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A GENERAL METHOD OF CONSTRUCTING THE SOLUTIONS OF SPATIAL AXISYMMETRIC STATIONARY PROBLEMS OF THE JET AND FILTRATION THEORIES WITH PARTIALLY UNKNOWN BOUNDARIES

Abstract. In the present work we consider a general mathematical method of constructing the solutions of spatial axisymmetric stationary problems of the jet and filtration theories with partially unknown boundaries. The x-axis coincides with the symmetry axis, and the distance to the x-axis is denoted by y. The use is made of the right coordinate system. Of infinitely many half-planes we arbitrarily select one passing through the symmetric axis. But for the sake of effectiveness sometimes it is more convenient to take two symmetric half-planes lying in one plane. The boundary of the domain under consideration consists of the known and unknown parts. The known ones consist of straight lines and their portions, while the unknown parts consists of curves. Every portion of the boundary is assigned two boundary conditions. The unknown functions (the velocity potential, the flow function) and their arguments on every portion of the boundary must satisfy two inhomogeneous boundary conditions.

The system of differential equations with respect to the velocity potential and flow function is reduced to a normal equation. Unknown functions are represented as sums of holomorphic and generalized analytic functions.

One problem of the jet theory and one problem of the filtration theory are solved.

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1. Axisymmetric Flows

If the velocity components u_x and u_y are functions of only x and y, whereas the velocity component u_z is equal to zero, then the motion takes place in the planes parallel to the plane x, y; the motion is the same in all such planes. This implies that there is a direction to which all velocities of the field are perpendicular. The investigation of the plane stationary liquid motion under the above assumptions is, as is known, characterized by certain analytic peculiarities, and many interesting problems can be solved effectively ([1]–[37]).

But, as is known, if the boundaries of the problems under consideration are partially unknown and the boundary conditions are mixed, then the solution of such problems becomes more complicated. The flow function in terms of which many problems are formulated is, as usual, introduced in the plane case, but it is very difficult to introduce it in the spatial case. In the plane problems, the velocity potential and the flow function form analytic functions, and the theory of such functions is well developed both from the qualitative and quantitative points of view ([1]–[6]). As it can be seen below, there exist spatial axisymmetric problems whose solution reduces to solution of plane problems ([23]–[25]).

The solution of spatial axisymmetric problems with partially unknown boundaries present great mathematical difficulties. Such problems are encountered in the theory of filtration, in the theory of jet flows, and in many parts of mathematical physics such, for example, as the mathematical theory of hydromechanics and some other sections of mechanics. The conditions are different in each case. For example, the liquid in the theory of jet flows is weightless, ideal and incompressible, capillary forces and vortices are absent, and the flow is stationary. In the problem of filtration the liquid has weight. Below we will describe a general method of solution of spatial axisymmetric problems of the jet and filtration theories with partially unknown boundaries.

The liquid motion is said to be spatial and axisymmetric, if all velocity vectors lie in half-planes passing through a straight line which is called the axis of symmetry, and the picture of the field of velocities is the same for all meridional half-planes. However, from the mechanical point of view the difference between the half-planes exists, and this is connected with the direction of velocity which can be determined according to the physical sense of the variables involved. The spatial field of velocities of an axisymmetric motion is completely described by the plane field of any of such half-planes. The symmetry axis is assumed to be the x-axis; the distance to the x-axis is denoted by y, and by u_x and u_y we denote, respectively, the components of the velocity vector $\vec{u}(u_x, u_y)$ which is connected with the velocity potential $\varphi(x,t)$ as follows: $\vec{u}\big[\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\big]$. On the plane, the use is made, as usual, of the right system of coordinates x, y; $z = x + iy$ ([1]–[3]).

The velocity potential φ and the flow function ψ are functions of only cylindrical coordinates x, y. Due to the axial symmetry, it suffices to study the flow in any arbitrarily taken meridional half-plane with the system of coordinates x, y.

We choose arbitrarily one half-plane passing through the symmetry axis x on which the moving liquid occupies certain simply connected domain $S(z)$, where $z = x + iy$, with the boundary $S(\ell)$; if some part of the boundary $\ell(z)$ of the domain $S(z)$ is unknown, we have to find it.

Here we present another definition of axisymmetry: the flow is axisymmetric, if the flow lines lie in the half-planes passing thorough the given axis; a picture of distribution of the flow lines is the same for every half-plane.

The lines of intersection of a surface and the planes passing through the symmetry axis x are called meridians, whereas the lines of intersection with the planes perpendicular to the x-axis are called parallels.

In the cylindrical system of coordinates x, θ, y, where from the definition of axisymmetry follows $u_\theta = 0$, the equation of continuity has the form

$$\frac{\partial(yu_x)}{\partial x} + \frac{\partial(yu_y)}{\partial y} = 0, \tag{1.1}$$

where $u_x = \frac{\partial\varphi}{\partial x}$ and $u_y = \frac{\partial\varphi}{\partial y}$ are the projections of velocities on the axes x and y.

As is known, the differential equation of any flow line for an axisymmetric flow, $u_y dx - u_x dy = 0$, multiplied by y, is the full differential of the flow function $d\psi = yu_y dx - yu_x dy$, since

$$u_x = \frac{1}{y}\,\frac{\partial\psi}{\partial y}\,, \quad u_y = -\frac{1}{y}\,\frac{\partial\psi}{\partial x}\,. \tag{1.2}$$

On the other hand,

$$u_x = \frac{\partial\varphi}{\partial x} = \frac{1}{y}\,\frac{\partial\psi}{\partial y}\,, \quad u_y = \frac{\partial\varphi}{\partial y} = -\frac{1}{y}\,\frac{\partial\psi}{\partial x}\,. \tag{1.3}$$

In [25] the reader can find a general method of solution of spatial axisymmetric stationary problems of filtration with partially unknown boundaries and mixed boundary conditions, where the porous medium is nondeformable, isotropic and homogeneous. Stationary motion of the liquid in the porous medium obeys the Darcy law.

Below we will present some statements of the well-known authors regarding solutions of spatial stationary axisymmetric problems with partially unknown boundaries ([3], [4], [6]).

Everywhere below, when solving the problems of the jet theory, the use will be made of the following assumptions. The liquid is weightless, ideal and incompressible. Capillary forces and vortices are absent, and the flow is stationary [3].

"Solution of spatial jet problems presents great mathematical difficulty. At present we are aware only of the works which are devoted to axisymmetric jet flows. However, even for that simple particular case of spatial problems no one succeeded in creation of a mathematical device which would be as convenient as that of the theory of functions of complex variable. The authors engaged in the axisymmetric jet flows either restrict themselves to approximate numerical solutions of the problems, or prove theorems of general nature" [3].

"Unfortunately, the methods of the theory of functions of complex variable applied to solution of plane problems have no effective analogue in the axisymmetric case, or, more precisely, analytic methods provide us with little information of physical interest" [6].

"The qualitative theory of solutions of the system of differential equations (1.3) can be constructed rather completely, whereas the quantitative theory is not as well developed as for the solutions of the (Cauchy–Riemann) system, i.e., for analytic functions" [4].

Below, we will give a general method of solution of spatial axisymmetric problems with unknown boundaries both in the theory of filtration and in the theory of jet flows.

Here we cite some rather frequently encountered boundary conditions for spatial axisymmetric problems of filtration.

1. On a free surface, the boundary conditions have the form

$$\varphi(x, y) - kx = const, \tag{1.4}$$

$$\psi(x, y) = const, \tag{1.5}$$

where $k = const$ is the coefficient of filtration;

2. Along the boundary of water basins:

$$\varphi(x, y) = const, \tag{1.6}$$

$$a_1 x + b_1 y + c_1 = 0, \quad a_1, b_1, c_1 = const; \tag{1.7}$$

3. Along the leaking intervals:

$$\varphi(x, y) - kx = const, \tag{1.8}$$

$$a_2 x + b_2 y + c_2 = 0, \quad a_2, b_2, c_2 = const; \tag{1.9}$$

4. Along the symmetry axis, when a segment of the symmetry axis x coincides with a portion of the boundary of $S(z)$, the boundary conditions are of the form

$$y = 0, \tag{1.10}$$

$$\psi(x, y) = 0, \tag{1.11}$$

but if the symmetry axis does not coincide with any part of the boundary of the flow domain $S(z)$, then

$$y = const, \quad const \neq 0, \tag{1.12}$$

$$\psi(x, y) = const, \quad const \neq 0; \tag{1.13}$$

5. Along nonpermeable boundaries there take place the following boundary conditions:

$$\psi(x,y) = const, \tag{1.14}$$

$$a_3x + b_3y + c_3 = 0, \quad a_3, b_3, c_3 = const; \tag{1.15}$$

6. Along the nonpermeable boundary, the velocity vector is directed along the boundary;

7. The velocity vector is perpendicular to the boundaries of water basins;

8. Along a free surface (depression curve) we have

$$u_x^2 + u_y^2 - ku_x = 0. \tag{1.16}$$

In our works [23], [24], [25] it is assumed that on the plane of complex velocity we have circular polygons of particular types. Despite this fact, this class of problem is still wide enough. There exist axisymmetric spatial problems with partially unknown boundaries, when the boundary of the domain does not contain the symmetry axis. But there are problems when the boundary of the domain involves, as is said above, the symmetry axis or its portions.

For circular polygons, in particular for linear ones, we are able to solve plane problems of filtration with partially unknown boundaries. The statement and solution of the corresponding plane problems of filtration with partially unknown boundaries can be found in [2], [12] and [18]–[25].

2. The Theory of Axisymmetric Flows

A flow of substance moving almost in a constant direction at a distance exceeding many times its cross-section size is called a jet. In order to get a jet, it suffices to make a hole in the reservoir whose local pressure exceeds that of the environment ([3], [5], [6]).

When flowing around an immovable obstacle or a wall protuberance, the flow, as usual, separates and forms the so-called isolated flow lines. The liquid between these flow lines forms a trace; right behind the obstacle the flow is quiet. The traces in the liquid are of dissimilar nature. A trace forms a chain of vortices stretching at a long distance behind the obstacle. The importance of traces is that they are the main source of resistance in the real liquid. As is known, the resistance in nonviscous liquids does not usually arise for subsonic velocities if the flow separation and the associated trace are absent ([3]–[6]).

If a body moves in a liquid with great velocity, the trace becomes gaseous; such a trace is called a cavity. If a ball moves in water at velocity about 8 m/sec or more, we obtain a cavity filled with air. Cavities arising at the velocity 30 m/sec or more are filled with steam ([3]–[6]).

Besides, there are still many questions of practical importance which are connected with formation of jets, traces and cavities ([3]–[6]).

3. Statement of the Problem in the Theory of Jets

The theory of jets considers flows which are bounded partially by rigid walls and unknown free surfaces of constant pressure ([3]–[6]).

The hydrodynamic problem is assumed to be solved if any of the two functions $\varphi(x,y)$ and $\psi(x,y)$ is known. Besides the equations (1.2) and (1.3), for finding $\varphi(x,y)$ and $\psi(x,y)$ we have the following boundary conditions. The normal velocity on the free and body surfaces is equal to zero ([1]–[7]),

$$\frac{\partial\varphi}{\partial n} = 0, \tag{3.1}$$

where n is the normal directed into the liquid. The flow function ψ on the free and body surfaces is a constant value [3],

$$\psi = const. \tag{3.2}$$

This condition for ψ is equivalent to the condition (3.1). On the boundaries, the constant in (3.2) may take different values.

For example, in Fig. 1 we can see one-half of the meridional plane $x0y$ for the problem concerning the flow round a circular cone in a tube. Since the flow function is defined within a constant summand, we can put $\psi = 0$ on the symmetry axis x, on the cone and on the free surface. But the difference between the values of ψ on the flow surface is equal to the liquid discharge between these surfaces divided by 2π; hence on the tube walls $\psi = \pi\mathbf{v}_\infty h^2/(2\pi)$, where h is the tube radius, and $\mathbf{v}_\infty$ is the velocity at infinity of the flow coming from the left ([3]–[6]).

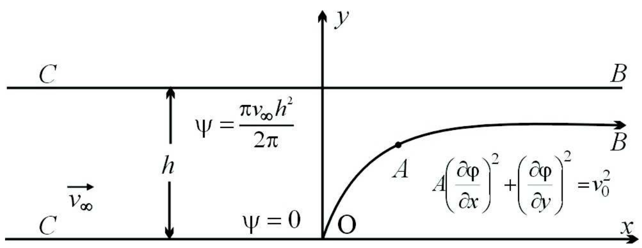

Figure 1

The form of the free surfaces is unknown, but here we have the supplementary condition of constancy of the velocity modulus $\mathbf{v}$, which is equivalent to the condition of pressure constancy. This condition can be written as ([3])

$$\frac{1}{\rho}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] = \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 = \mathbf{v}_0^2, \tag{3.3}$$

where $\mathbf{v}_0$ is equal to $\mathbf{v}$ on the free surface ([3]–[6]).

4. The Flow Function for the Axisymmetric Flow

If the flow is irrotational, then the flow function ψ should satisfy the equation

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x}, \quad \text{then} \quad \frac{\partial}{\partial x}\left(\frac{1}{y}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial \psi}{\partial y}\right) = 0. \tag{4.1}$$

Recall that the function $\varphi(x,t)$ is harmonic in the cylindrical coordinate system. Unlike the plane case, the flow function $\psi(x,y)$ is not harmonic. It follows from (1.3) that

$$\frac{\partial \varphi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial \psi}{\partial y} = 0. \tag{4.2}$$

The system (1.1), (1.3) can be rewritten as follows:

$$\Delta\varphi(x,t) + \frac{1}{y}\frac{\partial \varphi}{\partial y} = 0, \tag{4.3}$$

$$\Delta\psi(x,t) - \frac{1}{y}\frac{\partial \psi}{\partial y} = 0, \tag{4.4}$$

where Δ is the Laplace operator.

We write the system (4.3), (4.4) in the form

$$\frac{\partial^2 \varphi}{\partial x^2} + 4\alpha\frac{\partial^2 \varphi}{\partial \alpha^2} + 4\frac{\partial \varphi}{\partial \alpha} = 0, \tag{4.5}$$

$$\frac{\partial^2 \psi}{\partial x^2} + 4\alpha\frac{\partial^2 \psi}{\partial \alpha^2} = 0, \tag{4.6}$$

where $\alpha = y^2$.

It can be seen from (4.5) and (4.6) that the given system for $\alpha = y^2 \neq 0$ is elliptic. Along the $0x$-axis, as $\alpha \to 0$, we have

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{4}\frac{\partial \varphi}{\partial \alpha} = 0, \tag{4.7}$$

$$\frac{\partial^2 \psi}{\partial x^2} = 0. \tag{4.8}$$

Along the symmetry axis $0x$, we have

$$\lim_{y\to 0}\frac{\partial \varphi}{\partial y} = 0, \quad \lim_{y\to 0}\frac{\partial \psi}{\partial y} = 0, \quad \lim_{y\to 0}\frac{1}{y}\frac{\partial \varphi}{\partial y} = \frac{\partial^2 \varphi}{\partial y^2}, \tag{4.9}$$

$$\lim_{y\to 0}\frac{1}{y}\frac{\partial \psi}{\partial y} = \frac{\partial^2 \varphi}{\partial y^2}. \tag{4.10}$$

5. Application of Analytic and Generalized Analytic Functions to Solution of Axisymmetric Problems

We map conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ (or $\mathrm{Im}(\zeta) < 0$) of an auxiliary complex plane $\zeta = \xi + i\eta$ onto the domain $S(z)$, where $z(\zeta) = x(\xi,\eta)+iy(\xi,\eta)$. A part of the boundary $S(\ell)$ of the domain $S(z)$ is unknown and should be defined. On the plane $\zeta = \xi + i\eta$, the system (1.3) takes the form

$$\frac{\partial\varphi}{\partial\xi} = \frac{1}{y(\xi,\eta)}\,\frac{\partial\psi}{\partial\eta}\,, \tag{5.1}$$

$$\frac{\partial\psi}{\partial\eta} = -\frac{1}{y(\xi,\eta)}\,\frac{\partial\psi}{\partial\xi}\,. \tag{5.2}$$

It can be seen from (5.1), (5.2) that $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$ are mutually connected, and this fact should always be taken into consideration.

We rewrite the system (5.1), (5.2) as follows:

$$\Delta\varphi(\xi,\eta) + \frac{1}{y}\,\frac{\partial y}{\partial\xi}\,\frac{\partial\varphi}{\partial\xi} + \frac{1}{y}\,\frac{\partial y}{\partial\eta}\,\frac{\partial\varphi}{\partial\eta} = 0, \tag{5.3}$$

$$\Delta\psi(\xi,\eta) - \frac{1}{y}\,\frac{\partial y}{\partial\xi}\,\frac{\partial\psi}{\partial\xi} - \frac{1}{y}\,\frac{\partial y}{\partial\eta}\,\frac{\partial\psi}{\partial\eta} = 0. \tag{5.4}$$

Suppose that we have solved the plane problem, i.e., we have constructed analytic functions mapping conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ (or $\mathrm{Im}(\zeta) < 0$) of the plane $\zeta = \xi + i\eta$ onto the circular polygon. For general discussion we assume that there is a circular polygon with m vertices. To find an analytic function in the general case, we have to solve a nonlinear third order Schwartz differential equation. Its solution is reduced to solution of a Fuchs class differential equation. The Schwartz equation, and hence the corresponding Fuchs class equation, contains $2(m-3)$ essential unknown parameters. The general solution of the Schwartz equation involves additionally six parameters of integration. We write the system of higher $2(m-3)$ transcendent equations and also the system of six equations for finding the integration parameters of the Schwartz equation. Next, we construct solutions $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$ for the system (5.3) and (5.4) with regard for (5.1), (5.2) ([18]–[25]).

Introduce a notation for three analytic functions:

$$\begin{gathered} z(\zeta) = x(\xi,\eta) + iy(\xi,\eta), \quad \omega_0(\zeta) = \varphi_0(\xi,\eta) + i\psi_0(\xi,\eta), \\ w_0(\zeta) = \omega_0'(\zeta)/z'(\zeta), \end{gathered} \tag{5.5}$$

$$\Delta x(\xi,\eta) = 0, \quad \Delta y(\xi,\eta) = 0, \quad \Delta\varphi_0(\xi,\eta) = 0, \quad \Delta\psi_0(\xi,\eta) = 0, \tag{5.6}$$

which map conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ respectively onto the domain $S(z(\zeta))$ of liquid motion, onto the domain of the complex potential $\varphi_0(\xi,\eta) + i\psi_0(\xi,\eta) = \omega_0(\zeta)$, and onto the domain of the complex velocity $S(\omega_0'(\zeta)/z'(\zeta))$. The above functions are unknown and due to be defined.

Below, we will consider the problem of solvability of the system of equations (5.1), (5.2).

A solution of (5.3), (5.4) will be sought with regard for (5.1) and (5.2) in the form

$$\varphi(\xi,\eta) = \varphi_0(\xi,\eta) + \varphi_1(\xi,\eta), \tag{5.7}$$

$$\psi(\xi,\eta) = \psi_0(\xi,\eta) + \psi_1(\xi,\eta), \tag{5.8}$$

where $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$ are self-conjugate harmonic functions satisfying all boundary conditions. Substituting (5.7) and (5.8) into (5.3) and (5.4), we obtain

$$\begin{aligned}\Delta\varphi_1(\xi,\eta)+\frac{1}{y}\frac{\partial y}{\partial\xi}\frac{\partial\varphi_1}{\partial\xi}+\frac{1}{y}\frac{\partial y}{\partial\eta}\frac{\partial\varphi_1}{\partial\eta}&=\\ =-\left[\Delta\varphi_0+\frac{1}{y}\frac{\partial y}{\partial\xi}\frac{\partial\varphi_0}{\partial\xi}+\frac{1}{y}\frac{\partial y}{\partial\eta}\frac{\partial\varphi_0}{\partial\eta}\right],\qquad(5.9)\\ \Delta\psi_1(\xi,\eta)-\frac{1}{y}\frac{\partial y}{\partial\xi}\frac{\partial\psi_1}{\partial\xi}-\frac{1}{y}\frac{\partial y}{\partial\eta}\frac{\partial\psi_1}{\partial\eta}&=\\ =-\left[\Delta\psi_0-\frac{1}{y}\frac{\partial y}{\partial\xi}\frac{\partial\psi_0}{\partial\xi}-\frac{1}{y}\frac{\partial y}{\partial\eta}\frac{\partial\psi_0}{\partial\eta}\right].\qquad(5.10)\end{aligned}$$

In the right-hand sides of (5.9) and (5.10) we retain $\Delta\varphi_0=0$ and $\Delta\psi_0=0$ deliberately.

We transform the unknown functions $\varphi_1(\xi,\eta)$, $\psi_1(\xi,\eta)$, $\varphi_0(\xi,\eta)$ and $\psi_0(\xi,\eta)$ as follows:

$$\varphi_1(\xi,\eta)=y^{-1/2}(\xi,\eta)\varphi_2(\xi,\eta),\quad \psi_1=y^{1/2}(\xi,\eta)\psi_2(\xi,\eta),\tag{5.11}$$

$$\varphi_0(\xi,\eta)=y^{-1/2}(\xi,\eta)\varphi_2^*(\xi,\eta),\quad \psi_0(\xi,\eta)=y^{1/2}(\xi,\eta)\psi_2^*(\xi,\eta).\tag{5.12}$$

After transformation the system (5.9), (5.10) takes the form

$$\Delta(\varphi_1+\varphi_2^*)=-\frac{1}{4}\,\rho_1(\varphi_2+\varphi_2^*),\tag{5.13}$$

$$\Delta(\psi_2+\psi_2^*)=\frac{3}{4}\,\rho_1(\psi_2+\psi_2^*),\tag{5.14}$$

where

$$\rho_1=\left(\frac{1}{y}\frac{\partial y}{\partial\xi}\right)^2+\left(\frac{1}{y}\frac{\partial y}{\partial\eta}\right)^2.\tag{5.15}$$

As is said above, the hydrodynamic problem is assumed to be solved if either of the functions $\varphi(x,y)$ and $\psi(x,y)$ is found with regard for (5.13). Next, on the plane ζ we have to take into consideration (5.1) and (5.2).

Using Green's formula, we can obtain from (5.13) and (5.14) the following Fredholm integral equations of second kind:

$$\varphi_2(\xi,\eta)+\frac{1}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0}G(\xi,\eta;x,y)\rho_1(x,y)\varphi_2(x,y)\,dx\,dy=f_1(\xi,\eta),\tag{5.16}$$

$$\psi_2(\xi,\eta)-\frac{3}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0}G(\xi,\eta;x,y)\rho_1(x,y)\psi_2(x,y)\,dx\,dy=f_2(\xi,\eta),\tag{5.17}$$

where

$$\begin{aligned}f_1(\xi,\eta)=&-\varphi_2(\xi,\eta)-\\ &-\frac{1}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0}G(\xi,\eta;x,y)\rho_1(x,y)\varphi_2^*(x,y)\,dx\,dy,\end{aligned}\tag{5.18}$$

$$f_2(\xi,\eta) = -\,\psi_2^*(\xi,\eta) + \\ + \frac{3}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\rho_1(x,y)\psi_2^*(x,y)\,dx\,dy \tag{5.19}$$

and

$$G(\xi,\eta;x,y) = \frac{1}{4\pi}\ln\frac{(\xi-x)^2+(\eta+y)^2}{(\xi-x)^2+(\eta-y)^2}\,.$$

Solutions of the integral equations (5.16) and (5.17) will be sought by using the method of successive approximations in the form of the following series:

$$\varphi_2(\xi,\eta) = \sum_{n=0}^{\infty}\lambda^n\varphi_{2(n)}(\xi,\eta), \tag{5.20}$$

$$\psi_2(\xi,\eta) = \sum_{n=0}^{\infty}\mu^n\psi_{2(n)}(\xi,\eta), \tag{5.21}$$

where $\lambda = \frac{1}{4}$, $\mu = \frac{3}{4}$.

Substituting the series (5.20) and (5.21) respectively into the integral equations (5.16) and (5.17), and then equating the coefficients at the same degrees of the parameters λ and μ, we will obtain

$$\varphi_{2(0)}(\xi,\eta) = f_1(\xi,\eta), \tag{5.22}$$

. . . .

$$\varphi_{2(n)}(\xi,\eta) = \iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\rho_1(x,y)\varphi_{2(n-1)}(x,y)\,dx\,dy, \tag{5.23}$$

. . . .

$$\psi_{2(0)}(\xi,\eta) = f_2(\xi,\eta), \tag{5.24}$$

. . . .

$$\psi_{2(n)}(\xi,\eta) = \iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\rho_1(x,y)\psi_{2(n-1)}(x,y)\,dx\,dy, \tag{5.25}$$

. . . .

$$n = 1, 2, 3, \dots .$$

6. On Solution of Some Fredholm Integral Equations ([31], [32])

Consider the simplest Fredholm integral equation of the second kind [32]

$$u(x) - \lambda\int_a^b K(x,t)u(t)\,dt = f(x), \tag{6.1}$$

where the unknown function $u(x)$ depends on the real variable x which varies in the same interval $[a,b]$ as the integration variable t. This requirement concerns without exception to all classes of integral equations under consideration. The interval $[a,b]$ may be finite or infinite. The functions $K(x,t)$ and $f(x)$ are assumed to be given and defined almost everywhere, respectively, in the square $a \le x \le b$, $a \le t \le b$ and in the interval $a \le x \le b$.

The function $K(x,t)$ is said to be the kernel of the integral equation. The kernel $K(x,t)$ of the Fredholm equation satisfies the inequality

$$\int_a^b \int_a^b |K(x,t)|^2 \, dx \, dt < \infty \tag{6.2}$$

and the free term $f(x)$ satisfies the inequality

$$\int_a^b |f(x)|^2 \, dx < \infty. \tag{6.3}$$

We have to consider Fredholm equations of more general type. Let Ω be a measurable set in the space of an arbitrary number of variables, x and t be points of that set, and μ be a nonnegative measure defined on Ω [32].

The equality

$$u(x) - \lambda \int_\Omega K(x,t)u(t) \, d\mu(t) = f(x), \tag{6.4}$$

whose kernel $K(x,t)$ and free terms $f(x)$ satisfy, respectively, the inequalities

$$\int_\Omega \int_\Omega |K(x,t)|^2 \, d\mu(x) \, d\mu(t) < \infty, \quad \int_\Omega |f(x)|^2 \, d\mu(x) < \infty, \tag{6.5}$$

is also called a Fredholm equation.

The kernel $K(x,t)$ satisfying (6.5) is called the Fredholm kernel. We denote the volume element by dx and the integral (6.5) by B_K^2:

$$\int_\Omega \int_\Omega |K(x,t)|^2 \, dx \, dt = B_K^2. \tag{6.6}$$

The unknown function $u(x)$ is quadratically summable in (a,b), and, consequently, belongs to the space $L_2(a,b)$. A solution of the equation (6.4) belongs to the space $L_2(\mu,\Omega)$ of functions which are quadratically summable in Ω in measure μ. The inequalities (6.3) and (6.5) mean that the free term of the equation belongs to the same space. The parameter λ may take both real and complex values.

The parameters λ and μ of the integral equations (5.16) and (5.17) are less than unity, hence the convergence of the series (5.20) and (5.21) is guaranteed.

As is known, the Fredholm equation of the second kind has either finite, or countable set of characteristic numbers. But there are kernels having no characteristic numbers at all, as, for example, Volterra kernels. A complete characteristic of such kernels is given in the following Lalesko theorem.

Let $K(x,t)$ be a Fredholm kernel, and $K_n(x,t)$ be its iterated kernel. For the kernel $K(x,t)$ to have no characteristic numbers, it is necessary and sufficient that

$$A_n = \int\limits_{\Omega} K_n(x,t)\,dx = 0, \quad n = 3, 4, \dots\,, \tag{6.7}$$

where the numbers A_n are called traces of the kernel $K(x,t)$. Lalesko has proved his theorem for the case of bounded kernels, while a general proof has been given by S. Krachkovskiĭ ([31], [32]).

The Fredholm determinant and minors are represented as quotients of two entire functions of λ, the poles of the resolvent, i.e., the characteristic numbers of the kernel $K(x,t)$, not depending on x and t. Thus the resolvent should have the form

$$R(x,t;\lambda) = D(x,t;\lambda)/D(\lambda), \tag{6.8}$$

where $D(x,t;\lambda)$ and $D(\lambda)$ are entire functions of λ ([31], [32]).

For the numerator and denominator of the fraction in (6.8) we give the representations in the form of the following series ([31], [32]):

$$D(x,t;\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} B_n(x,t)\lambda^n, \quad D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} c_n \lambda^n, \tag{6.9}$$

where

$$c_0 = 1, \quad B_0(x,t) = K(x,t), \quad c_n = \int\limits_{\Omega} B_{n-1}(x,x)\,dx, \;\; n > 0, \tag{6.10}$$

$$B_n(x,t) = c_n - n\int\limits_{\Omega} K(x,t) B_{n-1}(\tau,t)\,d\tau, \tag{6.11}$$

which makes it possible to calculate the coefficients $B_n(x,t)$ and c_n recursively.

Below we will need the well-known formula ([31], [32])

$$D'(\lambda)/D(\lambda) = -\sum_{n=1}^{\infty} A_n \lambda^{n-1}, \tag{6.12}$$

where

$$A_n = \int\limits_{\Omega} K_n(x,x)\,dx, \quad n = 1, 2, 3, \dots\,, \tag{6.13}$$

are the above-mentioned traces of the kernel $K(x,t)$.

If the kernel $K(x,t)$ is noncontinuous and, more so, has discontinuities of the second kind, then the integrals in (6.10) defining the coefficients $c_1, c_2, \dots$ become meaningless.

The Fredholm kernel may sometimes have Green's function $G(P,Q)$ as a multiplier. As is known, this function is defined as a harmonic function symmetric with respect to P and Q, equal on the boundary to zero, and analytic at all points P of the domain D, except for the points $P = Q$ at which it has logarithmic singularity.

The kernel $K(x,t)$ may have logarithmic singularity. Then the integral

$$\int\limits_{\Omega} K(x,x)\,dx \tag{6.14}$$

defining the coefficient c_1 becomes meaningless. This difficulty can be overcome successfully by putting, for example, $c_1 = 0$ ([31], [32]).

The iterated kernel $K_2(s,t)$ has the form

$$K_2(s,t) = \int\limits_{\Omega} K(s,t_1)K(t_1,t)\,dt_1. \tag{6.15}$$

The integral $K_2(s,t)$ is meaningful for any positions of s and t in $[a,b]$ because in the most unfavorable case, when s and t coincide, the integrand admits the following estimate ([27]–[30]):

$$|K(s.t_1)K(t_1,t)| \le \frac{M_1}{|s-t_1|^{\varepsilon_1}}\,, \quad \varepsilon_1 > 0. \tag{6.16}$$

It is proved that $K_2(s,t)$ is a function continuous in the square $a \le x \le b$, $a \le t_1 \le b$, and the functions

$$K_n(s,t) = \int\limits_{\Omega} K(s,t_1)K_{n-1}(t_1,t)\,dt_1, \ \ n = 1,2,3,\dots, \tag{6.17}$$

are estimated analogously:

$$|K(s,t_1)K_{n-1}(t_1,t)| \le \frac{M_{n-1}}{|s-t_1|^{\varepsilon_{n-1}}}\,, \ \ \varepsilon_{n-1} > 0. \tag{6.18}$$

The integral $K_n(s,t)$, $n = 1,2,\dots$, is meaningful for any positions of s and t in $[a,b]$, and the estimates of the integrands have the form (6.18). Consequently, we have to put

$$K_n(s,s) = 0, \ \ n = 1,2,\dots, \tag{6.19}$$

$$A_n = \int\limits_{\Omega} K_n(s,s)\,ds = 0, \ \ n = 1,2,\dots. \tag{6.20}$$

Then

$$c_n = 0, \quad n = 1,2,\dots,n,\dots. \tag{6.21}$$

Taking into account (6.19), from (6.12) we get

$$D'(\lambda) = 0, \tag{6.22}$$

and in its turn, from (6.22) it follows that

$$D(\lambda) = 1. \tag{6.23}$$

Consequently, the kernel of the integral equation (6.4) has no characteristic numbers. In a complete analogy we can prove that the kernel of the integral equation (3.36), considered by us in [24], has no characteristic numbers.

7. Spatial Axisymmetric Jet Flows with Partially Unknown Boundaries

Below, the use will frequently be made of the works [3], [6]. Let us consider the stationary axisymmetric flow of an ideal, weightless, incompressible liquid. Let the x-axis coincide with the symmetry axis. The velocity potential $\varphi(x,y)$ and the flow function $\psi(x,y)$ are functions of only cylindrical coordinates x and y, where y is the distance to the axis x. Owing to the axial symmetry, it suffices to study the flow in an arbitrarily chosen meridional half-plane with the coordinate system x, y ([1]–[6]). By $w(x,y) = \varphi(x,y) + i\psi(x,y)$ we denote the complex potential, and by $z = x + iy$ the complex coordinate. As is known, these functions should satisfy the conditions (1.2) and (1.3).

In Fig. 1 we can see one half of the meridional plane $x0y$ for the problem of flow round a circular cone in a circular tube. Since the flow function $\psi(x,y)$ is defined to within a constant summand, we can put $\psi(x,y) = 0$ along the symmetry axis x both on the cone and on the free surface. But the difference between the values of ψ on the flow surfaces is equal to the liquid discharge between these surfaces divided by 2π, and hence on the tube walls $\psi = \pi \mathbf{v}_\infty h^2/(2\pi)$, where h is the tube radius, and $\mathbf{v}_\infty$ is velocity at infinity coming from the left [3].

The form of the free surfaces is unknown, but the supplementary condition for steady pressure is given. This condition can be written in the form (3.3), where v_0 is equal to v on the free surface [3].

To solve the problem, we map conformally the domains of variation of $\frac{dw}{(v_0 dz)}$ and w onto the semi-circle of unit radius (Fig. 2) of the parametric variable t ($|t| \le 1$, $\mathrm{Im}(t) \ge 0$), where $t = \xi + i\eta$. Having chosen arbitrarily three points on the mapped contour according to the Riemann theorem, we assume that to the singular points a_1, a_2, a_3, a_4, a_5 there correspond the points $\xi = a_1 = 0$, $a_2 = 1$, $a_3 = -1$, $a_4 = -h_0$ and $a_5 = -\frac{1}{h_0}$, where a_4 and a_5 are the source, and a_3 is the sink. The complex potential can be written either as

$$w(t) = \frac{q}{\pi} \ln\Big\{\big[(t-a_4)(t-1/a_4)\big]\big/(t-a_3)^2\Big\}, \tag{7.1}$$

or as

$$w(t) = \frac{q}{\pi} \ln\Big\{\big[(\xi-a_4)+i\eta\big]\,\big[(\xi-1/a_4)+i\eta\big]\big/\big[(\xi-a_3)+i\eta\big]^2\Big\}. \tag{7.2}$$

In the hodograph domain, the filtration velocity $dw/(v_0 dx)$ does get equal to infinity and it vanishes only at the point $\xi = a_1$.

Analyzing the behavior of the function $dw/(v_0 dz)$ [3], we obtain

$$\frac{dw}{(v_0 dx)} = t^\mu, \quad t > 0. \tag{7.3}$$

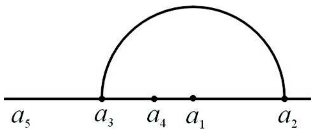

FIGURE 2

We can easily see that the formula (7.3) is valid. Inside the upper half of the semi-circle $|t| \le 1$, the function t^μ is holomorphic. On the circumference we have the equality $|t|^\mu = 1$. On the real axis $0 < t \le 1$, the function t^μ takes real positive values. Moving in the upper half-plane t around the point $\xi = a_1$ counterclockwise, we can see that the argument t^μ on OA $(-1 \le t \le 0)$ is equal to $\pi\mu$, that is, the boundary conditions are fulfilled everywhere.

To see that the formula (7.2) is valid, it suffices to verify that the boundary conditions are fulfilled. Suppose that $\eta = 0$. Then

$$\begin{aligned} w(t) &= \frac{q}{\pi}\ \ln\Big\{\big[(\xi - a_4)(\xi - 1/a_4)\big]\big/(\xi - a_3)^2\Big\} = \\ &= \frac{q}{\pi}\ \ln\Big\{\big[(\xi + h_0)(\xi + 1/h_0)\big]\big/(\xi + 1)^2\Big\}. \end{aligned} \tag{7.4}$$

It follows from (7.4) that $\psi = q$. Since the expression $(t - a_4)(t - 1/a_4)$ in the interval $(\frac{1}{a_4}, a_4)$ is negative, we have $\operatorname{Im} w(t) = q$, $a_3 < \xi < a_3$, and it is positive in the intervals $t < \frac{1}{a_4}$, $t > a_4$. This implies that in the interval $a_4 < \xi < a_2$

$$\operatorname{Im} w(t) = 0, \quad a_4 < \xi < a_2. \tag{7.5}$$

Assuming on the arc $t = e^{i\alpha}$, we obtain

$$\begin{gathered} \operatorname{Im} w(t) = \\ = \frac{q}{\pi} \operatorname{Im} \ln \Big[(e^{i\alpha} - a_4)(e^{-i\alpha} - a_4)\Big(\frac{1}{-a_4}\Big)\Big]\Big/(e^{-\alpha/2} + e^{-i\alpha/2})^2 = 0. \end{gathered} \tag{7.6}$$

Now find the velocity v_{a_4} of the flow in the vessel at infinity:

$$\frac{v_{a_4}}{v_0} = \Big(\frac{dw}{v_0 dx}\Big)_{a_4} = h_0^\mu. \tag{7.7}$$

Thus the value h_0 defines the velocity in the vessel at infinity. Obviously, $q = hv_{a_4}$, whence according to (7.7) we obtain

$$q = h \cdot v_0 h_0^\mu. \tag{7.7$_1$}$$

From (7.4) and (7.3) we can find $z(t)$. Thus we have

$$z(t) = \frac{e^{i\pi\mu}}{v_0} \int\limits_0^t t^{-\mu} w'(t)\, dt. \tag{7.8}$$

When $t \to a_2$, the equality (7.8) allows us to obtain the formulas

$$z(a_2) = \frac{e^{i\pi\mu}}{v_0} \int_0^{a_2} t^{-\mu} w'(t)\, dt, \quad a_2 = 1. \tag{7.9}$$

When $t < 0$, we define $z(t)$ by the formula

$$z(t) = -\frac{1}{v_0} \int_t^0 (-t)^{-\mu} w'(t)\, dt, \quad t < 0. \tag{7.10}$$

It follows from (7.9) that

$$x(a_2) = \cos(\pi\mu)\, \frac{1}{v_0} \int_0^{a_2} t^{-\mu} w'(t)\, dt, \tag{7.11}$$

$$y(a_2) = \sin(\pi\mu)\, \frac{1}{v_0} \int_0^{a_2} t^{-\mu} w'(t)\, dt, \tag{7.12}$$

$$\sqrt{[x(a_2)]^2 + [y(a_2)]^2} = \frac{1}{v_0} \int_0^{a_2} t^{-\mu} w'(t)\, dt. \tag{7.13}$$

Using the formula (7.10) and moving around the singular point $\zeta = a_4$ on an infinitesimal semi-circumference K with center $t = a_4$, we obtain

$$h = \frac{q}{v_0}\, h_0^{-\mu}, \quad q = h v_0 h_0^{\mu}, \tag{7.14}$$

where $h_0 = -a_4$, h is the radius of the cylinder.

The formula (7.14) coincides with (7.7_1).

In calculating the integral (7.10), when integration involves the singular point $\xi = -a_4 = h_0$, we have to apply the principal value of the Cauchy type integral, while when moving around the point $t = a_3 = -1$, we act as follows:

$$\left(\frac{dw}{(v_0 dx)}\right)_{a_3} = (+1)^{\mu} = 1. \tag{7.15}$$

At infinity and at the point $a_3 = -1$, the direction of the jets coincide with that of the x-axis.

Next, our main task is to obtain a complete exact solution of the plane problem by means of analytic functions which should be used to obtain a complete solution of the corresponding axisymmetric problem.

Using the functions

$$\ln t = 2i \arccos \frac{1}{\zeta}, \quad \zeta^2 > 1; \quad \ln t = -2i \ln \left| \frac{1 + \sqrt{1-\zeta^2}}{\zeta} \right|, \quad \zeta < 1, \tag{7.16}$$

we map conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ of the auxiliary plane $\zeta = \xi + i\eta$ (Fig. 4) onto a triangle (Fig. 3), and then, using the function $\ln t$, we map conformally the triangle of the type as in Fig. 3 onto the upper semi-circle of unit radius ($|t| \le 1$, $\mathrm{Im}(t) > 0$).

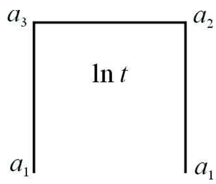

FIGURE 3

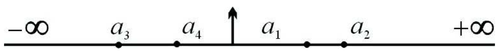

FIGURE 4

Thus the functions (7.1), (7.2) and (7.6) are defined in that domain, so we have obtained the solution of the plane problem on a liquid flowing out of a skew-walled vessel (Fig. 1). Using the above-obtained functions, we pass to solution of the spatial problem of flow around the circular cone in the tube. Using the functions (7.1), (7.2) and (7.3), we assume that the functions $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$ are the first approximations of the unknown functions $\varphi(\xi,\eta)$, $\psi(\xi,\eta)$. The functions $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$, $x(\xi,\eta)$ and $y(\xi,\eta)$ should satisfy all boundary conditions. Thus the above-defined functions $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$, $x(\xi,\eta)$ and $y(\xi,\eta)$ are pairwise self-conjugate harmonic. Note that the conditions of compatibility (5.1), (5.2) should be taken into account. The hydrodynamic problem is assumed to be solved if either of the functions $\varphi(x,y)$ and $\psi(x,y)$ is known.

Finally, we proceed to finding the functions $\varphi_2(\xi,\eta)$, $\psi_2(\xi,\eta)$. When solving the integral equation (5.16) or (5.17), we use the method of successive approximations and the fact that the right-hand sides of (5.16) and (5.17) involve the known functions. On the symmetry axis x of the cone and on the free surface we put $\psi = 0$. But the difference between the values of ψ on the flow surfaces is equal to 2π, hence on the tube walls $\psi = \pi v_\infty h^2/(2\pi)$, where h is the tube radius, v_∞ is velocity at infinity of the flow coming from the left.

As is said above, the form of free surfaces is unknown, but there is a complementary condition of constancy of the velocity modulus v which is equivalent to the condition of pressure constancy. This condition can be written in the form (3.3), where v_0 is equal to v on the free surface [3].

8. The Problem on the Ground Water Influx to A Spatial Axisymmetric Basin with Trapezoidal Axial Cross-Section

Under a water permeable ground layer is laid a ground layer of greater (theoretically infinite) water permeability, the pressure on the upper horizontal surface of the lower layer being constant. The depth of the water in the basin is neglected; if water is deep, the solution of the problem becomes more complicated. The basin is given in Fig. 5.

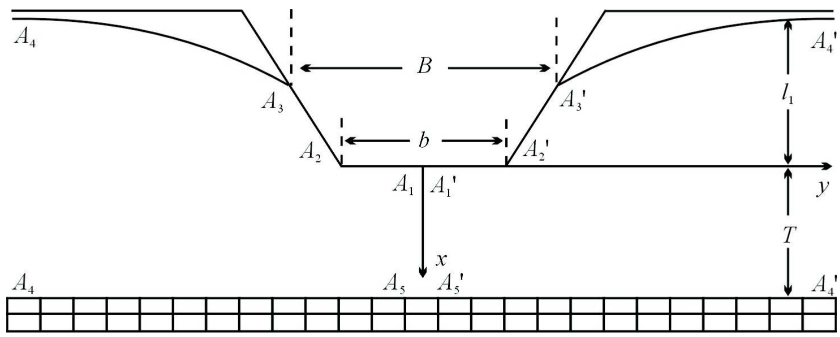

Figure 5

In solving this spatial axisymmetric problem the use will be made of the solution of the corresponding plane problem. The plane axisymmetric problem on the ground water influx to a drainage ditch with trapezoidal cross-section has been solved by V. V. Vedernikov [33], and his investigation was complemented by Yu. D. Sokolov [34]. Here we generalize the problem solved by V. V. Vedernikov [33]. Our generalization consists in the following: under the water permeable ground layer we lay ground layer of greater (theoretically, infinite) water permeability, and the pressure on the upper horizontal surface of the layer is constant. In its turn, we generalize this generalized plane problem to the spatial axisymmetric problem.

We direct the x-axis vertically downwards along the symmetry axis, and the y-axis we direct horizontally; here y is the distance to the x-axis.

Along the whole contour of the domain of liquid motion we have the conditions $\varphi - kx = 0$ and $\varphi - ky = T$. Hence on the Zhukovski plane we have a strip of length T. To solve the problem under consideration, it is convenient to use Zhukovski's function

$$\theta = \theta_1 + i\theta_2, \quad \theta_1 = \varphi - kx, \quad \theta_2 = \psi - ky. \tag{8.1}$$

The boundaries of the velocity hodograph consists of a circumference arc and straight lines which intersect each other at one point F, where $u = -u_0$, $v_0 = 0$.

The plane case under consideration is represented schematically in Fig. 5. Note that

$$\theta = \omega(z) - kz; \quad \omega(z) = \varphi(x,y) + i\psi(x,y), \quad z = x + iy, \\ \frac{d\theta}{dz} = w - k. \tag{8.2}$$

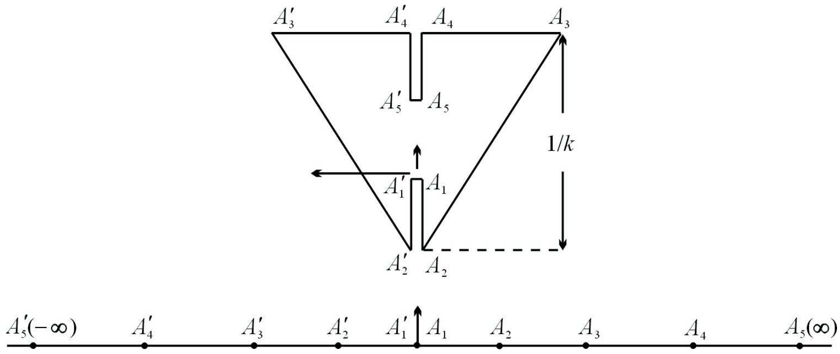

FIGURE 6

The use will be made of the formula

$$u = \frac{dz}{d\theta} = \frac{1}{w-k}\,, \tag{8.3}$$

which corresponds to that function whose domain is obtained after inversion (see Fig. 5). We transfer the vertices of this polygonal domain to the points of the plane ζ as in Fig. 6, and obtain

$$u(\zeta) = M\int\limits_{a_4}^{\zeta} \zeta(\zeta^2 - a_2^2)^{\alpha-1}(\zeta^2 - a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2 - a_4^2)^{-\frac{1}{2}}\,d\zeta + u(a_4), \tag{8.4}$$

where

$$u(a_4) = 1/k, \quad M \ \text{ is a real number.} \tag{8.5}$$

From (8.4) it follows that

$$u(a_5) = M\int\limits_{a_4}^{a_5(+\infty)} \zeta(\zeta^2 - a_2^2)^{\alpha-1}(\zeta^2 - a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2 - a_4^2)^{-\frac{1}{2}}\,d\zeta + u(a_4), \tag{8.6}$$

where

$$u(a_5) = u_5 \ \text{ is a real number.} \tag{8.7}$$

$$u_5 = M\int\limits_{a_4}^{a_5(+\infty)} \zeta(\zeta^2 - a_2^2)^{\alpha-1}(\zeta^2 - a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2 - a_4^2)^{-\frac{1}{2}}\,d\zeta + \frac{1}{k}, \tag{8.8}$$

$$u(\zeta) = -Mi\int\limits_{a_3}^{\zeta}(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta + u(a_3), \quad (8.9)$$

where

$$u(a_3) = -\frac{1}{k}\ \mathrm{tg}(\pi\alpha) + \frac{i}{k}\,. \quad (8.10)$$

$$u(a_4) = -Mi\int\limits_{a_3}^{a_4}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta + u(a_3), \quad (8.11)$$

where $u(a_4) = \frac{1}{k}$.

From (8.11) we have

$$M\int\limits_{a_4}^{a_4}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta + \frac{1}{k}\ \mathrm{tg}\,\pi\alpha = 0, \quad (8.12)$$

$$u(\zeta) = (-1)Me^{-i\pi\alpha}\times$$
$$\times\int\limits_{a_2}^{\zeta}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta + u(a_2), \quad (8.13)$$

where $u(a_2) = 0$,

$$u(a_3) = (-1)Me^{-i\pi\alpha}\int\limits_{a_2}^{a_3}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta, \quad (8.14)$$

$$u(a_3) = \frac{-i}{k}\ \mathrm{tg}\,\pi\alpha + \frac{1}{k}\,, \quad \frac{1}{k} - M\cos\pi\alpha\times$$
$$\times\int\limits_{a_2}^{a_3}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\ d\zeta = 0, \quad (8.15)$$

$$u(\zeta) = M\int\limits_{a_1}^{\zeta}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta + u(a_1), \quad (8.16)$$

$$M\int\limits_{a_1}^{a_2}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta - \frac{1}{u_0+k} = 0, \quad (8.17)$$

$$u(a_2) = \frac{-1}{u_0+k}\,, \quad u(a_2) = 0. \quad (8.18)$$

Of the parameters a_2, a_3 and a_4, we fix one as $a_2 = 1$, and the parameters u_0, a_3, a_4, M are to be defined by means of the system of equations (8.6), (8.8), (8.12), (8.15) and (8.17).

Now we define Zhukovski's function. We have

$$\theta(\zeta) = \frac{T}{\pi} \ln\left(\frac{\zeta - a_4}{\zeta + a_4}\right) + T. \tag{8.19}$$

For finding the function $z(\theta)$, we use the following formulas:

$$z(\zeta) = \int_{a_4}^{\zeta} u(\zeta)\theta'(\zeta)\, d\zeta + z(a_4), \quad z(a_5) = \int_{a_4}^{a_5} u(\zeta)\theta'(\zeta)\, d\zeta + z(a_4), \tag{8.20}$$

$$z(\zeta) = \int_{a_3}^{\zeta} u(\zeta)\theta'(\zeta)\, d\zeta + z(a_3), \quad z(a_4) = \int_{a_3}^{a_4} u(\zeta)\theta'(\zeta)\, d\zeta + z(a_3), \tag{8.21}$$

$$z(\zeta) = \int_{a_2}^{\zeta} u(\zeta)\theta'(\zeta)\, d\zeta + z(a_2), \quad z(a_3) = \int_{a_2}^{a_3} u(\zeta)\theta'(\zeta)\, d\zeta + z(a_2), \tag{8.22}$$

$$z(\zeta) = \int_{a_1}^{\zeta} u(\zeta)\theta'(\zeta)\, d\zeta + z(a_1), \quad z(a_2) = \int_{a_1}^{a_2} u(\zeta)\theta'(\zeta)\, d\zeta + z(a_1). \tag{8.23}$$

The system (8.20)–(8.23) allows us to define the coordinates of the leaking interval, and then using the function $\theta(\zeta)$, we find parametric equations of depression curves. In solving the problem (Fig. 5) we have considered two symmetric half-planes. Owing to the symmetry, we could have considered arbitrarily one half of the two half-planes. But because of the fact that on the boundary of the hodograph velocity, along the symmetry axis, we have two cuts to the ends of which there correspond two unknown parameters, for their determination we have to write two equations. Determination of another unknown parameters needs another equations, and this exactly has been done in the present work.

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