Memoirs on Differential Equations and Mathematical Physics
Volume 41, 2007, 97–114

Josef Rebenda

ASYMPTOTIC PROPERTIES OF SOLUTIONS OF REAL TWO-DIMENSIONAL DIFFERENTIAL SYSTEMS WITH A FINITE NUMBER OF CONSTANT DELAYS

Abstract. In this article stability and asymptotic properties of a real two-dimensional system $x'(t) = \mathbf{A}(t)x(t) + \sum\limits_{j=1}^{n} \mathbf{B}_j(t)x(t-r_j) + \mathbf{h}(t, x(t), x(t-r_1), \dots, x(t-r_n))$ are studied, where $r_1 > 0, \dots, r_n > 0$ are constant delays, $\mathbf{A}, \mathbf{B}_1, \dots, \mathbf{B}_n$ are matrix functions and $\mathbf{h}$ is a vector function. A generalization of results on stability of a two-dimensional differential system with one constant delay is obtained by using the methods of complexification and Lyapunov–Krasovskiĭ functional and some new types of corollaries are presented. The case $\liminf\limits_{t\to\infty}(|a(t)| - |b(t)|) > 0$ is studied.

2000 Mathematics Subject Classification. 34K20.

Key words and phrases. Stability, asymptotic behaviour, two-dimensional system with delay.

რეზიუმე. ნაშრომში შესწავლილია ნამდვილი ორგანზომილებიანი

$$x'(t) = \mathbf{A}(t)x(t) + \sum_{j=1}^{n} \mathbf{B}_j(t)x(t-r_j) + \mathbf{h}(t, x(t), x(t-r_1), \dots, x(t-r_n))$$

სისტემის მდგრადობა და ასიმპტოტური თვისებები, სადაც $r_1 > 0, \dots, r_n > 0$ მუდმივი დაგვიანებებია, $\mathbf{A}, \mathbf{B}_1, \dots, \mathbf{B}_n$ მატრიცული ფუნქციებია, ხოლო $\mathbf{h}$ ვექტორ-ფუნქციაა. კომპლექსიფიკაციისა და ლიაპუნოვ–კრასოვსკის ფუნქციონალის მეთოდების გამოყენებით მიღებულია ერთი დაგვიანების შემცველი ორგანზომილებიანი სისტემის მდგრადობის შესახებ ცნობილი შედეგების განზოგადება, აგრეთვე; მოყვანილია ზოგიერთი ახალი ტიპის შედეგი. განხილულია შემთხვევა, როცა $\liminf\limits_{t\to\infty}(|a(t)| - |b(t)|) > 0$.

1. Introduction

The subject of our study is the real two-dimensional system

$$x'(t) = \mathbf{A}(t)x(t)+\sum_{j=1}^{n}\mathbf{B}_j(t)x(t-r_j)+\mathbf{h}(t,x(t),x(t-r_1),\dots,x(t-r_n)), \quad (0)$$

where $\mathbf{A}(t) = (a_{ik}(t))$, $\mathbf{B}_j(t) = (b_{jik}(t))$ $(i,k = 1,2)$ for $j \in \{1,\dots,n\}$ are real square matrices and

$$\mathbf{h}(t,x,y_1,\dots,y_n) = \big(h_1(t,x,y_1,\dots,y_n),h_2(t,x,y_1,\dots,y_n)\big)$$

is a real vector function. We suppose that the functions a_{ik} are locally absolutely continuous on $[t_0,\infty)$, b_{jik} are locally Lebesgue integrable on $[t_0,\infty)$ and the function $\mathbf{h}$ satisfies the Carathéodory conditions on

$$\begin{aligned}[t_0,\infty) \times \big\{[x_1,x_2] \in \mathbb{R}^2 \colon x_1^2+x_2^2 < R^2\big\}\times\\ \times \big\{[y_{11},y_{12}] \in \mathbb{R}^2 \colon y_{11}^2+y_{12}^2 < R^2\big\} \times \dots \times\\ \times \big\{[y_{n1},y_{n2}] \in \mathbb{R}^2 \colon y_{n1}^2+y_{n2}^2 < R^2\big\},\end{aligned}$$

where $0 < R \le \infty$ is a real constant.

The investigation of the problem is based on the combination of the method of complexification and the method of Lyapunov–Krasovskiĭ functional, which is to a great extent effective for two-dimensional systems. This combination was successfully used in papers [2] and [3] and leads to interesting results.

The following notation will be used throughout the article:
$\mathbb{R}$ the set of all real numbers;
$\mathbb{R}_+$ the set of all positive real numbers;
$\mathbb{C}$ the set of all complex numbers;
$\mathbb{N}$ the set of all positive integers;
$\operatorname{Re} z$ the real part of z;
$\operatorname{Im} z$ the imaginary part of z;
$\overline{z}$ the complex conjugate of z;
$AC_{\text{loc}}(I,M)$ the class of all locally absolutely continuous functions $I \to M$;
$L_{\text{loc}}(I,M)$ the class of all locally Lebesgue integrable functions $I \to M$;
$K(I \times \Omega, M)$ the class of all functions $I \times \Omega \to M$ satisfying the Carathéodory conditions on $I \times \Omega$.

Introducing the complex variables $z = x_1 + ix_2$, $w_1 = y_{11} + iy_{12}$, …, $w_n = y_{n1} + iy_{n2}$, we can rewrite the system (0) as an equivalent equation with complex-valued coefficients:

$$\begin{aligned}z'(t) = a(t)z(t) + b(t)\overline{z}(t) + \sum_{j=1}^{n}\big[A_j(t)z(t-r_j) + B_j(t)\overline{z}(t-r_j)\big]+\\ + g\big(t,z(t),z(t-r_1),\dots,z(t-r_n)\big),\end{aligned} \quad (1)$$

where the functions a, b, A_j, B_j and g arise in similar way as in [2] and [3]. Conversely, the equation (1) can be written in the real form (0) as well.

2. Results

We study the equation

$$\begin{aligned} z'(t) = a(t)z(t) + b(t)\overline{z}(t) + \sum_{j=1}^{n}\big[A_j(t)z(t-r_j) + B_j(t)\overline{z}(t-r_j)\big] + \\ + g(t, z(t), z(t-r_1), \dots, z(t-r_n)), \end{aligned} \tag{1}$$

where r_j are positive constants for $j = 1, \dots, n$, A_j, $B_j \in L_{\mathrm{loc}}(J, \mathbb{C})$, a, $b \in AC_{\mathrm{loc}}(J, \mathbb{C})$, $g \in K(J \times \Omega, \mathbb{C})$, where $J = [t_0, \infty)$, $\Omega = \big\{(z, w_1, \dots, w_n) \in \mathbb{C}^{n+1}\colon\ |z| < R,\ |w_j| < R,\ j = 1, \dots, n\big\}$, $R > 0$. Denote $r = \max\{r_j\colon\ j = 1, \dots, n\}$.

In this article we consider the case

$$\liminf_{t\to\infty}\big(|a(t)| - |b(t)|\big) > 0 \tag{2'}$$

and study the behavior of solutions of (1) under this assumption.

Obviously, the inequality (2′) is equivalent to the existence of $T \geq t_0 + r$ and $\mu > 0$ such that

$$|a(t)| > |b(t)| + \mu \ \text{ for } \ t \geq T - r. \tag{2}$$

Denote

$$\gamma(t) = |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2}\,, \quad c(t) = \frac{\overline{a}(t)b(t)}{|a(t)|}\,. \tag{3}$$

Since $\gamma(t) > |a(t)|$ and $|c(t)| = |b(t)|$, the inequality

$$\gamma(t) > |c(t)| + \mu \tag{4}$$

is true for all $t \geq T - r$. It is easy to verify that γ, $c \in AC_{\mathrm{loc}}([T - r, \infty), \mathbb{C})$.

In the text we will often consider the following three conditions:

(i) The numbers $T \geq t_0 + r$ and $\mu > 0$ are such that (2) holds.

(ii) There are functions $\kappa_0, \kappa_1, \dots, \kappa_n, \lambda\colon [T, \infty) \mapsto \mathbb{R}$ such that

$$\begin{aligned} \big|\gamma(t)g(t, z, w_1, \dots, w_n) + c(t)\overline{g}(t, z, w_1, \dots, w_n)\big| \leq \\ \leq \kappa_0(t)|\gamma(t)z(t) + c(t)\overline{z}(t)| + \sum_{j=1}^{n}\kappa_j(t)\big|\gamma(t-r_j)w_j + c(t-r_j)\overline{w_j}\big| + \lambda(t) \end{aligned}$$

for $t \geq T$, $|z| < R$ and $|w_j| < R$ for $j = 1, \dots, n$, where κ_0, $\lambda \in L_{\mathrm{loc}}([T, \infty), \mathbb{R})$.

(iii) The function $\beta \in AC_{\mathrm{loc}}([T, \infty), \mathbb{R}_+)$ satisfies

$$\beta(t) \geq \psi(t) \ \text{ a.e. on } \ [T, \infty), \tag{5}$$

where ψ is defined for every $t \geq T$ by

$$\psi(t) = \max_{j=1,\dots,n}\left\{\kappa_j(t) + \big(|A_j(t)| + |B_j(t)|\big)\frac{\gamma(t) + |c(t)|}{\gamma(t-r_j) - |c(t-r_j)|}\right\}. \tag{6}$$

Clearly, if A_j, B_j, κ_j are absolutely continuous on $[T,\infty)$ for $j=1,\dots,n$ and $\psi(t)>0$ on $[T,\infty)$, we may choose $\beta(t)=\psi(t)$.

For the rest of the paper we denote

$$\begin{aligned}
\alpha(t) &= 1+\Big|\frac{b(t)}{a(t)}\Big| \operatorname{sgn}\operatorname{Re} a(t),\\
\vartheta(t) &= \frac{\operatorname{Re}(\gamma(t)\gamma'(t)-\overline{c}(t)c'(t))+|\gamma(t)c'(t)-\gamma'(t)c(t)|}{\gamma^2(t)-|c(t)|^2}\,,\\
\theta(t) &= \alpha(t)\operatorname{Re} a(t)+\vartheta(t)+\kappa_0(t)+n\beta(t),\\
\Lambda(t) &= \max\Big(\theta(t),\frac{\beta'(t)}{\beta(t)}\Big).
\end{aligned}\tag{7}$$

From the assumption (i) we get

$$\begin{aligned}
|\vartheta| &\le \frac{|\operatorname{Re}(\gamma\gamma'-\overline{c}c')|+|\gamma c'-\gamma' c|}{\gamma^2-|c|^2} \le \frac{(|\gamma'|+|c'|)(|\gamma|+|c|)}{\gamma^2-|c|^2} =\\
&= \frac{|\gamma'|+|c'|}{\gamma-|c|} \le \frac{1}{\mu}\big(|\gamma'|+|c'|\big),
\end{aligned}$$

so the functions ϑ, θ and Λ are locally Lebesgue integrable on $[T,\infty)$.

Notice that the condition (ii) implies that the functions $\kappa_j(t)$ are non-negative on $[T,\infty)$ for $j=0,\dots,n$, and due to this, $\psi(t)\ge 0$ on $[T,\infty)$. Moreover, if $\lambda(t)\equiv 0$ in (ii), then the equation (1) has the trivial solution $z(t)\equiv 0$.

Before we get to the main results, we prove

Lemma 1. *Let a_1, a_2, b_1, $b_2\in\mathbb{C}$ and $|a_2|>|b_2|$. Then*

$$\operatorname{Re}\frac{a_1 z+b_1\overline{z}}{a_2 z+b_2\overline{z}} \le \frac{\operatorname{Re}(a_1\overline{a_2}-b_1\overline{b_2})+|a_1 b_2-a_2 b_1|}{|a_2|^2-|b_2|^2}$$

for $z\in\mathbb{C}$, $z\ne 0$.

Proof. Firstly we will prove that $w=a_2 z+b_2\overline{z}$ is a bijective transformation of $\mathbb{C}$ onto itself. Indeed, since $\overline{w}=\overline{b_2}z+\overline{a_2 z}$, from $\overline{a_2}w-b_2\overline{w}=|a_2|^2 z-|b_2|^2 z$ we obtain

$$z=\frac{\overline{a_2}w-b_2\overline{w}}{|a_2|^2-|b_2|^2}\,.$$

Substituting this into $\operatorname{Re}\frac{a_1 z+b_1\overline{z}}{a_2 z+b_2\overline{z}}$, we have

$$\begin{aligned}
\operatorname{Re}\frac{a_1 z+b_1\overline{z}}{a_2 z+b_2\overline{z}} &= \operatorname{Re}\frac{\frac{a_1(\overline{a_2}w-b_2\overline{w})+b_1(a_2\overline{w}-\overline{b_2}w)}{|a_2|^2-|b_2|^2}}{w} =\\
&= \frac{1}{|a_2|^2-|b_2|^2}\operatorname{Re}\frac{w(a_1\overline{a_2}-b_1\overline{b_2})+\overline{w}(a_2 b_1-a_1 b_2)}{w} =\\
&= \frac{1}{|a_2|^2-|b_2|^2}\Big(\operatorname{Re}(a_1\overline{a_2}-b_1\overline{b_2})+\operatorname{Re}\Big[(a_2 b_1-a_1 b_2)\,\frac{\overline{w}}{w}\Big]\Big) \le\\
&\le \frac{\operatorname{Re}(a_1\overline{a_2}-b_1\overline{b_2})+|a_2 b_1-a_1 b_2|}{|a_2|^2-|b_2|^2}\,. \qquad \square
\end{aligned}$$

Theorem 1. *Let the conditions* (i), (ii) *and* (iii) *hold and* $\lambda(t) \equiv 0$.

a) *If*

$$\limsup_{t\to\infty} \int^{t} \Lambda(s)\, ds < \infty, \tag{8}$$

then the trivial solution of (1) *is stable on* $[T, \infty)$*;*

b) *if*

$$\lim_{t\to\infty} \int^{t} \Lambda(s)\, ds = -\infty, \tag{9}$$

then the trivial solution of (1) *is asymptotically stable on* $[T, \infty)$.

Proof. Choose arbitrary $t_1 \geq T$. Let $z(t)$ be any solution of (1) satisfying the condition $z(t) = z_0(t)$ for $t \in [t_1 - r, t_1]$, where $z_0(t)$ is a continuous complex-valued initial function defined on $t \in [t_1 - r, t_1]$. Consider the function

$$V(t) = U(t) + \beta(t) \sum_{j=1}^{n} \int_{t-r_j}^{t} U(s)\, ds, \tag{10}$$

where

$$U(t) = \big|\gamma(t)z(t) + c(t)\overline{z}(t)\big|.$$

To simplify the following computation, denote $w_j(t) = z(t-r_j)$ and write the functions of the variable t without brackets, for example, z instead of $z(t)$.

From (10) we get

$$\begin{aligned} V' = U' &+\beta' \sum_{j=1}^{n} \int_{t-r_j}^{t} U(s)\, ds + n\beta|\gamma z + c\overline{z}| - \\ &-\beta \sum_{j=1}^{n} \big|\gamma(t - r_j)w_j + c(t - r_j)\overline{w_j}\big| \end{aligned} \tag{11}$$

for almost all $t \geq t_1$ for which $z(t)$ is defined and $U'(t)$ exists.

Denote $\mathcal{K} = \big\{t \geq t_1\colon\ z(t) \text{ exists, } U(t) \neq 0\big\}$ and $\mathcal{M} = \big\{t \geq t_1\colon\ z(t)$ exists, $U(t) = 0\big\}$. It is clear that the derivative $U'(t)$ exists for almost all $t \in \mathcal{K}$, so let us focus on the set $\mathcal{M}$.

In view of (4) we have $z(t) = 0$ for $t \in \mathcal{M}$ and for almost all $t \in \mathcal{M}$ we compute

$$\begin{aligned} U'_{\pm}(t) &= \lim_{\tau\to t\pm} \frac{U(\tau) - U(t)}{\tau - t} = \lim_{\tau\to t\pm} \frac{U(\tau)}{\tau - t} = \\ &= \lim_{\tau\to t\pm} \frac{\big|\gamma(\tau)[z(\tau) - z(t)] - c(\tau)[\overline{z}(\tau) - \overline{z}(t)]\big|}{\tau - t} = \\ &= \pm\big|\gamma(t)z'(t) + c(t)\overline{z}'(t)\big| = \pm\big|\gamma(t)g^*(t) + c(t)\overline{g^*}(t)\big|, \end{aligned}$$

where

$$g^*(t)=\sum_{j=1}^{n}\big(A_j(t)w_j(t)+B_j(t)\overline{w_j}(t)\big)+g\big(t,0,w_1(t),\dots,w_n(t)\big).$$

Hence U has one-sided derivatives almost everywhere in $\mathcal{M}$. According to [4], chapter IX, Theorem 1.1, or [1], the set of all t such that $U'_+(t)\neq U'_-(t)$ can be at most countable, so the derivative U' exists for almost all $t\in\mathcal{M}$, and for these t, $U'(t)=0$.

In particular, the derivative U' exists for almost all $t\geq t_1$ for which $z(t)$ is defined. Thus (11) holds for almost all $t\geq t_1$ for which $z(t)$ is defined.

Now return to the set $\mathcal{K}$. Since

$$az+b\overline{z}=\frac{a}{2|a|}\,(\gamma z+c\overline{z})+\frac{b}{2\gamma}\,(\gamma\overline{z}+\overline{c}z),$$

the equation (1) can be written in the form

$$\begin{aligned} z'=&\ \frac{a}{2|a|}\,(\gamma z+c\overline{z})+\frac{b}{2\gamma}\,(\gamma\overline{z}+\overline{c}z)+\\ &+\sum_{j=1}^{n}\big(A_jw_j+B_j\overline{w_j}\big)+g(t,z,w_1,\dots,w_n). \end{aligned} \tag{12}$$

Short computation leads to

$$\operatorname{Re}\Big[\frac{\gamma a}{2|a|}+\frac{c\overline{b}}{2\gamma}\Big]=\operatorname{Re}a,\quad \frac{b}{2}+\frac{c\overline{a}}{2|a|}=b\,\frac{\operatorname{Re}a}{a}.$$

In view of this and (12) we have

$$\begin{aligned} UU'=&\ U\big(\sqrt{(\gamma z+c\overline{z})(\overline{\gamma z}+\overline{c}z)}\big)'=\\ &=\operatorname{Re}\big[(\gamma\overline{z}+\overline{c}z)(\gamma' z+\gamma z'+c'\overline{z}+c\overline{z}')\big]=\\ &=\operatorname{Re}\bigg\{(\gamma\overline{z}+\overline{c}z)\bigg[\gamma' z+c'\overline{z}+\gamma\Big(\frac{a}{2|a|}\,(\gamma z+c\overline{z})+\frac{b}{2\gamma}\,(\gamma\overline{z}+\overline{c}z)+\\ &\qquad+\sum_{j=1}^{n}\big(A_jw_j+B_j\overline{w_j}\big)+g\Big)+\\ &+c\Big(\frac{\overline{a}}{2|a|}\,(\gamma\overline{z}+\overline{c}z)+\frac{\overline{b}}{2\gamma}\,(\gamma z+c\overline{z})+\sum_{j=1}^{n}\big(\overline{A_j}\overline{w_j}+\overline{B_j}w_j\big)+\overline{g}\Big)\bigg]\bigg\}\leq\\ &\leq|\gamma z+c\overline{z}|^2\Big(\operatorname{Re}a+|b|\,\frac{|\operatorname{Re}a|}{|a|}\Big)+\\ &+\operatorname{Re}\bigg\{(\gamma\overline{z}+\overline{c}z)\bigg[\gamma' z+c'\overline{z}+\gamma\Big(\sum_{j=1}^{n}\big(A_jw_j+B_j\overline{w_j}\big)+g\Big)+\\ &+c\Big(\sum_{j=1}^{n}\big(\overline{A_j}\overline{w_j}+\overline{B_j}w_j\big)+\overline{g}\Big)\bigg]\bigg\} \end{aligned}$$

for almost all $t\in\mathcal{K}$.

If we recall the definition of $\alpha(t)$, then

$$\begin{aligned}
UU' &\le U^2\alpha \operatorname{Re} a + \\
&+ \operatorname{Re}\Big\{(\gamma\overline{z}+\overline{c}z)\Big[\gamma\sum_{j=1}^{n}\big(A_jw_j+B_j\overline{w_j}\big)+c\sum_{j=1}^{n}\big(\overline{A_j}\,\overline{w_j}+\overline{B_j}w_j\big)\Big]\Big\}+ \\
&+ \operatorname{Re}\big[(\gamma\overline{z}+\overline{c}z)(\gamma g+c\overline{g})\big]+\operatorname{Re}\big[(\gamma\overline{z}+\overline{c}z)(\gamma' z+c'\overline{z})\big] \le \\
&\le U^2\alpha \operatorname{Re} a + U(\gamma+|c|)\Big(\sum_{j=1}^{n}|A_jw_j+B_j\overline{w_j}|\Big)+ \\
&+ U|\gamma g+c\overline{g}|+U^2\operatorname{Re}\frac{\gamma' z+c'\overline{z}}{\gamma z+c\overline{z}}\,.
\end{aligned}$$

Applying Lemma 1 to the last term, we obtain

$$\operatorname{Re}\frac{\gamma' z+c'\overline{z}}{\gamma z+c\overline{z}} \le \vartheta.$$

Using this inequality together with (6) and the assumption (ii), we get

$$\begin{aligned}
UU' &\le U^2(\alpha\operatorname{Re} a+\vartheta+\kappa_0)+U\sum_{j=1}^{n}\big(\kappa_j\big|\gamma(t-r_j)w_j+c(t-r_j)\overline{w_j}\big|\big)+ \\
&+ U(\gamma+|c|)\Big(\sum_{j=1}^{n}\frac{|A_j|\,|w_j|+|B_j|\,|\overline{w_j}|}{\gamma(t-r_j)-|c(t-r_j)|}\,\big(\gamma(t-r_j)-|c(t-r_j)|\big)\Big) \le \\
&\le U^2(\alpha\operatorname{Re} a+\vartheta+\kappa_0)+ \\
&+ U\Big\{\sum_{j=1}^{n}\Big[\kappa_j+(|A_j|+|B_j|)\,\frac{\gamma+|c|}{\gamma(t-r_j)-|c(t-r_j)|}\Big]\times \\
&\times \big|\gamma(t-r_j)w_j+c(t-r_j)\overline{w_j}\big|\Big\} \le \\
&\le U^2(\alpha\operatorname{Re} a+\vartheta+\kappa_0)+U\psi\sum_{j=1}^{n}\big|\gamma(t-r_j)w_j+c(t-r_j)\overline{w_j}\big|
\end{aligned}$$

for almost all $t\in\mathcal{K}$.

Consequently,

$$U' \le U(\alpha\operatorname{Re} a+\vartheta+\kappa_0)+\psi\sum_{j=1}^{n}\big|\gamma(t-r_j)w_j+c(t-r_j)\overline{w_j}\big| \tag{13}$$

for almost all $t\in\mathcal{K}$.

Recalling that $U'(t)=0$ for almost all $t\in\mathcal{M}$, we can see that the inequality (13) is valid for almost all $t\ge t_1$ for which $z(t)$ is defined.

From (11) and (13) we have

$$V' \le U(\alpha\operatorname{Re} a+\vartheta+\kappa_0+n\beta)+(\psi-\beta)\sum_{j=1}^{n}\big|\gamma(t-r_j)w_j+c(t-r_j)\overline{w_j}\big|+$$

$$+\beta' \sum_{j=1}^{n} \int_{t-r_j}^{t} \big|\gamma(s)z(s) + c(s)\overline{z}(s)\big|\, ds.$$

As $\beta(t)$ fulfills the condition (5), we obtain

$$V'(t) \leq U(t)\theta(t) + \beta'(t) \sum_{j=1}^{n} \int_{t-r_j}^{t} \big|\gamma(s)z(s) + c(s)\overline{z}(s)\big|\, ds.$$

Hence

$$V'(t) - \Lambda(t)V(t) \leq 0 \tag{14}$$

for almost all $t \geq t_1$ for which the solution $z(t)$ exists.

Notice that, with respect to (4),

$$V(t) \geq (\gamma(t) - |c(t)|)|z(t)| \geq \mu|z(t)| \tag{15}$$

for all $t \geq t_1$ for which $z(t)$ is defined.

Suppose that the condition (8) holds, and choose an arbitrary $0 < \varepsilon < R$. Put

$$\Delta = \max_{s\in[t_1-r,t_1]} (\gamma(s) + |c(s)|), \quad L = \sup_{T\leq t<\infty} \int_{T}^{t} \Lambda(s)\, ds$$

and

$$\delta = \mu\varepsilon\Delta^{-1}\Big(1 + \beta(t_1)\sum_{j=1}^{n} r_j\Big)^{-1} \exp\Big\{\int_{T}^{t_1} \Lambda(s)\, ds - L\Big\},$$

where μ is the number from the condition (i).

If the initial function $z_0(t)$ of the solution $z(t)$ satisfies $\max\limits_{s\in[t_1-r,t_1]} |z_0(s)| < \delta$, then the multiplication of (14) by $\exp\big\{-\int_{t_1}^{t} \Lambda(s)\, ds\big\}$ and the integration over $[t_1, t]$ yield

$$V(t)\exp\Big\{-\int_{t_1}^{t} \Lambda(s)\, ds\Big\} - V(t_1) \leq 0 \tag{16}$$

for all $t \geq t_1$ for which $z(t)$ is defined. From (15) and (16) we obtain

$$\mu|z(t)| \leq V(t) \leq V(t_1)\exp\Big\{\int_{t_1}^{t} \Lambda(s)\, ds\Big\} \leq \Big[(\gamma(t_1) + |c(t_1)|)|z(t_1)| +$$

$$+\beta(t_1) \max_{s\in[t_1-r,t_1]} |z(s)| \Big(\sum_{j=1}^{n} \int_{t_1-r_j}^{t_1} (\gamma(s) + |c(s)|)\, ds\Big)\Big] \exp\Big\{\int_{t_1}^{t} \Lambda(s)\, ds\Big\} \leq$$

$$\leq \Big[\Delta \max_{s\in[t_1-r,t_1]} |z_0(s)| + \beta(t_1) \max_{s\in[t_1-r,t_1]} |z_0(s)| \Delta \sum_{j=1}^{n} r_j\Big] \exp\Big\{\int_{t_1}^{t} \Lambda(s)\, ds\Big\},$$

i.e.,

$$\mu|z(t)| \leq \Delta \max_{s\in[t_1-r,t_1]} |z_0(s)| \Big(1 + \beta(t_1) \sum_{j=1}^{n} r_j\Big) \exp\Big\{L - \int_{T}^{t_1} \Lambda(s)\, ds\Big\} < \mu\varepsilon.$$

Thus we have $|z(t)| < \varepsilon$ for all $t \geq t_1$ and we conclude that the trivial solution of the equation (1) is stable.

Now suppose that the condition (9) is valid. Then, in view of the first part of Theorem 1, for $R > 0$ there is a $\rho > 0$ such that $\max_{s\in[t_1-r,t_1]} |z_0(s)| < \rho$ implies that the solution $z(t)$ of (1) exists for all $t \geq t_1$ and satisfies $|z(t)| < R$, where R is from the definition of the set Ω. Hence,

$$|z(t)| \leq \mu^{-1} V(t) \leq \mu^{-1} V(t_1) \exp\Big\{\int_{t_1}^{t} \Lambda(s)\, ds\Big\}$$

for all $t \geq t_1$. This inequality along with the condition (9) gives

$$\lim_{t\to\infty} z(t) = 0,$$

which completes the proof. □

Remark 1. Since

$$\vartheta = \frac{\mathrm{Re}(\gamma\gamma' - \bar{c}c') + |\gamma c' - \gamma' c|}{\gamma^2 - |c|^2} \leq \frac{(|\gamma'| + |c'|)(|\gamma| + |c|)}{\gamma^2 - |c|^2} = \frac{|\gamma'| + |c'|}{\gamma - |c|},$$

it follows from (4) that we can replace the function ϑ in (7) by $\frac{1}{\mu}(|\gamma'| + |c'|)$.

Corollary 1. *Let the assumptions* (i), (ii) *and* (iii) *be fulfilled and* $\lambda(t) \equiv 0$*. If for some* $K \in \mathbb{R}_+$ *and* $T_1 \geq T$ *the function* $\beta(t)$ *satisfies* $\beta(T_1) = K$*,* $\beta(t) \leq K$ *for all* $t \geq T_1$ *and*

$$\lim_{t\to\infty} \int^{t} [\theta^*(s)]_+\, ds < \infty,$$

where $\theta^*(t) = \theta(t) - n\beta(t) + nK$ *and* $[\theta^*(t)]_+ = \max\{\theta^*(t), 0\}$*, then the trivial solution of* (1) *is stable.*

Proof. Put

$$\beta^*(t) = \begin{cases} \beta(t) & \text{on } [T, T_1]; \\ K & \text{for } t \geq T_1. \end{cases}$$

Then $\beta^*(t) \in AC_{\mathrm{loc}}([T,\infty), \mathbb{R}_+)$ and it is easy to see that $\beta^*(t) \geq \psi(t)$ a. e. on $[T,\infty)$.

Now $(\beta^*)'(t) \equiv 0$ on $[T_1,\infty)$, and also $\frac{(\beta^*)'(t)}{\beta^*(t)} \equiv 0$ on $[T_1,\infty)$. Clearly

$$\Lambda^*(t) = \max\{\theta^*(t), 0\} = [\theta^*(t)]_+$$

on $[T_1,\infty)$, and then

$$\limsup_{t\to\infty}\int^t \Lambda^*(s)\,ds=\limsup_{t\to\infty}\int^t[\theta^*(s)]_+\,ds=\lim_{t\to\infty}\int^t[\theta^*(s)]_+\,ds<\infty.$$

The assertion now follows from Theorem 1. □

Corollary 2. *Assume that the conditions* (i), (ii) *and* (iii) *are valid with* $\lambda(t)\equiv 0$. *If* $\beta(t)$ *is monotone and bounded on* $[T,\infty)$ *and if*

$$\lim_{t\to\infty}\int^t[\theta(s)]_+\,ds<\infty,$$

where $[\theta(t)]_+=\max\{\theta(t),0\}$, *then the trivial solution of* (1) *is stable.*

Proof. Suppose firstly that β is non-increasing on $[T,\infty)$. Then $\beta'\le 0$ a.e. on $[T,\infty)$. Since $\beta(t)>0$ on $[T,\infty)$, it follows that $\frac{\beta'}{\beta}\le 0$ a.e. on $[T,\infty)$. Hence

$$\Lambda(t)=\max\big\{\theta(t),\frac{\beta'(t)}{\beta(t)}\big\}\le\max\{\theta(t),0\}=[\theta(t)]_+,$$

and then

$$\limsup_{t\to\infty}\int^t \Lambda(s)\,ds\le\limsup_{t\to\infty}\int^t[\theta(s)]_+\,ds=\lim_{t\to\infty}\int^t[\theta(s)]_+\,ds<\infty.$$

Now assume that β is non-decreasing on $[T,\infty)$. Then $\beta'\ge 0$ a.e. on $[T,\infty)$ and it follows that $\frac{\beta'}{\beta}\ge 0$ a.e. on $[T,\infty)$. Hence

$$\Lambda(t)=\max\Big\{\theta(t),\frac{\beta'(t)}{\beta(t)}\Big\}\le\Big\{[\theta(t)]_+,\frac{\beta'(t)}{\beta(t)}\Big\}\le[\theta(t)]_+ +\frac{\beta'(t)}{\beta(t)}\,,$$

and then

$$\limsup_{t\to\infty}\int^t \Lambda(s)\,ds\le\limsup_{t\to\infty}\int^t[\theta(s)]_+\,ds+\limsup_{t\to\infty}\int^t\frac{\beta'(t)}{\beta(t)}\,ds\le$$

$$\le\lim_{t\to\infty}\int^t[\theta(s)]_+\,ds+\limsup_{t\to\infty}\big(\ln(\beta(t))\big)-\ln(\beta(T))<\infty$$

since β is bounded on $[T,\infty)$.

The statement follows from Theorem 1. □

Corollary 3. *Let* $a(t)\equiv a\in\mathbb{C}$, $b(t)\equiv b\in\mathbb{C}$, $|a|>|b|$. *Suppose that* $\rho_0,\rho_1,\dots,\rho_n\colon[T,\infty)\to\mathbb{R}$ *are such that*

$$\big|g(t,z,w_1,\dots,w_n)\big|\le\rho_0(t)|z|+\sum_{j=1}^n\rho_j(t)|w_j| \tag{17}$$

for $t\ge T$, $|z|<R$, $|w_j|<R$ *for* $j=1,\dots,n$ *and* $\rho_0\in L_{\rm loc}([T,\infty),\mathbb{R})$. *Let* $\beta\in AC_{\rm loc}([T,\infty),\mathbb{R}_+)$ *satisfy*

$$\beta(t)\ge\Big(\frac{|a|+|b|}{|a|-|b|}\Big)^{\frac12}\max_j\big(\rho_j(t)+|A_j(t)|+|B_j(t)|\big)\ \ a.e.\ on\ [T,\infty).$$

If

$$\limsup_{t\to\infty}\int^t \max\Big(\frac{|a|-|b|}{|a|}\operatorname{Re}a+\Big(\frac{|a|+|b|}{|a|-|b|}\Big)^{\frac{1}{2}}\rho_0(s)+n\beta(s),\frac{\beta'(s)}{\beta(s)}\Big)\,ds<\infty, \quad (18)$$

then the trivial solution of the equation (1) *is stable. If*

$$\lim_{t\to\infty}\int^t \max\Big(\frac{|a|-|b|}{|a|}\operatorname{Re}a+\Big(\frac{|a|+|b|}{|a|-|b|}\Big)^{\frac{1}{2}}\rho_0(s)+n\beta(s),\frac{\beta'(s)}{\beta(s)}\Big)\,ds=-\infty, \quad (19)$$

then the trivial solution of (1) *is asymptotically stable.*

Proof. Denote again $z = z(t)$ and $w_j = z(t - r_j)$. Since $a, b \in \mathbb{C}$ are constants, then also γ and c are constants and we have $\vartheta(t) \equiv 0$. Using the condition (17), we get

$$\begin{aligned}
\big|\gamma g(t,z,w_1,\dots,w_n)+c\overline{g}(t,z,w_1,\dots,w_n)\big| &\le \\
\le (\gamma+|c|)\Big(\rho_0(t)|z|+\sum_{j=1}^n\rho_j(t)|w_j|\Big) &= \\
=\frac{\gamma+|c|}{\gamma-|c|}(\gamma-|c|)\Big(\rho_0(t)|z|+\sum_{j=1}^n\rho_j(t)|w_j|\Big) &\le \\
\le \frac{\gamma+|c|}{\gamma-|c|}\Big(\rho_0(t)|\gamma z+c\overline{z}|+\sum_{j=1}^n\rho_j(t)|\gamma w_j+c\overline{w_j}|\Big)&,
\end{aligned}$$

and it follows that the condition (ii) holds with

$$\kappa_0(t)=\frac{\gamma+|c|}{\gamma-|c|}\rho_0(t),\quad \kappa_j(t)=\frac{\gamma+|c|}{\gamma-|c|}\rho_j(t)$$

and $\lambda(t) \equiv 0$.

The condition (18) implies that $\operatorname{Re}a \le 0$. Since

$$\alpha = 1+\frac{|b|}{|a|}\operatorname{sgn}\operatorname{Re}a=\frac{|a|+|b|\operatorname{sgn}\operatorname{Re}a}{|a|}\ge\frac{|a|-|b|}{|a|}$$

and

$$\frac{\gamma+|c|}{\gamma-|c|}=\frac{|a|+\sqrt{|a|^2-|b|^2}+|b|}{|a|+\sqrt{|a|^2-|b|^2}-|b|}=\Big(\frac{|a|+|b|}{|a|-|b|}\Big)^{\frac{1}{2}},$$

in view of (7) we obtain

$$\begin{aligned}
\psi(t)&=\max_j\psi_j(t)=\Big(\frac{|a|+|b|}{|a|-|b|}\Big)^{\frac{1}{2}}\max_j\big\{\rho_j(t)+|A_j(t)|+|B_j(t)|\big\},\\
\theta(t)&=\alpha\operatorname{Re}a+\frac{\gamma+|c|}{\gamma-|c|}\rho_0(t)+n\beta(t)\le\\
&\le\frac{|a|-|b|}{|a|}\operatorname{Re}a+\Big(\frac{|a|+|b|}{|a|-|b|}\Big)^{\frac{1}{2}}\rho_0(t)+n\beta(t),
\end{aligned}$$

and the assertion follows from Theorem 1. $\square$

In the following corollary, we denote

$$H_1(t) = \sqrt{\frac{(|a|-|b|)^3}{|a|+|b|}\,\frac{\operatorname{Re} a}{|a|}} + \rho_0(t) + n\max_j\bigl\{\rho_j(t)+|A_j|+|B_j|\bigr\},$$

$$H_2(t) = \sqrt{\frac{|a|-|b|}{|a|+|b|}}\,\frac{\rho_i'(t)}{\max\limits_j\{\rho_j(t)+|A_j|+|B_j|\}},$$

where, for every t, the index i in H_2 is such that $\rho_i(t)+|A_i|+|B_i| = \max\limits_j\{\rho_j(t)+|A_j|+|B_j|\}$.

Corollary 4. *Let $a(t)\equiv a\in\mathbb{C}$, $b(t)\equiv b\in\mathbb{C}$, $|a|>|b|$ and $A_j(t)\equiv A_j\in\mathbb{C}$, $B_j(t)\equiv B_j\in\mathbb{C}$ for all $j\in\{1,\dots,n\}$. Let there exist $\rho_0,\rho_1,\dots,\rho_n : [T,\infty)\to\mathbb{R}$, ρ_0 locally Lebesgue integrable and $\rho_1,\dots,\rho_n$ locally absolutely continuous, such that* (17) *holds for $t\ge T$, $|z|<R$, $|w_j|<R$, $j\in\{1,\dots,n\}$. Suppose $\max\limits_j\bigl\{\rho_j(t)+|A_j|+|B_j|\bigr\}>0$ on $[T,\infty)$. If*

$$\limsup_{t\to\infty}\int^t \max\bigl(H_1(s),H_2(s)\bigr)\,ds<\infty,$$

then the trivial solution of the equation (1) *is stable; if*

$$\lim_{t\to\infty}\int^t \max\bigl(H_1(s),H_2(s)\bigr)\,ds=-\infty,$$

then the trivial solution of (1) *is asymptotically stable.*

Proof. Since

$$\Bigl(\frac{|a|+|b|}{|a|-|b|}\Bigr)^{\frac12}\max_j\bigl\{\rho_j(t)+|A_j|+|B_j|\bigr\}$$

is locally absolutely continuous on $[T,\infty)$, we can choose

$$\beta(t)=\Bigl(\frac{|a|+|b|}{|a|-|b|}\Bigr)^{\frac12}\max_j\bigl\{\rho_j(t)+|A_j|+|B_j|\bigr\}$$

in Corollary 3. Then

$$\frac{\beta'(t)}{\beta(t)}=\frac{\rho_i'(t)}{\max\limits_j\bigl\{\rho_j(t)+|A_j|+|B_j|\bigr\}}$$

and

$$\Bigl(\frac{|a|+|b|}{|a|-|b|}\Bigr)^{\frac12}\rho_0(t)+n\beta(t)=\Bigl(\frac{|a|+|b|}{|a|-|b|}\Bigr)^{\frac12}\Bigl(\rho_0(t)+n\max_j\bigl\{\rho_j(t)+|A_j|+|B_j|\bigr\}\Bigr).$$

Substitution into (18) and (19) and multiplication by

$$\Bigl(\frac{|a|+|b|}{|a|-|b|}\Bigr)^{-\frac12}$$

gives the result. □

Theorem 2. *Let the assumptions* (i), (ii) *and* (iii) *hold and*

$$V(t) = \left|\gamma(t)z(t) + c(t)\overline{z}(t)\right| + \beta(t)\sum_{j=1}^{n}\int\limits_{t-r_j}^{t}\left|\gamma(s)z(s) + c(s)\overline{z}(s)\right| ds, \quad (20)$$

where $z(t)$ is any solution of (1) *defined on $[t_1,\infty)$, where $t_1 \geq T$. Then*

$$\mu|z(t)| \leq V(s)\exp\left(\int\limits_{s}^{t}\Lambda(\tau)\,d\tau\right) + \int\limits_{s}^{t}\lambda(\tau)\exp\left(\int\limits_{\tau}^{t}\Lambda(\sigma)\,d\sigma\right)d\tau \quad (21)$$

for $t \geq s \geq t_1$.

Proof. Following the proof of Theorem 1, we have

$$V'(t) \leq \left|\gamma(t)z(t) + c(t)\overline{z}(t)\right|\theta(t) +$$

$$+\beta'(t)\sum_{j=1}^{n}\int\limits_{t-r_j}^{t}\left|\gamma(s)z(s) + c(s)\overline{z}(s)\right| ds + \lambda(t) \leq \Lambda(t)V(t) + \lambda(t)$$

a.e. on $[t_1,\infty)$. Using this inequality, we get

$$V'(t) - \Lambda(t)V(t) \leq \lambda(t) \quad (22)$$

a.e. on $[t_1,\infty)$. Multiplying (22) by $\exp\left(-\int\limits_{s}^{t}\Lambda(\tau)\,d\tau\right)$, we obtain

$$\left[V(t)\exp\left(-\int\limits_{s}^{t}\Lambda(\tau)\,d\tau\right)\right]' \leq \lambda(t)\exp\left(-\int\limits_{s}^{t}\Lambda(\tau)\,d\tau\right)$$

a.e. on $[t_1,\infty)$. Integration over $[s,t]$ yields

$$V(t)\exp\left(-\int\limits_{s}^{t}\Lambda(\tau)\,d\tau\right) - V(s) \leq \int\limits_{s}^{t}\lambda(\tau)\exp\left(-\int\limits_{s}^{\tau}\Lambda(\sigma)\,d\sigma\right)d\tau, \quad (23)$$

and multiplying (23) by $\exp\left(\int\limits_{s}^{t}\Lambda(\tau)\,d\tau\right)$ we obtain

$$V(t) \leq V(s)\exp\left(\int\limits_{s}^{t}\Lambda(\tau)\,d\tau\right) + \int\limits_{s}^{t}\lambda(\tau)\exp\left(\int\limits_{\tau}^{t}\Lambda(\sigma)\,d\sigma\right)d\tau.$$

The statement now follows from (15). □

From Theorem 2 we obtain several consequences.

Corollary 5. *Let the conditions* (i), (ii) *and* (iii) *be fulfilled and*

$$\int\limits_{s}^{t}\lambda(\tau)\exp\left(-\int\limits_{s}^{\tau}\Lambda(\sigma)\,d\sigma\right)d\tau < \infty.$$

If $z(t)$ is any solution of (1) *defined for $t \to \infty$, then*

$$z(t) = O\left[\exp\left(\int_s^t \Lambda(\tau)\, d\tau\right)\right].$$

Proof. From the assumptions and (23) we have

$$V(t) \exp\left(-\int_s^t \Lambda(\tau)\, d\tau\right) - V(s) \le \int_s^t \lambda(\tau) \exp\left(-\int_s^\tau \Lambda(\sigma)\, d\sigma\right) d\tau = K < \infty.$$

Then

$$\mu|z(t)| \le V(t) \le (K + V(s)) \exp\left(\int_s^t \Lambda(\tau)\, d\tau\right),$$

so

$$z(t) = O\left[\exp\left(\int_s^t \Lambda(\tau)\, d\tau\right)\right]. \qquad \square$$

Corollary 6. *Let the assumptions* (i), (ii) *and* (iii) *hold and let*

$$\limsup_{t\to\infty} \Lambda(t) < \infty \quad \text{and} \quad \lambda(t) = O(\mathrm{e}^{\eta t}), \tag{24}$$

where $\eta > \limsup_{t\to\infty} \Lambda(t)$. If $z(t)$ is any solution of (1) *defined for $t \to \infty$, then $z(t) = O(\mathrm{e}^{\eta t})$.*

Proof. In view of (24), there are $L > 0$, $\eta^* < \eta$ and $s > T$ such that $\eta^* > \Lambda(t)$ for $t \ge s$ and $\lambda(t)\, \mathrm{e}^{-\eta t} < L$ for $t \ge s$. From (21) we get

$$\begin{aligned} \mu|z(t)| &\le V(s)\, \mathrm{e}^{\eta^*(t-s)} + L \int_s^t \mathrm{e}^{\eta\tau}\, \mathrm{e}^{\eta^*(t-\tau)}\, d\tau \le \\ &\le V(s)\, \mathrm{e}^{\eta^*(t-s)} + L\, \mathrm{e}^{\eta^* t} \frac{\mathrm{e}^{(\eta-\eta^*)t} - \mathrm{e}^{(\eta-\eta^*)s}}{\eta - \eta^*} \le \\ &\le V(s)\, \mathrm{e}^{\eta^*(t-s)} + \frac{L}{\eta - \eta^*}\, \mathrm{e}^{\eta t} = O(\mathrm{e}^{\eta t}). \end{aligned} \tag{25}$$

The proof is complete. $\square$

Remark 2. If $\lambda(t) \equiv 0$, we can take $L = 0$ in the proof of Corollary 6, and taking the inequalities (25) into account, we obtain the following statement: there is an $\eta^* < \eta_0 < \eta$ such that $z(t) = o(\mathrm{e}^{\eta_0 t})$ holds for the solution $z(t)$ defined for $t \to \infty$.

Consider now a special case of the equation (1) with $g(t,z,w_1,\dots,w_n)\equiv h(t)$:

$$z'(t)=a(t)z(t)+b(t)\overline{z}(t)+\sum_{j=1}^{n}\big(A_j(t)z(t-r_j)+B_j(t)\overline{z}(t-r_j)\big)+h(t), \quad (26)$$

where $h(t)\in L_{\mathrm{loc}}([t_0,\infty),\mathbb{C})$.

Corollary 7. *Let the assumption* (i) *be satisfied and suppose*

$$\limsup_{t\to\infty}(\gamma(t)+|c(t)|)<\infty. \quad (27)$$

Let $\widetilde{\beta}\in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_+)$ *be such that*

$$\widetilde{\beta}(t)\geq\max_j\left\{\big(|A_j(t)|+|B_j(t)|\big)\frac{\gamma(t)+|c(t)|}{\gamma(t-r_j)-|c(t-r_j)|}\right\}\ \text{a.e. on } [T,\infty). \quad (28)$$

If h *is bounded,*

$$\limsup_{t\to\infty}\big[\alpha(t)\operatorname{Re}a(t)+\vartheta(t)+n\widetilde{\beta}(t)\big]<0 \quad (29)$$

and

$$\limsup_{t\to\infty}\frac{\widetilde{\beta}'(t)}{\widetilde{\beta}(t)}<0, \quad (30)$$

then any solution of the equation (26) *is bounded.*

If $h(t)=O(\mathrm{e}^{\eta t})$ *for any* $\eta>0$,

$$\limsup_{t\to\infty}\big[\alpha(t)\operatorname{Re}a(t)+\vartheta(t)+n\widetilde{\beta}(t)\big]\leq 0 \ \text{ and } \ \limsup_{t\to\infty}\frac{\widetilde{\beta}'(t)}{\widetilde{\beta}(t)}\leq 0,$$

then any solution of (26) *satisfies* $z(t)=o(\mathrm{e}^{\eta t})$ *for any* $\eta>0$.

Proof. Choose $R=\infty$, $\kappa_0(t)\equiv 0$, $\kappa_j(t)\equiv 0$ for $j\in\{1,\dots,n\}$, $\lambda(t)\equiv|h(t)|\sup_{t\geq T}(\gamma(t)+|c(t)|)$ and $\beta(t)\equiv\widetilde{\beta}(t)$. Then $g(t,z,w_1,\dots,w_n)\equiv h(t)$ satisfies the condition (ii) and $\beta(t)$ satisfies (iii). The assumptions (29) and (30) give the estimate

$$\limsup_{t\to\infty}\Lambda(t)<0.$$

Hence the first statement of Corollary 7 follows from Corollary 6.

The second statement follows from Corollary 6 as well, since

$$\limsup_{t\to\infty}\Lambda(t)\leq 0$$

and $z(t)=o(\mathrm{e}^{\eta t})$ for any $\eta>0$ if and only if $z(t)=O(\mathrm{e}^{\eta t})$ for any $\eta>0$. $\square$

Remark 3. If $h(t)\equiv 0$ in Corollary 7, then, with respect to Corollary 6 and Remark 2, we obtain the following assertion.

Suppose that the assumptions (i) and (27) hold and for $\widetilde{\beta}$ from Corollary 7 the inequality (28) is valid. If (29) and (30) are satisfied, then there is $\eta_0 < 0$ such that $z(t) = o(\mathrm{e}^{\eta_0 t})$ for any solution $z(t)$ of

$$z'(t) = a(t)z(t) + b(t)\overline{z}(t) + \sum_{j=1}^{n}\bigl(A_j(t)z(t-r_j) + B_j(t)\overline{z}(t-r_j)\bigr)$$

defined for $t \to \infty$.

Theorem 3. *Let the assumptions* (i), (ii) *and* (iii) *be satisfied. Let* $\Lambda(t) \le 0$ *a.e. on* $[T^*, \infty)$, *where* $T^* \in [T, \infty)$. *If*

$$\lim_{t\to\infty} \int^{t} \Lambda(s)\, ds = -\infty \quad \textit{and} \quad \lambda(t) = o(\Lambda(t)), \tag{31}$$

then any solution $z(t)$ *of the equation* (1) *defined for* $t \to \infty$ *satisfies*

$$\lim_{t\to\infty} z(t) = 0.$$

Proof. Choose an arbitrary $\varepsilon > 0$. According to (31), there is $s \ge T^*$ such that $\lambda(t) \le \frac{\mu\varepsilon}{2}|\Lambda(t)|$ for $t \ge s$ and

$$\int_s^t \lambda(\tau)\exp\left(\int_\tau^t \Lambda(\sigma)\, d\sigma\right) d\tau \le$$

$$\frac{\mu\varepsilon}{2}\int_s^t [-\Lambda(\tau)]\exp\left(\int_\tau^t \Lambda(\sigma)\, d\sigma\right) d\tau =$$

$$= \frac{\mu\varepsilon}{2}\int_s^t \left(\frac{d}{d\tau}\left[\exp\left(\int_\tau^t \Lambda(\sigma)\, d\sigma\right)\right]\right) d\tau = \frac{\mu\varepsilon}{2}\left[\exp\left(\int_\tau^t \Lambda(\sigma)\, d\sigma\right)\right]_s^t =$$

$$= \frac{\mu\varepsilon}{2}\left[1 - \exp\left(\int_s^t \Lambda(\tau)\, d\tau\right)\right] < \frac{\mu\varepsilon}{2}$$

for $t \ge s$. From (31) we have $\exp\bigl(\int_s^t \Lambda(\tau)\, d\tau\bigr) \to 0$ as $t \to \infty$, hence there is $S \ge s$ such that $\exp\bigl(\int_s^t \Lambda(\tau)\, d\tau\bigr) < \frac{\mu\varepsilon}{2V(s)}$ for $t \ge S$. Considering this fact and (21), we get

$$\mu|z(t)| < V(s)\frac{\mu\varepsilon}{2V(s)} + \frac{\mu\varepsilon}{2} = \mu\varepsilon$$

for $t \ge S$. This completes the proof. $\square$

Corollary 8. *Let the assumptions* (i) *and* (27) *hold and* $\widetilde{\beta} \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_+)$ *satisfy* (28)*. If the conditions* (29) *and* (30) *are fulfilled and* $h \in L_{\mathrm{loc}}([t_0,\infty),\mathbb{C})$ *satisfies* $\lim\limits_{t\to\infty} h(t) = 0$*, then*

$$\lim_{t\to\infty} z(t) = 0$$

for any solution $z(t)$ *of the equation* (26)*.*

Proof. Choose $R = \infty$, $\kappa_0(t) \equiv 0$, $\kappa_j(t) \equiv 0$ for $j \in \{1,\dots,n\}$, $\lambda(t) \equiv |h(t)| \sup\limits_{t\geq T}(\gamma(t) + |c(t)|)$ and $\beta(t) \equiv \widetilde{\beta}(t)$ in the same way as in the proof of Corollary 7. This yields $\theta(t) = \alpha(t)\operatorname{Re} a(t) + \vartheta(t) + n\widetilde{\beta}(t)$.

From (29) and (30) we have $\limsup\limits_{t\to\infty} \Lambda(t) < 0$, i.e., for $L < 0$, $L > \limsup\limits_{t\to\infty} \Lambda(t)$ there is $s \geq T$ such that $\Lambda(t) \leq L$ for all $t \geq s$. In particular, $\Lambda(t) \neq 0$ for $t \geq s$, hence

$$\lim_{t\to\infty} \frac{\lambda(t)}{\Lambda(t)} = \lim_{t\to\infty} \frac{|h(t)| \sup\limits_{t\geq T}\bigl(\gamma(t) + |c(t)|\bigr)}{\Lambda(t)} = 0,$$

which gives $\lambda(t) = o(\Lambda(t))$.

Since $\Lambda(t) \leq L$ for all $t \geq s$, we get

$$\lim_{t\to\infty} \int_s^t \Lambda(\tau)\, d\tau \leq \lim_{t\to\infty} \int_s^t L\, d\tau = -\infty.$$

Thus (31) holds and we can apply Theorem 3 to the equation (26). □

References

1. V. Jarník, Differential calculus II. *NČSAV Praha*, 1956.
2. J. Kalas and L. Baráková, Stability and asymptotic behaviour of a two-dimensional differential system with delay. *J. Math. Anal. Appl.* **269**(2002), No. 1, 278–300.
3. M. Ráb and J. Kalas, Stability of dynamical systems in the plane. *Differential Integral Equations* **3**(1990), No. 1, 127–144.
4. S. Saks, Theory of the integral. Monografie Matematyczne, Tom 7. *Warszawa–Lwów*, 1937.

(Received 9.03.2007)

Author's address:

Department of Mathematics and Statistics
Masaryk University
Janáčkovo nám. 2a, 662 95 Brno
Czech Republic
E-mail: rebenda@math.muni.cz