

Memoirs on Differential Equations and Mathematical Physics
VOLUME 44, 2008, 23-44

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**ANISOTROPIC ROTATING MHD SYSTEM
IN CRITICAL ANISOTROPIC SPACES**

Abstract. The three-dimensional mixed (parabolic-hyperbolic) nonlinear magnetohydrodynamic system is investigated in the whole space \mathbb{R}^3 . Uniqueness is proved in the anisotropic Sobolev space $H^{0, \frac{1}{2}}$. Existence and uniqueness are proved in the anisotropic mixed Besov–Sobolev space $\mathcal{B}^{0, \frac{1}{2}}$. Asymptotic behavior is investigated as the Rossby number goes to zero. Energy methods, Freidrichs scheme, compactness arguments, anisotropic Littlewood–Paley theory, dispersive methods and Strichartz inequality are used.

2000 Mathematics Subject Classification. 35A05, 35A07, 35B40.

Key words and phrases. Mixed (parabolic-hyperbolic) *MHD* system, existence, uniqueness, critical spaces, Strichartz inequality, asymptotic behavior.

რეზიუმე. ნამრთველი ჰამილტონიანი მურეული (პარაბოლ-ჰიპერბოლიკი) არაწრფივი მაგნიტოჰიდროდინამიკური სისტემა მთელი \mathbb{R}^3 სივრცეში. უნიკუზიტეტი აჩვენებულია ანიზოტროპული სობოლევის $H^{0, \frac{1}{2}}$ სივრცეში. არსებობა და უნიკუზიტეტი აჩვენებულია ანიზოტროპული მურეული ბეზოვ–სობოლევის $\mathcal{B}^{0, \frac{1}{2}}$ სივრცეში. ასიმპტოტიკური ქოტეტირება, როდესაც როსბის რიცხვი ნულიდან იქნება ნული. ენერჯიტიკული მეთოდები, ფრიდრიხის სქემა, კომპაქტურობის არგუმენტები, ანიზოტროპული ლიტლიუდ–პეილის თეორემა, დისპერსიული მეთოდები და სტრიჩარტის უტოლობა.

1. INTRODUCTION

This paper deals with an incompressible mixed magnetohydrodynamic system with anisotropic diffusion with the small in the limit Rossby number. Namely, we consider the following system denoted by $(MHD_{\nu_h}^\varepsilon)$:

$$\begin{cases} \partial_t u - \nu_h \Delta_h u + u \cdot \nabla u - b \cdot \nabla b + \frac{1}{\varepsilon} \partial_3 b + \frac{1}{\varepsilon} u \times e_3 = -\nabla p & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \partial_t b - \nu_h \Delta_h b + u \cdot \nabla b - b \cdot \nabla u + \frac{1}{\varepsilon} \partial_3 u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} b = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ (u, b)|_{t=0} = (u_0, b_0) & \text{in } \mathbb{R}^3, \end{cases}$$

where the velocity field u , the induced magnetic perturbation b and p are unknown functions of time t and the space variable $x = (x_1, x_2, x_3) = (x_h, x_3)$, e_3 is the third vector of the Cartesian coordinate system and ν_h is a positive constant which represents both the cinematic viscosity and the magnetic diffusivity. Δ_h denotes the horizontal Laplace operator defined by $\Delta_h = \partial_1^2 + \partial_2^2$ and ε is a small positive parameter destined to go to zero. It is clear that the system is hyperbolic with respect to the direction x_3 called the vertical direction. About the physical motivations, we refer the reader to [3] and references therein.

If we denote by $U = (u, b)$, then U is a solution of the following abstract system:

$$(\mathcal{S}^\varepsilon) \quad \begin{cases} \partial_t U + a_{2,h}(D)U + Q(U, U) + L^\varepsilon(U) = (-\nabla p, 0) & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} b = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ U|_{t=0} = U_0 & \text{in } \mathbb{R}^3, \end{cases}$$

where the quadratic term, the linear perturbation and the viscous term are respectively defined by

$$Q(U, U) = \begin{pmatrix} u \cdot \nabla u - b \cdot \nabla b \\ u \cdot \nabla b - b \cdot \nabla u \end{pmatrix},$$

$$L^\varepsilon(U) = \frac{1}{\varepsilon} L(U) = \frac{1}{\varepsilon} \begin{pmatrix} \partial_3 b + u \times e_3 \\ \partial_3 u \end{pmatrix}$$

and

$$a_{2,h}(D)U = -\nu_h \Delta_h U.$$

In the isotropic case, that is, when the global Laplace operator is taken instead of the horizontal one, some isotropic magnetohydrodynamic systems were studied by several authors ([1], [2], [11]). However, according to our knowledge, the first paper dealing with the anisotropic case is due to the authors in [3]. It deals with existence, uniqueness in $H^{0,s}$ for $s > \frac{1}{2}$ and asymptotic behavior of the solution as $\varepsilon \rightarrow 0$ for the same $(MHD_{\nu_h}^\varepsilon)$. Nevertheless, in [10], the author studied the case of anisotropic pure fluid

and proved the uniqueness in the critical Sobolev space and both existence and uniqueness in the anisotropic Besov–Sobolev space.

In this paper, we extend those results to the rotating $(MHD_{\nu_h}^\varepsilon)$ system, which presents the difficulty to be coupled in a nonlinear way. In addition, as the considered system is a perturbed one, it is quite natural to ask about the asymptotic behavior of the solution as the Rossby number ε tends to zero. We note that this perturbation presents the difficulty of being singular. Precisely, we establish uniqueness results in $H^{0,\frac{1}{2}}(\mathbb{R}^3)$ and $\mathcal{B}^{0,\frac{1}{2}}(\mathbb{R}^3)$, uniform local existence for arbitrary initial data and global existence for small initial data in $\mathcal{B}^{0,\frac{1}{2}}(\mathbb{R}^3)$. Moreover, we establish a convergence result as $\varepsilon \rightarrow 0$.

Let us first say that $H^{0,\frac{1}{2}}(\mathbb{R}^3)$ is the space of regularity $L^2(\mathbb{R}^2)$ in x_h and $H^{\frac{1}{2}}(\mathbb{R})$ in x_3 , and $\mathcal{B}^{0,\frac{1}{2}}(\mathbb{R}^3)$ is also $L^2(\mathbb{R}^2)$ in x_h but $B_{2,1}^{\frac{1}{2}}(\mathbb{R})$ in x_3 . As in [7], for $H^{\frac{1}{2}}(\mathbb{R}^3)$ in the case of Navier–Stokes equation, we say here that $H^{0,\frac{1}{2}}$ and $\mathcal{B}^{0,\frac{1}{2}}$ are critical spaces for the system $(\mathcal{S}^\varepsilon)$. This means that they are invariant by the following scaling of $(\mathcal{S}^\varepsilon)$: if $U(t, x)$ is a solution of $(\mathcal{S}^\varepsilon)$ with the data $U_0(x)$, then $U_\lambda(t, x) = \lambda U(\lambda^2 t, \lambda x)$ is also a solution of $(\mathcal{S}^\varepsilon)$ with the data $\lambda U_0(\lambda x)$.

The main idea is to use the structure of the convection operator together with the incompressibility condition to compensate the lack of information due to the incomplete diffusion operator that describes the anisotropy effect. This very fine analysis is performed with the help of Littlewood–Paley decomposition in order to deal with scale invariant spaces such as $H^{0,\frac{1}{2}}$ and $\mathcal{B}^{0,\frac{1}{2}}$.

The uniqueness result in the anisotropic homogenous Sobolev space $H^{0,\frac{1}{2}}$ is dealt with by the following theorem:

Theorem 1. *The system $(MHD_{\nu_h}^\varepsilon)$ has at most one solution U^ε such that $U^\varepsilon \in L_T^\infty(H^{0,\frac{1}{2}}(\mathbb{R}^3))$ and $\nabla_h U^\varepsilon \in L_T^2(H^{0,\frac{1}{2}}(\mathbb{R}^3))$.*

The proof of this theorem is partially based on a technical lemma inspired from [10] and adapted here for the case of $(MHD_{\nu_h}^\varepsilon)$. By this lemma, we establish a doubly logarithmic estimate for the $H^{0,-\frac{1}{2}}$ norm of W^ε , the difference of two solutions, and we use Osgood lemma to finish the proof. Though W^ε belongs to $H^{0,\frac{1}{2}}$, it will be estimated in $H^{0,-\frac{1}{2}}$. This is due to the fact that the equation satisfied by W^ε is hyperbolic in the variable x_3 .

As in [10], we are not able to establish existence in $H^{0,\frac{1}{2}}$ but only uniqueness. This is due to the noninclusion of $H^{0,\frac{1}{2}}$ in $L_v^\infty(L_h^2)$. Such inclusion holds for $\mathcal{B}^{0,\frac{1}{2}}(\mathbb{R}^3)$ and plays an essential role in proving the existence result. In order to state such result, we introduce Besov type spaces that take into account Lebesgue regularity in time on the dyadic blocs. These spaces are denoted for $p \geq 1$ by $\widetilde{L}_T^p(\mathcal{B}^{0,\frac{1}{2}})$ and defined as in [6] by

$$\|u\|_{\widetilde{L}_T^p(\mathcal{B}^{0,\frac{1}{2}})} = \sum_{q \in \mathbb{Z}} 2^{\frac{q}{2}} \|\Delta_q^v u\|_{L^p([0,T]; L^2(\mathbb{R}^3))}.$$

In the case of critical anisotropic Besov–Sobolev space $\mathcal{B}^{0, \frac{1}{2}}$, existence and uniqueness results are given by the following theorem:

Theorem 2. *Let $U_0 = (u_0, b_0) \in \mathcal{B}^{0, \frac{1}{2}}(\mathbb{R}^3)$ be a divergence free vector fields. There exists a positive time T such that for all $\varepsilon > 0$ there exists a unique solution U^ε of $(MHD_{\nu_h}^\varepsilon)$, where $U^\varepsilon \in \widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}}(\mathbb{R}^3))$ with $\nabla_h U^\varepsilon \in \widetilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}}(\mathbb{R}^3))$ and satisfies the following energy estimate:*

$$\|U^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}}(\mathbb{R}^3))} + \sqrt{\nu_h} \|\nabla_h U^\varepsilon\|_{\widetilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}}(\mathbb{R}^3))} \leq \sqrt{2} \|U_0\|_{\mathcal{B}^{0, \frac{1}{2}}}. \quad (1)$$

Moreover, if the maximal time of existence T^* is finite, then

$$\|U^\varepsilon\|_{\widetilde{L}_{T^*}^\infty(\mathcal{B}^{0, \frac{1}{2}}(\mathbb{R}^3))} = +\infty, \quad (2)$$

and if there exists a constant c such that $\|U_0\|_{\mathcal{B}^{0, \frac{1}{2}}(\mathbb{R}^3)} \leq c\nu_h$, then the solution is global.

We use Friedrichs’s scheme to prove global in time existence result in $L^2(\mathbb{R}^3)$. To establish global in time existence result in $\mathcal{B}^{0, \frac{1}{2}}(\mathbb{R}^3)$, we use again Friedrichs’s scheme and Littlewood–Paley theory. A suitable rearrangement of the nonlinear term allows to apply a technical lemma due to [10]. Then, absorption techniques yield an estimate of the approximate solution. Using standard compactness argument, we finish the proof. To prove local in time existence result in $\mathcal{B}^{0, \frac{1}{2}}(\mathbb{R}^3)$, we decompose the initial data into low and high frequency parts. The low frequency part will be the initial data of a linear problem and the high one will be the initial data of the remainder which is nonlinear. For the former, classical arguments give explicitly the result. For the latter, Littlewood–Paley theory, and especially Bony decomposition, plays a crucial role for estimation of the nonlinear part. The target is to establish an estimate where the norm of the solution will be bounded by an expression that depends on the life span T of the solution and the high frequency part of the initial data. Thus, since we are looking for a local in time result, we can choose, in the appropriate order, the cut-off integer N as big as needed and T as small as needed. This is the idea behind the frequency decomposition of the initial data. We note that the uniform life span of the solution will not depend, as usual, on the norm of the initial data but only on its frequency repartition.

Concerning the asymptotic behavior of the solution as the Rossby number ε tends to zero, we prove the following convergence result:

Theorem 3. *Let $U_0 = (u_0, b_0) \in \mathcal{B}^{0, \frac{1}{2}}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ be a divergence-free vector field and (U^ε) the family of solutions given by Theorem 2. If we set for $\chi \in \mathcal{D}(\mathbb{R})$ $U_R^\varepsilon = \chi(\frac{\cdot}{R})U^\varepsilon$ and $\widetilde{U}_R^\varepsilon = U^\varepsilon - U_R^\varepsilon$, then*

1. $\forall \alpha \in]0, \frac{1}{4}[$, $\varepsilon > 0$ and $R > 0$ there exists $N_\alpha(\varepsilon, R) \in \mathbb{N}$ such that

$$N_\alpha(\varepsilon, R) \xrightarrow{\varepsilon \rightarrow 0} +\infty$$

and

$$\sup_{|p| \leq N_\alpha(\varepsilon, R)} \|\Delta_q^v U_R^\varepsilon\|_{L_T^4(L^\infty)} = o(\varepsilon^\alpha), \quad \varepsilon \rightarrow 0.$$

2. $\forall \eta > 0$,

$$\limsup_{\varepsilon \rightarrow 0} \left\| |\nabla_h|^{1-\eta} \tilde{U}_R^\varepsilon \right\|_{L_T^2(\mathcal{B}^{0, \frac{1}{2}})} \xrightarrow{R \rightarrow +\infty} 0.$$

The proof uses a Strichartz inequality and Fourier analysis. In fact, dispersive effects are of great importance in the study of nonlinear partial differential equations, since they yield decay estimates on waves in \mathbb{R}^3 .

The structure of this paper is as follows. The next section is devoted to introduction of anisotropic Lebesgue spaces and anisotropic Littlewood–Paley theory. In the third section, we prove Theorem 1. The fourth section deals with the proof of Theorem 2. In the last section, we prove Theorem 3.

2. NOTATION AND TECHNICAL LEMMAS

2.1. Anisotropic Lebesgue spaces. Let us define anisotropic Lebesgue spaces and recall some of their properties which are useful in the sequel.

Definition 1. We define $L_h^p(L_v^r)$ to be the space $L^p(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}; L^r(\mathbb{R}_{x_3}))$ endowed with the norm

$$\|f\|_{L_h^p(L_v^r)} = \left\| \|f(x_h, \cdot)\|_{L^r(\mathbb{R}_{x_3})} \right\|_{L^p(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2})}.$$

Similarly, $L_v^r(L_h^p)$ is the space $L^r(\mathbb{R}_{x_3}; L^p(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}))$ endowed with the norm

$$\|f\|_{L_v^r(L_h^p)} = \left\| \|f(\cdot, x_3)\|_{L^p(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2})} \right\|_{L^r(\mathbb{R}_{x_3})}.$$

In the frame of anisotropic Lebesgue spaces, the Hölder inequality reads

$$\|fg\|_{L_v^r(L_h^p)} \leq \|f\|_{L_v^{r'}(L_h^{p'})} \|g\|_{L_v^{r''}(L_h^{p''})},$$

where $\frac{1}{r} = \frac{1}{r'} + \frac{1}{r''}$ and $\frac{1}{p} = \frac{1}{p'} + \frac{1}{p''}$.

Young's convolution inequality takes the following form:

$$\|f \star g\|_{L_v^r(L_h^p)} \leq \|f\|_{L_v^{r'}(L_h^{p'})} \|g\|_{L_v^{r''}(L_h^{p''})},$$

where $1 + \frac{1}{r} = \frac{1}{r'} + \frac{1}{r''}$ and $1 + \frac{1}{p} = \frac{1}{p'} + \frac{1}{p''}$.

The following lemma will be useful in the sequel

Lemma 1. *Let $1 \leq p \leq q$ and $f : X_1 \times X_2 \rightarrow \mathbb{R}$ be a function belonging to $L^p(X_1; L^q(X_2))$, where $(X_1; d\mu_1)$ and $(X_2; d\mu_2)$ are measurable spaces. Then $f \in L^q(X_2; L^p(X_1))$ and*

$$\|f\|_{L^q(X_2; L^p(X_1))} \leq \|f\|_{L^p(X_1; L^q(X_2))}.$$

2.2. Anisotropic Littlewood–Paley theory. The basic idea of Littlewood–Paley theory consists in a localization procedure in the frequency space. The powerful point of this theory is that the derivatives and, more generally, the Fourier multipliers act on distributions whose Fourier transform is supported in a ball or a ring in a very special way that we will return on. Such theory in its anisotropic form allows to introduce anisotropic Sobolev and Besov spaces. To do so, we use an anisotropic dyadic decomposition of the frequency space. We begin by defining for any function a the following operators of localization:

$$\begin{aligned}\Delta_j^h a &= \mathcal{F}^{-1}(\varphi(2^{-j}|\xi_h|)\mathcal{F}(a)) \quad \text{for } j \in \mathbb{Z}, \\ \Delta_q^v a &= \mathcal{F}^{-1}(\vartheta(2^{-q}|\xi_3|)\mathcal{F}(a)) \quad \text{for } q \in \mathbb{N}, \\ \Delta_{-1}^v a &= \mathcal{F}^{-1}(\vartheta(|\xi_3|)\mathcal{F}(a))\end{aligned}$$

and

$$\Delta_q^v a = 0 \quad \text{for } q \leq -2.$$

The functions φ and ϑ represent a dyadic partition of unity in \mathbb{R} ; they are regular non-negative functions and satisfy $\text{supp}(\vartheta) \subset B(0, \frac{4}{3})$, $\text{supp}(\varphi) \subset \mathcal{C}(0, \frac{3}{4}, \frac{8}{3})$. Moreover, for all $t \in \mathbb{R}$,

$$\vartheta(t) + \sum_{q \geq 0} \varphi(2^{-q}t) = 1.$$

Furthermore, we define the operators S_q^v and S_j^h by

$$S_q^v u = \sum_{q' \leq q-1} \Delta_{q'}^v u$$

and

$$S_j^h u = \sum_{j' \leq j-1} \Delta_{j'}^h u.$$

In this way, we are considering a homogeneous decomposition in the horizontal variable and an inhomogeneous one in the vertical one. We define respectively the corresponding Sobolev space and the mixed Besov–Sobolev space by the following definitions:

Definition 2. Let s and s' be two real numbers such that $s < 1$, u be a tempered distribution and

$$\|u\|_{H^{s,s'}} = \left(\sum_{j,q} 2^{2(j s + q s')} \|\Delta_j^h \Delta_q^v u\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

The space $H^{s,s'}(\mathbb{R}^3)$ is the closure of $\mathcal{D}(\mathbb{R}^3)$ in the above semi-norm.

Definition 3. The anisotropic Besov space $\mathcal{B}^{0, \frac{1}{2}}$ is the closure of $\mathcal{D}(\mathbb{R}^3)$ in the following norm

$$\|u\|_{\mathcal{B}^{0, \frac{1}{2}}} = \sum_{q \in \mathbb{Z}} \left(\int_{\xi_h} \int_{2^{q-1} \leq |\xi_3| \leq 2^q} |\xi_3| |\mathcal{F}u(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

The above norm is equivalent to the one defined by

$$\|u\|_{\mathcal{B}^{0, \frac{1}{2}}} = \sum_{q \in \mathbb{Z}} 2^{q/2} \|\Delta_q^v u\|_{L^2(\mathbb{R}^3)}.$$

The interest of this decomposition resides in the fact that any vertical derivative of a function localized in vertical frequencies of size 2^q acts as a multiplication by 2^q .

The following lemma is an anisotropic Bernstein type inequality (see [10]).

Lemma 2. *Let u be a function such that $\text{supp}(\mathcal{F}^v u) \subset \mathbb{R}_h^2 \times 2^q \mathcal{C}$, where \mathcal{C} is a dyadic ring. Let $p \geq 1$ and $r \geq r' \geq 1$ be real numbers. The following holds:*

$$2^{qk} C^{-k} \|u\|_{L_h^p(L_v^r)} \leq \|\partial_{x_3}^k u\|_{L_h^p(L_v^r)} \leq 2^{qk} C^k \|u\|_{L_h^p(L_v^r)}, \quad (3)$$

$$2^{qk} C^{-k} \|u\|_{L_v^r(L_h^p)} \leq \|\partial_{x_3}^k u\|_{L_v^r(L_h^p)} \leq 2^{qk} C^k \|u\|_{L_v^r(L_h^p)}, \quad (4)$$

$$\|u\|_{L_h^p(L_v^r)} \leq C 2^{q(\frac{1}{r'} - \frac{1}{r})} \|u\|_{L_h^p(L_v^{r'})} \quad (5)$$

and

$$\|u\|_{L_v^r(L_h^p)} \leq C 2^{q(\frac{1}{r'} - \frac{1}{r})} \|u\|_{L_v^{r'}(L_h^p)}. \quad (6)$$

It is well known that the dyadic decomposition is useful to define the product of two distributions. That is,

$$uv = \sum_{q \in \mathbb{Z}, q' \in \mathbb{Z}} \Delta_q^v u \cdot \Delta_{q'}^v v = T_v u + T_u v + R(u, v),$$

where

$$\begin{aligned} T_v u &= \sum_{q' \leq q-2} \Delta_{q'}^u v \cdot \Delta_q^v u = \sum_q S_{q-1}^v v \cdot \Delta_q^v u, \\ T_u v &= \sum_{q' \leq q-2} \Delta_{q'}^v u \cdot \Delta_q^v v = \sum_q S_{q-1}^v u \cdot \Delta_q^v v \end{aligned}$$

and

$$R(u, v) = \sum_q \sum_{i \in \{0, \pm 1\}} \Delta_q^v u \cdot \Delta_{q+i}^v v.$$

The two first sums are said to be the paraproducts and the third sum is the remainder. This is known to be the Bony's decomposition in the vertical variable (see [4], [5], [9]). In this framework, we have the following properties

$$\Delta_q^v (S_{q-1}^v u \cdot \Delta_{q'}^v v) = 0 \quad \text{if } |q - q'| \geq 5$$

and

$$\Delta_q^v (S_{q+1}^v u \cdot \Delta_{q'}^v v) = 0 \quad \text{if } q' \leq q - 4.$$

For the sake of simplification, we will denote by (a_q) , (b_q) and (c_q) generic positive sequences (depending possibly on t) such that $\sum_{q \in \mathbb{Z}} \sqrt{a_q} \leq 1$, $\sum_{q \in \mathbb{Z}} b_q \leq 1$

and $\sum_{q \in \mathbb{Z}} c_q^2 \leq 1$.

Notice that u belongs to $H^{0,s}(\mathbb{R}^3)$ if and only if

$$\|\Delta_q^v u\|_{L^2} \leq C2^{-qs} c_q \|u\|_{H^{0,s}}, \quad (7)$$

and that u belongs to $\mathcal{B}^{0,\frac{1}{2}}(\mathbb{R}^3)$ if and only if

$$\|\Delta_q^v u\|_{L^2} \leq C2^{-q/2} b_q \|u\|_{\mathcal{B}^{0,\frac{1}{2}}}. \quad (8)$$

In the sequel, it will be useful to introduce mixed Besov–Sobolev type spaces that take into account Lebesgue regularity in time on the dyadic blocs. Those spaces will be denoted, for $p \geq 1$, by $\widetilde{L}_T^p(\mathcal{B}^{0,\frac{1}{2}})$ and defined as in [6] by

$$\|u\|_{\widetilde{L}_T^p(\mathcal{B}^{0,\frac{1}{2}})} = \sum_{q \in \mathbb{Z}} 2^{q/2} \|\Delta_q^v u\|_{L^p([0,T];L^2(\mathbb{R}^3))} < +\infty.$$

Remark that we have

$$\|u\|_{L_T^p(\mathcal{B}^{0,\frac{1}{2}})} \leq \|u\|_{\widetilde{L}_T^p(\mathcal{B}^{0,\frac{1}{2}})},$$

where

$$\|u\|_{L_T^p(\mathcal{B}^{0,\frac{1}{2}})}^p = \int_0^t \|u(t)\|_{\mathcal{B}^{0,\frac{1}{2}}}^p dt.$$

3. UNIQUENESS

Following the ideas in [10], we establish Lemma 3 to prove uniqueness in $H^{0,\frac{1}{2}}$.

Lemma 3. *Let $U = (u, b)$ and $V = (v, c)$ be two divergence free vector fields, which belong to $L_T^\infty(H^{0,\frac{1}{2}})$, such that $\nabla_h U$ and $\nabla_h V$ in $L_T^2(H^{0,\frac{1}{2}})$.*

Let $W = (w, \beta) \in L_T^\infty(H^{0,\frac{1}{2}})$ with $\nabla_h W \in L_T^2(H^{0,\frac{1}{2}})$ be a solution of

$$\begin{cases} \partial_t W + \nu_h \Delta_h W + Q(W, W + 2U) + \frac{1}{\varepsilon} L(W) = (-\nabla p, 0) \\ \operatorname{div} w = \operatorname{div} \beta = 0 \\ W|_{t=0} = (0, 0). \end{cases}$$

For all $0 < t < T$, if $\|W\|_{H^{0,-\frac{1}{2}}} \leq \frac{1}{e}$, then

$$\frac{d}{dt} \|W\|_{H^{0,-\frac{1}{2}}}^2 \leq C f(t) \|W\|_{H^{0,-\frac{1}{2}}}^2 (1 - \ln \|W\|_{H^{0,-\frac{1}{2}}}) \ln (1 - \ln \|W\|_{H^{0,-\frac{1}{2}}}),$$

where f is a time-locally integrable function defined by

$$f(t) = (1 + 2\|U\|_{H^{0,\frac{1}{2}}}^2 + 2\|V\|_{H^{0,\frac{1}{2}}}^2)(1 + 2\|\nabla_h U\|_{H^{0,\frac{1}{2}}}^2 + 2\|\nabla_h V\|_{H^{0,\frac{1}{2}}}^2).$$

Proof. We begin by noting that $Q(W, W + 2U)$ is explicitly given by

$$Q(W, W + 2U) = \begin{pmatrix} u \cdot \nabla w + w \cdot \nabla v - b \cdot \nabla \beta - \beta \cdot \nabla c \\ u \cdot \nabla \beta + w \cdot \nabla c - \beta \cdot \nabla u - c \cdot \nabla w \end{pmatrix}.$$

If we take the scalar product in $H^{0, -\frac{1}{2}}$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|W\|_{H^{0, -\frac{1}{2}}}^2 + \nu_h \|\nabla_h W\|_{H^{0, -\frac{1}{2}}}^2 \leq \sum_{q \geq -1} 2^{-q} (\Delta_q^v Q(W, W + 2U) | \Delta_q^v W)_{L^2},$$

where

$$\begin{aligned} (\Delta_q^v Q(W, W + 2U) | \Delta_q^v W)_{L^2} &= \\ &= (\Delta_q^v (u \cdot \nabla w) | \Delta_q^v w)_{L^2} + (\Delta_q^v (w \cdot \nabla v) | \Delta_q^v w)_{L^2} - \\ &\quad - (\Delta_q^v (b \cdot \nabla \beta) | \Delta_q^v w)_{L^2} - (\Delta_q^v (\beta \cdot \nabla c) | \Delta_q^v w)_{L^2} + \\ &\quad + (\Delta_q^v (u \cdot \nabla \beta) | \Delta_q^v \beta)_{L^2} + (\Delta_q^v (w \cdot \nabla c) | \Delta_q^v \beta)_{L^2} - \\ &\quad - (\Delta_q^v (\beta \cdot \nabla u) | \Delta_q^v \beta)_{L^2} - (\Delta_q^v (c \cdot \nabla w) | \Delta_q^v \beta)_{L^2}. \end{aligned}$$

We mention that the above nonlinearities are of two types: those where two variables are the same, for example $(\Delta_q^v (u \cdot \nabla w) | \Delta_q^v w)_{L^2}$, and those where the three variables are different like $(\Delta_q^v (\beta \cdot \nabla c) | \Delta_q^v w)_{L^2}$. The former are estimated in [10], for the latter it suffices to note, after applying Cauchy–Schwarz or Hölder inequality, that $\|w\|, \|\beta\| \leq \|W\|$ and $\|\nabla_h w\|, \|\nabla_h \beta\| \leq \|\nabla_h W\|$. The same holds for U and V compared to their components (u, v) and (b, c) . \square

We return to the proof of the uniqueness result and suppose that U^ε and V^ε are two solutions of $(\mathcal{S}^\varepsilon)$ with the same initial data, such that U^ε and V^ε belong to $L_{loc}^\infty(H^{0, \frac{1}{2}})$ with $\nabla_h U^\varepsilon$ and $\nabla_h V^\varepsilon$ belonging to $L_{loc}^2(H^{0, \frac{1}{2}})$. We will prove that $W^\varepsilon = U^\varepsilon - V^\varepsilon$ is such that $W^\varepsilon = 0$ in $L_T^\infty(H^{0, -\frac{1}{2}})$ with $\nabla_h W^\varepsilon = 0$ in $L_T^2(H^{0, \frac{1}{2}})$.

W^ε satisfies the following equation

$$\partial_t W^\varepsilon + \nu_h \Delta_h W^\varepsilon + Q^\varepsilon(W^\varepsilon, W^\varepsilon + 2U^\varepsilon) + \frac{1}{\varepsilon} L^\varepsilon(W^\varepsilon) = (-\nabla p^\varepsilon, 0).$$

Lemma 3 implies that for all $0 < t < T$ if $\|W^\varepsilon\|_{H^{0, -\frac{1}{2}}} \leq \frac{1}{e}$, then

$$\begin{aligned} \frac{d}{dt} \|W^\varepsilon\|_{H^{0, -\frac{1}{2}}}^2 &\leq \\ &\leq C f(t) \|W^\varepsilon\|_{H^{0, -\frac{1}{2}}}^2 (1 - \ln \|W^\varepsilon\|_{H^{0, -\frac{1}{2}}}) \ln (1 - \ln \|W^\varepsilon\|_{H^{0, -\frac{1}{2}}}), \end{aligned}$$

where f is a locally time-integrable function defined by

$$f(t) = (1 + 2\|U^\varepsilon\|_{H^{0, \frac{1}{2}}}^2 + 2\|V^\varepsilon\|_{H^{0, \frac{1}{2}}}^2) (1 + 2\|\nabla_h U^\varepsilon\|_{H^{0, \frac{1}{2}}}^2 + 2\|\nabla_h V^\varepsilon\|_{H^{0, \frac{1}{2}}}^2).$$

By the Osgood lemma, one infers the uniqueness in $H^{0, \frac{1}{2}}$.

To investigate uniqueness in $\mathcal{B}^{0, \frac{1}{2}}$, note that W^ε will be estimated in the norm

$$\|S_0^v W^\varepsilon(t)\|_{L_v^\infty L_h^2}^2 + \sum_{q \geq 0} 2^{-q} \|\Delta_q^v W^\varepsilon(t)\|_{L^2}^2 +$$

$$\begin{aligned}
& + \nu_h \sup_{x_3} \int_0^t \|S_0^v \nabla_h W^\varepsilon(\tau, \cdot, x_3)\|_{L_h^2}^2 d\tau + \\
& + \nu_h \sum_{q \geq 0} 2^{-q} \int_0^t \|\nabla_h \Delta_q^v W^\varepsilon(\tau)\|_{L^2}^2 d\tau.
\end{aligned}$$

The exact estimations are easy to obtain, since we use functions localized in low vertical frequencies.

4. PROOF OF EXISTENCE RESULTS

4.1. Proof of the existence result in $\mathcal{B}^{0, \frac{1}{2}}$.

4.1.1. *Global existence for small initial data.* For a strictly positive integer n , we define Friedrichs's operators by

$$\begin{aligned}
J_n(u) &= \mathcal{F}^{-1}(1_{B(0,n)} \mathcal{F}u(\xi)), \\
J_n^v(u) &= \mathcal{F}^{-1}(1_{\{\xi, |\xi_3| \leq n\}} \mathcal{F}u(\xi))
\end{aligned}$$

and

$$\tilde{J}_n(u) = (J_n - J_{1/n}^v)(u).$$

Let us consider the following approximate magnetohydrodynamic system denoted $(MHD_{\nu_h}^n)$

$$\left\{ \begin{array}{l}
\partial_t u - \nu_h \Delta_h \tilde{J}_n u + \tilde{J}_n(\tilde{J}_n u \cdot \nabla \tilde{J}_n u) - \tilde{J}_n(\tilde{J}_n b \cdot \nabla \tilde{J}_n b) + \frac{1}{\varepsilon}(\tilde{J}_n u \times e_3) + \\
\quad + \frac{1}{\varepsilon} \partial_3 \tilde{J}_n b = \nabla \sum_{i,j} \Delta^{-1} \partial_i \partial_j \tilde{J}_n(\tilde{J}_n u_i \tilde{J}_n u_j + \tilde{J}_n b_i \tilde{J}_n b_j) + \\
\quad + \frac{1}{\varepsilon} \nabla \sum_i \Delta^{-1} \partial_i (\partial_3 \tilde{J}_n b - \tilde{J}_n u \times e_3)_i, \\
\partial_t b - \nu_h \Delta_h \tilde{J}_n b + \tilde{J}_n(\tilde{J}_n u \cdot \nabla \tilde{J}_n b) - \tilde{J}_n(\tilde{J}_n b \cdot \nabla \tilde{J}_n u) + \frac{1}{\varepsilon} \partial_3 \tilde{J}_n u = 0, \\
\operatorname{div} u = 0, \\
\operatorname{div} b = 0, \\
U|_{t=0} = \tilde{J}_n U^0.
\end{array} \right.$$

The above system is an ODE that can be rewritten in the following abstract form:

$$\partial_t U = F_n(U),$$

where $U = (u, b)$ and the expression of F_n is given by the system $(MHD_{\nu_h}^n)$. Note that since $U^0 \in \mathcal{B}^{0, \frac{1}{2}}$, $U(0) = \tilde{J}_n U^0$ belongs to L^2 . Moreover, F_n is a continuous function from L^2 into L^2 and the Cauchy–Lipschitz theorem implies that $(MHD_{\nu_h}^n)$ has a unique local solution U_n^ε in $\mathcal{C}^1([0, T_n(\varepsilon)], L^2)$. On the other hand, the fact that \tilde{J}_n is a projector implies that $\tilde{J}_n U_n^\varepsilon$ is also a solution of $(MHD_{\nu_h}^n)$. By uniqueness, it follows that

$$\tilde{J}_n U_n^\varepsilon = U_n^\varepsilon$$

and U_n^ε is a solution of the following system also denoted $(MHD_{\nu_h}^n)$:

$$(MHD_{\nu_h}^n) \begin{cases} \partial_t u_n^\varepsilon - \nu_h \Delta_h u_n^\varepsilon + \tilde{J}_n(u_n^\varepsilon \cdot \nabla u_n^\varepsilon) - \tilde{J}_n(b_n^\varepsilon \cdot \nabla b_n^\varepsilon) + \frac{1}{\varepsilon} \partial_3 b_n^\varepsilon + \\ \quad + \frac{1}{\varepsilon} (u_n^\varepsilon \times e_3) = \nabla \sum_{i,j} \tilde{J}_n \Delta^{-1} \partial_i \partial_j (u_{i,n}^\varepsilon u_{j,n}^\varepsilon + b_{i,n}^\varepsilon b_{j,n}^\varepsilon) + \\ \quad + \frac{1}{\varepsilon} \nabla \sum_i \Delta^{-1} \partial_i (\partial_3 b_n^\varepsilon - u_n^\varepsilon \times e_3)_i, \\ \partial_t b_n^\varepsilon - \nu_h \Delta_h b_n^\varepsilon + \tilde{J}_n(u_n^\varepsilon \cdot \nabla b_n^\varepsilon) - \tilde{J}_n(b_n^\varepsilon \cdot \nabla u_n^\varepsilon) + \frac{1}{\varepsilon} \partial_3 u_n^\varepsilon = 0, \\ \operatorname{div} u_n^\varepsilon = 0, \\ \operatorname{div} b_n^\varepsilon = 0, \\ U_n^\varepsilon|_{t=0} = \tilde{J}_n U^0. \end{cases}$$

The L^2 energy estimate implies that

$$\frac{1}{2} \frac{d}{dt} \|U_n^\varepsilon(t)\|_{L^2}^2 + \nu_h \|\nabla_h U_n^\varepsilon(t)\|_{L^2}^2 = 0.$$

So, one deduces the global existence in L^2 .

To prove the existence result in $\mathcal{B}^{0, \frac{1}{2}}$, we introduce the following lemma due to [10].

Lemma 4. *Let u and v be two vector fields defined on \mathbb{R}^3 such that $u(t)$ is divergence-free for all $t \in [0, T]$. There exists a real sequence (a_q) satisfying $a_q = a_q(u, v, T) > 0$ and $\sum_{q \in \mathbb{Z}} \sqrt{a_q} < 1$ such that*

$$\begin{aligned} & \int_0^T |(\Delta_q^v(u \cdot \nabla v)|\Delta_q^v v)_{L^2}| dt \leq \\ & \leq C a_q 2^{-q} \left(\|\nabla_h u\|_{\widetilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})} \|v\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})} \|\nabla_h v\|_{\widetilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})} + \right. \\ & \quad \left. + \|u\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h u\|_{\widetilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|v\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{1}{2}} \|\nabla_h v\|_{\widetilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^{\frac{3}{2}} \right). \end{aligned}$$

We apply the operator Δ_q^v and use the L^2 energy estimate to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta_q^v U_n^\varepsilon(t)\|_{L^2}^2 + \nu_h \|\nabla_h \Delta_q^v U_n^\varepsilon(t)\|_{L^2}^2 \leq \\ & \leq (\Delta_q^v(u_n^\varepsilon \cdot \nabla u_n^\varepsilon)|\Delta_q^v u_n^\varepsilon)_{L^2} + |(\Delta_q^v(u_n^\varepsilon \cdot \nabla b_n^\varepsilon)|\Delta_q^v b_n^\varepsilon)_{L^2}| + \\ & \quad + |(\Delta_q^v(b_n^\varepsilon \cdot \nabla b_n^\varepsilon)|\Delta_q^v u_n^\varepsilon)_{L^2} + (\Delta_q^v(b_n^\varepsilon \cdot \nabla u_n^\varepsilon)|\Delta_q^v b_n^\varepsilon)_{L^2}|. \quad (9) \end{aligned}$$

We note the following rearrangement:

$$\begin{aligned} & (\Delta_q^v(b_n^\varepsilon \cdot \nabla b_n^\varepsilon)|\Delta_q^v u_n^\varepsilon)_{L^2} + (\Delta_q^v(b_n^\varepsilon \cdot \nabla u_n^\varepsilon)|\Delta_q^v b_n^\varepsilon)_{L^2} = \\ & = \left(\Delta_q^v(b_n^\varepsilon \cdot \nabla(u_n^\varepsilon + b_n^\varepsilon))|\Delta_q^v(u_n^\varepsilon + b_n^\varepsilon) \right)_{L^2} - \\ & - (\Delta_q^v(b_n^\varepsilon \cdot \nabla u_n^\varepsilon)|\Delta_q^v u_n^\varepsilon)_{L^2} - (\Delta_q^v(b_n^\varepsilon \cdot \nabla b_n^\varepsilon)|\Delta_q^v b_n^\varepsilon)_{L^2}. \quad (10) \end{aligned}$$

Then lemma 4 leads to

$$\begin{aligned} \|\Delta_q^v U_n^\varepsilon\|_{L_T^\infty(L^2)}^2 + 2\nu_h \|\nabla_h \Delta_q^v U_n^\varepsilon\|_{L_T^2(L^2)}^2 &\leq \\ &\leq \|\Delta_q^v U_n^0\|_{L^2}^2 + 2^{-q} a_q C \nu_h \|U_n^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})} \|\nabla_h U_n^\varepsilon\|_{\widetilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}. \end{aligned}$$

Since $a^2 + b^2$ is equivalent to $(a + b)^2$, we deduce that

$$\begin{aligned} 2^{q/2} \|\Delta_q^v U_n^\varepsilon\|_{L_T^\infty(L^2)} + \sqrt{2\nu_h} 2^{q/2} \|\nabla_h \Delta_q^v U_n^\varepsilon\|_{L_T^2(L^2)} &\leq \\ &\leq 2^{q/2} \sqrt{2} \|\Delta_q^v U_n^0\|_{L^2} + a_q^{1/2} \sqrt{2} C^{1/2} \|U_n^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^{1/2} \|\nabla_h U_n^\varepsilon\|_{\widetilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}. \end{aligned}$$

We take the sum over q . Then, we reapply the same equivalence property to infer that

$$\begin{aligned} \|U_n^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^2 + 2\nu_h \|\nabla_h U_n^\varepsilon\|_{\widetilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^2 &\leq \\ &\leq 4 \|U_n^0\|_{\mathcal{B}^{0, \frac{1}{2}}}^2 + C \|U_n^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})} \|\nabla_h U_n^\varepsilon\|_{\widetilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}. \quad (11) \end{aligned}$$

Let $U_0 \in \mathcal{B}^{0, \frac{1}{2}}$ be such that $\|U_0\|_{\mathcal{B}^{0, \frac{1}{2}}} < c\nu_h$, where $c < \frac{1}{C}$. Let U_n^ε be the regular solution of $(MHD_{\nu_h}^n)$. If we set

$$T_n^* = \sup \left\{ T > 0, \|U_n^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})} < 2c\nu_h \right\},$$

then the estimate (11) implies that for all T such that $0 < T < T_n^*$,

$$\|U_n^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})} \leq 2 \|U_0\|_{\mathcal{B}^{0, \frac{1}{2}}} < 2c\nu_h.$$

Since the function $T \rightarrow \|U_n^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}$ is continuous, we obtain that $T_n^* = +\infty$ and the sequence of global solutions $(U_n^\varepsilon)_{n \in \mathbb{N}}$ is such that U_n^ε belongs to $\widetilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^{0, \frac{1}{2}})$ and $\nabla_h U_n^\varepsilon$ belongs to $\widetilde{L}^2(\mathbb{R}^+, \mathcal{B}^{0, \frac{1}{2}})$. In particular, the facts that $S_0^v U_n^\varepsilon$ belongs to $L_v^\infty L_h^2$ and $(I - S_0^v) U_n^\varepsilon$ belongs to L^2 allow to deduce that $(U_n^\varepsilon)_{n \in \mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^+, L_{loc}^2)$. By the system $(MHD_{\nu_h}^n)$, one has that $(\partial_t U_n^\varepsilon)_{n \in \mathbb{N}}$ is bounded in $L_{loc}^\infty(\mathbb{R}^+, H_{loc}^{-N})$, where N is a sufficiently large integer. By Arzela–Ascoli theorem, there exists a subsequence denoted also by $(U_n^\varepsilon)_n$ such that

$$U_n^\varepsilon \rightarrow U^\varepsilon \text{ in } L_{loc}^\infty(\mathbb{R}^+, H_{loc}^{-N}).$$

Since $(U_n^\varepsilon)_n$ is bounded in $L^\infty(\mathbb{R}^+, L_{loc}^2)$, an interpolation inequality implies that

$$U_n^\varepsilon \rightarrow U^\varepsilon \text{ in } L_{loc}^\infty(\mathbb{R}^+, H_{loc}^{-\sigma}) \quad \forall \sigma > 0.$$

Since $\forall \rho < \frac{1}{2}$, $(U_n^\varepsilon)_n$ is bounded in $L_{loc}^2(\mathbb{R}^+, H_{loc}^\rho)$, classical product laws in Sobolev spaces imply that for $\sigma < \rho$

$$Q(U_n^\varepsilon, U_n^\varepsilon) \rightarrow Q(U^\varepsilon, U^\varepsilon) \text{ in } L_{loc}^2(\mathbb{R}^+, H_{loc}^{\rho - \sigma - 3/2}).$$

In particular,

$$Q(U_n^\varepsilon, U_n^\varepsilon) \rightarrow Q(U^\varepsilon, U^\varepsilon) \text{ in } \mathcal{D}'.$$

Taking the limit in $(MHD_{\nu_h}^n)$, we obtain the global solution for small initial data.

4.1.2. *Continuity in time of the solution.* The equation verified by $\Delta_q^v U$ is

$$\partial_t \Delta_q^v U^\varepsilon = \nu_h \Delta_h \Delta_q^v U^\varepsilon - \Delta_q^v Q_h(U^\varepsilon, U^\varepsilon) - \Delta_q^v Q_3(U^\varepsilon, U^\varepsilon) - (\Delta_q^v \nabla p^\varepsilon, 0).$$

The divergence-free condition implies that

$$Q_h(U^\varepsilon, U^\varepsilon) = \begin{pmatrix} \operatorname{div}_h(u^\varepsilon \otimes u^\varepsilon) - \operatorname{div}_h(b^\varepsilon \otimes b^\varepsilon) \\ \operatorname{div}_h(u^\varepsilon \otimes b^\varepsilon) - \operatorname{div}_h(b^\varepsilon \otimes u^\varepsilon) \end{pmatrix}$$

and

$$Q_3(U^\varepsilon, U^\varepsilon) = \begin{pmatrix} \partial_3(u^\varepsilon \otimes u^\varepsilon) - \partial_3(b^\varepsilon \otimes b^\varepsilon) \\ \partial_3(u^\varepsilon \otimes b^\varepsilon) - \partial_3(b^\varepsilon \otimes u^\varepsilon) \end{pmatrix}.$$

Note that $\nu_h \Delta_h \Delta_q^v U^\varepsilon - \Delta_q^v Q_h(U^\varepsilon, U^\varepsilon)$ belongs to $L^2([0, T], L_v^2(\dot{H}_h^{-1}))$ and $\Delta_q^v U^\varepsilon$ belongs to $L^2([0, T], L_v^2(\dot{H}_h^1))$ to deduce that

$$(\nu_h \Delta_h \Delta_q^v U^\varepsilon - \Delta_q^v Q_h(U^\varepsilon, U^\varepsilon) | \Delta_q^v U^\varepsilon)_{L^2(\mathbb{R}^3)} \in L^1([0, T]). \quad (12)$$

Moreover, by the fact that U^ε belongs to $L^2([0, T], L_h^4 L_v^2)$ we have that $\Delta_q^v Q_3(U^\varepsilon, U^\varepsilon)$ belongs to $L^1([0, T], L_h^2 L_v^1)$. Furthermore, the fact that U^ε belongs to $L^\infty([0, T], L_h^2(L_v^\infty))$ leads to

$$(\Delta_q^v Q_3(U^\varepsilon, U^\varepsilon) | \Delta_q^v U^\varepsilon)_{L^2(\mathbb{R}^3)} \in L^1([0, T]). \quad (13)$$

By (12) and (13), one obtains that $\partial_t \|\Delta_q^v U^\varepsilon\|_{L^2}^2$ belongs to L_T^1 and, in particular for fixed q , $\|\Delta_q^v U^\varepsilon\|_{L^2}^2 \in \mathcal{C}([0, T])$. On the other hand, note that $t \rightarrow \Delta_q^v U^\varepsilon(t)$ is weakly continuous. Finally, we obtain that $t \rightarrow \Delta_q^v U^\varepsilon(t)$ is strongly continuous on $[0, T]$ with values in L^2 .

Since U^ε belongs to $\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})$, for $\zeta > 0$ there exists N such that

$$\sum_{|q| \geq N} 2^{q/2} \|\Delta_q^v U^\varepsilon\|_{L_T^\infty(L^2)} \leq \zeta.$$

Since Δ_q^v belongs to $\mathcal{C}_T(L^2)$, there exists $\delta > 0$ such that for $|t - t'| \leq \delta$ one has

$$\sum_{|q| \leq N} 2^{q/2} \|\Delta_q^v(U^\varepsilon(t) - U^\varepsilon(t'))\|_{L^2} \leq \zeta.$$

Thus, for $|t - t'| \leq \delta$ one obtains that

$$\|\Delta_q^v(U^\varepsilon(t) - U^\varepsilon(t'))\|_{\mathcal{B}^{0, \frac{1}{2}}} \leq \zeta.$$

Proposition 1. *Let $U^\varepsilon \in \widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})$ be a solution of $(MHD_{\nu_h}^\varepsilon)$. Then*

1. $U^\varepsilon \in \mathcal{C}_b([0, T], \mathcal{B}^{0, \frac{1}{2}})$.
2. The set $\{U^\varepsilon(t), t \in [0, T]\}$ is relatively compact in $\mathcal{B}^{0, \frac{1}{2}}$.

4.1.3. *Local existence result for arbitrary initial data.* To prove the local existence result for arbitrary initial data, we decompose the initial data into low and high frequency. Then we look for a local solution in the form

$$U = U_{1,N} + U_{2,N},$$

where $U_{1,N}$ is the solution of the linear problem below:

$$\begin{cases} \partial_t V - \nu_h \Delta_h V + \frac{1}{\varepsilon} L(V) = (-\nabla p, 0), \\ \operatorname{div} V = 0, \\ V|_{t=0} = S_N U_0. \end{cases}$$

It follows that for all t

$$\|U_{1,N}(t)\|_{L^2}^2 + 2\nu_h \int_0^t \|\nabla_h U_{1,N}(\tau)\|_{L^2}^2 d\tau = \|S_N U_0\|_{L^2}^2. \quad (14)$$

In this way, U^ε will be a solution of $(MHD_{\nu_h}^\varepsilon)$ if $U_{2,N}^\varepsilon$ is a solution of the system below:

$$\begin{cases} \partial_t W - \nu_h \Delta_h W + Q(W, W) + \frac{1}{\varepsilon} L(W) + Q(U_{1,N}^\varepsilon, W) + Q(W, U_{1,N}^\varepsilon) = \\ \quad = (-\nabla p, 0) - Q(U_{1,N}^\varepsilon, U_{1,N}^\varepsilon), \\ \operatorname{div} W = 0, \\ W|_{t=0} = (I - S_N)U_0. \end{cases}$$

We use the Freidrichs scheme. Thus, the L^2 energy estimate leads to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q^v U_{2,N}^\varepsilon(t)\|_{L^2}^2 + \nu_h \|\nabla_h \Delta_q^v U_{2,N}^\varepsilon(t)\|_{L^2}^2 \leq \\ \left| (\Delta_q^v Q(U_{2,N}^\varepsilon, U_{2,N}^\varepsilon) | \Delta_q^v U_{2,N}^\varepsilon)_{L^2} \right| + \left| (\Delta_q^v Q(U_{1,N}^\varepsilon, U_{2,N}^\varepsilon) | \Delta_q^v U_{2,N}^\varepsilon)_{L^2} \right| + \\ \left| (\Delta_q^v Q(U_{2,N}^\varepsilon, U_{1,N}^\varepsilon) | \Delta_q^v U_{2,N}^\varepsilon)_{L^2} \right| + \\ \left| (\Delta_q^v Q(U_{1,N}^\varepsilon, U_{1,N}^\varepsilon) | \Delta_q^v U_{2,N}^\varepsilon)_{L^2} \right|. \quad (15) \end{aligned}$$

By Lemma 4, we can estimate directly the sums

$$\int_0^t \left| (\Delta_q^v Q(U_{2,N}^\varepsilon, U_{2,N}^\varepsilon) | \Delta_q^v U_{2,N}^\varepsilon)_{L^2} \right| d\tau$$

and

$$\int_0^t \left| (\Delta_q^v Q(U_{1,N}^\varepsilon, U_{2,N}^\varepsilon) | \Delta_q^v U_{2,N}^\varepsilon)_{L^2} \right| d\tau.$$

About the other terms, we note that

$$\int_0^t \left| (\Delta_q^v Q(U_{2,N}^\varepsilon, U_{1,N}^\varepsilon) | \Delta_q^v U_{2,N}^\varepsilon)_{L^2} \right| \leq$$

$$\begin{aligned}
&\leq \|\Delta_q^v(u_{2,N}^\varepsilon \cdot \nabla u_{1,N}^\varepsilon)\|_{L_T^1(L^2)} \|\Delta_q^v u_{2,N}^\varepsilon\|_{L_T^\infty(L^2)} + \\
&+ \|\Delta_q^v(b_{2,N}^\varepsilon \cdot \nabla b_{1,N}^\varepsilon)\|_{L_T^1(L^2)} \|\Delta_q^v u_{2,N}^\varepsilon\|_{L_T^\infty(L^2)} + \\
&+ \|\Delta_q^v(u_{2,N}^\varepsilon \cdot \nabla b_{1,N}^\varepsilon)\|_{L_T^1(L^2)} \|\Delta_q^v b_{2,N}^\varepsilon\|_{L_T^\infty(L^2)} + \\
&+ \|\Delta_q^v(b_{2,N}^\varepsilon \cdot \nabla u_{1,N}^\varepsilon)\|_{L_T^1(L^2)} \|\Delta_q^v b_{2,N}^\varepsilon\|_{L_T^\infty(L^2)}.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
&\int_0^t |(\Delta_q^v Q(U_{2,N}^\varepsilon, U_{1,N}^\varepsilon) | \Delta_q^v U_{2,N}^\varepsilon)_{L^2}| \leq \\
&\leq 2^{-q/2} b_q CT \|\Delta_q^v(u_{2,N}^\varepsilon \cdot \nabla u_{1,N}^\varepsilon)\|_{L_T^\infty(L^2)} \|u_{2,N}^\varepsilon\|_{\widetilde{L_T^\infty}(\mathcal{B}^{0, \frac{1}{2}})} + \\
&+ 2^{-q/2} b_q CT \|\Delta_q^v(b_{2,N}^\varepsilon \cdot \nabla b_{1,N}^\varepsilon)\|_{L_T^\infty(L^2)} \|u_{2,N}^\varepsilon\|_{\widetilde{L_T^\infty}(\mathcal{B}^{0, \frac{1}{2}})} + \\
&+ 2^{-q/2} b_q CT \|\Delta_q^v(u_{2,N}^\varepsilon \cdot \nabla b_{1,N}^\varepsilon)\|_{L_T^\infty(L^2)} \|b_{2,N}^\varepsilon\|_{\widetilde{L_T^\infty}(\mathcal{B}^{0, \frac{1}{2}})} + \\
&+ 2^{-q/2} b_q CT \|\Delta_q^v(b_{2,N}^\varepsilon \cdot \nabla u_{1,N}^\varepsilon)\|_{L_T^\infty(L^2)} \|b_{2,N}^\varepsilon\|_{\widetilde{L_T^\infty}(\mathcal{B}^{0, \frac{1}{2}})}.
\end{aligned}$$

Using the vertical Bony decomposition, one can write that

$$\begin{aligned}
\Delta_q^v(u_{2,N}^\varepsilon \cdot \nabla u_{1,N}^\varepsilon) &= \Delta_q^v \left(\sum_{|q'-q| \leq 4} S_{q'-1}^v u_{2,N}^\varepsilon \cdot \Delta_{q'}^v \nabla u_{1,N}^\varepsilon \right) + \\
&+ \Delta_q^v \left(\sum_{q' \geq q-N_0} S_{q'+1}^v \nabla u_{1,N}^\varepsilon \cdot \Delta_{q'}^v u_{2,N}^\varepsilon \right).
\end{aligned}$$

This leads to

$$\begin{aligned}
&\|\Delta_q^v(u_{2,N}^\varepsilon \cdot \nabla u_{1,N}^\varepsilon)\|_{L_T^\infty(L^2)} \leq \\
&\leq \sum_{|q'-q| \leq 4} \|S_{q'-1}^v u_{2,N}^\varepsilon\|_{L_T^\infty(L^2)} \|\Delta_{q'}^v \nabla u_{1,N}^\varepsilon\|_{L_T^\infty(L^\infty)} + \\
&+ \sum_{q' \geq q-N_0} \|S_{q'+1}^v \nabla u_{1,N}^\varepsilon\|_{L_T^\infty(L^\infty)} \|\Delta_{q'}^v u_{2,N}^\varepsilon\|_{L_T^\infty(L^2)} \leq \\
&\leq 2^{5N/2} \|u_{2,N}^\varepsilon\|_{L_T^\infty(\mathcal{B}^{0, \frac{1}{2}})} \sum_{|q'-q| \leq 4} \|\Delta_{q'}^v u_{1,N}^\varepsilon\|_{L_T^\infty(L^2)} + \\
&+ 2^{5N/2} \|u_{1,N}^\varepsilon\|_{L_T^\infty(L^2)} 2^{-q/2} \widetilde{b}_q \|u_{2,N}^\varepsilon\|_{L_T^\infty(\mathcal{B}^{0, \frac{1}{2}})},
\end{aligned}$$

where $\widetilde{b}_q = \sum_{q' \geq q-N_0} 2^{(q-q')/2} b_{q'}$. \widetilde{b}_q belongs to l^1 since $b_{q'}$ does. Then, applying the inequality $ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{\frac{4}{3}}$, we deduce that

$$\|u_{1,N}^\varepsilon\|_{L_T^\infty(\mathcal{B}^{0, \frac{1}{2}})} \leq \|u_0\|_{\mathcal{B}^{0, \frac{1}{2}}}.$$

The same holds for $\Delta_q^v(b_{2,N}^\varepsilon \cdot \nabla b_{1,N}^\varepsilon)$, $\Delta_q^v(u_{2,N}^\varepsilon \cdot \nabla b_{1,N}^\varepsilon)$ and $\Delta_q^v(b_{2,N}^\varepsilon \cdot \nabla u_{1,N}^\varepsilon)$. So, it follows that

$$\|\Delta_q^v Q(U_{2,N}^\varepsilon, U_{1,N}^\varepsilon)\|_{L_T^\infty(L^2)} \leq 2^{-q/2} b_q C 2^{5N/2} \|U_{2,N}^\varepsilon\|_{L_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}.$$

Thus,

$$\int_0^t |(\Delta_q^v Q(U_{2,N}^\varepsilon, U_{1,N}^\varepsilon) | \Delta_q^v U_{2,N}^\varepsilon)_{L^2}| \leq 2^{-q} a_q C 2^{5N/2} T \|U_{2,N}^\varepsilon\|_{L_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^2. \quad (16)$$

To estimate $\int_0^t |(\Delta_q^v Q(U_{1,N}^\varepsilon, U_{1,N}^\varepsilon) | \Delta_q^v U_{2,N}^\varepsilon)_{L^2}| dt$, we proceed as follows:

$$\begin{aligned} \int_0^t |(\Delta_q^v Q(U_{1,N}^\varepsilon, U_{1,N}^\varepsilon) | \Delta_q^v U_{2,N}^\varepsilon)_{L^2}| &\leq \\ &\leq C 2^N T \|\Delta_q^v(Q(U_{1,N}^\varepsilon, U_{1,N}^\varepsilon))\|_{L_T^\infty(L^2)} \|\Delta_q^v U_{2,N}^\varepsilon\|_{L_T^\infty(L^2)} \leq \\ &\leq C 2^N T 2^{-q/2} b_q \|U_{1,N}^\varepsilon\|_{L_T^\infty(L^\infty)} \|U_{1,N}^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})} \|\Delta_q^v U_{2,N}^\varepsilon\|_{L_T^\infty(L^2)} \leq \\ &\leq C 2^{-q/2} b_q 2^{5N/2} T \|\Delta_q^v U_{2,N}^\varepsilon\|_{L_T^\infty(L^2)}. \end{aligned}$$

This leads to

$$\int_0^t |(\Delta_q^v Q(U_{1,N}^\varepsilon, U_{1,N}^\varepsilon) | \Delta_q^v U_{2,N}^\varepsilon)_{L^2}| dt \leq C 2^{-q} a_q 2^{5N/2} T \|U_{2,N}^\varepsilon\|_{L_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^2. \quad (17)$$

Finally, we integrate (15) and take the sum over q to obtain

$$\begin{aligned} \|U_{2,N}^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^2 + \nu_h \|\nabla_h U_{2,N}^\varepsilon\|_{\widetilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^2 &\leq \\ &\leq 4 \|U_{2,N}^\varepsilon(0)\|_{\mathcal{B}^{0, \frac{1}{2}}}^2 + C \|\nabla_h U_{2,N}^\varepsilon\|_{\widetilde{L}_T^2(\mathcal{B}^{0, \frac{1}{2}})}^2 \|U_{2,N}^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \\ &\quad + CT(1 + 2^{2N} + 2^{5N/2}) \|U_0\|_{\mathcal{B}^{0, \frac{1}{2}}} \|U_{2,N}^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})}^2. \end{aligned} \quad (18)$$

Let N be a sufficiently big integer such that

$$\|U_{2,N}^\varepsilon(0)\|_{\mathcal{B}^{0, \frac{1}{2}}} \leq c \frac{\nu_h}{2},$$

where c is sufficiently small. We consider T such that

$$CT 2^{5N/2} \|U_0\|_{\mathcal{B}^{0, \frac{1}{2}}} \leq \frac{1}{2}$$

and

$$\|U_{2,N}^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})} \leq c \nu_h$$

to obtain by (18)

$$\|U_{2,N}^\varepsilon\|_{\widetilde{L}_T^\infty(\mathcal{B}^{0, \frac{1}{2}})} \leq \frac{c \nu_h}{2}.$$

Then there exists $T = T(N, U_0)$ such that $U_{2,N}^\varepsilon \in \widetilde{L_T^\infty}(\mathcal{B}^{0, \frac{1}{2}})$ for all $\varepsilon > 0$.

To prove (2), we proceed by contraposition. To do so, we begin by noting that Proposition 1 allows to deduce that $U^\varepsilon(t_n)$ has only one limit in $\mathcal{B}^{0, \frac{1}{2}}$ as t_n tends to T^* . One solves locally in time the (MHD_{ν_h}) , where the initial data is chosen to be $U^\varepsilon(T^*) = \lim_{t \rightarrow T^*} U^\varepsilon(t)$. By this way, the life span of the solution can be extended and will be defined on $[0, T^* + \delta)$ and so on. This contradicts the fact that T^* is finite.

Remark 1. The life span of the local solution depends only on the frequency repartition of the initial data and is not a function, as in classical cases, of its norm $\|U_0\|_{\mathcal{B}^{0, \frac{1}{2}}}$.

5. PROOF OF CONVERGENCE RESULTS

The “linearized” equation associated to the system $(\mathcal{S}^\varepsilon)$ is

$$(LS^\varepsilon) \begin{cases} \partial_t U^\varepsilon - \nu_h \Delta_h U^\varepsilon + \frac{1}{\varepsilon} L(U^\varepsilon) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x^3, \\ \operatorname{div} u^\varepsilon = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x^3, \\ \operatorname{div} b^\varepsilon = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x^3, \\ U^\varepsilon|_{t=0} = U_0(x) & \text{in } \mathbb{R}_x^3. \end{cases}$$

In Fourier variables $\xi \in \mathbb{R}^3$ we obtain

$$\partial_t \mathcal{F}(U^\varepsilon) + \nu_h |\xi_h|^2 \mathcal{F}(U^\varepsilon) + \frac{1}{\varepsilon} A(\xi) \mathcal{F}(U^\varepsilon) = 0.$$

Hence, we are led to study the following family of operators

$$\mathcal{G}^\varepsilon : f \longmapsto \int_{\mathbb{R}_y^3 \times \mathbb{R}_\xi^3} f(y) e^{-t(\nu_h |\xi_h|^2 + i \frac{a(\xi)}{\varepsilon}) + i(x-y) \cdot \xi} d\xi dy,$$

where the phase function a is such that

$$a(\xi) \in \left\{ \pm \frac{\xi_3}{|\xi|} (1 + \sqrt{1 + 4|\xi|^2}), \pm \frac{\xi_3}{|\xi|} (1 - \sqrt{1 + 4|\xi|^2}) \right\}.$$

So it is almost stationary when ξ_3 is almost equal to 0 as well as when $|\xi|$ goes to $+\infty$.

For some $0 < r < \min(R, R')$, we define the domain $\mathcal{C}_{r, R, R'}$ by

$$\mathcal{C}_{r, R, R'} = \left\{ \xi \in \mathbb{R}^3; R' \geq |\xi_3| \geq r \text{ and } |\xi_h| \leq R \right\}. \quad (19)$$

We consider a cut-off function ψ which is radial with respect to ξ_h and whose value is 1 near $\mathcal{C}_{r, R, R'}$. First, to study the case where $\mathcal{F}(f)$ is supported in $\mathcal{C}_{r, R, R'}$, we write

$$\mathcal{G}^\varepsilon f(t, x) = (K(t/\varepsilon, \nu t, \cdot) \star f)(x),$$

where the kernel K is defined by

$$K(t, \tau, z) = \int_{\mathbb{R}^3} \psi(\xi) e^{it a(\xi) + iz \cdot \xi - \tau |\xi_h|^2} d\xi.$$

As in [5], we recall the following property of K .

Lemma 5. *For all r , R and R' satisfying $0 < r < \min(R, R')$, there exists a constant $C(r, R, R')$, such that*

$$\|K(t, \tau, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C(r, R, R') \min \{1, t^{-\frac{1}{2}}\}.$$

Proof. The proof follows the lines of the stationary phase method. Using the rotation invariance in ξ_h , we restrict ourself to the case $z_2 = 0$. If we denote $\alpha(\xi) = -\partial_{\xi_2} a(\xi)$, then

$$|\alpha(\xi)| \geq C(r, R, R') |\xi_2|,$$

where C is a strictly positive constant depending only on r , R and R' . Next, for all ξ that belongs to $\mathcal{C}_{r, R, R'}$, we introduce the differential operator \mathcal{L} defined by

$$\mathcal{L} = \frac{1}{1 + t\alpha^2(\xi)} (1 + i\alpha(\xi)\partial_{\xi_2}).$$

This operator acts on the ξ_2 variables and satisfies $\mathcal{L}(e^{ita}) = e^{ita}$. Integrating by parts, we obtain

$$K(t, \tau, z) = \int_{\mathbb{R}^3} {}^t\mathcal{L}(\psi(\xi)e^{-\tau|\xi_h|^2}) e^{ita(\xi) + iz \cdot \xi} d\xi.$$

Direct computation yields

$$\begin{aligned} {}^t\mathcal{L}(\psi(\xi)e^{-\tau|\xi_h|^2}) &= \left(\frac{1}{1 + t\alpha^2} - i(\partial_{\xi_2}\alpha) \frac{1 - t\alpha^2}{(1 + t\alpha^2)^2} \right) \psi(\xi)e^{-\tau|\xi_h|^2} - \\ &\quad - \frac{i\alpha}{1 + t\alpha^2(\xi)} \partial_{\xi_2}(\psi(\xi)e^{-\tau|\xi_h|^2}). \end{aligned}$$

Finally, we use the fact that ξ is in a fixed annulus of \mathbb{R}^3 and $\psi \in \mathcal{D}(\mathbb{R}^3)$ to infer that

$$\|K(t, \tau, \cdot)\|_{L^\infty} \leq C(r, R, R') \int_{\mathbb{R}} \frac{d\xi_2}{1 + t\xi_2^2}. \quad \square$$

Let us denote by w^ε the solution of the free linear system (PLF^ε) associated with $(\mathcal{S}^\varepsilon)$

$$(PLF^\varepsilon) \begin{cases} \partial_t w^\varepsilon - \nu_h \Delta_h w^\varepsilon + \frac{1}{\varepsilon} L(w^\varepsilon) = f & \text{in } \mathbb{R}_t \times \mathbb{R}_x^3, \\ w^\varepsilon(0) = w_0 & \text{in } \mathbb{R}_x^3. \end{cases}$$

Lemma 5 yields, in a standard way, the following Strichartz estimate (see [8]).

Corollary 1. *For all constants r , R and R' such that $0 < r < \min(R, R')$, let $\mathcal{C}_{r, R, R'}$ be the domain defined by (19). A constant $C'(r, R, R')$ exists such that if*

$$\text{supp}(\mathcal{F}(w_0)) \cup \text{supp}(\mathcal{F}(f)) \subset \mathcal{C}_{r, R, R'},$$

then the solution w^ε of (PLF^ε) with the forcing term f and initial data w_0 satisfies

$$\|w^\varepsilon\|_{L^4(\mathbb{R}_+, L^\infty)} \leq C'(r, R, R')\varepsilon^{\frac{1}{4}} (\|w_0\|_{L^2} + \|f\|_{L^1(\mathbb{R}_+, L^2)}).$$

Notice that the constant $C'(r, R, R')$ does not depend on ε .

Proof of Theorem 3. The first equation of the system $(\mathcal{S}^\varepsilon)$ can be rewritten in the form

$$\partial_t U^\varepsilon - \nu_h \Delta_h U^\varepsilon + \frac{1}{\varepsilon} L(U^\varepsilon) = F^\varepsilon \text{ in } \mathbb{R}_t \times \mathbb{R}_x^3.$$

Applying, consecutively, the operators $\chi(\frac{|\nabla_h \cdot|}{R})$ and Δ_q^v , we obtain for all $R > 0$ and $q \in \mathbb{Z}$ that

$$\partial_t \Delta_q^v U_R^\varepsilon - \nu_h \Delta_h \Delta_q^v U_R^\varepsilon + \frac{1}{\varepsilon} L(\Delta_q^v U_R^\varepsilon) = \Delta_q^v F_R^\varepsilon.$$

By Corollary 1, we infer that

$$\|\Delta_q^v U_R^\varepsilon\|_{L_T^4(L^\infty)} \leq C'(2^{q+1}, R, 2^q) \varepsilon^{\frac{1}{4}} (\|\Delta_q^v U_{0,R}\|_{L^2} + \|\Delta_q^v F_R^\varepsilon\|_{L_T^1(L^2)}).$$

To estimate the right hand side of the above inequality, notice that

$$\|\Delta_q^v U_{0,R}\|_{L^2} \leq C(q, R). \quad (20)$$

If we denote by $\mu_3(\mathcal{C}_{2^q, R, 2^{q+1}})$ the Lebesgue measure of the set $\mathcal{C}_{2^q, R, 2^{q+1}}$, then the Plancherel formula and classical analysis imply that

$$\begin{aligned} \|\Delta_q^v F_R^\varepsilon\|_{L_T^1(L^2)} &= \int_0^T \left(\int_{\mathbb{R}^3} |\mathcal{F}(\Delta_q^v F_R^\varepsilon)(\tau, \xi)|^2 d\xi \right)^{\frac{1}{2}} d\tau \leq \\ &\leq (2^{q+1} + R) \int_0^T \left(\int_{\mathcal{C}_{2^q, R, 2^{q+1}}} |\mathcal{F}(U^\varepsilon \otimes U^\varepsilon)(\tau, \xi)|^2 d\xi \right)^{\frac{1}{2}} d\tau \leq \\ &\leq (2^{q+1} + R) T (\mu_3(\mathcal{C}_{2^q, R, 2^{q+1}}))^{\frac{1}{2}} \|\mathcal{F}(U^\varepsilon \otimes U^\varepsilon)\|_{L^\infty(L^\infty)} \leq \\ &\leq (2^{q+1} + R) T (\mu_3(\mathcal{C}_{2^q, R, 2^{q+1}}))^{\frac{1}{2}} \|U^\varepsilon \otimes U^\varepsilon\|_{L^\infty(L^1)} \leq \\ &\leq (2^{q+1} + R) T (\mu_3(\mathcal{C}_{2^q, R, 2^{q+1}}))^{\frac{1}{2}} \|U^\varepsilon\|_{L^\infty(L^2)}^2. \end{aligned}$$

Since the energy estimate

$$\|U^\varepsilon(t, \cdot)\|_{L^2}^2 + 2\nu_h \int_0^t \|\nabla_h U^\varepsilon(\tau, \cdot)\|_{L^2}^2 d\tau \leq \|U_0\|_{L^2}^2 \quad (21)$$

holds, it follows that

$$\|\Delta_q^v F_R^\varepsilon\|_{L_T^1(L^2)} \leq (2^{q+1} + R) T (\mu_3(\mathcal{C}_{2^q, R, 2^{q+1}}))^{\frac{1}{2}} \|U_0\|_{L^2}^2. \quad (22)$$

By the inequalities (20) and (22), we infer that

$$\|\Delta_q^v U_R^\varepsilon\|_{L_T^4(L^\infty)} \leq (1 + T) C(q, R) \varepsilon^{\frac{1}{4}}.$$

For $\alpha \in]0, \frac{1}{4}[$, $R > 0$ and $\varepsilon > 0$, we set

$$N_{\alpha,R}(\varepsilon) = \sup \left\{ p \in \mathbb{Z}; \forall |q| \leq p, (1+T)C(q,R)\varepsilon^{\frac{1}{4}-\alpha} \leq 1 \right\}.$$

It is clear that $N_{\alpha,R}(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} +\infty$ and

$$\sup_{|q| \leq N_{\alpha,R}(\varepsilon)} \|\Delta_q^v U_R^\varepsilon\|_{L_T^4(L^\infty)} \leq \varepsilon^\alpha. \quad (23)$$

Let $\eta > 0$ and $R > 1$. By the energy estimate (21) we have

$$\begin{aligned} \|\|\nabla_h|^{1-\eta}\tilde{U}_R^\varepsilon\|_{L_T^2(\mathcal{B}^{0,\frac{1}{2}})}^2 &\leq R^{-2\eta} \int_0^T \|\|\nabla_h\tilde{U}_R^\varepsilon(t,\cdot)\|_{\mathcal{B}^{0,\frac{1}{2}}}^2 dt \leq \\ &\leq R^{-2\eta} \int_0^T \|\|\nabla_h\tilde{U}^\varepsilon(t,\cdot)\|_{\mathcal{B}^{0,\frac{1}{2}}}^2 dt \leq R^{-2\eta} \sqrt{2/\nu_h} \|U_0\|_{\mathcal{B}^{0,\frac{1}{2}}}^2. \end{aligned}$$

Consequently, it follows that

$$\limsup_{\varepsilon \rightarrow 0} \|\|\nabla_h|^{1-\eta}\tilde{U}_R^\varepsilon\|_{L_T^2(\mathcal{B}^{0,\frac{1}{2}})} \xrightarrow{R \rightarrow +\infty} 0. \quad (24)$$

The equations (23) and (24) finish the proof of Theorem 3. \square

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(Received 13.02.2008)

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