i \in \widetilde{C}_{loc}(I) \cap C(I), \ \ u_i(t) \ge 0 \ \text{ for } \ t \in I \ \ (i=1,\dots,n)$$

and almost everywhere on I the inequalities (1.1) are satisfied.

Theorem 1.1. *Let $-\infty < a < b < +\infty$, $I=[a,b]$, $t_i = a$ for $\sigma_i = 1$, $t_i = b$ for $\sigma_i = -1$,*

$$\begin{gathered} h_{ik}(t) = \bigg| \int\limits_{t_i}^{t} \exp\Big(\sigma_i \int\limits_{s}^{t} p_i(x)\,dx\Big) p_{ik}(s)\,ds \bigg|, \\ h_i(t) = \bigg| \int\limits_{t_i}^{t} \exp\Big(\sigma_i \int\limits_{s}^{t} p_i(x)\,dx\Big) q_i(s)\,ds \bigg| \ \ (i,k=1,\dots,n) \end{gathered} \tag{1.2}$$

and

$$r(H) < 1, \ \ \text{where} \ \ H = \big(\|h_{ik}\|_{C(I)}\big)_{i,k=1}^n. \tag{1.3}$$

Then an arbitrary nonnegative solution $(u_i)_{i=1}^n$ of the system (1.1) *admits the estimate*

$$\sum_{i=1}^n \|u_i\|_{C(I)} \le \rho \sum_{i=1}^n \big(u_i(t_i) + \|h_i\|_{C(I)}\big), \tag{1.4}$$

where

$$\rho = \big\|(E-H)^{-1}\big\|. \tag{1.5}$$

Proof. According to (1.2), every nonnegative solution $(u_i)_{i=1}^n$ of the system (1.1) on the interval I satisfies the inequalities

$$\begin{aligned} u_i(t) \le \exp\Big(\int\limits_a^t p_i(x)\,dx\Big) u_i(a) + &\\ + \sum_{k=1}^n \Big(\int\limits_a^t \exp\Big(\int\limits_s^t p_i(x)\,dx\Big) p_{ik}(s)\,ds\Big) \|u_k\|_{C(I)} + &\\ + \int\limits_a^t \exp\Big(\int\limits_s^t p_i(x)\,dx\Big) q_i(s)\,ds \le &\\ \le u_i(t_i) + \sum_{k=1}^n h_{ik}(t) \|u_k\|_{C(I)} + h_i(t) \ \text{ for } \ \sigma_i = 1; & \end{aligned} \tag{1.6}$$

$$
\begin{aligned}
u_i(t) \leq \exp\Big(\int\limits_t^b p_i(x)\,dx\Big)u_i(b)+ \\
&+\sum_{k=1}^n\bigg(\int\limits_t^b \exp\Big(\int\limits_t^s p_i(x)\,dx\Big)p_{ik}(s)\,ds\bigg)\|u_k\|_{C(I)}+ \\
&+\int\limits_t^b \exp\Big(\int\limits_t^s p_i(x)\,dx\Big)q_i(s)\,ds \leq \\
\leq u_i(t_i)+\sum_{k=1}^n h_{ik}(t)\|u_k\|_{C(I)}+h_i(t) \ \text{ for } \ \sigma_i=-1. \qquad (1.7)
\end{aligned}
$$

If we assume

$$
\overline{u}=\big(\|u_i\|_{C(I)}\big)_{i=1}^n, \quad \overline{u}_0=\big(u_i(t_i)\big)_{i=1}^n, \quad \overline{h}=\big(\|h_i\|_{C(I)}\big)_{i=1}^n,
$$

then (1.6) and (1.7) yield

$$
\overline{u}\leq H\overline{u}+\overline{u}_0+\overline{h},
$$

i.e.,

$$
(E-H)\overline{u}\leq \overline{u}_0+\overline{h}. \tag{1.8}
$$

This, by virtue of the condition (1.3), implies that

$$
\overline{u}\leq (E-H)^{-1}\big(\overline{u}_0+\overline{h}\big). \tag{1.9}
$$

Consequently, the estimate (1.4) is valid. □

If $I=\mathbb{R}_+$, then we assume that

$$
h_{ik}\in C(\mathbb{R}_+), \quad h_i\in C(\mathbb{R}_+) \ \ (i,k=1,\dots,n), \tag{1.10}
$$

where

$$
\begin{aligned}
h_{ik}(t)&=\int\limits_0^t \exp\Big(\int\limits_s^t p_i(x)\,dx\Big)p_{ik}(s)\,ds, \\
h_i(t)&=\int\limits_0^t \exp\Big(\int\limits_s^t p_i(x)\,dx\Big)q_i(s)\,ds \ \text{ for } \ \sigma_i=1;
\end{aligned} \tag{1.11}
$$

$$
\begin{aligned}
h_{ik}(t)&=\int\limits_t^{+\infty} \exp\Big(\int\limits_t^s p_i(x)\,dx\Big)p_{ik}(s)\,ds, \\
h_i(t)&=\int\limits_t^{+\infty} \exp\Big(\int\limits_t^s p_i(x)\,dx\Big)q_i(s)\,ds \ \text{ for } \ \sigma_i=-1.
\end{aligned} \tag{1.12}
$$

We are interested in the case where, along with (1.10), one of the following three conditions

$$\sigma_i = 1 \quad (i = 1, \dots, n); \tag{1.13$_1$}$$

$$\begin{gathered} m \in \{1, \dots, n-1\}, \quad \sigma_i = 1 \quad (i = 1, \dots, m), \\ \sigma_i = -1, \quad \int\limits_0^{+\infty} p_i(s)\,ds = -\infty \quad (i = m+1, \dots, n), \end{gathered} \tag{1.13$_2$}$$

and

$$\sigma_i = -1, \quad \int\limits_0^{+\infty} p_i(s)\,ds = -\infty \quad (i = 1, \dots, n) \tag{1.13$_3$}$$

is fulfilled.

In these cases we will, respectively, establish the following a priori estimates:

$$\sum_{i=1}^n \|u_i\|_{C(\mathbb{R}_+)} \le \rho \sum_{i=1}^n \big(u_i(0) + \|h_i\|_{C(\mathbb{R}_+)}\big); \tag{1.14$_1$}$$

$$\sum_{i=1}^n \|u_i\|_{C(\mathbb{R}_+)} \le \rho \Big(\sum_{i=1}^m u_i(0) + \sum_{i=1}^n \|h_i\|_{C(\mathbb{R}_+)}\Big); \tag{1.14$_2$}$$

$$\sum_{i=1}^n \|u_i\|_{C(\mathbb{R}_+)} \le \rho \sum_{i=1}^n \|h_i\|_{C(\mathbb{R}_+)}. \tag{1.14$_3$}$$

Theorem 1.2. *Let $I = \mathbb{R}_+$ and, along with* (1.3) *and* (1.10), *for some $j \in \{1, 2, 3\}$ the condition* (1.13$_j$) *be fulfilled. Then an arbitrary nonnegative solution of the system* (1.1) *admits the estimate* (1.14$_j$), *where ρ is the number given by the equality* (1.5).

Proof. We will prove the theorem only for the case $j = 2$, because the case $j \in \{1, 3\}$ is considered analogously.

By virtue of (1.11) and (1.12), for an arbitrary $b \in]0, +\infty[$ every nonnegative solution $(u_i)_{i=1}^n$ of the system (1.1) on the interval $[0, b]$ satisfies the inequalities

$$\begin{aligned} u_i(t) \le{}& \exp\Big(\int\limits_0^t p_i(x)\,dx\Big) u_i(0) + \\ &+ \sum_{k=1}^n \|h_{ik}\|_{C(\mathbb{R}_+)} \|u_k\|_{C(\mathbb{R}_+)} + \|h_i\|_{C(\mathbb{R}_+)} \quad (i = 1, \dots, m), \\ u_i(t) \le{}& \exp\Big(\int\limits_t^b p_i(x)\,dx\Big) u_i(b) + \end{aligned}$$

$$+\sum_{k=1}^{n}\|h_{ik}\|_{C(\mathbb{R}_+)}\|u_k\|_{C(\mathbb{R}_+)}+\|h_i\|_{C(\mathbb{R}_+)}\quad(i=m+1,\dots,n).$$

If in these inequalities we pass to the limit as $b\to+\infty$, then taking into account the condition (1.13_2) we obtain

$$u_i(t)\le u_i(0)+\sum_{k=1}^{n}\|h_{ik}\|_{C(\mathbb{R}_+)}\|u_k\|_{C(\mathbb{R}_+)}+\|h_i\|_{C(\mathbb{R}_+)}\quad(i=1,\dots,m),$$

$$u_i(t)\le \sum_{k=1}^{n}\|h_{ik}\|_{C(\mathbb{R}_+)}\|u_k\|_{C(\mathbb{R}_+)}+\|h_i\|_{C(\mathbb{R}_+)}\quad(i=m+1,\dots,n).$$

Consequently, the inequality (1.8) is valid, where

$$\overline{u}=\big(\|u_i\|_{C(\mathbb{R}_+)}\big)_{i=1}^{n},\quad \overline{u}_0=(u_{0i})_{i=1}^{n},$$
$$u_{0i}=u_i(0)\ \ (i=1,\dots,m),\quad u_{0i}=0\ \ (i=m+1,\dots,n),$$
$$\overline{h}=\big(\|h_i\|_{C(\mathbb{R}_+)}\big)_{i=1}^{n},\quad H=\big(\|h_{ik}\|_{C(\mathbb{R}_+)}\big)_{i,k=1}^{n}.$$

However, according to the above-said, by means of the condition (1.3), from (1.8) we obtain the inequality (1.9). Thus we have proved that the estimate (1.14_2) is valid. □

In case $I=\mathbb{R}$, the system (1.1) is investigated under the assumptions that

$$\int_{-\infty}^{0}p_i(s)\,ds=-\infty\ \text{ for }\ \sigma_i=1,\quad \int_{0}^{+\infty}p_i(s)\,ds=-\infty\ \text{ for }\ \sigma_i=-1,\tag{1.15}$$

$$h_{ik}\in C(\mathbb{R}),\quad h_i\in C(\mathbb{R})\ \ (i,k=1\dots,n),\tag{1.16}$$

where

$$h_{ik}(t)=\int_{-\infty}^{t}\exp\Big(\int_{s}^{t}p_i(x)\,dx\Big)p_{ik}(s)\,ds,$$

$$h_i(t)=\int_{-\infty}^{t}\exp\Big(\int_{s}^{t}p_i(x)\,dx\Big)q_i(s)\,ds\ \ \text{for}\ \ \sigma_i=1,$$

while for $\sigma_i=-1$ the functions h_{ik} and h_i are defined by the equalities (1.12).

Analogously to Theorem 1.2, we prove

Theorem 1.3. *Let $I=\mathbb{R}$, and the conditions* (1.3), (1.15) *and* (1.16) *be fulfilled. Then an arbitrary nonnegative solution $(u_i)_{i=1}^{n}$ of the system* (1.1) *admits the estimate*

$$\sum_{i=1}^{n}\|u_i\|_{C(\mathbb{R})}\le\rho\sum_{i=1}^{n}\|h_i\|_{C(\mathbb{R})},$$

where ρ is the number given by the equality (1.5).

2. Boundedness and Stability of Solutions of Differential Systems with Delay

Consider the differential systems

$$x_i'(t) + g_i\big(t, x_1(\tau_{i1}(t)), \dots, x_n(\tau_{in}(t))\big)x_i(t) = \\ = f_i\big(t, x_1(\tau_{i1}(t)), \dots, x_n(\tau_{in}(t))\big) \ \ (i = 1, \dots, n) \tag{2.1}$$

and

$$x_i'(t) + g_{0i}(t)x_i(\tau_i(t)) = f_i\big(t, x_1(\tau_{i1}(t)), \dots, x_n(\tau_{in}(t))\big) \ \ (i = 1, \dots, n). \tag{2.2}$$

Here $g_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, $f_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \dots, n)$ are functions satisfying the local Carathéodory conditions,

$$g_{0i} \in L_{loc}(\mathbb{R}_+), \quad g_{0i}(t) \geq 0 \ \text{ for } \ t \in \mathbb{R}_+ \ \ (i = 1, \dots, n),$$

and $\tau_i : \mathbb{R}_+ \to \mathbb{R}$, $\tau_{ik} : \mathbb{R}_+ \to \mathbb{R}$ $(i, k = 1, \dots, n)$ are measurable on every finite interval functions such that

$$\tau_i(t) \leq t, \quad \tau_{ik}(t) \leq t \ \text{ for } \ t \in \mathbb{R}_+ \ \ (i, k = 1, \dots, n).$$

Let

$$a \in \mathbb{R}_+, \quad c_i \in C(]-\infty, a[), \quad c_{0i} \in \mathbb{R} \ \ (i = 1, \dots, n).$$

For the systems (2.1) and (2.2), we consider the Cauchy problem

$$x_i(t) = c_i(t) \ \text{ for } \ t < a, \quad x_i(a) = c_{0i} \ \ (i = 1, \dots, n). \tag{2.3}$$

Suppose

$$\chi_a(t) = \begin{cases} 1 & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}, \quad \tau_{aik}(t) = \begin{cases} \tau_{ik}(t) & \text{for } t \geq a \\ a & \text{for } t < a \end{cases} \ \ (i, k = 1, \dots, n),$$

$$\tau_{ai}(t) = \begin{cases} \tau_i(t) & \text{for } t \geq a \\ a & \text{for } t < a \end{cases} \ \ (i = 1, \dots, n),$$

and introduce the following

Definition 2.1. Let $-\infty < a < b \leq +\infty$ and $I = [a, b[$, or $-\infty < a < b < +\infty$ and $I = [a, b]$. The vector function $(x_i)_{i=1}^n : I \to \mathbb{R}^n$ is said to be a **solution of the problem (2.1), (2.3)** (of the **problem (2.2), (2.3)**) defined on I, if

$$x_i \in \widetilde{C}_{loc}(I), \quad x_i(a) = c_0 \ \ (i = 1, \dots, n)$$

and almost everywhere on I the equality (2.1) (the equality (2.2)) is fulfilled, where

$$x_i(\tau_{ik}(t)) = \big(1 - \chi_a(\tau_{ik}(t))\big)c_i(\tau_{ik}(t)) + \\ + \chi_a(\tau_{ik}(t))x_i(\tau_{aik}(t)) \ \ (i, k = 1, \dots, n), \tag{2.4}$$

$$x_i(\tau_i(t)) = \big(1 - \chi_a(\tau_i(t))\big)c_i(\tau_i(t)) + \chi_a(\tau_i(t))x_i(\tau_{ai}(t)) \ \ (i = 1, \dots, n), \tag{2.5}$$

and

$$c_i(t) = 0 \ \text{ for } \ t \geq a \ \ (i = 1, \dots, n).$$

Definition 2.2. Let $-\infty < a < b < +\infty$ and $I = [a, b[$ ($I = [a, b]$). A solution $(x_i)_{i=1}^n$ of the problem (2.1), (2.3) or of the problem (2.2), (2.3) is said to be **continuable** if there exist $\overline{b} \in [b, +\infty[$ ($\overline{b} \in]b, +\infty[$) and a solution $(\overline{x}_i)_{i=1}^n$ of that problem defined on $[a, \overline{b}]$ and such that

$$\overline{x}_i(t) = x_i(t) \ \text{ for } \ t \in I \ \ (i = 1, \dots, n).$$

A solution $(x_i)_{i=1}^n$ is otherwise called **noncontinuable**.

If $f_i(t, 0, \dots, 0) \equiv 0$ $(i = 1, \dots, n)$, then the system (2.1) (the system (2.2)) under the initial conditions

$$x_i(t) = 0 \ \text{ for } \ t \le 0$$

has a trivial solution. Following [10], we introduce

Definition 2.3. A trivial solution of the system (2.1) (of the system (2.2)) is said to be **uniformly stable** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for arbitrary numbers and functions $a \in \mathbb{R}_+$, $c_{0i} \in \mathbb{R}$ and $c_i \in C(]-\infty, a[)$ $(i = 1, \dots, n)$ satisfying the condition

$$\sum_{i=1}^n \Big(|c_{0i}| + \|c_i\|_{C(]-\infty,a[)} \Big) < \delta, \tag{2.6}$$

every noncontinuable solution of the problem (2.1), (2.3) (of the problem (2.2), (2.3)) is defined on $[a, +\infty[$ and admits the estimate

$$\sum_{i=1}^n \|x_i\|_{C([a,+\infty[)} < \varepsilon. \tag{2.7}$$

Definition 2.4. A trivial solution of the system (2.1) (of the system (2.2)) is said to be **uniformly asymptotically stable** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for arbitrary numbers and functions $a \in \mathbb{R}_+$, $c_{0i} \in \mathbb{R}$ and $c_i \in C(]-\infty, a[\,)$ $(i = 1, \dots, n)$ satisfying the condition (2.6), every noncontinuable solution of the problem (2.1), (2.3) (of the problem (2.2), (2.3)) is defined on $[a, +\infty[$, admits the estimate (2.7) and is vanishing at infinity, i.e.,

$$\lim_{t \to +\infty} x_i(t) = 0 \ \ (i = 1, \dots, n). \tag{2.8}$$

Theorem 2.1. *Let there exist nonnegative numbers ℓ_{ik}, ℓ_i $(i, k = 1, \dots, n)$ and nonnegative functions f_{0i} and $g_{0i} \in L_{loc}([a, +\infty[)$ $(i = 1, \dots, n)$ such that*

$$r(\mathcal{L}) < 1, \ \ \textit{where} \ \ \mathcal{L} = (\ell_{ik})_{i,k=1}^n, \tag{2.9}$$

$$\ell_{0i} = \sup \left\{ \int_a^t \exp\Big(- \int_s^t g_{0i}(x)\, dx \Big) f_{0i}(s)\, ds : \ t \ge a \right\} < +\infty \tag{2.10}$$

$$(i = 1, \dots, n)$$

and on $[a, +\infty[\, \times \mathbb{R}^n$ the inequalities

$$g_i(t, x_1, \dots, x_n) \ge g_{0i}(t), \quad \big| f_i(t, x_1, \dots, x_n) \big| \le$$

$$\leq g_i(t, x_1, \dots, x_n)\Big(\sum_{k=1}^n \ell_{ik}|x_k| + \ell_i\Big) + f_{0i}(t) \ \ (i = 1, \dots, n) \tag{2.11}$$

are satisfied. Then every noncontinuable solution of the problem (2.1), (2.3) *is defined on* $[a, +\infty[$ *, is bounded and admits the estimate*

$$\sum_{i=1}^n \|x_i\|_{C([a,+\infty[)} \leq \rho\Big(\sum_{i,k=1}^n \ell_{ik}\|c_k\|_{C(]-\infty,a[)} + \sum_{i=1}^n \big(|c_{0i}| + \ell_{0i} + \ell_i\big)\Big), \tag{2.12}$$

where $\rho = \|(E - \mathcal{L})^{-1}\|$.

Proof. Let

$$\ell = \sum_{i,k=1}^n \ell_{ik}\|c_i\|_{C(]-\infty,a[)} + \sum_{i=1}^n \big(|c_{0i}| + \ell_{0i} + \ell_i\big). \tag{2.13}$$

To prove the theorem, it suffices to verify that for every $b \in]a, +\infty[$ an arbitrary solution of the problem (2.1), (2.3) defined on $[a, b]$ admits the estimate

$$\sum_{i=1}^n \|x_i\|_{C([a,b])} \leq \rho\ell. \tag{2.14}$$

Suppose $c_i(t) = 0$ for $t \geq a$ $(i = 1, \dots, n)$,

$$\begin{gathered} p_i(t) = -g_i\big(t, x_1(\tau_{i1}(t)), \dots, x_n(\tau_{in}(t))\big), \\ p_{ik}(t) = \ell_{ik}|p_i(t)| \ \ (i, k = 1, \dots, n), \end{gathered} \tag{2.15}$$

$$q_i(t) = \Big(\sum_{k=1}^n \ell_{ik}\big|c_k(\tau_{ik}(t))\big| + \ell_i\Big)|p_i(t)| + f_{0i}(t) \ \ (i = 1, \dots, n) \tag{2.16}$$

and

$$u_i(t) = |x_i(t)| \ \ (i = 1, \dots, n).$$

Then by the condition (2.11), almost everywhere on $[a, b]$ the inequalities

$$p_i(t) \leq -g_{0i}(t) \leq 0 \ \ (i = 1, \dots, n) \tag{2.17}$$

and

$$\begin{gathered} u_i'(t) = p_i(t)u_i(t) + f_i\big(t, x(\tau_{i1}(t)), \dots, x(\tau_{in}(t))\big)\operatorname{sgn}(x_i(t)) \leq \\ \leq p_i(t)u_i(t) + \sum_{k=1}^n p_{ik}(t)u_k(\tau_{aik}(t)) + q_i(t) \ \ (i = 1, \dots, n) \end{gathered}$$

are satisfied. Consequently, $(u_i)_{i=1}^n$ is a nonnegative solution of the system (1.1), where $\sigma_i = 1$ $(i = 1, \dots, n)$, $I = [a, b]$.

Let h_{ik} and h_i $(i, k = 1, \dots, n)$ be the functions given by the equalities (1.2), where $t_i = a$ $(i = 1, \dots, n)$. Then by virtue of the conditions (2.10), (2.17) and the notation (2.13), (2.15), (2.16), we have

$$\|h_{ik}\|_{C(I)} = \ell_{ik} \ \ (i, k = 1, \dots, n), \tag{2.18}$$

$$h_i(t) \le \Big(\sum_{k=1}^{n} \ell_{ik} \|c_k\|_{C(]-\infty,a[)} + \ell_i\Big) \int_a^t \exp\Big(\int_s^t p_i(x)\,dx\Big) |p_i(s)|\,ds +$$
$$+ \int_a^t \exp\Big(-\int_s^t g_{0i}(x)\,dx\Big) f_{0i}(s)\,ds \le$$
$$\le \sum_{k=1}^{n} \ell_{ik} \|c_k\|_{C(]-\infty,a[)} + \ell_i + \ell_{0i} \ \text{ for } \ a \le t \le b \ \ (i = 1, \dots, n)$$

and

$$\sum_{i=1}^{n} \|h_i\|_{C(I)} \le \ell - \sum_{i=1}^{n} |c_{0i}|. \tag{2.19}$$

From the conditions (2.9) and (2.18) we obtain the condition (1.3). Using now Theorem 1.1 and the inequality (2.19), it becomes clear that

$$\sum_{i=1}^{n} \|u_i\|_{C(I)} \le \rho \sum_{i=1}^{n} \big(u_i(a) + \|h_i\|_{C(I)}\big) = \rho \sum_{i=1}^{n} \big(|c_{0i}| + \|h_i\|_{C(I)}\big) \le \rho \ell.$$

Consequently, the estimate (2.14) is valid. □

Corollary 2.1. *Let on* $[a, +\infty[\times \mathbb{R}^n$ *the inequalities*

$$g_i(t, x_1, \dots, x_n) - g_0(t) \ge g_{0i}(t), \quad \exp\Big(\int_a^t g_0(x)\,dx\Big) \big|f_i(t, x_1, \dots, x_n)\big| \le$$
$$\le \big(g_i(t, x_1, \dots, x_n) - g_0(t)\big) \Big(\sum_{k=1}^{n} \ell_{ik} \eta_{ik}(t) |x_k| + \ell_i\Big) + f_{0i}(t) \tag{2.20}$$
$$(i = 1, \dots, n)$$

be satisfied, where ℓ_{ik} *and* ℓ_i $(i, k = 1, \dots, n)$ *are nonnegative numbers,* g_0, g_{0i} *and* $f_{0i} \in L_{loc}([a, +\infty[)$ $(i = 1, \dots, n)$ *are nonnegative functions, and*

$$\eta_{ik}(t) = \exp\Big(\int_a^{\tau_{aik}(t)} g_0(x)\,dx\Big) \ \ (i, k = 1, \dots, n). \tag{2.21}$$

If, moreover, along with (2.9) *and* (2.10) *the condition*

$$\int_a^{+\infty} g_0(x)\,dx = +\infty \tag{2.22}$$

is fulfilled, then every noncontinuable solution of the problem (2.1), (2.3) *is defined on* $[a, +\infty[$ *and is vanishing at infinity.*

Proof. After the transformation

$$x_i(t) = y_i(t) \quad \text{for} \quad t < a,$$
$$x_i(t) = \exp\Big(-\int\limits_a^t g_0(x)\,dx\Big) y_i(t) \quad \text{for} \quad t \geq a \quad (i = 1,\dots,n), \tag{2.23}$$

the problem (2.1), (2.3) takes the form

$$y_i'(t) + \widetilde{g}_i\big(t, y_1(\tau_{i1}(t)),\dots,y_n(\tau_{in}(t))\big) y_i(t) =$$
$$= \widetilde{f}_i\big(t, y_1(\tau_{i1}(t)),\dots,y_n(\tau_{in}(t))\big) \quad (i = 1,\dots,n), \tag{2.24}$$
$$y_i(t) = c_i(t) \quad \text{for} \quad t < a, \quad y_i(a) = c_{0i} \quad (i = 1,\dots,n), \tag{2.25}$$

where

$$\widetilde{g}_i\big(t, x_1,\dots,x_n\big) = g_i\Big(t, \frac{x_1}{\eta_{i1}(t)},\dots,\frac{x_n}{\eta_{in}(t)}\Big) - g_0(t),$$
$$\widetilde{f}_i\big(t, x_1,\dots,x_n\big) = \exp\Big(\int\limits_a^t g_0(x)\,dx\Big) f_i\Big(t, \frac{x_1}{\eta_{i1}(t)},\dots,\frac{x_n}{\eta_{in}(t)}\Big) \quad (i=1,\dots,n).$$

On the other hand, by virtue of (2.20), on $[a,+\infty[\,\times\mathbb{R}^n$ the inequalities

$$\widetilde{g}_i\big(t, x_1,\dots,x_n\big) \geq g_{0i}(t), \quad \big|\widetilde{f}_i\big(t, x_1,\dots,x_n\big)\big| \leq$$
$$\leq \widetilde{g}_i\big(t, x_1,\dots,x_n\big)\Big(\sum_{i=1}^n \ell_{ik}|x_k| + \ell_i\Big) + f_{0i}(t) \quad (i = 1,\dots,n)$$

are satisfied. However, by Theorem 2.1, it follows from these inequalities and the conditions (2.9) and (2.10) that every noncontinuable solution of the problem (2.24), (2.25) is defined and bounded on $[a,+\infty[$.

Taking now into account the equalities (2.22) and (2.23), one easily sees that an arbitrary noncontinuable solution of the problem (2.1), (2.3) is defined on $[a,+\infty[$ and satisfies the condition (2.8). □

Corollary 2.2. *Let for some $\delta_0 > 0$ on the set*

$$\Big\{(t, x_1,\dots,x_n) : \; t \in \mathbb{R}_+, \; |x_k| \leq \delta_0 \; (k = 1,\dots,n)\Big\} \tag{2.26}$$

the inequalities

$$\big|f_i(t, x_1,\dots,x_n)\big| \leq g_i(t, x_1,\dots,x_n)\sum_{k=1}^n \ell_{ik}|x_k| \quad (i = 1,\dots,n) \tag{2.27}$$

be satisfied, where ℓ_{ik} $(i,k = 1,\dots,n)$ are nonnegative constants satisfying the condition (2.9). *Then the trivial solution of the system* (2.1) *is uniformly stable.*

Proof. Suppose

$$v(x) = \begin{cases} x & \text{for} \;\; |x| \leq \delta_0 \\ \delta_0 \operatorname{sgn} x & \text{for} \;\; |x| > \delta \end{cases},$$

$$\begin{aligned}&\widetilde{g}_i(t,x_1,\dots,x_n)=g_i\big(t,v(x_1),\dots,v(x_n)\big),\\&\widetilde{f}_i(t,x_1,\dots,x_n)=f_i\big(t,v(x_1),\dots,v(x_n)\big)\ \ (i=1,\dots,n),\end{aligned}\tag{2.28}$$

and consider the differential system

$$\begin{aligned}x_i'(t)+\widetilde{g}_i\big(t,x_1(\tau_{i1}(t)),\dots,x_n(\tau_{in}(t))\big)x_i(t)=\\=\widetilde{f}_i\big(t,x_1(\tau_{i1}(t)),\dots,x_n(\tau_{in}(t))\big)\ \ (i=1,\dots,n).\end{aligned}\tag{2.29}$$

By virtue of (2.28), on the set (2.26) the equalities

$$\widetilde{g}_i(t,x_1,\dots,x_n)=g_i(t,x_1,\dots,x_n),$$
$$\widetilde{f}_i(t,x_1,\dots,x_n)=f_i(t,x_1,\dots,x_n)\ \ (i=1,\dots,n)$$

are satisfied. Therefore for the trivial solution of the system (2.1) to be uniformly stable, it is necessary and sufficient that the trivial solution of the system (2.29) be uniformly stable.

According to (2.27), on $\mathbb{R}_+\times\mathbb{R}^n$ the inequalities

$$\big|\widetilde{f}_i(t,x_1,\dots,x_n)\big|\le\widetilde{g}_i(t,x_1,\dots,x_n)\sum_{k=1}^n\ell_{ik}|x_k|\ \ (i=1,\dots,n)\tag{2.30}$$

are satisfied. By Theorem 2.1, these inequalities and the condition (2.9) imply that for arbitrary $a\in\mathbb{R}_+$, $c_{0i}\in\mathbb{R}$ and $c_i\in C(]-\infty,a[)$ $(i=1,\dots,n)$ every noncontinuable solution of the problem (2.29), (2.3) is defined on $[a,+\infty[$, is bounded and admits the estimate

$$\sum_{i=1}^n\|x_i\|_{C([a,+\infty[)}\le\rho_0\sum_{i=1}^n\big(|c_{0i}|+\|c_i\|_{C([a,+\infty[)}\big),\tag{2.31}$$

where

$$\rho_0=\big\|(E-H)^{-1}\big\|\Big(1+\sum_{i,k=1}^n\ell_{ik}\Big).$$

For an arbitrarily given $\varepsilon>0$ we assume

$$\delta=\varepsilon/\rho_0.$$

Then by virtue of the estimate (2.31), the fulfilment of the inequality (2.6) ensures that of the inequality (2.7). Thus the trivial solution of the system (2.29) is uniformly stable. $\square$

Corollary 2.3. *Let for some $\delta_0>0$ on the set* (2.26) *the inequalities*

$$\begin{aligned}&g_i(t,x_1,\dots,x_n)\ge g_0(t),\quad \exp\Big(\int_0^t g_0(x)\,dx\Big)\big|f_i(t,x_1,\dots,x_n)\big|\le\\&\le\big(g_i(t,x_1,\dots,x_n)-g_0(t)\big)\sum_{k=1}^n\ell_{ik}\gamma_{ik}(t)|x_k|\ \ (i=1,\dots,n)\end{aligned}\tag{2.32}$$

be satisfied, where

$$\gamma_{ik}(t) = \exp\Big(\int\limits_0^{\tau_{0ik}(t)} g_0(x)\,dx\Big) \ \ (i,k=1,\dots,n) \tag{2.33}$$

and ℓ_{ik} $(i,k=1,\dots,n)$ *and* $g_0 \in L_{loc}(\mathbb{R}_+)$ *are, respectively, nonnegative constants and a nonnegative function satisfying the condition* (2.9) *and*

$$\int\limits_0^{+\infty} g_0(s)\,ds = +\infty. \tag{2.34}$$

Then the trivial solution of the system (2.1) *is uniformly asymptotically stable.*

Proof. Let $\widetilde{g}_i$ and $\widetilde{f}_i$ $(i=1,\dots,n)$ be the functions given by the equalities (2.28). To prove the corollary, it suffices to verify that the trivial solution of the system (2.29) is uniformly asymptotically stable.

According to (2.33),

$$\gamma_{ik}(t) \le \exp\Big(\int\limits_0^t g_0(x)\,dx\Big) \ \text{ for } \ t \ge 0 \ \ (i,k=1,\dots,n). \tag{2.35}$$

On the other hand, for an arbitrary $a \in \mathbb{R}_+$ we have

$$\gamma_{ik}(t) = \eta_{ik}(t)\exp\Big(\int\limits_0^a g_0(x)\,dx\Big) \ \text{ for } \ t \ge a \ \ (i,k=1,\dots,n), \tag{2.36}$$

where η_{ik} $(i,k=1,\dots,n)$ are the functions given by the equalities (2.21).

By (2.28), (2.32), (2.35), and (2.36), the inequalities (2.30) are satisfied on $\mathbb{R}_+ \times \mathbb{R}^n$, while the inequalities

$$\widetilde{g}_i(t,x_1,\dots,x_n) \ge g_0(t), \quad \exp\Big(\int\limits_a^t g_0(x)\,dx\Big)\big|\widetilde{f}_i(t,x_1,\dots,x_n)\big| \le$$
$$\le \big(\widetilde{g}_i(t,x_1,\dots,x_n) - g_0(t)\big)\sum_{k=1}^n \ell_{ik}\eta_{ik}(t)|x_k| \ \ (i,k=1,\dots,n) \tag{2.37}$$

are satisfied on $[a,+\infty[\,\times\mathbb{R}^n$.

Owing to Corollary 2.2, the inequalities (2.9) and (2.30) guarantee the uniform stability of the trivial solution of the system (2.29). On the other hand, by Corollary 2.1, it follows from (2.9), (2.33) and (2.37) that an arbitrary noncontinuable solution of the problem (2.29), (2.3) is defined on $[a,+\infty[$ and satisfies the equalities (2.8). Consequently, the trivial solution of the system (2.29) is uniformly asymptotically stable. □

Corollary 2.4. *Let*

$$\operatorname{vrai\,max}\big\{t - \tau_{ik}(t):\ t \in \mathbb{R}_+\big\} < +\infty \ \ (i,k=1,\dots,n) \tag{2.38}$$

and for some $\delta_0 > 0$ on the set (2.26) *the inequalities*

$$
\begin{aligned}
&g_i(t, x_1, \dots, x_n) \geq g_0(t), \\
&\big|f_i(t, x_1, \dots, x_n)\big| \leq g_i(t, x_1, \dots, x_n) \sum_{k=1}^{n} \ell_{ik} |x_k| \;\; (i = 1, \dots, n)
\end{aligned}
\tag{2.39}
$$

be satisfied, where ℓ_{ik} $(i, k = 1, \dots, n)$ and $g_0 \in L_{loc}(\mathbb{R}_+)$ are, respectively, nonnegative numbers and a nonnegative function satisfying the conditions (2.9) *and* (2.34). *Then the trivial solution of the system* (2.1) *is uniformly asymptotically stable.*

Proof. By (2.9) and (2.38), there exist numbers $\eta > 1$ and $\gamma > 0$ such that

$$
r(\mathcal{L}_\eta) < 1, \;\; \text{where} \;\; \mathcal{L}_\eta = (\eta \ell_{ik})_{i,k=1}^{n} \tag{2.40}
$$

and

$$
0 \leq t - \tau_{ik}(t) \leq \gamma \;\; \text{for} \;\; t \in \mathbb{R}_+ \;\; (i, k = 1, \dots, n). \tag{2.41}
$$

We choose $\varepsilon > 0$ such that

$$
(1 + \varepsilon) \exp(\varepsilon) < \eta. \tag{2.42}
$$

Without loss of generality it can be assumed that

$$
\int_{t}^{t+\gamma} g_0(s)\, ds \leq 1 \;\; \text{for} \;\; t \in \mathbb{R}_+.
$$

Set

$$
g_\varepsilon(t) = \frac{\varepsilon}{1 + \varepsilon}\, g_0(t).
$$

Then, by virtue of (2.34), (2.41) and (2.42), we have

$$
\int_{0}^{+\infty} g_\varepsilon(t)\, dt = +\infty, \tag{2.43}
$$

$$
\int_{\tau_{0ik}(t)}^{t} g_\varepsilon(x)\, dx < \varepsilon \;\; \text{for} \;\; t \in \mathbb{R}_+ \;\; (i, k = 1, \dots, n),
$$

and

$$
\begin{aligned}
\exp\Big(\int_{0}^{t} g_\varepsilon(x)\, dx \Big) &= \exp\Big(\int_{\tau_{0ik}(t)}^{t} g_\varepsilon(x)\, dx \Big) \exp\Big(\int_{0}^{\tau_{0ik}(t)} g_\varepsilon(x)\, dx \Big) < \\
&< \frac{\eta}{1 + \varepsilon}\, \gamma_{\varepsilon ik}(t) \;\; \text{for} \;\; t \in \mathbb{R}_+ \;\; (i, k = 1, \dots, n),
\end{aligned}
\tag{2.44}
$$

where

$$
\gamma_{\varepsilon ik}(t) = \exp\Big(\int_{0}^{\tau_{0ik}(t)} g_\varepsilon(x)\, dx \Big) \;\; (i, k = 1, \dots, n).
$$

Moreover, it is clear that on $\mathbb{R}_+ \times \mathbb{R}^n$ the inequalities

$$\begin{aligned} g_i(t,x_1,\dots,x_n) = (1+\varepsilon)g_i(t,x_1,\dots,x_n) - \varepsilon g_i(t,x_1,\dots,x_n) \le \\ \le (1+\varepsilon)g_i(t,x_1,\dots,x_n) - \varepsilon g_0(t) = \\ = (1+\varepsilon)\big(g_i(t,x_1,\dots,x_n) - g_\varepsilon(t)\big) \ \ (i=1,\dots,n) \end{aligned} \tag{2.45}$$

are satisfied. Taking into account (2.44) and (2.45), from (2.39) we find

$$g_i(t,x_1,\dots,x_n) \ge g_\varepsilon(t), \quad \exp\Big(\int\limits_0^t g_\varepsilon(x)\,dx\Big)\big|f_i(t,x_1,\dots,x_n)\big| \le$$

$$\le \big(g_i(t,x_1,\dots,x_n) - g_\varepsilon(t)\big)\sum_{k=1}^n \eta\ell_{ik}\gamma_{\varepsilon ik}(t)|x_k| \ \ (i=1,\dots,n).$$

However, by Corollary 2.3, the above inequalities together with the conditions (2.40) and (2.43) guarantee the asymptotic stability of the trivial solution of the system (2.1). □

We will now proceed by considering the system (2.2). The following theorem holds.

Theorem 2.2. *Let there exist nonnegative constants ℓ_{ik} $(i,k=1,\dots,n)$ and nonnegative functions $g_{ik} \in L_{loc}([a,+\infty[)$ and $f_{0i} \in L_{loc}([a,+\infty[)$ $(i,k=1,\dots,n)$ such that the inequalities*

$$\big|f_i(t,x_1,\dots,x_n)\big| \le \sum_{k=1}^n g_{ik}(t)|x_k| + f_{0i}(t) \ \ (i=1,\dots,n), \tag{2.46}$$

$$g_{ik}(t)+g_{0i}(t)\int\limits_{\tau_{ai}(t)}^t \big(g_{ik}(s)+\delta_{ik}g_{0k}(s)\big)\,ds \le \ell_{ik}g_{0i}(t) \ \ (i,k=1,\dots,n) \tag{2.47}$$

are satisfied, respectively, on $[a,+\infty[\times\mathbb{R}^n$ and $[a,+\infty[$. If, moreover,

$$\liminf_{t\to+\infty}\tau_i(t) > a \ \ (i=1,\dots,n), \tag{2.48}$$

$$\begin{aligned} \ell_{0i} = \sup\Big\{\int\limits_a^t \exp\Big(-\int\limits_s^t g_{0i}(x)\,dx\Big)f_{0i}(s)\,ds + \int\limits_{\tau_{ai}(t)}^t f_{0i}(s)\,ds : \ t\ge a\Big\} < \\ < +\infty \ \ (i=1,\dots,n) \end{aligned} \tag{2.49}$$

and the condition (2.9) is fulfilled, then every noncontinuable solution of the problem (2.2), (2.3) is defined on $[a,+\infty[$, is bounded and admits the estimate

$$\sum_{i=1}^n \|x_i\|_{C([a,+\infty[)} \le \rho\sum_{i=1}^n \big(\|c_i\|_{C(]-\infty,a[)} + \|c_{0i}\| + \ell_{0i}\big), \tag{2.50}$$

where ρ is a positive constant depending only on g_{0i}, g_{ik} and ℓ_{ik} $(i,k=1,\dots,n)$.

Proof. By (2.48), there exists $b_0 \in]a,+\infty[$ such that

$$\tau_{ai}(t) = \tau_i(t) > a \quad \text{for} \quad t \geq b_0. \tag{2.51}$$

Assume

$$\rho_0 = \Big(1 + \sum_{i,k=1}^{n} \ell_{ik}\Big) \sum_{i=1}^{n} \exp\Big(\int_a^{b_0} g_{0i}(s)\,ds\Big), \tag{2.52}$$

$$g(t) = \sum_{i,k=1}^{n} \big(g_{ik}(t) + \delta_{ik} g_{0k}(t)\big), \tag{2.53}$$

$$\rho_1 = \rho_0 \exp\Big(\int_a^{b_0} g(s)\,ds\Big), \tag{2.54}$$

$$\rho = \Big[n + 4\big\|(E - \mathcal{L})^{-1}\big\| \sum_{i,k=1}^{n} (\ell_{ik} + \delta_{ik})\Big]\rho_1, \tag{2.55}$$

and

$$\ell = \sum_{i=1}^{n} \big(\|c_i\|_{C(]-\infty,a[)} + |c_{0i}| + \ell_{0i}\big). \tag{2.56}$$

To prove the theorem, it suffices to state that for an arbitrary $b \in]a+\infty[$, every solution of the problem (2.2), (2.3), defined on $[a,b]$ admits the estimate (2.14).

According to the conditions (2.46), (2.47) and the equalities (2.4) and (2.5), almost everywhere on $[a,b]$ the inequalities

$$\begin{gathered}\Big|x_i'(t) + \chi_a(\tau_i(t)) g_{0i}(t) x_i(\tau_{ai}(t))\Big| \leq \\ \leq \sum_{k=1}^{n} g_{ik}(t)\big|x_k(\tau_{aik}(t))\big| + q_{0i}(t) \quad (i = 1,\dots,n)\end{gathered} \tag{2.57}$$

and

$$|x_i'(t)| \leq g_{0i}(t)\big|x_i(\tau_{ai}(t))\big| + \sum_{k=1}^{n} g_{ik}(t)\big|x_k(\tau_{aik}(t))\big| + q_{0i}(t) \quad (i = 1,\dots,n) \tag{2.58}$$

are satisfied, where

$$q_{0i}(t) = g_{0i}(t) \sum_{k=1}^{n} (\ell_{ik} + \delta_{ik}) \|c_k\|_{C(]-\infty,a[)} + f_{0i}(t) \quad (i = 1,\dots,n). \tag{2.59}$$

Suppose first that $b \in]a, b_0]$ and put

$$y(t) = \sum_{i=1}^{n} |c_{0i}| + \sum_{i=1}^{n} \int_a^t |x_i'(s)|\,ds \quad (i = 1,\dots,n).$$

Then on $[a,b]$ the inequality

$$\sum_{i=1}^{n} |x_i(t)| \le y(t)$$

is satisfied. If along with this fact we take into account the notation (2.53), then from (2.58) we can find

$$y(t) \le \sum_{i=1}^{n} \bigg(|c_{0i}| + \int_a^{b_0} q_{0i}(s)\,ds\bigg) + \int_a^{t} g(s)y(s)\,ds.$$

On the other hand, by (2.49), (2.52), (2.56), and (2.59) we have

$$\sum_{i=1}^{n} \bigg(|c_{0i}| + \int_a^{b_0} q_{0i}(s)\,ds\bigg) \le$$

$$\le \sum_{i=1}^{n}|c_{0i}| + \sum_{i=1}^{n}\exp\Big(\int_a^{b_0} g_{0i}(s)\,ds\Big)\int_a^{b_0}\exp\Big(-\int_s^{b_0} g_{0i}(x)\,dx\Big)q_{0i}(s)\,ds \le \rho_0\ell.$$

Therefore,

$$y(t) \le \rho_0\ell + \int_a^{t} g(s)y(s)\,ds \ \text{ for } \ a \le t \le b,$$

whence by the Gronwall lemma and the notation (2.54) it follows that

$$y(t) \le \rho_0\ell \exp\Big(\int_a^{t} g(s)\,ds\Big) \le \rho_1\ell \ \text{ for } \ a \le t \le b$$

and, consequently,

$$\sum_{i=1}^{n} |x_i(t)| \le \rho_1\ell \ \text{ for } \ a \le t \le b.$$

Thus we have proved that if $b \in \,]a, b_0]$, then the estimate (2.14) is valid since owing to (2.55) we have $n\rho_1 \le \rho$.

Let us now pass to the consideration of the case where $b \in \,]b_0, +\infty[\,$. According to the above-proven, we have

$$\|x_i\|_{C([a,b_0])} \le \rho_1\ell \ \ (i = 1, \dots, n). \tag{2.60}$$

Therefore,

$$|x_i(b_0)| \le \rho_1\ell \ \ (i = 1, \dots, n), \tag{2.61}$$

$$\big|x_i(\tau_{ai}(t))\big| \le \rho_1\ell + \|x_i\|_{C(I)}, \quad \big|x_i(\tau_{aik}(t))\big| \le \rho_1\ell + \|x_i\|_{C(I)} \quad \text{for } \ t \in [a,b] \ \ (i,k = 1, \dots, n), \tag{2.62}$$

where $I = [b_0, b]$.

Taking into account the inequalities (2.47) and (2.62), from (2.58) and (2.59) we obtain

$$
\begin{gathered}
\big|x_i(t) - x_i(\tau_{ai}(t))\big| \le \int\limits_{\tau_{ai}(t)}^{t} |x_i'(s)|\,ds \le \\
\le \sum_{k=1}^{n} \Bigg(\int\limits_{\tau_{ai}(t)}^{t} \big(g_{ik}(s) + \delta_{ik} g_{0k}(s)\big)\,ds \Bigg) \|x_k\|_{C(I)} + \\
+ \sum_{k=1}^{n} (\ell_{ik} + \delta_{ik}) \big(\|c_k\|_{C(]-\infty,a[)} + \rho_1 \ell\big) + \ell_{0i} \ \text{ for a.a. } \ t \in I \qquad (2.63) \\
(i = 1, \dots, n).
\end{gathered}
$$

By virtue of (2.51) and (2.57), almost everywhere on I the inequalities

$$
\begin{aligned}
|x_i(t)|' = x_i'(t) \operatorname{sgn} x_i(t) \le &- g_{0i}(t)|x_i(t)| + g_{0i}(t)\big|x_i(t) - x_i(\tau_{ai}(t))\big| + \\
&+ \sum_{k=1}^{n} g_{ik}(t) \big|x_k(\tau_{aik}(t))\big| + q_{0i}(t) \ \ (i = 1, \dots, n)
\end{aligned}
$$

are satisfied. This together with (2.47), (2.59), and (2.63) imply that the vector function $(u_i)_{i=1}^{n}$ with the components

$$
u_i(t) = |x_i(t)| \ \text{ for } \ t \in I \ \ (i = 1, \dots, n)
$$

is a solution of the system of functional differential inequalities

$$
u_i'(t) \le -g_{0i}(t) u_i(t) + \sum_{k=1}^{n} \ell_{ik} g_{0i}(t) \|u_i\|_{C(I)} + q_i(t) \ \ (i = 1, \dots, n),
$$

where

$$
q_i(t) = \Bigg[\sum_{k=1}^{n} (\ell_{ik} + \delta_{ik}) \big(2\|c_k\|_{C(]-\infty,a[)} + \rho_1 \ell\big) + \ell_{0i} \Bigg] g_{0i}(t) + f_{0i}(t) \quad (2.64)
$$
$$
(i = 1, \dots, n).
$$

These inequalities by virtue of Theorem 1.1 and the condition (2.9) yield

$$
\sum_{i=1}^{n} \|x_i\|_{C(I)} = \sum_{i=1}^{n} \|u_i\|_{C(I)} \le \big\|(E - H)^{-1}\big\| \ell^*, \qquad (2.65)
$$

where

$$
\ell^* = \sum_{i=1}^{n} u_i(b_0) + \sum_{i=1}^{n} \max \Bigg\{ \int\limits_{b_0}^{t} \exp\Bigg(- \int\limits_{s}^{t} g_{0i}(x)\,dx \Bigg) q_i(s)\,ds : \ b_0 \le t \le b \Bigg\}.
$$

On the other hand, according to (2.49), (2.56), (2.61), and (2.64), we have

$$\ell^* \leq n\rho_1\ell + \sum_{i,k=1}^{n} (\ell_{ik} + \delta_{ik})\big(2\|c_k\| + \rho_1\ell\big) + 2\sum_{i=1}^{n} \ell_{0i} \leq$$
$$\leq 4\sum_{i,k=1}^{n} (\ell_{ik} + \delta_{ik})\rho_1\ell.$$

Taking this inequality and the notation (2.55) into account, we find from (2.60) and (2.65) that

$$\sum_{i=1}^{n} \|x_i\|_{C([a,b])} \leq \sum_{i=1}^{n} \big(\|x_i\|_{C([a,b_0])} + \|x_i\|_{C(I)}\big) \leq \rho\ell.$$

Consequently, the estimate (2.14) is valid. □

Below, we will apply a somewhat more general than Theorem 2.2 proposition concerning the boundedness of solutions of the differential system

$$x_i'(t) + g_{0i}(t)x_i(\tau_i(t)) =$$
$$= f_i\big(t, x_1(\tau_{i11}(t)), \dots, x_1(\tau_{i1m}(t)), \dots, x_n(\tau_{in1}(t)), \dots, x_n(\tau_{inm}(t))\big) \quad (2.66)$$
$$(i = 1, \dots, n).$$

Here $f_i : \mathbb{R}_+ \times \mathbb{R}^{mn} \to \mathbb{R}$ $(i = 1, \dots, n)$ are functions satisfying the local Carathéodory conditions, $g_{0i} \in L_{loc}(\mathbb{R}_+)$ $(i = 1, \dots, n)$ are nonnegative functions, and $\tau_i : \mathbb{R}_+ \to \mathbb{R}$, $\tau_{ikj} : \mathbb{R}_+ \to \mathbb{R}$ $(i, k = 1, \dots, n;\ j = 1, \dots, m)$ are measurable on every finite interval functions such that

$$\tau_i(t) \leq t, \quad \tau_{ikj}(t) \leq t \ \text{ for } \ t \in \mathbb{R}_+ \ \ (i, k = 1, \dots, n; \ \ j = 1, \dots, m).$$

Theorem 2.2′. *Let there exist nonnegative constants ℓ_{ik} $(i, k = 1, \dots, n)$ and nonnegative functions $g_{ikj} \in L([a, +\infty[)$ $(i, k = 1, \dots, n;\ j = 1, \dots, m)$ and $f_{0i} \in L_{loc}([a, +\infty[)$ $(i=1, \dots, n)$ such that, respectively, on $[a, +\infty[\, \times \mathbb{R}^{mn}$ and $[a, +\infty[$ the inequalities*

$$\big|f_i(t, x_{11}, \dots, x_{1m}, \dots, x_{m1}, \dots, x_{nm})\big| \leq$$
$$\leq \sum_{k=1}^{n}\sum_{j=1}^{m} g_{ikj}(t)|x_{ikj}| + f_{0i}(t) \ \ (i = 1, \dots, n)$$

and

$$\sum_{j=1}^{m} g_{ikj}(t) + g_{0i}(t)\int\limits_{\tau_{ai}(t)}^{t} \Big(\sum_{j=1}^{m} g_{ikj}(s) + \delta_{ik}g_{0k}(s)\Big)\, ds \leq \ell_{ik}g_{0i}(t)$$
$$(i = 1, \dots, n)$$

are satisfied. If, moreover, the conditions (2.9), (2.48) *and* (2.49) *are fulfilled, then every noncontinuable solution of the problem* (2.66), (2.3) *is defined on* $[a, +\infty[$, *is bounded and admits the estimate* (2.50), *where* ρ *is* *a positive constant depending only on* g_{0i}, g_{ikj} *and* ℓ_{ik} $(i,k = 1,\dots,n;$ $j = 1,\dots,m)$.

We omit the proof of this theorem since it is analogous to that of Theorem 2.2.

Corollary 2.5. *Let there exist nonnegative constants* ℓ_{ik} $(i,k = 1,\dots,n)$ *and* γ *and nonnegative functions* $g_0 \in L_{loc}([a,+\infty[)$, $g_{ik} \in L_{loc}([a,+\infty[)$ *and* $f_i \in L_{loc}([a,+\infty[)$ $(i,k = 1,\dots,n)$ *such that the inequalities* (2.46) *are satisfied on* $[a,+\infty[\,\times\mathbb{R}^n$ *and the inequalities*

$$t-\tau_i(t) \le \gamma, \quad t-\tau_{ik}(t) \le \gamma \;\; (i,k = 1,\dots,n), \tag{2.67}$$

$$g_{0i}(t) \ge g_0(t) \;\; (i = 1,\dots,n) \tag{2.68}$$

along with (2.47) *are satisfied on* $[a,+\infty[$. *Let, moreover,*

$$\sup\bigg\{\int_0^t \exp\Big(-\int_s^t g_{0i}(x)\,dx\Big)\widetilde{f}_{0i}(s)\,ds + \int_{\tau_{ai}(t)}^t \widetilde{f}_{0i}(s)\,ds : \; t \ge a\bigg\} < +\infty \tag{2.69}$$

$$(i = 1,\dots,n),$$

where

$$\widetilde{f}_i(t) = \exp\Big(\int_a^t g_0(s)\,ds\Big) f_{0i}(t) \;\; (i = 1,\dots,n), \tag{2.70}$$

and let the conditions (2.9) *and* (2.22) *be fulfilled. Then every noncontinuable solution of the problem* (2.2), (2.3) *is defined on* $[a,+\infty[$ *and is vanishing at infinity.*

Proof. Without loss of generality, we can assume that $\ell_{ik} > 0$ $(i,k = 1,\dots,n)$ and

$$\int_t^{t+\gamma} g_0(s)\,ds \le 1 \;\text{ for }\; t \ge a. \tag{2.71}$$

On the other hand, by virtue of (2.9), there exists $\eta > 1$ such that the inequality (2.40) is fulfilled. We choose $\varepsilon > 0$ so small that

$$\varepsilon_{ik} = (1+\varepsilon)\exp(\varepsilon) + \varepsilon/\ell_{ik} < \eta \;\; (i,k = 1,\dots,n). \tag{2.72}$$

By the transformation

$$\begin{aligned} &x_i(t) = y_i(t) \;\text{ for }\; t < a, \\ &x_i(t) = \exp\Big(-\varepsilon\int_a^t g_0(x)\,dx\Big) y_i(t) \;\text{ for }\; t \ge a \;\; (i = 1,\dots,n), \end{aligned} \tag{2.73}$$

the problem (2.2), (2.3) is reduced to the system

$$y_i'(t) + \widetilde{g}_{0i}(t) y_i(\tau_i(t)) = \widetilde{f}_i\big(t, y_i(t), y_1(\tau_{i1}(t)),\dots,y_n(\tau_{in}(t))\big) \tag{2.74}$$

$$(i = 1,\dots,n)$$

with the initial conditions (2.25), where

$$\widetilde{g}_{0i}(t) = \exp\Big(\varepsilon \int\limits_{\tau_{ai}(t)}^{t} g_0(s)\,ds\Big) g_{0i}(t) \ \ (i=1,\dots,n), \tag{2.75}$$

$$\widetilde{f}_i(t,x,x_1,\dots,x_n) = \varepsilon g_0(t)x +$$

$$+\exp\Big(\varepsilon \int\limits_{a}^{t} g_0(s)\,ds\Big) f_i\big(t,\zeta_{i1}(t)x_1,\dots,\zeta_{in}(t)x_n\big) \ \ (i=1,\dots,n) \tag{2.76}$$

and

$$\zeta_{ik}(t) = \exp\Big(-\varepsilon \int\limits_{\tau_{aik}(t)}^{t} g_0(s)\,ds\Big) \ \ (i=1,\dots,n). \tag{2.77}$$

By the inequalities (2.46) and the notation (2.70) and (2.77), we find

$$\big|\widetilde{f}_i(t,x,x_1,\dots,x_n)\big| =$$

$$= \varepsilon g_0(t)|x| + \sum_{k=1}^{n} \widetilde{g}_{ik}(t)|x_k| + \widetilde{f}_{0i}(t) \ \ (i=1,\dots,n), \tag{2.78}$$

where

$$\widetilde{g}_{ik}(t) = \exp\Big(\varepsilon \int\limits_{\tau_{aik}(t)}^{t} g_0(s)\,ds\Big) g_{ik}(t) \ \ (i,k=1,\dots,n). \tag{2.79}$$

On the other hand, according to (2.68), it follows from (2.75) that

$$\widetilde{g}_{0i}(t) \ge g_{0i}(t) \ge g_0(t) \ \ (i=1,\dots,n). \tag{2.80}$$

By virtue of (2.67) and (2.71), the inequalities

$$\int\limits_{\tau_{ai}(t)}^{t} g_0(s)\,ds \le \int\limits_{\tau_{ai}(t)}^{\tau_{ai}(t)+\gamma} g_0(s)\,ds \le 1,$$

$$\int\limits_{\tau_{aik}(t)}^{t} g_0(s)\,ds \le \int\limits_{\tau_{aik}(t)}^{\tau_{aik}(t)+\gamma} g_0(s)\,ds \le 1, \ \ (i=1,\dots,n)$$

are satisfied on $[a,+\infty[$. Therefore from (2.75) and (2.79) we have

$$\widetilde{g}_{0i}(t) \le \exp(\varepsilon) g_{0i}(t), \quad \widetilde{g}_{ik}(t) \le \exp(\varepsilon) g_{ik}(t) \ \text{ for } \ t \ge a \ \ (i,k=1,\dots,n).$$

If along with the above estimates we take into account the inequalities (2.47), (2.72), and (2.80), we obtain

$$\widetilde{g}_{ik}(t) + \varepsilon\delta_{ik} g_0(t) + \widetilde{g}_{0i}(t) \int\limits_{\tau_{ai}(t)}^{t} \big(\widetilde{g}_{ik}(s) + \varepsilon\delta_{ik} g_0(s) + \delta_{ik}\widetilde{g}_{ik}(s)\big)\,ds \le$$

$$\le \exp(\varepsilon) g_{ik}(t) + \varepsilon \widetilde{g}_{0i}(t) + (1+\varepsilon)\exp(\varepsilon)\widetilde{g}_{oi}(t) \int\limits_{\tau_{ai}(t)}^{t} \big(g_{ik}(s) + \delta_{ik} g_{0k}(s)\big)\, ds \le$$

$$\le (1+\varepsilon)\exp(\varepsilon)\exp\Big(\int\limits_{\tau_{ai}(t)}^{t} g_0(s)\, ds\Big)\Big[g_{ik}(t) + \int\limits_{\tau_{ai}(t)}^{t} \big(g_{ik}(s)+\delta_{ik} g_{0k}(s)\big)\, ds\Big] +$$

$$+\varepsilon \widetilde{g}_{0i}(t) \le \varepsilon_{ik} \ell_{ik} \widetilde{g}_{0i}(t) \le \eta \ell_{ik} \widetilde{g}_{0i}(t) \ \ (i=1,\dots,n). \tag{2.81}$$

By Theorem 2.2$'$, it follows from the conditions (2.40), (2.67), (2.69), (2.78), and (2.81) that every noncontinuable solution $(y_i)_{i=1}^n$ of the problem (2.76), (2.25) is defined on $[a,+\infty[$ and is bounded.

On the other hand, every noncontinuable solution $(x_i)_{i=1}^n$ of the problem (2.2), (2.3) admits the representation (2.73). Owing to (2.22) and the boundedness of $(y_i)_{i=1}^n$, it is clear that $(x_i)_{i=1}^n$ is vanishing at infinity. □

Corollary 2.6. *Let there exist constants $\delta_0 > 0$, $\ell_{ik} \ge 0$ $(i,k =,\dots,n)$ and nonnegative functions $g_{ik} \in L_{loc}(\mathbb{R}_+)$ $(i,k = 1,\dots,n)$ such that, respectively, on the set* (2.26) *and on the interval $\mathbb{R}_+$ the inequalities*

$$\big|f_i(t,x_1,\dots,x_n)\big| \le \sum_{k=1}^{n} g_{ik}(t)|x_k| \ \ (i=1,\dots,n) \tag{2.82}$$

and (2.47) *are satisfied. If, moreover,*

$$\liminf_{t\to+\infty} \tau_i(t) > 0 \ \ (i=1,\dots,n)$$

and the condition (2.9) *is fulfilled, then the trivial solution of the system* (2.2) *is uniformly stable.*

Corollary 2.7. *Let there exist constants $\delta_0 > 0$, $\ell_{ik} \ge 0$ $(i,k = 1,\dots,n)$ and nonnegative functions $g_{ik} \in L_{loc}(\mathbb{R}_+)$ $(i,k = 1,\dots,n)$ such that on the set* (2.26) *the inequalities* (2.82) *are fulfilled, while on the interval $\mathbb{R}_+$ the inequalities* (2.47) *and* (2.67) *hold. If, moreover, the conditions* (2.9) *and* (2.34) *are fulfilled, where*

$$g_0(t) = \min\big\{g_{0i}(t): \ i=1,\dots,n\big\},$$

then the trivial solution of the system (2.2) *is uniformly asymptotically stable.*

Corollary 2.6 (Corollary 2.7) is proved analogously to Corollary 2.2 (Corollary 2.3). The only difference is that instead of Theorem 2.1 we use Theorem 2.2 (Theorem 2.2 and Corollary 2.5).

As an example, let us consider the linear differential system

$$x_i'(t) = \sum_{k=1}^{n} p_{ik}(t) x_i(\tau_{ik}(t)) \ \ (i=1,\dots,n), \tag{2.83}$$

where $p_{ik} \in L_{loc}(\mathbb{R}_+)$ $(i,k=1,\dots,n)$, and $\tau_{ik} : \mathbb{R}_+ \to \mathbb{R}$ $(i,k=1,\dots,n)$ are measurable on every finite segment functions satisfying the inequalities

$$\tau_{ik}(t) \le t \ \ (i,k=1,\dots,n).$$

The system (2.83) is said to be **uniformly stable** (**uniformly asymptotically stable**) if its trivial solution is uniformly stable (uniformly asymptotically stable).

Suppose

$$\tau_{0i}(t) = \begin{cases} \tau_{ii}(t) & \text{for } \ \tau_{ii}(t) \ge 0 \\ 0 & \text{for } \ \tau_{ii}(t) < 0 \end{cases} \quad (i=1,\dots,n).$$

From Corollary 2.6 we have

Corollary 2.8. *Let almost everywhere on* $\mathbb{R}_+$ *the inequalities*

$$p_{ii}(t) \le 0, \quad \int\limits_{\tau_{0i}(t)}^{t} |p_{ii}(s)|\, ds \le \ell_{ii} \ \ (i=1,\dots,n), \tag{2.84}$$

$$|p_{ik}(t)|+|p_{ii}(t)| \int\limits_{\tau_{0i}(t)}^{t} |p_{ik}(s)|\, ds \le \ell_{ik} |p_{ii}(t)| \ \ (i,k=1,\dots,n; \ \ i \ne k) \tag{2.85}$$

be satisfied, where ℓ_{ik} $(i,k=1,\dots,n)$ *are nonnegative constants satisfying the condition* (2.9)*. If, moreover,*

$$\liminf_{t\to+\infty} \tau_{ii}(t) > 0 \ \ (i=1,\dots,n),$$

then the system (2.83) *is uniformly stable.*

Corollary 2.7 results in

Corollary 2.9. *Let almost everywhere on* $\mathbb{R}_+$ *the inequalities* (2.84) *and* (2.85) *be satisfied, where* ℓ_{ik} $(i,k=1,\dots,n)$ *are nonnegative constants satisfying the condition* (2.9)*. If, moreover,*

$$\operatorname{vrai\,max}\big\{t-\tau_{ik}(t) : \ t \in \mathbb{R}_+\big\} < +\infty \ \ (i,k=1,\dots,n), \quad \int\limits_0^{+\infty} p(t)\, dt = +\infty,$$

where

$$p(t) = \min\big\{|p_{ii}(t)| : \ i=1,\dots,n\big\},$$

then the system (2.83) *is uniformly asymptotically stable.*

For $\tau_{ik}(t) \equiv t$ $(i,k=1,\dots,n)$, results analogous to Corollaries 2.8 and 2.9 have been obtained in [8].

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