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ON SOME NONLINEAR BOUNDARY VALUE PROBLEMS FOR HIGH ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. Sufficient conditions for solvability and unique solvability are established for the problems of the type

$$u^{(2n)}(t) = g(u)(t);$$

$$u^{(i-1)}(a) = u^{(i-1)}(b) = 0 \ \ (i = 1, \dots, n);$$

$$\sum_{k=1}^{2n} \big(\alpha_{jk}(u)u^{(n+k-1)}(a) + \beta_{jk}(u)u^{(n+k-1)}(b)\big) = 0 \ \ (j = 1, \dots, 2n)$$

where $g : C^n \to L$ is a continuous operator and $\alpha_{jk} : C^n \to R$ and $\beta_{jk} : C^n \to R$ are continuous functionals.

[illegible]

$$u^{(2n)}(t) = g(u)(t);$$

$$u^{(i-1)}(a) = u^{(i-1)}(b) = 0 \ \ (i = 1, \dots, n);$$

$$\sum_{k=1}^{2n} \big(\alpha_{jk}(u)u^{(n+k-1)}(a) + \beta_{jk}(u)u^{(n+k-1)}(b)\big) = 0 \ \ (j = 1, \dots, 2n)$$

[illegible] $g : C^n \to L$ [illegible] $\alpha_{jk} : C^n \to R$ [illegible] $\beta_{jk} : C^n \to R$ [illegible]

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Let $-\infty < a < b < +\infty$, n be a natural number, C^n be the space of n times continuously differentiable functions $u : [a, b] \to R$ with the norm

$$\|u\|_{C^n} = \max\Big\{\sum_{k=1}^{n} |u^{(k-1)}(t)| : \ a \le t \le b\Big\},$$

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L be the space of Lebesgue integrable functions $v : [a, b] \to R$ with the norm

$$\|v\|_L = \int\limits_a^b |v(t)|\,dt,$$

and $g : C^n \to L$ be a continuous operator such that

$$g^*_\rho \in L \ \text{ for any } \ \rho \in \,]0, +\infty[\,,$$

where

$$g^*_\rho(t) = \sup\Big\{|g(u)(t)| : \ u \in C^n, \ \ \|u\|_{C^n} \le \rho\Big\}.$$

Consider the functional differential equation

$$u^{(4n)}(t) = g(u)(t) \tag{1}$$

with the boundary conditions

$$\begin{gathered} u^{(i-1)}(a) = u^{(i-1)}(b) = 0 \ \ (i = 1, \dots, n), \\ \sum_{k=1}^{2n} \big(\alpha_{jk}(u)u^{(n+k-1)}(a) + \beta_{jk}(u)u^{(n+k-1)}(b)\big) = 0 \ \ (j = 1, \dots, 2n), \end{gathered} \tag{2}$$

where $\alpha_{jk} : C^n \to R$, $\beta_{jk} : C^n \to R$ $(j, k = 1, \dots, 2n)$ are functionals continuous and bounded on every bounded set of the space C^n.

We are interested in the case where for arbitrary $v \in C^n$, $x_k \in R$, $y_k \in R$ $(k = 1, \dots, 2n)$ the condition

$$\sum_{j=1}^{2n} \Big| \sum_{k=1}^{2n} \big(\alpha_{jk}(v)x_k + \beta_{jk}(v)y_k\big)\Big| > 0$$

$$\text{for } \ \sum_{k=1}^{n}(y_{2n-k+1}y_k - x_{2n-k+1}x_k) > 0 \tag{3}$$

holds.

The particular case of (1) is the differential equation

$$u^{(4n)}(t) = f\big(t, u(t), \dots, u^{(n)}(t)\big), \tag{4}$$

and the particular cases of (2) are the boundary conditions

$$\begin{gathered} u^{(i-1)}(a) = u^{(i-1)}(b) = 0, \quad \gamma_{1i}u^{(n+i-1)}(a) + \gamma_{2i}u^{(3n-i)}(a) = 0, \\ \eta_{1i}u^{(n+i-1)}(b) + \eta_{2i}u^{(3n-i)}(b) = 0 \ \ (i = 1, \dots, n); \end{gathered} \tag{2_1}$$

$$\begin{gathered} u^{(i-1)}(a) = u^{(i-1)}(b) = 0, \quad u^{(n+i-1)}(a) = \gamma_i u^{(n+i-1)}(b), \\ u^{(3n-i)}(b) = \gamma_i u^{(3n-i)}(a) \ \ (i = 1, \dots, n); \end{gathered} \tag{2_2}$$

and

$$\begin{gathered} u^{(i-1)}(a) = u^{(i-1)}(b) = 0 \ \ (i = 1, \dots, n), \\ u^{(n+j-1)}(a) = u^{(n+j-1)}(b) \ \ (i = 1, \dots, n). \end{gathered} \tag{2_3}$$

Here $f : [a,b] \times R^{n+1} \to R$ is a function satisfying the local Carathéodory conditions, and γ_{1i}, γ_{2i}, η_{1i}, η_{2i}, γ_i are constants such that

$$\gamma_{1i}\gamma_{2i} \le 0, \ \ \eta_{1i}\eta_{2i} \ge 0, \ \ |\gamma_{1i}| + |\gamma_{2i}| > 0, \ \ |\eta_{1i}| + |\eta_{2i}| > 0 \ \ (i = 1, \dots, n)$$

and

$$\gamma_i \ne 0 \ \ (i = 1, \dots, n).$$

By $\widetilde{C}^{4n-1}$ we denote the space of functions $u : [a,b] \to R$ absolutely continuous along with their first $4n-1$ derivatives.

By a solution of Eq. (1) we mean a function $u \in \widetilde{C}^{4n-1}$ satisfying this equation almost everywhere on $[a,b]$.

A solution of Eq. (1) satisfying the conditions (2) is called **a solution of the problem** (1), (2).

Definition 1. We will say that a function $u : [a,b] \to R$ belongs to the set D_0^n, if $u \in \widetilde{C}^{4n-1}$ and

$$u^{(i-1)}(a) = u^{(i-1)}(b) = 0 \quad (i = 1, \dots, n).$$

Definition 2. We will say that a function u belongs to the set D^n, if $u \in D_0^n$ and there exists a function $v \in C^n$, such that

$$\sum_{k=1}^{2n} \big(\alpha_{jk}(v)u^{(n+k-1)}(a) + \beta_{jk}(v)u^{(n+k-1)}(b)\big) = 0 \ \ (j = 1, \dots, 2n).$$

Theorem 1. *Let there exist $l \in]0,1[$ and $l_0 \ge 0$ such that for an arbitrary $u \in D^n$ the inequality*

$$\int_a^b g(u)(t)\, u(t)\, dt \le l \int_a^b [u^{(2n)}(t)]^2\, dt + l_0 \tag{5}$$

is fulfilled. Then the problem (1), (2) *has at least one solution.*

Corollary 1. *Let for an arbitrary $u \in D_0^n$ the inequality* (5) *hold, where $l \in]0,1[$ and $l_0 \ge 0$. Then for every $k \in \{1,2,3\}$ the problem* (1), (2_k) *has at least one solution.*

Theorem 2. *Let there exist $l \in]0,1[$ such that for an arbitrary u and $v \in D^n$ the inequality*

$$\int_a^b \big(g(u)(t) - g(v)(t)\big)\big(u(t) - v(t)\big)\, dt \le l \int_a^b \big|u^{(2n)}(t) - v^{(2n)}(t)\big|^2\, dt \tag{6}$$

is fulfilled. Then the problem (1), (2) *has one and only one solution.*

Corollary 2. *If for arbitrary u and $v \in D_0^n$ the inequality* (6) *holds, where $l \in]0,1[$, then for every $k \in \{1,2,3\}$ the problem* (1), (2_k) *has one and only one solution.*

Theorems 1 and 2 and their corollaries are new not only in the general case, but also in the case where g is Nemytski's operator, i.e., when Eq. (1) is of the form (4) (see [1]–[5] and the references therein). We will now proceed to the consideration just of that case.

Theorem 3. *Let on the set* $[a,b]\times R^{n+1}$ *the inequality*

$$f(t,x_1,\dots,x_{n+1})\operatorname{sgn}x_1\le\sum_{k=1}^{n+1}l_k|x_k|+h(t) \tag{7}$$

hold, where $h\in L$ *and* l_k $(k=1,\dots,n+1)$ *are nonnegative constants such that*

$$\sum_{k=1}^{n+1}\Big(\frac{b-a}{\pi}\Big)^{4n-k+1}l_k<1. \tag{8}$$

Then the problem (4), (2) *has at least one solution.*

Corollary 3. *If the conditions of Theorem* 3 *hold, then for every* $k\in\{1,2\}$ *the problem* (4), (2_k) *has at least one solution.*

Theorem 4. *Let on the set* $[a,b]\times R^{n+1}$ *the condition*

$$\big[f(t,x_1,\dots,x_{n+1})-f(t,y_1,\dots,y_{n+1})\big]\operatorname{sgn}(x_1-y_1)\le\sum_{k=1}^{n+1}l_k|x_k-y_k| \tag{9}$$

hold, where l_k $(k=1,\dots,n+1)$ *are nonnegative constants satisfying the inequality* (8)*. Then the problem* (4), (2) *has one and only one solution.*

Corollary 4. *If the conditions of Theorem* 4 *hold, then for every* $k\in\{1,2\}$ *the problem* (4), (2_k) *has one and only one solution.*

The following two theorems deal with the problem (4), (2_3).

Theorem 5. *Let on the set* $[a,b]\times R^{n+1}$ *the inequality* (7) *hold, where* $h\in L$ *and* l_k $(k=1,\dots,n+1)$ *are nonnegative constants such that*

$$\sum_{k=1}^{n+1}\Big(\frac{b-a}{\pi}\Big)^{4n-k+1}l_k<4^n. \tag{10}$$

Then the problem (4), (2_3) *has at least one solution.*

Theorem 6. *Let on the set* $[a,b]\times R^{n+1}$ *the condition* (9) *hold, where* l_k $(k=1,\dots,n+1)$ *are nonnegative constants satisfying the inequality* (10)*. Then the problem* (4), (2_3) *has one and only one solution.*

As an example, we consider the linear differential equation

$$u^{(4n)}(t)=\sum_{k=1}^{n+1}p_k(t)u^{(k-1)}(t)+q(t), \tag{11}$$

where

$$p_k\in L\ \ (k=1,\dots,n),\ \ q\in L.$$

From Theorems 4 and 6 we have

Corollary 5. *Let almost everywhere on* $[a, b]$ *the inequalities*

$$p_1(t) \le l_1, \quad |p_k(t)| \le l_k \quad (k = 2, \dots, n + 1)$$

hold, where l_k $(k = 1, \dots, n + 1)$ *are nonnegative constants satisfying the inequality* (8) (*the inequality* (10)). *Then each of the problems* (11), (2); (11), (2_1) *and* (11), (2_2) (*the problem* (11), (2_3)) *has one and only one solution.*

In the case $n = 1$ the above theorems and corollaries generalize the results of the paper [6].

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