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TWO VERSIONS OF THE W-METHOD FOR QUADRATIC VARIATIONAL PROBLEMS WITH MANY LINEAR CONSTRAINTS

Dedicated to the blessed memory of Professor N. V. Azbelev

Abstract. Quadratic variational problems are considered in the space of functions on the segment $[a, b]$. They are transformed to extremal problems in the space $\mathbf{L}_2$ by the W-substitution $x = \mathbf{W}z + X\alpha$, where $\mathbf{W}$ is Green's operator of some boundary value problem for differential equation of the n-th order, and $(X\alpha)(t) = \alpha^1 x_1(t) + \cdots + \alpha^n x_n(t)$, $x_i(t)$ being suitable fundamental system of solutions of the corresponding homogeneous equation. This substitution allows one to satisfy n constraints.

If the number of linear constraints exceeds the order n, the transformed extremal problem in $\mathbf{L}_2$ contains constraints not satisfied by the substitution. For such a case, two ways are considered to satisfy all constraints and, hence, to deal with a problem without constraints at all. Those are the modified W-substitution, and the so called double W-substitution.

We show that they both give a quadratic extremal problems in subspaces of $\mathbf{L}_2$, which are easy to study and solve. The paper, mainly, is devoted to the comparison of techniques based on these substitutions and relation between them.

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[illegible] $[a, b]$ [illegible] $x = \mathbf{W}z + X\alpha$ W-[illegible] $\mathbf{L}_2$ [illegible] $\mathbf{W}$ n-[illegible] $(X\alpha)(t) = \alpha^1 x_1(t) + \cdots + \alpha^n x_n(t)$, [illegible] $x_i(t)$ [illegible] n [illegible].

[illegible] n-[illegible] $\mathbf{L}_2$-[illegible] W-[illegible] W-[illegible].

[illegible] $\mathbf{L}_2$-[illegible].

1. Introduction

Variational problems were the last devotion of N. V. Azbelev. The general idea of W-substitution turned out a very effective tool for studying variational problems with quadratic functionals. In the last years, the study in this direction was carried out more intensively. As a result, the book [2] was published.

This paper may be considered as a continuation of the research presented in [2]. In order that our presentation be somewhat independent of the book, we describe here shortly the application of W-substitution to quadratic variational problems.

1. Let $\mathbf{D}$ be a space of functions $[a,b] \to \mathbb{R}$ that is *naturally isomorphic* to the product $\mathbf{L}_2 \times \mathbb{R}^n$ ($\mathbf{L}_2$ is the Hilbert space of square summable functions $[a,b] \to \mathbb{R}$ with the inner product $\langle y, z\rangle = \int\limits_a^b y(t)z(t)\,dt$). For example, $\mathbf{D}$ may be the Sobolev space $\mathbf{H}^n$ of the functions x represented as

$$x(t) = \int\limits_a^t \frac{(t-s)^{n-1}}{(n-1)!}\, z(s)\,ds + \sum_{k=0}^{n-1} \beta^{k+1} \frac{(t-a)^k}{k!}\,,$$

where $z \in \mathbf{L}_2$ and $\beta^k \in \mathbb{R}$.

We consider the quadratic variational problem

$$\begin{gathered} \mathcal{I}(x) \stackrel{\text{def}}{=} \int\limits_a^b \frac{1}{2}\sum_{i=1}^m (T_{1i}x)(t)(T_{2i}x)(t) + (T_0x)(t)\,dt \to \inf, \quad x \in \mathbf{D}, \\ \ell^i x = \alpha^i, \quad i = 1, 2, \ldots, N, \quad N \geq n, \end{gathered} \tag{1}$$

where $T_{ji} : \mathbf{D} \to \mathbf{L}_2$ and $T_0 : \mathbf{D} \to \mathbf{L}_1$ are continuous linear operators, $\ell^i : \mathbf{D} \to \mathbb{R}$ are linearly independent continuous linear functionals.

Define the vector functional $\ell : \mathbf{D} \to \mathbb{R}^N$ as $\ell x = [\ell^1 x, \ldots, \ell^N x]$ and the vector $\alpha = (\alpha^1, \ldots, \alpha^N)$. Then the constraints of the problem (1) may be written in the form

$$\ell x = \alpha.$$

To solve such problems, a new approach based on the W-substitution was suggested in [5], [1], [4].

2. Define the vector functional $\ell^{[n]} : \mathbf{D} \to \mathbb{R}^n$ and the vector $\alpha^{[n]} \in \mathbb{R}^n$ by the equalities $\ell^{[n]}x = (\ell^1 x, \ldots, \ell^n x)$ and $\alpha^{[n]} = (\alpha^1, \ldots, \alpha^n)$.

Let $\delta : \mathbf{D} \to \mathbf{L}_2$ be a linear continuous operator such that ***the abstract boundary value problem*** (abstract BVP) [5], [4]

$$\delta x = z, \quad \ell^{[n]}x = \alpha^{[n]}, \tag{2}$$

has, for every couple $(z, \alpha^{[n]}) \in \mathbf{L}_2 \times \mathbb{R}^n$, a unique solution continuously dependent on $(z, \alpha^{[n]})$. Then the solution is representable in the form

$$x = \mathbf{W}z + X\alpha^{[n]}, \tag{3}$$

where $\mathbf{W} : \mathbf{L}_2 \to \mathbf{D}$ and $X : \mathbb{R}^n \to \mathbf{D}$ are continuous linear operators. The operator $\mathbf{W}$ is known as ***Green's operator*** for the BVP (2) (see [5]). Besides,

$$X\alpha^{[n]} = \sum_{i=1}^{n} \alpha^i x_i, \tag{4}$$

where $x_1, \dots, x_n \in \mathbf{D}$ is a ***fundamental system of solutions*** of the homogeneous equation $\delta x = 0$, ***normal with respect to the vector functional*** $\ell^{[n]}$.

We define $\mathbf{D}_\alpha = \{x \in \mathbf{D} : \ \ell x = \alpha\}$ and $\mathbf{D}_\alpha^{[n]} = \{x \in \mathbf{D} : \ \ell^{[n]} x = \alpha^{[n]}\}$. So we have the isomorphisms $\{\mathbf{W}, X\} : \mathbf{L}_2 \times \mathbb{R}^n \to \mathbf{D}$ and $\mathbf{W} : \mathbf{L}_2 \to \mathbf{D}_0^{[n]}$ with the inverse operators $[\delta, \ell^{[n]}] : \mathbf{D} \to \mathbf{L}_2 \times \mathbb{R}^n$ and $\delta : \mathbf{D}_0^{[n]} \to \mathbf{L}_2$.

3. In the case $N = n$, the variational problem for $x \in \mathbf{D}$

$$\mathcal{I}(x) \to \inf,$$
$$\ell x = \alpha$$

is converted by the W-***substitution*** (3) into the extremal problem

$$\mathcal{J}(z) \stackrel{\text{def}}{=} \mathcal{I}(\mathbf{W}z + X\alpha) - \mathcal{I}(X\alpha) \to \inf \tag{5}$$

in the space $\mathbf{L}_2$. This problem without constraints is equivalent to (1) in the sense that

> there exists a one-to-one correspondence between the admissible set $\mathbf{D}_\alpha$ of the problem (1) and the space $\mathbf{L}_2$;
> the values of the functionals at the corresponding points differ by a constant;
> so, minimum points, if any, $\widehat{z} \in \mathbf{L}_2$ and $\widehat{x} = \mathbf{W}\widehat{z} + X\alpha \in \mathbf{D}_\alpha$ also correspond to each other.

Note that if a minimum point $\widehat{z}$ of the problem (5) is found, then all the constraints of the variational problem (1) for the solution $\widehat{x} = \mathbf{W}\widehat{z} + X\alpha$ are satisfied by the properties of Green's operator $\mathbf{W}$.

We refer to the problem with $N = n$ as ***well-determined problem***.

4. Nevertheless, problems with $N > n$, so called ***overdetermined problems***, are widely met. For example, the simplest problems of the classic calculus of variations, such as the brachistochrone problem, or the problem (18) below, are overdetermined – the highest order of derivatives is $n = 1$, but we have $N = 2$ boundary conditions.

In such a case, after the substitution (3) we have several constraints not satisfied. The book [2] provides two ways to solve the problem:

> a modified W-substitution, proposed by S. Yu. Kultyshev; and
> solving the constrained extremal problem in the space $\mathbf{L}_2$ by the Lagrange multipliers rule.

The examples of applied problems solved by these methods may be found in the papers [3], [6]–[8].

It was encountered that the modified W-substitution has some shortcomings described below. In particular, we have not a one-to-one correspondence between $\mathbf{D}_\alpha$ and $\mathbf{L}_2$.

The aesthetic idea of N. V. Azbelev for the overdetermined situation was, avoiding these shortcomings, to find an abstract BVP generating the W-substitution that converts the problem (1) to an extremal problem without constraints. In [9] we have achieved this goal: the BVP (12) below gives a desired one-to-one correspondence between $\mathbf{D}_\alpha$ and some subspace of the space $\mathbf{L}_2$ with codimension $N-n$. The corresponding substitution is referred to as *double W-substitution*.

We also discuss the nature of the shortcomings of the modified W-substitution. In this case, we restore the one-to-one correspondence by restricting the range of definition of the converted extremal problem to some subspace of $\mathbf{L}_2$, also with codimension $N-n$.

Then we study the connections between the mentioned two methods. A special construction of the modified W-substitution is found which involves the reduced extremal problem coinciding with the problem obtained by the double W-substitution.

2. Modified W-Substitution

This idea is due to S. Yu. Kultyshev [6], [2, § 2.3].

Let the family of functions $v_1, \ldots, v_N \in \mathbf{D}$ be biorthogonal to the system of functionals $\ell^1, \ldots, \ell^N$, that is,

$$\ell^i v_k = \delta^i_k \overset{\text{def}}{=} \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k. \end{cases}$$

Define the operator $\mathbf{\Lambda} : \mathbf{L}_2 \to \mathbf{D}_0^{[n]}$ as follows:

$$\mathbf{\Lambda} y = \mathbf{W} y - \sum_{k=n+1}^{N} (\ell^k \mathbf{W} y) \cdot v_k.$$

1. *For*

$$x = \mathbf{\Lambda} y + \sum_{i=1}^{N} \alpha^i v_i, \tag{6}$$

all the boundary conditions $\ell^i x = \alpha^i$, $i = 1, \ldots, N$, are satisfied.

Since $\ell^i \sum_{j=1}^{N} \alpha^j v_j = \alpha^i$, for $i \leq n$ we have $\ell^i \mathbf{\Lambda} y = \ell^i \mathbf{W} y = 0$. If $i > n$, then $\ell^i \mathbf{\Lambda} y = \ell^i \mathbf{W} y - \sum_{k=n+1}^{N} (\ell^k \mathbf{W} y) \delta^i_k = 0$. □

2. We denote $l^i = \mathbf{W}^* \ell^i$, that is, $\ell^i \mathbf{W} z = \langle l^i, z \rangle$, for $i = n+1, \ldots, N$. Note that

$$\langle l^i, \delta v_k \rangle = \ell^i \mathbf{W} \delta v_k = \ell^i v_k = \delta^i_k \ \text{ for } \ k > n. \tag{7}$$

Now we construct an operator in the space $\mathbf{L}_2$ which plays an important role in all the considerations below. The operator $\mathbf{V} : \mathbf{L}_2 \to \mathbf{L}_2$ is defined by the equality

$$\mathbf{V}y = y - \sum_{k=n+1}^{N} \langle l^k, y\rangle \cdot \delta v_k;$$

so the adjoint operator is

$$\mathbf{V}^* y = y - \sum_{k=n+1}^{N} \langle y, \delta v_k\rangle \cdot l^k,$$

and $\mathbf{\Lambda} = \mathbf{WV}$.

Let Y_0 be the linear hull of the system of the vectors $\{\delta v_{n+1}, \dots, \delta v_N\}$, and Y_1 be its orthogonal complement. Let Z_0 be the linear hull of the system of vectors $\{l_{n+1}, \dots, l_N\}$, and Z_1 be its orthogonal complement.

Due to (7), we have the following properties:

a) *the kernel* $\operatorname{Ker}\mathbf{V} = Y_0$;
b) $\operatorname{Ker}\mathbf{V}^* = Z_0$.

Besides,

c) $\mathbf{V}$ *is a projector to the subspace* Z_1;
d) $\mathbf{V}^*$ *is a projector to the subspace* Y_1.

To prove the property c), we first obtain the equality $\langle \mathbf{V}y, l^i\rangle = 0$ by direct calculation. Hence, the image $\operatorname{Im}\mathbf{V} \subset Z_1$.

On the other hand, if $z \in Z_1$, then $\langle l^k, z\rangle = 0$ for $k > n$; therefore $\mathbf{V}z = z$.

The property d) is shown analogously. □

3. *The equality*

$$\mathbf{D}_\alpha = \mathbf{\Lambda}\mathbf{L}_2 + \sum_{i=1}^{N} \alpha^i v_i$$

holds.

It is sufficient to show that $\operatorname{Im}\mathbf{\Lambda} = \mathbf{D}_0$. We have got above that $\operatorname{Im}\mathbf{\Lambda} \subset \mathbf{D}_0$.

To prove the inverse inclusion $\mathbf{D}_0 \subset \operatorname{Im}\mathbf{\Lambda} = \operatorname{Im}\mathbf{WV}$, let $x \in \mathbf{D}_0$. Then $\mathbf{W}\delta x = x$. Therefore, it suffices to show that $\delta x \in \operatorname{Im}\mathbf{V}$.

For $i = n+1, \dots, N$ we have $\langle l^i, \delta x\rangle = \ell^i \mathbf{W}\delta x = \ell^i x = 0$. So, according to the property c), $\delta x \in Z_1 = \operatorname{Im}\mathbf{V}$. □

4. Define the operators

$$A_{ji} = T_{ji}\mathbf{\Lambda} : \mathbf{L}_2 \to \mathbf{L}_2, \quad A_0 = T_0\mathbf{\Lambda} : \mathbf{L}_2 \to \mathbf{L}_1$$

(then $A_{ji}^* : \mathbf{L}_2 \to \mathbf{L}_2$ and $A_0^* : \mathbf{L}_\infty \to \mathbf{L}_2$). Also define the operators $G : \mathbf{L}_2 \to \mathbf{L}_2$ and $\mathcal{L} : \mathbf{D} \to \mathbf{L}_2$ by the equalities

$$G = \frac{1}{2}\sum_{i=1}^{m}(A_{1i}^* A_{2i} + A_{2i}^* A_{1i}) \text{ and } \mathcal{L} = \frac{1}{2}\sum_{i=1}^{m}(A_{1i}^* T_{2i} + A_{2i}^* T_{1i}).$$

Then $\mathcal{L}\mathbf{\Lambda} = G$.

The modified W-substitution (6) converts the variational problem (1) into the extremal problem

$$\mathcal{I}_1(y) = \frac{1}{2}\langle Gy, y\rangle - \langle \theta, y\rangle \to \min, \tag{8}$$

where $\theta = -\mathcal{L}\big(\sum_{j=1}^{N} \alpha^j v_j\big) - A_0^*\mathbf{1}$, and $\mathbf{1}(t) \equiv 1$.

Differentiating the functional $\mathcal{I}_1$, we get the equation in $\mathbf{L}_2$:

$$Gy = \theta. \tag{9}$$

If $\widehat{y} \in \mathbf{L}_2$ is a solution of this equation, then $\widehat{x} = \mathbf{\Lambda}\widehat{y} + \sum_{i=1}^{N} \alpha^i v_i$ satisfies the following BVP in the space $\mathbf{D}$:

$$\begin{aligned} \mathcal{L}x &= -A_0^*\mathbf{1}, \\ \ell x &= \alpha. \end{aligned} \tag{10}$$

We ought to name it ***Euler boundary value problem***.

Theorem 1 ([5], [6])**.** *The following conditions are equivalent:*

a) *the problem* (8) *has a minimum point* $\widehat{y} \in \mathbf{L}_2$*;*

b) *the problem* (1) *has a minimum point* $\widehat{x} = \mathbf{\Lambda}\widehat{y} + \sum_{j=1}^{N} \alpha^j v_j \in \mathbf{D}$*;*

c) $\widehat{y}$ *satisfies the equation* (9) *and the operator* G *is positive definite;*

d) $\widehat{x}$ *is a solution of the problem* (10) *and the operator* G *is positive definite.*

Remark 1. We can say nothing on the uniqueness of the minimum points, because there is no one-to-one correspondence between the admissible set $\mathbf{D}_\alpha$ and the space $\mathbf{L}_2$.

How to construct the biorthogonal system of functions $v_1, \dots, v_N \in \mathbf{D}$? This question has yet no answer.

At last, this biorthogonal system may be chosen with some uncertainty. So the operator G is defined ambiguously.

3. Double W-Substitution

The W-substitution (3) provides a one-to-one correspondence between $\mathbf{D}_\alpha^{[n]}$ and $\mathbf{L}_2$. Using this, we convert the variational problem (1) into the equivalent extremal problem in $\mathbf{L}_2$:

$$\begin{aligned} \mathcal{J}(z) &= \frac{1}{2}\langle Hz, z\rangle - \langle \rho, z\rangle \to \min, \\ \langle l^i, z\rangle &= \beta^i \stackrel{\text{def}}{=} \alpha^i - \ell^i X \alpha^{[n]}, \quad i = n+1, \dots, N, \end{aligned} \tag{11}$$

with $N-n$ constraints (first n constraints are satisfied by the properties of Green's operator $\mathbf{W}$). Here

$$H = \frac{1}{2}\sum_{i=1}^{m}(B_{1i}^{*}B_{2i} + B_{2i}^{*}B_{1i}) \text{ and } \rho = -\mathcal{M}X\alpha^{[n]} - B_0^{*}\mathbf{1},$$

where

$$B_{ji} = T_{ji}\mathbf{W} : \mathbf{L}_2 \to \mathbf{L}_2, \quad \mathcal{M} = \frac{1}{2}\sum_{i=1}^{m}(B_{1i}^{*}T_{2i} + B_{2i}^{*}T_{1i}),$$
$$B_0 = T_0\mathbf{W} : \mathbf{L}_2 \to \mathbf{L}_1, \quad \mathbf{1}(t) \equiv 1.$$

Note that $\mathcal{M}\mathbf{W} = H$.

Due to (7), the system $\{l^{n+1}, \dots, l^{N}\}$ is linearly independent.

To solve the problem (11), instead of the Lagrange multipliers rule we apply the trick of W-substitution repeatedly [9].

Denote by $\mathbf{P}$ the orthogonal projector to Z_0. Then $I-\mathbf{P}$ is the orthogonal projector to Z_1. The projector $\mathbf{P}$ is an integral operator,

$$(\mathbf{P}z)(t) = \int_a^b P(t,s)z(s)\,ds,$$

with the symmetric kernel $P(t,s) = \sum_{i=n+1}^{N}\sum_{j=n+1}^{N}\gamma_{ij}l^i(t)l^j(s)$, where the matrix $(\gamma_{ij})_{i,j=n+1}^{N}$ of coefficients is inverse to the Gramian matrix $(\langle l^i, l^j\rangle)_{i,j=n+1}^{N}$.

The *"boundary value problem"*

$$(I-\mathbf{P})z = z_1 \ \ (\text{where } z_1 \in Z_1),$$
$$\langle l^i, z\rangle = \beta^i, \ \ i = n+1, \dots, N,$$

is uniquely solvable. Its solution may be represented as

$$z = z_1 + z_0, \quad z_0 \stackrel{\text{def}}{=} \sum_{i=n+1}^{N}\sum_{j=n+1}^{N}\gamma_{ij}\beta^j l^i.$$

So the BVP

$$\begin{aligned}(I-\mathbf{P})\delta x &= z_1, \ \ z_1 \in Z_1,\\ \ell x &= \alpha\end{aligned} \tag{12}$$

has the unique solution

$$x = \mathbf{W}\Big(z_1 + \sum_{i=n+1}^{N}\sum_{j=n+1}^{N}\gamma_{ij}\big(\alpha^j - \ell^j X\alpha^{[n]}\big)l^i\Big) + X\alpha^{[n]} \tag{13}$$

and generates a one-to-one correspondence between the admissible set $\mathbf{D}_\alpha$ and the Hilbert space Z_1.

Since H is a self-adjoint operator, this substitution converts the problem (11) into the following equivalent problem in the Hilbert space Z_1:

$$\mathcal{F}(z_1) = \frac{1}{2}\big\langle (I-\mathbf{P})Hz_1, z_1\big\rangle - \big\langle (I-\mathbf{P})\rho - (I-\mathbf{P})Hz_0, z_1\big\rangle \to \min . \quad (14)$$

Thus we have the equation

$$(I-\mathbf{P})Hz_1 = (I-\mathbf{P})\rho - (I-\mathbf{P})Hz_0 \quad (15)$$

for $z_1 \in Z_1$.

Substituting $z_0 = \mathbf{P}\delta x$, $z_1 = (I-\mathbf{P})\delta x$, we get the equivalent condition

$$H\delta x - \rho \in \operatorname{Ker}(I-\mathbf{P}) = Z_0.$$

Since $H\delta x = \mathcal{M}\mathbf{W}\delta x = \mathcal{M}(x - X\alpha^{[n]})$ and $\rho = -\mathcal{M}X\alpha^{[n]} - B_0^*\mathbf{1}$, the last condition is equivalent to the inclusion

$$\mathcal{M}x + B_0^*\mathbf{1} \in Z_0. \quad (16)$$

Thus, there exist scalars, the so called Lagrange multipliers, $\lambda_{n+1}, \ldots, \lambda_N$, such that

$$\begin{aligned} \mathcal{M}x &= -B_0^*\mathbf{1} + \sum_{i=n+1}^{N} \lambda_i l^i, \\ \ell x &= \alpha. \end{aligned} \quad (17)$$

This problem will be referred to as ***Euler–Lagrange boundary value problem***.

Theorem 2. *The following conditions are equivalent:*

a) *the problem* (14) *has a minimum point* $\widehat{z}_1 \in Z_1$*;*

b) *the problem* (1) *has a minimum point*

$$\widehat{x} = \mathbf{W}\Big(\widehat{z}_1 + \sum_{i=n+1}^{N}\sum_{j=n+1}^{N} \gamma_{ij}\big(\alpha^j - \ell^j X\alpha^{[n]}\big)l^i\Big) + X\alpha^{[n]} \in \mathbf{D}_\alpha;$$

c) $\widehat{z}_1$ *satisfies the equation* (15) *and the operator* $(I-\mathbf{P})H$ *is positive definite on the space* Z_1*;*

d) *for some* $\lambda_{n+1}, \ldots, \lambda_N$*, the function* $\widehat{x}$ *satisfies the system* (17) *and the operator* $(I-\mathbf{P})H$ *is positive definite on the space* Z_1*.*

Remark 2. The condition

the operator $(I-\mathbf{P})H$ *is positive definite on the space* Z_1

may be replaced by the equivalent condition

the operator $(I-\mathbf{P})H(I-\mathbf{P})$ *is positive definite on* $\mathbf{L}_2$*.*

Theorem 3 ([2], [9])**.** *Suppose that the conditions of Theorem* 2 *are fulfilled. The following conditions are equivalent:*

e) *the problem* (1) *has a unique minimum point;*

f) *the problem* (14) *has a unique minimum point* $\widehat{z}_1 \in Z_1$*;*

g) *the equation* (15) *has a unique solution* $\widehat{z}_1 \in Z_1$*;*

h) *the system* (17) *has a unique solution* $\widehat{x}$; $\lambda_{n+1}, \dots, \lambda_N$;
i) *the operator* $(I - \mathbf{P})H$ *is strictly positive definite on the space* Z_1.

4. Discussion

We have some freedom in choosing the functions v_i. So, even if the solution of the problem (1) is unique, the operator G depends on this choice.

Besides, the correspondence between $x \in \mathbf{D}_\alpha$ and $y \in \mathbf{L}_2$ given by the equation (6) is not one-to-one.

Therefore, we ought to study the nature of these uncertainties. Of course, we are also interested in the connections between the operators G and H, the problems (8) and (14), etc.

The facts fundamental for the comparison are that

$$\mathbf{\Lambda} = \mathbf{W}\mathbf{V}$$

and, therefore,

$$A_{ji} = B_{ji}\mathbf{V}, \quad A_{ji}^* = \mathbf{V}^* B_{ji}^*, \quad A_0 = B_0\mathbf{V}, \quad A_0^* = \mathbf{V}^* B_0^*.$$

So

$$G = \mathbf{V}^* H \mathbf{V} \ \text{ and } \ \mathcal{L} = \mathbf{V}^* \mathcal{M}.$$

1. *The solutions of the problem* (10) *are independent of the choice of functions* v_i, $i = 1, \dots, N$.

Indeed, the first equation of (10) is equivalent to the inclusion

$$\mathcal{M}x + B_0^*\mathbf{1} \in \operatorname{Ker}\mathbf{V}^* = Z_0.$$

The subspace Z_0 is defined independently of the functions v_i. □

Remark 3. This inclusion coincides with (16), the one obtained by the double W-substitution.

2. Since $\operatorname{Ker}\mathbf{V} = Y_0$, we have

$$\mathcal{I}_1(y + u) = \mathcal{I}_1(y)$$

for every $u \in Y_0$; *the subspace* $\operatorname{Ker}\mathbf{\Lambda} = Y_0$. *So we conclude that the problem* (8) *is solvable disregarding the subspace* Y_0.

To prove this, note that $\langle G(y+u), y+u\rangle = \langle H\mathbf{V}(y+u), \mathbf{V}(y+u)\rangle = \langle H\mathbf{V}y, \mathbf{V}y\rangle = \langle Gy, y\rangle$, because $\mathbf{V}u = 0$. Besides, if $\mathbf{W}z = 0$, then $z = \delta\mathbf{W}z = 0$. So $\operatorname{Ker}\mathbf{\Lambda} = \operatorname{Ker}(\mathbf{W}\mathbf{V}) = \operatorname{Ker}\mathbf{V} = Y_0$. □

3. To obtain the uniqueness of the minimum point for the problem (8) and the one-to-one correspondence in the equation (6), we should restrict the domain of the operator $\mathbf{\Lambda}$ and, accordingly, of the operator G. Note that $\theta = -\mathbf{V}^*\mathcal{M}\big(\sum_{j=1}^{N} \alpha^j v_j\big) - \mathbf{V}^* B_0^*\mathbf{1} \in \operatorname{Im}\mathbf{V}^* = Y_1$ and $Gy = \mathbf{V}^* H \mathbf{V} y \in Y_1$. So the restriction is done to the subspace Y_1.

The equation (6) *defines the homeomorphism* $Y_1 \to \mathbf{D}_\alpha$.

Theorem 4. *Suppose that the conditions of Theorem 1 are fulfilled. The following conditions are equivalent:*

e) *the problem* (1) *has a unique minimum point;*
f) *the problem* (8), *considered on* Y_1, *has a unique minimum point* $\widehat{y}_1$*;*
g) *the equation* (9) *has a unique solution* $\widehat{y}_1 \in Y_1$*;*
h) *the operator* G *is strictly positive definite on the space* Y_1.

4. Due to the equality

$$\langle Gy, y\rangle = \langle H\mathbf{V}y, \mathbf{V}y\rangle,$$

the operator G *is positive definite if and only if* H *is positive definite on the subspace* $\operatorname{Im}\mathbf{V} = Z_1$.

Since

$$\langle Gy, y\rangle = \langle H\mathbf{V}y, \mathbf{V}y\rangle = \langle (I - \mathbf{P})H\mathbf{V}y, \mathbf{V}y\rangle$$

for $y \in Y_1$, *the operator* G *is strictly positive definite on the subspace* Y_1 *if and only if* $(I - \mathbf{P})H$ *is strictly positive definite on the subspace* Z_1.

5. So the conjecture arises that $\mathcal{F}(\mathbf{V}y) = \mathcal{I}_1(y)$ for $y \in Y_1$ and, therefore, the equation (15) is converted into the equation (9) by the substitution $z_1 = \mathbf{V}y$.

The following example shows that this conjecture is not substantiated in the general case.

Example 1. For the problem

$$\begin{gathered} \mathcal{I}(x) = \frac{1}{2}\int_0^1 \dot{x}^2(t)\, dt \to \min, \\ x(0) = 0, \quad x(1) = 1 \end{gathered} \tag{18}$$

we have $\delta x = \dot{x}$, $(\mathbf{W}z)(t) = \int_0^t z(s)\, ds$, $X\alpha^{[1]} = 0$. Besides, $l^2 = \mathbf{1}$, so Z_0 is the subspace of constants and $\mathbf{P}z = \langle \mathbf{1}, z\rangle \mathbf{1}$ (that is, $P(t,s) = 1$). The function $z_0 = \mathbf{1}$, and $(I - \mathbf{P})z_0 = 0$. So, the problem (14) takes the form

$$\mathcal{F}(z_1) = \frac{1}{2}\langle z_1, z_1\rangle \to \min.$$

To obtain the solution, we get $z_1 = 0$, $z = z_1 + z_0 = \mathbf{1}$, and $x(t) = t$. This is a minimum point, because the operator $H = I$ is strictly positive definite.

Let $v_1(t) = 1 - t$, $v_2(t) = t^2$ (we do not choose $v_2(t) = t$, such a choice will be considered later). So $\mathbf{V}y = y - \langle \mathbf{1}, y\rangle \delta v_2$, $\delta v_2(t) = 2t$, and $\langle \delta v_2, \delta v_2\rangle = \frac{4}{3}$.

Since $\langle \delta v_2, y\rangle = 0$ for $y \in Y_1$, we have

$$\mathcal{F}(\mathbf{V}y) = \frac{1}{2}\langle \mathbf{V}y, \mathbf{V}y\rangle = \frac{1}{2}\langle y, y\rangle + \frac{2}{3}\langle \mathbf{1}, y\rangle^2.$$

If we use the modified W-substitution $\dot{x} = y - \langle \mathbf{1}, y\rangle \delta v_2 + \delta v_2$, we get

$$\mathcal{I}_1(y) = \frac{1}{2}\langle y, y\rangle + \frac{2}{3}\langle \mathbf{1}, y\rangle^2 - \frac{4}{3}\langle \mathbf{1}, y\rangle.$$

Thus, $\mathcal{F}(\mathbf{V}y) \neq \mathcal{I}_1(y)$ for y being an orthogonal projection of $\mathbf{1}$ to Y_1, i.e., for $y(t) = 1 - \frac{3}{2}t$.

6. Given the fundamental system $x^1, \dots, x^n$ of solutions of the equation $\delta x = 0$, normal with respect to $\ell^{[n]}$, and the functions $l^{n+1}, \dots, l^N$, we may construct a biorthogonal family $v_1, \dots, v_N \in \mathbf{D}$ in the following special way. Recall that the matrix $(\gamma_{ij})_{i,j=n+1}^N$ is inverse to the Gramian matrix $\big(\langle l^i, l^j\rangle\big)_{i,j=n+1}^N$. For $i = n+1, \dots, N$, put

$$v_i = \mathbf{W}\Big(\sum_{j=n+1}^{N} \gamma_{ij} l^j\Big). \tag{19}$$

If $k \leq n$, then $\ell^k v_i = 0$ directly by the definition of the operator $\mathbf{W}$. For $k > n$ we have $\ell^k v_i = \sum\limits_{j=n+1}^{N} \gamma_{ij}\ell^k \mathbf{W} l^j = \sum\limits_{j=n+1}^{N} \gamma_{ij}\langle l^k, l^j\rangle = \delta_i^k$.

The functions $v_1, \dots, v_n \in \mathbf{D}$ are chosen as follows:

$$v_i = x_i - \sum_{j=n+1}^{N} \ell^j x_i \cdot v_j. \tag{20}$$

In the case $\alpha = 0$, or when studying the solvability of problems without calculation of solutions, we do not need them at all.

Remark 4. Following this way, we take for the example (18) $v_1(t) = 1 - t$ and $v_2(t) = t$.

For such a choice of v_i, we have $\delta v_i = \sum\limits_{j=n+1}^{N} \gamma_{ij} l^j$ for $i = n+1, \dots, N$, so $Y_0 = Z_0$ and $Y_1 = Z_1$.

Hence, by direct calculation we get that $\mathbf{V} = I - \mathbf{P}$. This is an orthogonal projector, so $\mathbf{V}^* = \mathbf{V} = I - \mathbf{P}$.

Apply (4) to (13), (20) and (19) to (6), and take into account that $\langle l^k, y\rangle = 0$ for $y \in Y_1$. So we get that

> *the substitutions* (6) *and* (13) *coincide*;
> *hence, the functionals* $\mathcal{I}_1$ *and* $\mathcal{F}$ *coincide on* Z_1;
> *and the equations* (9) *and* (15) *coincide as well.*

7. Looking through the examples (in particular, [2, examples 2.3, 2.4, 3.2]), the author concludes that the double W-substitution is more convenient than the modified W-substitution for its briefness.

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