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ON SOLVABILITY OF THE PROBLEM WITH A DIRECTIONAL DERIVATIVE FOR THE EQUATION $\Delta^3 v = 0$

Abstract. We consider the problem of directional derivative for the equation $\Delta^3 v = 0$ when the direction of the derivative belongs to the tangent plane. It is proved that if the boundary functions belong to a certain class, then the problem has a solution.

[illegible] $\Delta^3 v = 0$ [illegible]

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The investigation of the problem with a directional derivative in an n-dimensional domain for $n \geq 3$ has been initiated in [1], [2] and continued in [3]–[8]. In this note we treat the problem for the equation $\Delta^3 v = 0$.

Let us introduce the necessary notation. Let Ω be a bounded smooth domain belonging to the class $C^{(5,\alpha)}$, ℓ_x be a smooth direction at $x \in \partial\Omega$, $\ell_x \in C^{(4,\alpha)}$, $|\ell_x| = 1$. For simplicity of presentation, we will consider the three-dimensional case.

The volume potential and the single-layer potential will be denoted as follows (see [9]):

$$V^{\mu}(x) = \frac{1}{4\pi} \int\limits_{\Omega} \frac{\mu(y)}{|x-y|}\, dy; \quad U^{\psi}(x) = \frac{1}{4\pi} \int\limits_{\partial\Omega} \frac{\psi(y)}{|x-y|}\, dSy,$$

where $\partial\Omega$ is the boundary of Ω, $\mu \in L_1(\Omega)$, $\psi \in L_1(\partial\Omega)$. If G is the Green function for the Dirichlet problem for Ω, then the solution to the Dirichlet problem has the form

$$v(x) = -\int\limits_{\partial\Omega} \frac{\partial G(x,y)}{\partial \nu_y} \varphi(y)\, dSy, \quad x \in \Omega, \quad \varphi \in C(\partial\Omega),$$

where ν_x is the outer normal at $x \in \partial\Omega$, $|\nu_x| = 1$.

The Green potential is denoted by

$$V_G^f(x) = \int\limits_{\Omega} G(x,y) f(y)\, dy, \quad f \in L_1(\Omega).$$

The density of a balayaged measure $\mu \in C(\overline{\Omega})$ (see [9]) in the volume potential case is defined as the linear operator

$$T\mu(y) = \mu'(y) = -\int\limits_{\Omega} \frac{\partial G(x,y)}{\partial \nu_y} \mu(y)\, dy, \quad y \in \partial\Omega.$$

It is well-known ([9], p. 260) that

$$V^{\mu}(x) = U^{\mu'}(x), \quad x \in R^3 \setminus \Omega.$$

Denote

$$E = \Big\{ x : \ (\nu_x, \ell_x) = \cos(\nu_x \widehat{\ } \ell_x) = 0, \ x \in \partial\Omega \Big\}.$$

We always assume that E consists of a finite number of closed curves. Denote by $\Gamma(x,y)$ the Newton kernel $(4\pi|x-y|)^{-1}$.

In the sequel we use the following auxiliary equality [10]

$$\frac{\partial G(x,y)}{\partial \ell_x} = \cos(\nu_x \widehat{\ } \ell_x)\, \frac{\partial G(x,y)}{\partial \nu_x}, \quad x \in \partial\Omega \setminus E, \quad y \in \Omega. \tag{1}$$

In the proof of the main assertion the following lemma will be used.

Lemma. *Let $\mu \in C^{\alpha}(\overline{\Omega})$, $\Omega \in C^{(2,\alpha)}$. Then the following equality holds*

$$\frac{\partial^2 V_i^{\mu}(x)}{\partial \ell_x^2} - \frac{\partial^2 V_e^{\mu}}{\partial \ell_x^2} = -\mu(x) \cos^2(\nu_x \widehat{\ } \ell_x), \quad x \in \partial\Omega \setminus E. \tag{2}$$

The outwards and inwards limits for V^{μ} are considered.

Proof. Let $x_0 \in \partial\Omega - E$. Obviously

$$\frac{\partial V^{\mu}(x)}{\partial \ell_{x_0}} = \sum_{k=1}^{3} \frac{\partial V^{\mu}(x)}{\partial x_k} \alpha_k^0, \quad \ell_{x_0} = \ell_0 = (\alpha_1^0, \alpha_2^0, \alpha_3^0).$$

It is easy to see that

$$\frac{\partial^2 V^{\mu}(x)}{\partial \ell_0^2} = \sum \frac{\partial^2 V^{\mu}(x)}{\partial x_j \partial x_k} \alpha_j^0 \alpha_k^0.$$

Consider the difference

$$\frac{\partial^2 V_i^{\mu}(x)}{\partial \ell_0^2} - \frac{\partial^2 V_e^{\mu}(x)}{\partial \ell_0^2} = \sum_{j,k=1}^{3} \Big(\frac{\partial^2 V_i^{\mu}(x)}{\partial x_j \partial x_k} - \frac{\partial^2 V_\ell^{\mu}(x)}{\partial x_j \partial x_k} \Big) \alpha_j^0 \alpha_k^0. \tag{3}$$

Let us now use the following relation ([11], p. 115)

$$\frac{\partial^2 V_i^{\mu}(x)}{\partial x_j \partial x_k} - \frac{\partial^2 V_e^{\mu}(x)}{\partial x_j \partial x_k} = -\mu(x) \nu_j \nu_k, \quad \nu = (\nu_1, \nu_2, \nu_3). \tag{4}$$

(4) along with (3) gives $(\nu_0 = \nu_{x_0})$

$$\frac{\partial^2 V_i^{\mu}(x_0)}{\partial \ell_0^2} - \frac{\partial^2 V_e^{\mu}(x_0)}{\partial \ell_0^2} = -\mu(x_0) \sum_{j,k=1}^{3} \alpha_j \alpha_k \nu_j \nu_k = -\mu(x) \cos^2(\nu_0 \widehat{\ } \ell_0).$$

The lemma is proved.

Let us proceed to the main assertion of this note.

Theorem. *There exists a solution to the equation $\Delta^3 v = 0$ in the domain Ω belonging to the space $C^{(4,\alpha)}(\overline{\Omega})$ and satisfying the following boundary conditions*

$$\begin{gathered}
v\big|_{\partial\Omega} = \varphi_0, \ \ \varphi_0 \in C^{(5,\alpha)}(\partial\Omega), \ \ H_0\big|_{\partial\Omega} = \varphi_0, \Delta H_0 = 0, \\
\frac{\partial v(x)}{\partial \ell_x} = \frac{\partial H_0(x)}{\partial \ell_x} + \varphi_1(x)(\nu_x, \ell_x), \ \ \varphi_1 \in C^{(4,\alpha)}, \ \ x \in \partial\Omega \setminus E, \\
\frac{\partial^2 v(x)}{\partial \ell_x^2} = \frac{\partial^2 H_0(x)}{\partial \ell_x^2} + \varphi_2(x)(\nu_x, \ell_x)^2 + \\
+ \frac{\partial^2 U_i^{\varphi_1}(x)}{\partial \ell_x^2} - \frac{\partial^2 U_e^{\varphi_1}(x)}{\partial \ell_x^2}, \ \ \varphi_2 \in C^{(3,\alpha)}, \ \ x \in \partial\Omega \setminus E.
\end{gathered} \tag{5}$$

Proof. The solution $v \in C^{(4,\alpha)}$ has the form

$$v(x) = H_0(x) - \int\limits_{\Omega} G(x,y) H_1(y)\, dy + \int\limits_{\Omega} G(x,y) \int\limits_{\Omega} G(y,z) H_2(z)\, dz\, dy,$$

where H_0, H_1 and H_2 are harmonic functions such that $\Delta v(x) = H_1(x)$, $\Delta^2 v(x) = H_2(x)$, $x \in \partial\Omega$. In order to find H_1, let us rewrite the solution as follows

$$v(x) = H_0(x) - \int\limits_{\Omega} G(x,y) \Phi_1(y)\, dy, \ \ \Phi_1(y) = H_1(y) - \int\limits_{\Omega} G(x,y) H_2(z)\, dz.$$

Due to the second boundary condition, we get

$$\frac{\partial H_0(x)}{\partial \ell_x} + \cos(\nu_x \widehat{\ } \ell_x)\Phi'(x) = \frac{\partial H_0(x)}{\partial \ell_x} + \varphi_1(x)\cos(\nu_x \widehat{\ } \ell_x).$$

Therefore, $\Phi_1'(x) = \varphi_1(x)$ $x \in \partial\Omega$ ($\Phi_1(x) = H_1(x)$ $x \in \partial\Omega$).

The third boundary condition also gives

$$\begin{aligned}
&\frac{\partial^2 H_0}{\partial \ell_x^2} + \Phi(x)\cos^2(\nu_x \widehat{\ } \ell_x) + \frac{\partial^2 U_i^{\varphi_1}(x)}{\partial \ell_x^2} - \frac{\partial^2 U_e^{\varphi_1}(x)}{\partial \ell_x^2} = \\
&\qquad = \frac{\partial^2 H_0(x)}{\partial \ell_x^2} + \varphi_2(x)\cos^2(\nu_x \widehat{\ } \ell_x) + \frac{\partial^2 U_i^{\varphi_1}(x)}{\partial \ell_x^2} - \frac{\partial^2 U_e^{\varphi_1}(x)}{\partial \ell_x^2}.
\end{aligned}$$

Hence, $\Phi(x) = H_1(x) = \varphi_2(x)$, $x \in \partial\Omega$.

In order to define H_2, we apply again the second boundary condition

$$\begin{aligned}
\frac{\partial v(x)}{\partial \ell_x} &= \frac{\partial H_0(x)}{\partial \ell_x} + \varphi_1 \cos(\nu_x \widehat{\ } \ell_x) = \\
&= \frac{\partial H_0(x)}{\partial \ell_x} + H_1'(x)\cos(\nu_x \widehat{\ } \ell_x) + (V_G^{h_2})' \cos(\nu_x \widehat{\ } \ell_x).
\end{aligned}$$

Hence, $\varphi_1(x) - H_1'(x) = (V_G^{H_2}(x))' x \in \partial\Omega$.

Therefore,

$$\int\limits_{\Omega} \frac{V_G^{H_2}(x)}{|x-y|} dx = \int\limits_{\partial\Omega} \frac{\varphi(x) dS_x}{|x-y|}, \ \ \varphi = \varphi_1 - H_1' \in C^{(4,\alpha)}, \ \ x \in R^3 - \Omega.$$

Due to the equality

$$\int\limits_{\Omega} \frac{V_G^{H_2}(y)dy}{|x-y|} = \int\limits_{\partial\Omega} \frac{\varphi(y)dS_y}{|x-y|}, \quad x \in R^3 \setminus \Omega \ \Big(v_1(x) = \frac{V_G^{H_2}(y)dy}{|x-y|} \Big),$$

we get the Dirichlet problem in the domain Ω for the equation $\Delta^3 v_1 = 0$

$$v_1(x) = U^{\varphi}(x), \quad x \in \partial\Omega,$$
$$\frac{\partial v_1(x)}{\partial \nu_x} = \frac{\partial U_\ell^{\varphi}(x)}{\partial \nu_x}, \quad x \in \partial\Omega,$$
$$\frac{\partial^2 v_1(x)}{\partial \nu_x^2} = \frac{\partial^2 U_\ell^{\varphi}(x)}{\partial \nu_x^2}, \quad x \in \partial\Omega.$$

Now H_2 can easily be found by solving the Dirichet problem ($\Delta^2 v_1 = H_2$).

References

1. A. V. Bitsadze, The homogeneous problem for the directional derivative for harmonic functions in three-dimensional regions. (Russian) *Dokl. Akad. Nauk SSSR* **148** (1963), 749–752.
2. A. V. Bitsadze, Boundary value problems for elliptic equations of second order. (Russian) *Nauka, Moscow,* 1966.
3. M. B. Maljutov, Oblique derivative problem in three-dimensional space. (Russian) *Dokl. Akad. Nauk SSSR* **172** (1967), 283–286.
4. A. Janušauskas, On the oblique derivative problem for harmonic functions of three independent variables. (Russian) *Sibirsk. Mat. Ž.* **8** (1967), 447–462.
5. Ju. V. Egorov and V. A. Kondrat'ev, The oblique derivative problem. (Russian) *Mat. Sb. (N.S.)* **78 (120)** (1969), 148–176.
6. V. G. Maz'ja, The degenerate problem with an oblique derivative. (Russian) *Mat. Sb. (N.S.)* **87 (129)** (1972), 417–454.
7. Sh. A. Alimov, On a problem with an oblique derivative. (Russian) *Differentsial'nye Uravneniya* **17** (1981), No. 10, 1738–1751.
8. S. Rempel and B.-V. Schulze, Index theory of elliptic boundary problems. *Akademie-Verlag, Berlin,* 1982; Russian transl.: *Mir, Moscow,* 1986.
9. N. S. Landkof, Fundamentals of modern potential theory. (Russian) *Nauka, Moscow,* 1966.
10. D. V. Kapanadze, On the uniqueness of a solution of the oblique derivative problem for the equation $\Delta^n v = 0$. (Russian) *Ukrain. Mat. Zh.* **58** (2006), No. 6, 835–841; English transl.: *Ukrainian Math. J.* **58** (2006), No. 6, 945–953
11. N. M. Gyunter, Theory of the potential and its application to the basic problems of mathematical physics. (Russian) *Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow,* 1953.

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