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ON UNIQUE SOLVABILITY OF THE INITIAL VALUE PROBLEM FOR NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. Some general conditions sufficient for unique solvability of the Cauchy problem for nonlinear functional differential equations are established. The class of equations considered covers, in particular, nonlinear equations with transformed argument, integro-differential equations, neutral equations and their systems of an arbitrary order.

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რეზიუმე. დადგენილია არაწრფივი ფუნქციონალურ-დიფერენციალური განტოლებებისათვის კოშის ამოცანის ცალსახა ამოხსნადობის ზოგადი საკმარისი პირობები. განხილული განტოლებების კლასი კერძოდ მოიცავს გარდაქმნილ არგუმენტიან არაწრფივ განტოლებებს, ინტეგრო-დიფერენციალურ განტოლებებს, ნეიტრალური ტიპის განტოლებებს და ნებისმიერი რიგის მათ სისტემებს.

The main goal of this paper is to establish new conditions sufficient for unique solvability of the Cauchy problem for certain classes of many-dimensional systems of nonlinear functional differential equations. Similar topics for linear and nonlinear problems with the Cauchy and more general conditions were addressed, in particular, in [2], [3, 4], [5], [6], [9], [10], [11].

In this work, for nonlinear functional differential systems determined by operators that may be defined on the space of the absolutely continuous functions only, we prove several new theorems close to some results of [5], [11] concerning existence and uniqueness of the Cauchy problem. The proof of the main results is based on application of Theorem 1 from [4] which, in its turn, was established using Theorem 3 from [7].

1. Problem Formulation

Here we consider the system of functional differential equations of the general form

$$u_k'(t) = (f_k u)(t), \quad t \in [a, b], \quad k = 1, 2, \dots, n, \tag{1}$$

subjected to the initial condition

$$u_k(a) = c_k, \quad k = 1, 2, \dots, n, \tag{2}$$

where $-\infty < a < b < +\infty$, $n \in \mathbb{N}$, $f_k : D([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R})$, $k = 1, 2, \dots, n$, are generally speaking nonlinear continuous operators, and $\{c_k \mid k = 1, 2, \dots, n\} \subset \mathbb{R}$ (see Section 2 for the notation). It should be noted that, in contrast to the case considered in [5], [11], setting (1) covers, in particular, neutral differential equations because the expressions for $f_k u$, $k = 1, 2, \dots, n$, in (1) may contain various terms with derivatives.

2. Notation and Definition

The following notation is used throughout the paper.

(1) $\mathbb{R} := (-\infty, \infty)$, $\mathbb{N} := \{1, 2, 3, \dots\}$.
(2) $\|x\| := \max_{1 \le k \le n} |x_k|$ for $x = (x_k)_{k=1}^n \in \mathbb{R}^n$.
(3) $C([a, b], \mathbb{R}^n)$ is the Banach space of continuous functions $[a, b] \to \mathbb{R}^n$ equipped with the norm

$$C([a, b], \mathbb{R}^n) \ni u \longmapsto \max_{s \in [a, b]} \|u(s)\|.$$

(4) $D([a, b], \mathbb{R}^n)$ is the Banach space of absolutely continuous functions $[a, b] \to \mathbb{R}^n$ equipped with the norm

$$D([a, b], \mathbb{R}^n) \ni u \longmapsto \|u(a)\| + \int_a^b \|u'(s)\| \, ds.$$

(5) $L_1([a,b],\mathbb{R}^n)$ is the Banach space of Lebesgue integrable vector functions $u:[a,b]\to\mathbb{R}^n$ with the standard norm

$$L_1([a,b],\mathbb{R}^n) \ni u \longmapsto \int\limits_a^b \|u(s)\|\, ds.$$

The notion of a solution of the problem under consideration is defined in the standard way (see, e.g., [1]).

Definition 1. We say that a vector function $u=(u_k)_{k=1}^n:[a,b]\to\mathbb{R}^n$ is a solution of the problem (1), (2) if it satisfies the system (1) almost everywhere on the interval $[a,b]$ and possesses the property (2) at the point a.

We will use in the sequel the natural notion of positivity of a linear operator.

Definition 2. A linear operator $l=(l_k)_{k=1}^n : D([a,b],\mathbb{R}^n)\to L_1([a,b],\mathbb{R}^n)$ is said to be positive if

$$\operatorname*{vrai\,min}_{t\in[a,b]} (l_k u)(t) \ge 0, \quad k=1,2,\dots,n,$$

for any $u=(u_k)_{k=1}^n$ from $D([a,b],\mathbb{R}^n)$ with non-negative components.

Consider the linear semihomogeneous problem for the functional differential equation

$$u_k' = (l_k u)(t) + q_k(t), \quad t\in[a,b], \quad k=1,2\dots,n, \tag{3}$$

with the initial value condition

$$u_k(a)=0, \quad k=1,2,\dots,n, \tag{4}$$

where $l_k : D([a,b],\mathbb{R}^n)\to L_1([a,b],\mathbb{R})$, $k=1,2,\dots,n$, are linear operators, $\{q_k \mid k=1,2,\dots,n\}\subset L_1([a,b],\mathbb{R})$. The following definition is motivated by a notion used, in particular, in [5], [11].

Definition 3. A linear operator $l=(l_k)_{k=1}^n : D([a,b],\mathbb{R}^n)\to L_1([a,b],\mathbb{R}^n)$ is said to belong to the set $\mathcal{S}_a([a,b],\mathbb{R}^n)$ if the semihomogeneous initial value problem (3), (4) has a unique solution $u=(u_k)_{k=1}^n$ for any $\{q_k \mid k=1,2,\dots,n\}\subset L_1([a,b],\mathbb{R})$ and, moreover, the solution of (3), (4) possesses the property

$$\min_{t\in[a,b]} u_k(t)\ge 0, \qquad k=1,2,\dots,n, \tag{5}$$

whenever the functions q_k, $k=1,2,\dots,n$, appearing in (3) are non-negative almost everywhere on $[a,b]$.

3. Main Results

The following statements are true.

Theorem 1. *Let there exist linear operators $\phi_0, \phi_1 : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $\phi_i = (\phi_{ik})_{k=1}^n$, $i = 0,1$, satisfying the inclusions*

$$\phi_0 \in \mathcal{S}_a([a,b],\mathbb{R}^n), \quad \phi_0 + \phi_1 \in \mathcal{S}_a([a,b],\mathbb{R}^n) \tag{6}$$

and such that the inequalities

$$\big|(f_k u)(t) - (f_k v)(t) - \phi_{0k}(u-v)(t)\big| \le \phi_{1k}(u-v)(t), \quad k = 1,2,\dots,n, \tag{7}$$

are true for arbitrary absolutely continuous functions $u = (u_k)_{k=1}^n$ and $v = (v_k)_{k=1}^n$ from $[a,b]$ to $\mathbb{R}^n$ possessing the properties

$$u_k(a) = v_k(a), \quad u_k(t) \ge v_k(t) \ \textit{for} \ t \in [a,b], \quad k = 1,2,\dots,n. \tag{8}$$

Then the Cauchy problem (1), (2) *is uniquely solvable for arbitrary real c_k, $k = 1,2,\dots,n$.*

Theorem 2. *Assume that there exist positive linear operators $g_i = (g_{ik})_{i=1}^n : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i = 0,1$, such that the inequalities*

$$\big|(f_k u)(t) - (f_k v)(t) + g_{1k}(u-v)(t)\big| \le g_{0k}(u-v)(t), \quad k = 1,2,\dots,n, \tag{9}$$

hold on $[a,b]$ for any vector functions $u = (u_k)_{k=1}^n$ and $v = (v_k)_{k=1}^n$ from $D([a,b],\mathbb{R}^n)$ with the properties (8). *Moreover, let the inclusions*

$$g_0 + (1-2\theta)g_1 \in \mathcal{S}_a([a,b],\mathbb{R}^n), \quad -\theta g_1 \in \mathcal{S}_a([a,b],\mathbb{R}^n) \tag{10}$$

be true.

Then the initial value problem (1), (2) *has a unique solution for all c_k, $k = 1,2,\dots,n$.*

Corollary 1. *Let there exist positive linear operators $g_i = (g_{ik})_{k=1}^n : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i = 0,1$, satisfying the condition* (9) *for arbitrary absolutely continuous functions $u = (u_k)_{k=1}^n$ and $v = (v_k)_{k=1}^n$ with the properties* (8) *and, moreover, such that the inclusions*

$$g_0 \in \mathcal{S}_a([a,b],\mathbb{R}^n), \quad -\frac{1}{2}g_1 \in \mathcal{S}_a([a,b],\mathbb{R}^n) \tag{11}$$

hold.

Then the initial value problem (1), (2) *has a unique solution for any c_k, $k = 1,2,\dots,n$.*

Corollary 2. *Let there exist positive linear operators $g_i = (g_{ik})_{k=1}^n : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i = 0,1$, satisfying the condition* (9) *for arbitrary absolutely continuous functions $u = (u_k)_{k=1}^n$ and $v = (v_k)_{k=1}^n$ with the properties* (8) *and, moreover, such that the inclusions*

$$g_0 + \frac{1}{2}g_1 \in \mathcal{S}_a([a,b],\mathbb{R}^n), \quad -\frac{1}{4}g_1 \in \mathcal{S}_a([a,b],\mathbb{R}^n) \tag{12}$$

are true.

Then the problem (1), (2) *has a unique solution for any c_k, $k = 1,2,\dots,n$.*

Remark 1. The conditions (6), (10), (11), (12) appearing in the theorems and corollaries presented above are unimprovable in a certain sense. More precisely, for example, the condition (11) cannot be replaced by its weaker versions

$$(1-\varepsilon)g_0 \in S_a([a,b],\mathbb{R}^n), \quad -\frac{1}{2}g_1 \in S_a([a,b],\mathbb{R}^n)$$

and

$$g_0 \in S_a([a,b],\mathbb{R}^n), \qquad -\frac{1}{2+\varepsilon}g_1 \in S_a([a,b],\mathbb{R}^n),$$

no matter how small the constant $\varepsilon \in (0,1)$ is. In order to show this, it is sufficient to use [11, Examples 6.1 and 6.4] (see also [8, Section 5]).

4. Auxiliary Statements

We need the following statement on unique solvability of the problem (1), (2) established in [4].

Theorem 3 ([4, Theorem 1])**.** *Let there exist linear operators $p_i = (p_{ik})_{k=1}^n : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i=1,2$, satisfying the inclusions*

$$p_1 \in \mathcal{S}_a([a,b],\mathbb{R}^n), \quad \frac{1}{2}(p_1+p_2) \in \mathcal{S}_a([a,b],\mathbb{R}^n) \tag{13}$$

and such that the inequalities

$$p_{2k}(u-v)(t) \le (f_k u)(t) - (f_k v)(t) \le \\ \le p_{1k}(u-v)(t), \quad t\in[a,b], \quad k=1,2,\dots,n, \tag{14}$$

are true for arbitrary absolutely continuous functions $u=(u_k)_{k=1}^n$ and $v=(v_k)_{k=1}^n$ from $D([a,b],\mathbb{R}^n)$ with the properties (8).

Then the Cauchy problem (1), (2) *is uniquely solvable for all real c_k, $k=1,2,\dots,n$.*

5. Proofs

In this section, we present the proofs of the results formulated above.

5.1. *Proof of Theorem* 1. Obviously, the condition (7) is equivalent to the relation

$$-\phi_{1k}(u-v)(t) + \phi_{0k}(u-v)(t) \le \\ \le (f_k u)(t) - (f_k v)(t) \le \phi_{1k}(u-v)(t) + \phi_{0k}(u-v)(t), \quad t\in[a,b], \tag{15}$$

for any functions $u=(u_k)_{k=1}^n$ and $v=(v_k)_{k=1}^n$ from $D([a,b],\mathbb{R}^n)$ with the property (8).

Let us put

$$(p_{ik}x)(t) := (\phi_{0k}x)(t) - (-1)^i(\phi_{1k}x)(t), \quad t\in[a,b], \quad i=1,2, \tag{16}$$

for any x from $D([a,b],\mathbb{R}^n)$ and $k=1,2,\dots,n$. Considering (15), we find that the operator f admits the estimate (14) with the operators p_1 and p_2 defined by the formulae (16). Therefore, it remains only to note that the assumption (6) ensures the validity of the inclusions (13).

Thus, applying Theorem 3, we arrive at the assertion of Theorem 1.

5.2. *Proof of Theorem* 2. One can verify that under the conditions (9), (10) the operators $\phi_i : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i = 0, 1$, defined by the formulae

$$\phi_0 := -\theta g_1, \quad \phi_1 := g_0 + (1-\theta)g_1 \tag{17}$$

satisfy the conditions (6), (7) of Theorem 1. Indeed, the estimate (9), the assumption $\theta \in (0,1)$, and the positivity of the operator g_1 imply that for any absolutely continuous functions $u = (u_k)_{k=1}^n$ and $v = (v_k)_{k=1}^n$ with the properties (8) the relations

$$\begin{gathered}
\big|(f_k u)(t) - (f_k v)(t) + \theta g_1(u-v)(t)\big| = \\
= \big|(f_k u)(t) - (f_k v)(t) + g_{1k}(u-v)(t) - (1-\theta)g_{1k}(u-v)(t)\big| \le \\
\le g_{0k}(u-v)(t) + \big|(1-\theta)g_{1k}(u-v)(t)\big| = \\
= g_{0k}(u-v)(t) + (1-\theta)(g_{1k}(u-v)(t)), \quad t \in [a,b], \quad k = 1, 2, \dots, n,
\end{gathered}$$

are true. This means that f admits the estimate (7) with the operators ϕ_0 and ϕ_1 defined by the formulae (17). Therefore, it remains only to note that the assumption (10) ensures the validity of the inclusions (6) for the operators (17). Applying Theorem 1, we arrive at the required assertion.

5.3. *Proof of Corollary* 1. This statement follows from Theorem 2 with $\theta = \frac{1}{2}$.

5.4. *Proof of Corollary* 2. It is sufficient to apply Theorem 2 with $\theta = \frac{1}{4}$.

6. Example of a Scalar Nonlinear Differential Equation with Argument Deviations

As an example, we consider the initial value problem for the nonlinear scalar differential equation with argument deviations

$$u'(t) = \mu_0(t)\sqrt{1 + \lambda(t)\sin u(\omega_0(t))} - \mu_1(t)u(\omega_1(t)), \quad t \in [a,b], \tag{18}$$

with the initial condition

$$u(a) = c, \tag{19}$$

where $c \in \mathbb{R}$ is an arbitrary constant, $\omega_i : [a,b] \to [a,b]$, $i = 0, 1$, are Lebesgue measurable functions, and $\{\lambda, \mu_i\} \subset L_1([a,b],\mathbb{R})$, $i = 0, 1$, are functions such that

$$0 \le \lambda(t) < 1 \ \text{ for a.e. } t \in [a,b]. \tag{20}$$

It is important to point out that the first term in (18) may not be of retarding type (i.e., the set $\{t \in [a,b] : \omega_0(t) > t\}$ may have non-zero measure) and, therefore, the solvability of the problem (18), (19) is not obvious.

6.1. **Existence of a solution of** (18)**,** (19)**.** To show the existence of a solution of (18), (19), it is sufficient to impose some conditions on μ_1 and ω_1 only. More precisely, the following statement is true.

Proposition 1. *The problem* (18), (19) *is solvable for any* $c \in \mathbb{R}$ *if at least one of the following conditions is satisfied:*

$$\omega_1(t) \le t \ \textit{for a.e.} \ t \in [a,b]; \tag{21a}$$

$$\int_a^b [\mu_1(s)]_- \, ds < 1, \quad \int_a^b [\mu_1(s)]_+ \, ds < 1 + 2\sqrt{1 - \int_a^b [\mu_1(s)]_- \, ds}\,. \tag{21b}$$

Here, by definition, $[u(t)]_+ := \max\{u(t), 0\}$ and $[u(t)]_- := \max\{-u(t), 0\}$ for $t \in [a,b]$ and $u : [a,b] \to \mathbb{R}$. To prove Proposition 1, the following lemmata can be used.

Let $f : D([a,b],\mathbb{R}) \to L_1([a,b],\mathbb{R})$ be an operator and $h : D([a,b],\mathbb{R}) \to \mathbb{R}$ be a continuous linear functional.

Lemma 1 ([6, Lemma 3.1])**.** *Let there exist a linear operator* $l : C([a,b],\mathbb{R}) \to L_1([a,b],\mathbb{R})$ *such that the problem*

$$u'(t) + (lu)(t) = 0, \quad u(a) = 0 \tag{22}$$

has only the trivial solution and for all $v \in C([a,b],\mathbb{R})$

$$|(lv)(t)| \le \eta(t) \max_{s \in [a,b]} |v(s)|, \quad t \in [a,b], \tag{23}$$

where $\eta \in L_1([a,b],\mathbb{R})$ *does not depend on* v*. Moreover, assume that there exists a positive number* ρ *such that for every* $\delta \in (0,1)$ *and for an arbitrary solution* $u \in D([a,b],\mathbb{R})$ *of the problem*

$$u'(t) + (lu)(t) = \delta[(fu)(t) + (lu)(t)], \quad u(a) = \delta h(u) \tag{24}$$

the estimate

$$\max_{t \in [a,b]} |u(t)| \le \rho \tag{25}$$

holds. Then the problem

$$u'(t) = (fu)(t), \quad t \in [a,b],$$
$$u(a) = h(u)$$

has at least one solution.

Lemma 1 is, in fact, [6, Theorem 1] in the formulation of [5].

Lemma 2 (Lemma 3.4 from [5])**.** *Let there exist a linear operator* l *such that the condition* (23) *is satisfied and the homogeneous problem* (22) *has only the trivial solution. Then there exists a positive number* r *such that for any* $q \in L([a,b],\mathbb{R})$ *and real* c *every solution* u *of the problem*

$$\begin{aligned} u'(t) &= (lu)(t) + q(t), \\ u(a) &= c \end{aligned} \tag{26}$$

admits the estimate

$$\max_{t\in[a,b]} |u(t)| \le r\bigg(|c| + \int\limits_a^b |q(s)|\,ds\bigg). \tag{27}$$

Lemma 3. *Problem* (18), (19) *is solvable if the problem*

$$u'(t) = -\mu_1(t)u(\omega_1(t)), \quad t \in [a,b], \tag{28}$$

$$u(a) = 0 \tag{29}$$

has no non-trivial solution.

Proof. Indeed, assume that the problem (28), (29) has no non-trivial solution. Let u be a solution of the problem (26). Then

$$u'(t) = (lu)(t) + Q(t), \quad u(a) = c, \tag{30}$$

where

$$Q(t) := \mu_0(t)\sqrt{1 + \lambda(t)\sin u(\omega_0(t))}, \quad t \in [a,b]. \tag{31}$$

Using the estimate

$$\big|\mu_0(t)\sqrt{1 + \lambda(t)\sin u(\omega_0(t))}\big| \le |\mu_0(t)|\sqrt{1+\lambda(t)}$$

valid for a.e. $t \in [a,b]$ and taking Lemma 2 into account, we conclude that an arbitrary solution u of the problem (26) satisfies the estimate

$$\max_{t\in[a,b]} |u(t)| \le r\bigg(|c| + \int\limits_a^b |\mu_0(s)|\sqrt{1+\lambda(s)}\,ds\bigg).$$

Let us put

$$\rho := r\bigg(|c| + \int\limits_a^b |\mu_0(s)|\sqrt{1+\lambda(s)}\,ds\bigg) \tag{32}$$

and

$$(lu)(t) := -\mu_1(t)u(\omega_1(t)), \quad t \in [a,b], \tag{33}$$

and return to the problem (24). In this notation, all solutions of the problem (24) satisfy the estimate (25). So, using the assumption that the problem (28), (29) has only the trivial solution and applying Lemmas 1 and 2, we prove Lemma 3. □

Lemma 4. *Let at least one of the conditions* (21a) *and* (21b) *be satisfied. Then the homogeneous Cauchy problem* (28), (29) *has no non-trivial solution.*

Proof. This statement follows from [11, Corollary 3.3] and [2, Theorem 1.3] with the operator l defined by the formula (33). □

Proof of Proposition 1. To obtain this statement, it is sufficient to apply Lemmata 3 and 4 with $h := 0$,

$$(fu)(t) := \mu_0(t)\sqrt{1 + \lambda(t)\sin u(\omega_0(t))} - \mu_1(t)u(\omega_1(t)), \quad t \in [a,b],$$

and l defined by the formula (33) for any u from $C([a,b],\mathbb{R})$. □

6.2. **Existence and uniqueness of a solution of** (18), (19)**.** We see that the conditions which are assumed in Proposition 1 and guarantee the solvability of the problem (18), (19) concern the second term of the equation (18) only. To guarantee the uniqueness of the solution, one has to impose some conditions on μ_0, λ, and ω_0. Along these lines, we have the following result.

Corollary 3. *Let the functions* λ*,* ω_1*, and* μ_i*,* $i = 0, 1$*, satisfy, respectively, the conditions* (20) *and* (21a)*,*

$$\mu_i(t) \geq 0, \quad i = 0, 1, \tag{34}$$

for almost all $t \in [a,b]$ *and, moreover, there exist some constants* $\alpha \in [1,\infty)$ *and* $\gamma \in (0,1)$ *for which the following estimate holds:*

$$\frac{\mu_0(t)\lambda(t)}{\sqrt{1-\lambda(t)}}(\omega_0(t) - a)^{\alpha} \leq 2\alpha\gamma(t-a)^{\alpha-1}, \quad t \in [a,b]. \tag{35}$$

In addition, suppose that the inequality

$$\int_{\omega_1(t)}^{t} \mu_1(s)\, ds \leq \frac{2}{e}, \quad t \in [a,b], \tag{36}$$

is satisfied.

Then the problem (18), (19) *is uniquely solvable for any* $c \in \mathbb{R}$*.*

It follows immediately from Corollary 3 that

Corollary 4. *The problem* (18), (19) *is uniquely solvable for any* c*, in particular, if* μ_i*,* $i = 0, 1$*, are non-negative,* μ_1 *and* λ *satisfy* (36) *and* (20)*, the condition* (21a) *holds, and the inequality*

$$\operatorname*{vrai\,max}_{t\in[a,b]} \frac{\mu_0(t)\lambda(t)}{\sqrt{1-\lambda(t)}}(\omega_0(t) - a) < 2$$

is true.

Proof. It is sufficient to put $\alpha = 1$ in Corollary 3. □

To prove Corollary 3, we need the following propositions.

Proposition 2 ([3], Corollary 13)**.** *Suppose that in the scalar functional differential equation*

$$u'(t) = r(t)u(\omega(t)) + q(t), \quad t \in [a,b], \tag{37}$$

the function $\omega : [a, b] \to [a, b]$ *is measurable and* $r : [a, b] \to \mathbb{R}$ *and* $q : [a, b] \to \mathbb{R}$ *are summable, the function* r *has the property*

$$r(t) \operatorname{sign}(t - \tau) \geq 0, \quad t \in [a, b], \tag{38}$$

where τ *is a given point from* $[a, b]$*, and, moreover, there exist constants* $\alpha \in [1, \infty)$*,* $\gamma \in (0, 1)$ *for which*

$$\alpha\gamma|t - \tau|^{\alpha-1} \geq |\omega(t) - \tau|^{\alpha} r(t) \operatorname{sign}(t - \tau) \tag{39}$$

for almost all t *from* $[a, b]$*.*

Then for an arbitrary real c *and an arbitrary summable function* $q : [a, b] \to \mathbb{R}$ *the Cauchy problem*

$$u(\tau) = c \tag{40}$$

for the equation (37) *is uniquely solvable. Moreover, if* q *and* c *satisfy the condition*

$$\int_{\tau}^{t} q(s)\, ds \geq -c,$$

then the unique solution of the problem (37), (40) *is non-negative.*

Proposition 3 ([2], Corollary 1.1 (iv))**.** *Assume that in the equation* (37) $\omega(t) \leq t$ *and* $r(t) \leq 0$ *for almost all* $t \in [a, b]$ *and, moreover,*

$$\int_{\omega(t)}^{t} |r(s)|\, ds \leq \frac{1}{e}, \quad t \in [a, b]. \tag{41}$$

Then for arbitrary real c *and nonnegative summable function* $q : [a, b] \to \mathbb{R}$ *the Cauchy problem* (37), (19) *is uniquely solvable, and its solution is non-negative for non-negative* q *and* c*.*

Proof of Corollary 3. The equation (18) is a particular case of (1), where $n = 1$ and the operator $f_1 : D([a, b], \mathbb{R}) \to L_1([a, b], \mathbb{R})$ is given by the formula

$$(f_1 u)(t) := \mu_0(t)\sqrt{1 + \lambda(t) \sin u(\omega_0(t))} - \mu_1(t) u(\omega_1(t)), \quad t \in [a, b], \tag{42}$$

for any u from $D([a, b], \mathbb{R})$. Using the Lagrange theorem and taking (20) into account, we easily get that the relations

$$\begin{aligned} \Big|\mu_0(t)\sqrt{1 + \lambda(t) \sin u(\omega_0(t))} - \mu_0(t)\sqrt{1 + \lambda(t) \sin v(\omega_0(t))}\,\Big| &\leq \\ \leq \sup_{\xi \in \mathbb{R}} \frac{\mu_0(t)\lambda(t)|\cos \xi|(u(\omega_0(t)) - v(\omega_0(t)))}{2\sqrt{1 + \lambda(t) \sin \xi}} &\leq \\ \leq \frac{\mu_0(t)\lambda(t)(u(\omega_0(t)) - v(\omega_0(t)))}{2\sqrt{1 - \lambda(t)}} & \end{aligned} \tag{43}$$

hold for almost all $t \in [a, b]$ and arbitrary absolutely continuous functions $u : [a, b] \to \mathbb{R}$ and $v : [a, b] \to \mathbb{R}$ possessing the properties

$$u(t) \geq v(t) \quad \text{for all} \quad t \in [a, b]. \tag{44}$$

Let us put

$$r(t) := \frac{\mu_0(t)\lambda(t)}{2\sqrt{1-\lambda(t)}}, \quad \tau := a, \quad \omega(t) := \omega_0(t), \quad t \in [a,b], \tag{45}$$

and consider the linear initial value problem (19) for the scalar differential equation with argument deviation

$$u'(t) = \frac{\mu_0(t)\lambda(t)}{2\sqrt{1-\lambda(t)}} u(\omega_0(t)) + q(t), \quad t \in [a,b], \tag{46}$$

where $q \in L_1([a,b],\mathbb{R})$. It follows from the assumption (35) that the function (45) has the property (39). Applying Proposition 2, we get that the problem (46), (19) with r given by (45) has a unique solution for any q and c and, moreover, u is non-negative for non-negative q. Consequently, according to Definition 3, the linear operator $g_0 : D([a,b],\mathbb{R}) \to L_1([a,b],\mathbb{R})$ defined by the formula

$$(g_0 u)(t) := \frac{\mu_0(t)\lambda(t)}{2\sqrt{1-\lambda(t)}} u(\omega_0(t)), \quad t \in [a,b], \tag{47}$$

belongs to the set $\mathcal{S}_a([a,b],\mathbb{R})$. Similarly, if we put in Proposition 3 $r(t) := -\frac{1}{2}\mu_1(t)$, $t \in [a,b]$, and $\omega := \omega_1$, and use the conditions (21a) and (36), we conclude that the linear initial value problem (19) for the scalar functional differential equation

$$u'(t) = -\frac{1}{2}\mu_1(t)u(\omega_1(t)) + q(t), \quad t \in [a,b], \tag{48}$$

is uniquely solvable for any $q \in L_1([a,b],\mathbb{R})$, and its solution is non-negative for the non-negative q. In view of Definition 3, this means that the linear operator $g_1 : D([a,b],\mathbb{R}) \to L_1([a,b],\mathbb{R})$ defined by the formula

$$(g_1 u)(t) := \mu_1(t)u(\omega_1(t)), \quad t \in [a,b], \tag{49}$$

has the property $-\frac{1}{2}g_1 \in S_a([a,b],\mathbb{R})$. Thus we have shown that under the conditions assumed in Corollary 3 the operators (47) and (49) have the property (11). Furthermore, taking the estimate (43) into account, we conclude that the operator (42) satisfies the condition (9) with g_0 and g_1 defined by the equalities (47) and (49).

Applying Corollary 1, we obtain the assertion of Corollary 3. □

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