Memoirs on Differential Equations and Mathematical Physics
Volume 51, 2010, 109–118

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POSITIVE PERIODIC SOLUTIONS FOR A NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATION

Abstract. In this paper, sufficient conditions have been obtained for the existence of at least two positive periodic solutions of the Nicholson's Blowflies model

$$x'(t) = -a(t)x(t) + p(t)x^m(t-\tau(t))e^{-\gamma(t)x^n(t-\tau(t))}.$$

The Leggett–Williams multiple fixed point theorem has been used to prove our results.

2010 Mathematics Subject Classification. 34K40, 34C10, 34C25.
Key words and phrases. Periodic solution, nonnegative solution.

რეზიუმე. ნაშრომში მიღებულია საკმარისი პირობები იმისათვის, რომ ნიკოლსონის მოდელს

$$x'(t) = -a(t)x(t) + p(t)x^m(t-\tau(t))e^{-\gamma(t)x^n(t-\tau(t))}$$

ჰქონდეს სულ მცირე ორი დადებითი პერიოდული ამონახსნი. შედეგების დასამტკიცებლად გამოყენებულია ლექეტ–ვილიამსის მრავალი უძრავი წერტილის თეორემა.

1. Introduction

In this paper, we study the existence of two positive periodic solutions of a nonlinear functional differential equation of the form

$$x'(t) = -a(t)x(t) + p(t)x^m(t-\tau(t))e^{-\gamma(t)x^n(t-\tau(t))}, \tag{1}$$

where a, p, γ and $\tau \in C(R, R^+)$ are T-periodic functions, $m > 1$ and $n > 0$ are reals and T is a positive constant.

If $m = 1$ and $n = 1$, then (1) yields the Nicholson's Blowflies model

$$x'(t) = -a(t)x(t) + p(t)x(t-\tau(t))e^{-\gamma(t)x(t-\tau(t))}. \tag{2}$$

When all the parameters are positive constants, (2) reduces to an original model developed by Gurney et al. [6] to describe the population of Australian sheep-blowfly that agrees well with the experimental data of Nicholson [11]. One may note that the equation explains Nicholson's data of blowfly quite accurately and hence we refer (2) as the Nicholson's Blowflies model.

The variation of the environment plays an important role in many biological and ecological dynamical systems. In particular, the effects of a periodically varying environment are important for evolutionary theories, as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of parameters of the system (in a way) incorporates the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.). In fact, it has been suggested by Nicholson [12] that any periodic change of climate tends to improve it's periodicity upon oscillations of internal origin or to cause such oscillations to have a harmonic relation to periodic climate changes. In view of the above fact, it is realistic to assume the periodicity on the parameters or on the coefficient functions of (1) and (2). Thus, the existence of periodic solutions of (1) or (2) are naturally expected.

Many authors have studied the existence of at least one positive periodic solution of (2). For this, one may refer the papers in [5], [7], [16], [23], [24], [27]–[29]. Krasnoselskiĭ fixed point theorem [3] have been used to prove the results. Although the existence of at least one periodic solution of (2) is largely studied in the literature, studies on the existence of at least two periodic solutions of (1) and (2) are relatively scarce.

In this paper, we have made an attempt to study the existence of at least two positive periodic solutions of (1). We have used Leggett–Williams multiple fixed point theorem [10] to prove our theorem. This theorem have been used by the authors in [19]–[22] to study the existence of three periodic solution of the following differential equations:

$$x'(t) = -a(t)x(t) + \lambda f(t, x(h(t))),$$

and

$$x'(t) = a(t)x(t) - \lambda f(t, x(h(t))),$$

where λ is a positive parameter. The results obtained for the above equations were applied to (1) with constant coefficients of the form

$$x'(t) = -ax(t) + px^m(t-\tau)e^{-\gamma x^n(t-\tau)}, \tag{3}$$

We state the results obtained in [20], [21] in the form of theorems.

Theorem 1.1 ([20])**.** *Let $m > 1$ and $2e(\delta-1)\delta^{m-1}\gamma^{\frac{(m-1)}{n}} \leq 1$. Then the equation (3) has at least three positive T-periodic solutions for $\frac{1}{2T} < p < \frac{1}{T}$.*

Theorem 1.2 ([21])**.** *Assume that $m > 1$ and that*

$$\int_0^T p(t)\,dt > \delta(\delta-1)\Big(\frac{\gamma\delta^2 e}{m-1}\Big)^{m-1}. \tag{4}$$

Then the equation

$$x'(t) = -a(t)x(t) + p(t)x^m(t-\tau(t))e^{-\gamma x(t-\tau(t))} \tag{5}$$

has at least three nonnegative T-periodic solutions, where $\gamma > 0$ is a constant and $\delta = \exp\big(\int_0^T a(s)\,ds\big)$.

For the last two decades, there has been a rich literature on the use of fixed point theorems on the existence of positive solutions of boundary value problems. The existence of periodic solutions of this type equation is closely related to the existence of solutions of general boundary value problems. The ideas in this paper have come from those for general boundary value problem.

In the next section, we will state the well known Leggett–Williams multiple fixed point theorem [10] and then we will apply the theorem to the model (1). The obtained result improves our previous result.

2. Main Results

From the periodicity of the solution and the assumption that x is known on the nonlinear parts of (1), one can construct a Green's Kernel. In fact, (1) is equivalent to

$$x(t) = \int_t^{t+T} G(t,s)p(s)x^m(s-\tau(s))e^{-\gamma(s)x^n(s-\tau(s))}\,ds,$$

where $G(t,s) = \dfrac{e^{\int_t^s a(\theta)\,d\theta}}{e^{\int_0^T a(\theta)\,d\theta} - 1}$ is Green's Kernel, which is bounded by

$$\alpha = \frac{1}{\delta-1} \leq G(t,s) \leq \frac{\delta}{\delta-1} = \beta, \quad \delta = e^{\int_0^T a(\theta)\,d\theta}.$$

The following concept from the Leggett–Williams multiple fixed point theorem [10] is needed. Let X be a Banach space and K be a cone in X.

For $a > 0$, define $K_a = \{x \in K; \|x\| < a\}$. A mapping ψ is said to be a concave nonnegative continuous functional on K if $\psi : K \to [0,\infty)$ is continuous and

$$\psi(\mu x + (1-\mu)y) \geq \mu\psi(x) + (1-\mu)\psi(y), \quad x, y \in K, \quad \mu \in [0,1].$$

Let $b, c > 0$ be constants with K and X as defined above. Define

$$K(\psi, b, c) = \big\{x \in K; \psi(x) \geq b, \|x\| \leq c\big\}.$$

Theorem 2.1 (Leggett–Williams multiple fixed point theorem [10, Theorem 3.3]). *Let $X = (X, \|\cdot\|)$ be a Banach space and $K \subset X$ a cone, and $c_4 > 0$ a constant. Suppose there exists a concave nonnegative continuous functional ψ on K with $\psi(u) \leq \|u\|$ for $u \in \overline{K}_{c_4}$ and let $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$ be a continuous compact map. Assume that there are numbers c_1, c_2 and c_3 with $0 < c_1 < c_2 < c_3 \leq c_4$ such that*

(i) $\big\{u \in K(\psi, c_2, c_3);\ \psi(u) > c_2\big\} \neq \phi$ *and* $\psi(Au) > c_2$ *for all* $u \in K(\psi, c_2, c_3)$*;*
(ii) $\|Au\| < c_1$ *for all* $u \in \overline{K}_{c_1}$*;*
(iii) $\psi(Au) > c_2$ *for all* $u \in K(\psi, c_2, c_4)$ *with* $\|Au\| > c_3$.

Then A has at least three fixed points u_1, u_2 and u_3 in $\overline{K}_{c_4}$. Furthermore, we have $u_1 \in \overline{K}_{c_1}$, $u_2 \in \{u \in K(\psi, c_2, c_4); \psi(u) > c_2\}$, $u_3 \in \overline{K}_{c_4} \setminus \{K(\psi, c_2, c_4) \cup \overline{K}_{c_1}\}$.

In this article, X will denote the set of continuous T-periodic functions, which forms a Banach space under the norm $\|x\| = \sup_{0\leq t\leq T} |x(t)|$. Define an operator A on X by

$$(Ax)(t) = \int_t^{t+T} G(t,s)p(s)x(s-\tau(s))e^{-\gamma(s)x(s-\tau(s))}\, ds$$

and a cone K on X by

$$K = \Big\{x \in X; x(t) \geq \frac{1}{\delta}\|x\|\Big\}.$$

It is easy to verify that $A(K) \subset K$ and A is a completely continuous operator on K. Further, the existence of a positive periodic solution of (1) is equivalent to the existence of a fixed point of A in K.

According to the localization of the fixed points in Theorem 2.1, one of them is possibly a zero (namely $u_1 \in \overline{K}_{c_1}$). Thus, the operator A has at least two positive fixed points and a zero fixed point as can be easily observed. Accordingly, (1) has two positive T-periodic solutions and a possible trivial solution (if the conditions of Theorem 1 are satisfied).

On the cone K, we define a nonnegative concave functional ψ as

$$\psi(x) = \inf_{0\leq t\leq T} x(t)$$

and let

$$\gamma = \max_{0 \le t \le T} \gamma(t).$$

Now, we are ready to prove our main results in this paper.

Theorem 2.2. *Let $m > 1$, $a(t) > 0$ and $\gamma(t) > 0$ for $t \in R$, and*

$$\int_0^T p(t)\,dt > e(\delta - 1)\delta^{m-1}\gamma^{\frac{m-1}{n}} \tag{6}$$

hold. Then (1) *has at least two positive T-periodic solutions.*

Proof. From

$$\limsup_{x\to\infty} \max_{0\le t\le T} \frac{p(t)x^{m-1}e^{-\gamma(t)x^n}}{a(t)} = 0$$

it follows that there exist constants $0 < \mu_1 < 1$ and $\eta > 0$ such that

$$\frac{p(t)x^m e^{-\gamma(t)x^n}}{a(t)} < \mu_1 x \ \text{ for } \ 0 \le t \le T, \ \ x \ge \eta.$$

Let

$$\mu_2 = \max_{0\le t\le T, 0\le x\le \eta} \frac{p(t)x^m e^{-\gamma(t)x^n}}{a(t)}.$$

Then

$$\frac{p(t)x^m e^{-\gamma(t)x^n}}{a(t)} < \mu_1 x + \mu_2, \ \text{ for } \ x \ge 0 \ \text{ and } \ 0 \le t \le T.$$

Choose $c_4 > 0$ such that

$$c_4 > \max\left\{\frac{\mu_2}{1-\mu_1}, \frac{1}{\gamma^{\frac{1}{n}}}\right\}.$$

Then for $x \in \overline{K}_{c_4}$, we have

$$\begin{aligned}
\|Ax\| &\le \sup_{0\le t\le T} \int_t^{t+T} G(t,s)p(s)x^m(s-\tau(s))e^{-\gamma(s)x^n(s-\tau(s))}\,ds \le \\
&\le \sup_{0\le t\le T} \int_t^{t+T} G(t,s)a(s)(\mu_1 x(s-\tau(s)) + \mu_2)\,ds \le \\
&\le \sup_{0\le t\le T} \int_t^{t+T} G(t,s)a(s)(\mu_1\|x\| + \mu_2)\,ds \le \\
&\le \mu_1 c_4 + \mu_2 \le c_4.
\end{aligned}$$

Hence $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$. Set $c_2 = \frac{1}{\delta\gamma^{\frac{1}{n}}}$ and $c_3 = \frac{1}{\gamma^{\frac{1}{n}}}$. Clearly $c_2 < \delta c_2 = c_3 \le c_4$. Setting $\phi_0(t) = \phi_0 = \frac{c_2+c_3}{2}$, we have that $\phi_0 \in \{x; x \in$

$K(\psi, c_2, c_3), \psi(x) > c_2\} \neq \phi$. Now, for $x \in K(\psi, c_2, c_3)$ we obtain

$$\psi(Ax) = \min_{0\le t\le T} \int\limits_t^{t+T} G(t,s)p(s)x^m(s-\tau(s))e^{-\gamma(s)x^n(s-\tau(s))} \ge$$

$$\ge \frac{1}{\delta - 1} c_2^m e^{-\gamma\delta^n c_2^n} \int\limits_0^T p(s)\,ds > c_2.$$

Hence the condition (i) of Theorem 2.1 is satisfied. Since $m > 1$, we have that

$$\limsup_{x\to 0} \max_{0\le t\le T} \frac{p(t)x^m e^{-\gamma(t)x^n}}{a(t)x} = 0$$

implies that there exists a constant $c_1 \in (0, c_2)$ small enough such that

$$\frac{p(t)x^m e^{-\gamma(t)x^n}}{a(t)x} < 1 \text{ for } 0 \le x \le c_1.$$

Thus for $x \in \overline{K}_{c_1}$, we have

$$\|Ax\| \le \sup_{0\le t\le T} \int\limits_t^{t+T} G(t,s)p(s)x^m(s-\tau(s))e^{-\gamma(s)x^n(s-\tau(s))}\,ds <$$

$$< \sup_{0\le t\le T} \int\limits_t^{t+T} G(t,s)a(s)\|x\|\,ds \le c_1,$$

that is, $A : \overline{K}_{c_1} \to \overline{K}_{c_1}$. Thus the property (ii) of Theorem 2.1 is satisfied.

Finally, for $x \in K(\psi, c_2, c_4)$ with $\|Ax\| > c_3$,

$$c_3 < \|Ax\| \le \frac{\delta}{\delta - 1} \int\limits_0^T p(s)x^m(s-\tau(s))e^{-\gamma(s)x^n(s-\tau(s))}\,ds$$

implies that

$$\psi(Ax) \ge \frac{1}{\delta - 1} \int\limits_0^T p(s)x^m(s-\tau(s))e^{-\gamma(s)x^n(s-\tau(s))}\,ds >$$

$$> \frac{1}{\delta}\, c_3 = c_2.$$

This shows that the condition (iii) of Theorem 2.1 is satisfied. By Theorem 2.1, the equation (1) has at least two positive T-periodic solutions. This completes the proof of the theorem. □

The following corollary can be obtained as an immediate consequence of Theorem 2.2.

Corollary 2.3. *If $m > 1$, $a > 0$, $\gamma > 0$ and*

$$pT > e(\delta - 1)\delta^{m-1}\gamma^{\frac{m-1}{n}} \tag{7}$$

hold, then (3) *has at least two positive T-periodic solutions, where $\delta = e^{aT}$.*

Remark 2.4. The conditions of Theorem 1.1 imply the conditions of Corollary 2.3. However, Corollary 2.3 gives two positive T-periodic solutions where as Theorem 1.1 yields three positive T-periodic solutions. Although the range on p defined in Theorem 1.1 forces us to assume that $pT < 1$ and $2e(\delta - 1)\delta^{m-1}\gamma^{\frac{m-1}{n}} \leq 1$ must hold. On the other hand, the condition (7) is sufficient in corollary 2.3 for the existence of two positive periodic solutions of (1).

In what follows, we prove another theorem on the existence of two positive periodic solutions of (1).

Theorem 2.5. *Let $m > 1$, $a(t) > 0$ and $\gamma(t) > 0$ for $t \in R$, and*

$$\min_{0 \leq t \leq T} \left\{ \frac{p(t)}{a(t)} \right\} > e\delta^{m-1}\gamma^{\frac{m-1}{n}} \tag{8}$$

hold. Then (1) *has at least two positive T-periodic solutions.*

Proof. Set $c_2 = \frac{1}{\delta\gamma^{\frac{1}{n}}}$ and $c_3 = \frac{1}{\gamma^{\frac{1}{n}}}$. Choose $c_4 > 0$ as in Theorem 2.2. One may proceed as in Theorem 2.2 to prove that $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$. Clearly, $\phi_0 = \phi_0(t) = \frac{c_2 + c_3}{2} \in \{x,\ x \in K(\psi, c_2, c_3), \psi(x) > c_2\} \neq 0$. For $x \in K(\psi, c_2, c_3)$, we have

$$\psi(Ax) > \min_{0 \leq t \leq T} \left\{ \frac{p(t)}{a(t)} \right\} c_2^m e^{-\gamma\delta^n c_2^n} \int_t^{t+T} G(t,s)a(s)\,ds > c_2.$$

Choose $c_1 = \frac{1}{\max\{\frac{p(t)}{a(t)}\}^{\frac{1}{m-1}}}$. Using (8) we have $c_1 < c_2$. Now, for $x \in \overline{K}_{c_1}$ we obtain

$$\|Ax\| < \max_{0 \leq t \leq T} \left\{ \frac{p(t)}{a(t)} \right\} c_1^m = c_1.$$

The third condition of Theorem 2.1 is easy to verify and hence we omit it. The theorem is proved. □

The following corollary follows from Theorem 2.5 as a direct application to equation (3).

Corollary 2.6. *Let $m > 1$, $a > 0$, $\gamma > 0$ and*

$$p > ae^{1+(m-1)aT}\gamma^{\frac{m-1}{n}} \tag{9}$$

hold. Then (3) *has at least two positive T-periodic solutions.*

Remark 2.7. Since $aT < e^{aT} - 1$, Corollary 2.6 gives a better sufficient condition than the one in Corollary 2.3.

3. Conclusion

In this paper, we have been able to find sufficient conditions for the existence of multiple periodic solutions of (1) when $m > 1$. We have not obtained any result concerning the existence of multiple periodic solutions of (1) when $0 \leq m \leq 1$. As mentioned earlier, many authors [5], [7], [16], [23], [24], [27]–[29] have used Krasnoselskiĭ and other fixed point theorems for the existence of one periodic solution of (1) when $m = 1$, that is, of equation (2). From the literature, it seems that no results have been obtained regarding the existence of multiple periodic solutions of (1) with $0 \leq m \leq 1$. Thus, it would be interesting to obtain sufficient conditions for the existence of multiple periodic solutions of (1) when $0 \leq m \leq 1$. This is left as an open problem.

Acknowledgments

This work is supported by National Board for Higher Mathematics, Department of Atomic Energy, Govt. of India, under sponsored research scheme vide grant no. 48/5/2006-R&D-II/1350 dated 26.02.2007.

The authors are thankful to the referee for valuable comments and suggestions in revising the manuscript to the present form.

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(Received 12.04.2009)

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