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Malkhaz Ashordia

ON TWO-POINT SINGULAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF LINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. The two-point boundary value problem is considered for the system of linear generalized ordinary differential equations with singularities on a non-closed interval. The constant term of the system is a vector-function with bounded total variations components on the closure of the interval, and the components of the matrix-function have bounded total variations on every closed interval from this interval.

The general sufficient conditions are established for the unique solvability of this problem in the case where the system has singularities. Singularity is understand in a sense the components of the matrix-function corresponding to the system may have unbounded variations on the interval.

Relying on these results the effective conditions are established for the unique solvability of the problem.

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რეზიუმე. განზოგადებულ ჩვეულებრივ წრფივ დიფერენციალურ განტოლებათა სისტემისათვის სინგულარობებით არაჩაკეტილ ინტერვალზე განხილულია ორწერტილოვანი სასაზღვრო ამოცანა. ამ სისტემის თავისუფალი წევრი სასრული ვარიაციის კომპონენტებიანი ვექტორული ფუნქციაა აღნიშნული ინტერვალის ჩაკეტვაზე, ხოლო მატრიცული ფუნქციის კომპონენტებს კი აქვს სასრული ვარიაციები ყოველ ჩაკეტილ სეგმენტზე ამ ინტერვალიდან.

მიღებულია ამ ამოცანის ცალსახად ამოხსნადობის ზოგადი საკმარისი პირობები, როცა სისტემას გააჩნია სინგულარობები. სინგულარობა გაიგება იმ აზრით, რომ სისტემის შესაბამისი მატრიცული ფუნქციის კომპონენეტებს შეიძლება ჰქონდეს შემოუსაზღვრელი ვარიაციები განსახილველ შუალედზე.

ამ შედეგებზე დაყრდნობით დადგენილია ამ ამოცანის ცალსახად ამოხსნადობის ეფექტური პირობები.

1. Statement of the Problem and Basic Notation

In the present paper, for a system of linear generalized ordinary differential equations with singularities

$$dx(t) = dA(t) \cdot x(t) + df(t) \tag{1.1}$$

we consider the two-point boundary value problem

$$x_i(a+) = 0 \ \ (i = 1, \dots, n_0), \ \ x_i(b-) = 0 \ \ (i = n_0 + 1, \dots, n), \tag{1.2}$$

where $-\infty < a < b < +\infty$, $n_0 \in \{1, \dots, n\}$, $x_1, \dots, x_n$ are the components of the desired solution x, $n_0 \in \{1, \dots, n\}$, $f : [a, b] \to \mathbb{R}^n$ is a vector-function with bounded total variation components, and $A :]a, b[\to \mathbb{R}^{n\times n}$ is a matrix-function with bounded total variation components on every closed interval from the interval $]a, b[$.

We investigate the question of unique solvability of the problem (1.1), (1.2), when the system (1.1) has singularities. Singularity is understand in a sense that the components of the matrix-function A may have unbounded variation on the closed interval $[a, b]$, in general. On the basis of this theorem we obtain effective criteria for the solvability of this problem.

Analogous and related questions are investigated in [17–24] and [26] (see also references therein) for the singular two-point and multipoint boundary value problems for linear and nonlinear systems of ordinary differential equations, and in [1, 3, 6, 8, 10] (see also references therein) for regular two-point and multipoint boundary value problems for systems of linear and nonlinear generalized differential equations. As for the two-point and multipoint singular boundary value problems for generalized differential systems, they are little studied and, despite some results given in [12] and [13] for two-point singular boundary value problem, their theory is rather far from completion even in the linear case. Therefore, the problem under consideration is actual.

To a considerable extent, the interest in the theory of generalized ordinary differential equations has been motivated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see e.g. [1–13, 15, 16, 25, 27–29] and references therein).

Throughout the paper, the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$; $R_+ = [0, +\infty[$; $[a, b]$, $]a, b[$ and $]a, b]$, $[a, b[$ are, respectively, closed, open and half-open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{il})_{i,l=1}^{n,m}$ with the norm

$$\|X\| = \sum_{i,l=1}^{n,m} |x_{il}|.$$

$\mathbb{R}_+^{n\times m} = \big\{(x_{il})_{i,l=1}^{n,m} : \ x_{il} \geq 0 \ (i = 1, \dots, n; \ l = 1, \dots, m)\big\}$.

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix.

If $X=(x_{il})_{i,l=1}^{n,m}\in\mathbb{R}^{n\times m}$, then $|X|=(|x_{il}|)_{i,l=1}^{n,m}$.

$\mathbb{R}^n=R^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n=\mathbb{R}_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times m}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n\times n$-matrix; δ_{il} is the Kroneker symbol, i.e., $\delta_{ii}=1$ and $\delta_{il}=1$ for $i\neq l$ $(i,l=1,\dots,n)$.

$\bigvee\limits_c^d(X)$, where $a<c<d<b$, is the variation of the matrix-function $X:\,]a,b[\,\to\mathbb{R}^{n\times m}$ on the closed interval $[c,d]$, i.e., the sum of total variations of the latter components x_{il} $(i=1,\dots,n;\ l=1,\dots,m)$ on this interval; if $d<c$, then $\bigvee\limits_c^d(X)=-\bigvee\limits_d^c(X)$; $V(X)(t)=(v(x_{il})(t))_{i,l=1}^{n,m}$, where $v(x_{il})(c_0)=0$, $v(x_{il})(t)=\bigvee\limits_{c_0}^t(x_{il})$ for $a<t<b$, and $c_0=(a+b)/2$.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X:\,]a,b[\,\to\mathbb{R}^{n\times m}$ at the point $t\in]a,b[$ (we assume $X(t)=X(a+)$ for $t\le a$ and $X(t)=X(b-)$ for $t\ge b$, if necessary).

$d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded variation $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X)<+\infty$);

$\|X\|_s=\sup\big\{\|X(t)\|:\ t\in[a,b]\big\}$, $\|X\|_v=\|X(a)\|+\bigvee\limits_a^b(X)$;

$\mathrm{BV}_s([a,b],\mathbb{R}^{n\times m})$ is the normed space $(\mathrm{BV}([a,b],\mathbb{R}^{n\times m}),\|\cdot\|_s)$;

$\mathrm{BV}_v([a,b],\mathbb{R}^{n\times m})$ is the Banach space $(\mathrm{BV}([a,b],\mathbb{R}^{n\times m}),\|\cdot\|_v)$.

$\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\,]a,b[\,\to\mathbb{R}^{n\times m}$ such that $\bigvee\limits_c^d(X)<+\infty$ for every $a<c<d<b$.

If $X\in\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$, $\det(I_n+(-1)^jd_jX(t))\neq 0$ for $t\in]a,b[$ $(j=1,2)$, and $Y\in\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times m})$, then $\mathcal{A}(X,Y)(t)\equiv\mathcal{B}(X,Y)(c_0,t)$, where $\mathcal{B}$ is the operator defined by

$$\mathcal{B}(X,Y)(t,t)=O_{n\times m}\ \text{ for }\ t\in]a,b[\,,$$

$$\mathcal{B}(X,Y)(s,t)=Y(t)-Y(s)+\sum_{s<\tau\le t}d_1X(\tau)\cdot\big(I_n-d_1X(\tau)\big)^{-1}d_1Y(\tau)-$$

$$-\sum_{s\le\tau<t}d_2X(\tau)\cdot\big(I_n+d_2X(\tau)\big)^{-1}d_2Y(\tau)\ \text{ for }\ a<s<t<b$$

and

$$\mathcal{B}(X,Y)(s,t)=-\mathcal{B}(X,Y)(t,s)\ \text{ for }\ a<t<s<b.$$

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If $\alpha \in \mathrm{BV}([a,b],\mathbb{R})$ has no more than a finite number of points of discontinuity, and $m \in \{1,2\}$, then $D_{\alpha m} = \{t_{\alpha m1}, \dots, t_{\alpha m n_{\alpha m}}\}$ ($t_{\alpha m1} < \dots < t_{\alpha m n_{\alpha m}}$) is the set of all points from $[a,b]$ for which $d_m\alpha(t) \neq 0$, and $\mu_{\alpha m} = \max\{d_m\alpha(t) : t \in D_{\alpha m}\}$ $(m = 1,2)$.

If $\beta \in \mathrm{BV}([a,b],\mathbb{R})$, then

$$\nu_{\alpha m\beta j} = \max\left\{d_j\beta(t_{\alpha ml}) + \sum_{t_{\alpha m\, l+1-m} < \tau < t_{\alpha m\, l+2-m}} d_j\beta(\tau) : \ l = 1, \dots, n_{\alpha m}\right\}$$

$(j, m = 1,2)$; here $t_{\alpha 20} = a - 1$, $t_{\alpha 1 n_{\alpha 1}+1} = b + 1$.

$s_j : \mathrm{BV}([a,b],\mathbb{R}) \to \mathrm{BV}([a,b],\mathbb{R})$ $(j = 0,1,2)$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,$$

$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \ \text{ and } \ s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \ \text{ for } \ a < t \le b,$$

and

$$s_0(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \ \text{ for } \ t \in [a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,ds_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_0(g))$ corresponding to the function $s_0(g)$; if $a = b$, then we assume $\int_a^b x(t)\,dg(t) = 0$. Moreover, we put

$$\int_{s+}^t x(\tau)\,dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int_{s+\varepsilon}^t x(\tau)\,dg(\tau)$$

and

$$\int_s^{t-} x(\tau)\,dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int_s^{t-\varepsilon} x(\tau)\,dg(\tau).$$

$L([a,b],\mathbb{R};g)$ is the space of all functions $x : [a,b] \to \mathbb{R}$ measurable and integrable with respect to the measure $\mu(g)$ with the norm

$$\|x\|_{L,g} = \int_a^b |x(t)|\,dg(t).$$

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\, dg(\tau) = \int_s^t x(\tau)\, dg_1(\tau) - \int_s^t x(\tau)\, dg_2(\tau) \ \text{ for } \ s \le t.$$

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to R^{l\times n}$ is a nondecreasing matrix-function and $D \subset \mathbb{R}^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to D$ such that $x_{kj} \in L([a,b],R;g_{ik})$ $(i = 1,\dots,l;\ k = 1,\dots,n;\ j = 1,\dots,m)$;

$$\int_s^t dG(\tau)\cdot X(\tau) = \bigg(\sum_{k=1}^n \int_s^t x_{kj}(\tau) dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m} \ \text{ for } \ a \le s \le t \le b,$$

$$S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \ \ (j = 0,1,2).$$

If $G_j : [a,b] \to \mathbb{R}^{l\times n}$ $(j = 1,2)$ are nondecreasing matrix-functions, $G(t) \equiv G_1(t) - G_2(t)$ and $X : [a,b] \to \mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \int_s^t dG_1(\tau)\cdot X(\tau) - \int_s^t dG_2(\tau)\cdot X(\tau) \ \text{ for } \ s \le t,$$

$$S_k(G) = S_k(G_1) - S_k(G_2) \quad (k = 0,1,2),$$

$$L([a,b],D;G) = \bigcap_{j=1}^2 L([a,b],D;G_j),$$

The inequalities between the vectors and between the matrices are understood componentwise.

We assume that the vector-function $f = (f_i)_{i=1}^n$ belongs to $\mathrm{BV}([a,b],\mathbb{R}^n)$, and the matrix-function $A = (a_{il})_{i,l=1}^n$ is such that $a_{il} \in \mathrm{BV}([a,b],\mathbb{R})$ $(i \ne l;\ i,l = 1,\dots,n)$, $a_{ii} \in \mathrm{BV}(]a,b],\mathbb{R})$ $(i = 1,\dots,n_0)$ and $a_{ii} \in \mathrm{BV}([a,b[\,,\mathbb{R})$ $(i = n_0+1,\dots,n)$.

A vector-function $x = (x_i)_{i=1}^n$ is said to be a solution of the system (1.1) if $x_i \in \mathrm{BV}_{loc}(]a,b],\mathbb{R})$ $(i = 1,\dots,n_0)$, $x_i \in \mathrm{BV}_{loc}([a,b[\,,\mathbb{R})$ $(i = n_0+1,\dots,n)$ and

$$x_i(t) = x_i(s) + \sum_{l=1}^n \int_s^t x_l(\tau)\, da_{il}(\tau) + f_i(t) - f_i(s)$$

for $a<s\le t\le b$ $(i=1,\dots,n_0)$ and for $a\le s<t<b$ $(i=n_0+1,\dots,n)$.

Under the solution of the problem (1.1), (1.2) we mean a solution $x(t) = (x_i(t))_{i=1}^n$ of the system (1.1) such that the one-sided limits $x_i(a+)$ $(i = 1,\dots,n_0)$ and $x_i(b-)$ $(i = n_0+1,\dots,n)$ exist and the equalities (1.2) are fulfilled. We assume $x_i(a) = 0$ $(i = 1,\dots,n_0)$ and $x_i(b) = 0$ $(i = n_0 + 1,\dots,n)$, if necessary.

A vector-function $x \in \mathrm{BV}([a,b],\mathbb{R}^n)$ is said to be a solution of the system of generalized differential inequalities

$$dx(t) - dB(t) \cdot x(t) - dq(t) \le 0 \ (\ge 0) \ \text{ for } \ t \in [a,b],$$

where $B \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$, $q \in \mathrm{BV}([a,b],\mathbb{R}^n)$, if

$$x(t) - x(s) + \int\limits_s^t dB(\tau) \cdot x(\tau) - q(t) + q(s) \le 0 \ (\ge 0) \ \text{ for } \ a \le s \le t \le b.$$

Without loss of generality we assume that $A(a) = O_{n\times n}$, $f(a) = 0$. Moreover, we assume

$$\det(I_n + (-1)^j d_j A(t)) \ne 0 \ \text{ for } \ t \in]a,b[\ \ (j=1,2). \tag{1.3}$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding system (see [29, Theorem III.1.4]).

If $s \in]a,b[$ and $\alpha \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R})$ are such that

$$1 + (-1)^j d_j\beta(t) \ne 0 \ \text{ for } \ (-1)^j(t-s) < 0 \ \ (j=1,2),$$

then by $\gamma_\beta(\cdot\,,s)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t) = \gamma(t)d\beta(t), \ \ \gamma(s) = 1.$$

It is known (see [15, 16]) that

$$\gamma_\alpha(t,s) = \begin{cases} \exp\big(s_0(\beta)(t) - s_0(\beta)(s)\big)\times \\ \qquad \times \prod\limits_{s<\tau\le t}(1-d_1\alpha(\tau))^{-1}\prod\limits_{s\le\tau<t}(1+d_2\beta(\tau)) & \text{for } \ t>s, \\ \exp\big(s_0(\beta(t) - s_0(\beta(s))\big)\times \\ \qquad \times \prod\limits_{t<\tau\le s}(1-d_1\beta(\tau))\prod\limits_{t\le\tau<s}(1+d_2\beta(\tau))^{-1} & \text{for } \ t<s, \\ 1 & \text{for } \ t=s. \end{cases} \tag{1.4}$$

It is evident that if the last inequalities are fulfilled on the whole interval $[a,b]$, then $\gamma_\alpha^{-1}(t)$ exists for every $t \in [a,b]$.

Definition 1.1. Let $n_0 \in \{1,\dots,n\}$. We say that a matrix-function $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ belongs to the set $\mathcal{U}(a+,b-;n_0)$ if the functions c_{il} $(i \ne l;\ i,l = 1,\dots,n)$ are nondecreasing on $[a,b]$ and the system

$$\operatorname{sgn}\Big(n_0 + \frac{1}{2} - i\Big)\, dx_i(t) \le \sum_{l=1}^n x_l(t)\, dc_{il}(t) \ \text{ for } \ t \in [a,b] \ \ (i=1,\dots,n) \tag{1.5}$$

has no nontrivial nonnegative solution satisfying the condition (1.2).

The similar definition of the set $\mathcal{U}$ has been introduced by I. Kiguradze for ordinary differential equations (see [20, 21]).

Theorem 1.1. *Let the components of the matrix-function* $A=(a_{il})_{i,l=1}^n\in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$ *satisfy the conditions*

$$\begin{gathered}\big(s_0(a_{ii})(t)-s_0(a_{ii})(s)\big)\operatorname{sgn}\Big(n_0+\frac{1}{2}-i\Big)\leq\\ \leq s_0(c_{ii}-\alpha_i)(t)-s_0(c_{ii}-\alpha_i)(s)\ \ \text{for}\ \ a<s<t<b\ \ (i=1,\dots,n),\end{gathered}\tag{1.6}$$

$$\begin{gathered}(-1)^j\big(\big|1+(-1)^j d_j a_{ii}(t)\big|-1\big)\operatorname{sgn}\Big(n_0+\frac{1}{2}-i\Big)\leq\\ \leq d_j\big(c_{ii}(t)-\alpha_i(t)\big)\ \ \text{for}\ \ t\in\,]a,b]\ \ (j=1,2;\ i=1,\dots,n_0)\\ \text{and for}\ \ t\in[a,b[\ \ (j=1,2;\ i=n_0+1,\dots,n),\end{gathered}\tag{1.7}$$

$$\begin{gathered}\big|s_0(a_{il})(t)-s_0(a_{il})(s)\big|\leq\\ \leq s_0(c_{il})(t)-s_0(c_{il})(s)\ \ \text{for}\ \ a<s<t<b\ \ (i\neq l;\ \ i,l=1,\dots,n)\end{gathered}\tag{1.8}$$

and

$$|d_j a_{il}(t)|\leq d_j c_{il}(t)\ \ \text{for}\ \ t\in[a,b]\ \ (i\neq l;\ i,l=1,\dots,n),\tag{1.9}$$

where

$$C=(c_{il})_{i,l=1}^n\in\mathcal{U}(a+,b-;\ n_0),\tag{1.10}$$

$\alpha_i:]a,b]\to\mathbb{R}$ $(i=1,\dots,n_0)$ *and* $\alpha_i:[a,b[\to\mathbb{R}$ $(i=n_0+1,\dots,n)$ *are nondecreasing functions such that*

$$\begin{gathered}\lim_{t\to a+} d_2\alpha_i(t)<1\ \ (i=1,\dots,n_0),\\ \lim_{t\to b-} d_1\alpha_i(t)<1\ \ (i=n_0+1,\dots,n)\end{gathered}\tag{1.11}$$

and

$$\begin{gathered}\lim_{t\to a+}\ \lim_{k\to\infty}\sup\gamma_{\beta_i}(t,a+1/k)=0\ \ (i=1,\dots,n_0),\\ \lim_{t\to b-}\ \lim_{k\to\infty}\sup\gamma_{\beta_i}(t,b-1/k)=0\ \ (i=n_0+1,\dots,n),\end{gathered}\tag{1.12}$$

where $\beta_i(t)\equiv\alpha_i(t)\operatorname{sgn}\big(n_0+\frac{1}{2}-i\big)$ $(i=1,\dots,n)$. *Then the problem* (1.1), (1.2) *has one and only one solution.*

Corollary 1.1. *Let the components of the matrix-function* $A=(a_{il})_{i,l=1}^n\in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$ *satisfy the conditions*

$$\begin{gathered}\big(s_0(a_{ii})(t)-s_0(a_{ii})(s)\big)\operatorname{sgn}\Big(n_0+\frac{1}{2}-i\Big)\leq-\big(s_0(\alpha_i)(t)-s_0(\alpha_i)(s)\big)\\ +\int_s^t h_{ii}(\tau)\,ds_0(\beta_i)(\tau)\ \ \text{for}\ \ a<s<t<b\ \ (i=1,\dots,n),\end{gathered}\tag{1.13}$$

$$\begin{gathered}(-1)^j\big(\big|1+(-1)^j d_j a_{ii}(t)\big|-1\big)\operatorname{sgn}\Big(n_0+\frac{1}{2}-i\Big)\leq\\ \leq h_{ii}(t)d_j\beta_i(t)-d_j\alpha_i(t)\big)\ \ \text{for}\ \ t\in\,]a,b]\ \ (j=1,2;\ i=1,\dots,n_0)\\ \text{and for}\ \ t\in[a,b[\ \ (j=1,2;\ i=n_0+1,\dots,n),\end{gathered}$$

$$\big|s_0(a_{il})(t)-s_0(a_{il})(s)\big|\leq$$
$$\leq\int\limits_s^t h_{il}(\tau)\,ds_0(\beta_l)(\tau)\ \ for\ \ a<s<t<b\ \ (i\neq l;\ \ i,l=1,\dots,n)\qquad(1.14)$$

and

$$|d_j a_{il}(t)|\leq h_{il}(t)d_j\beta_l(t)\ \ for\ \ t\in[a,b]\ \ (i\neq l;\ i,l=1,\dots,n),\qquad(1.15)$$

where $\alpha_i:]a,b]\to\mathbb{R}$ $(i=1,\dots,n_0)$ *and* $\alpha_i:[a,b[\to\mathbb{R}$ $(i=n_0+1,\dots,n)$ *are nondecreasing functions satisfying the conditions* (1.11) *and* (1.12), β_l $(l=1,\dots,n)$ *are functions nondecreasing on* $[a,b]$ *and having not more than a finite number of points of discontinuity,* $h_{ii}\in L^\mu([a,b],\mathbb{R};\beta_i)$, $h_{il}\in L^\mu([a,b],\mathbb{R}_+;\beta_l)$ $(i\neq l;\ i,l=1,\dots,n)$, $1\leq\mu\leq+\infty$. *Let, moreover,*

$$r(\mathcal{H})<1,\qquad(1.16)$$

where the $3n\times 3n$*-matrix* $\mathcal{H}=(\mathcal{H}_{j+1\,m+1})_{j,m=0}^2$ *is defined by*

$$\mathcal{H}_{j+1\,m+1}=\big(\lambda_{kmij}\|h_{ik}\|_{\mu,s_m(\beta_i)}\big)_{i,k=1}^n\quad(j,m=0,1,2),$$
$$\xi_{ij}=\big(s_j(\beta_i)(b)-s_j(\beta_i)(a)\big)^{\frac{1}{\nu}}\quad(j=0,1,2,;\ \ i=1,\dots,n);$$
$$\lambda_{k0i0}=\begin{cases}\Big(\dfrac{4}{\pi^2}\Big)^{\frac{1}{\nu}}\xi_{k0}^2 & if\ \ s_0(\beta_i)(t)\equiv s_0(\beta_k)(t),\\ \xi_{k0}\xi_{i0} & if\ \ s_0(\beta_i)(t)\not\equiv s_0(\beta_k)(t)\ \ (i,k=1,\dots,n);\end{cases}$$
$$\lambda_{kmij}=\xi_{km}\xi_{ij}\ \ if\ \ m^2+j^2>0,\ \ mj=0\ \ (j,m=0,1,2;\ \ i,k=1,\dots,n),$$
$$\lambda_{kmij}=\Big(\frac{1}{4}\,\mu_{\alpha_k m}\nu_{\alpha_k m\alpha_i j}\sin^{-2}\frac{\pi}{4n_{\alpha_k m}+2}\Big)^{\frac{1}{\nu}}\ \ (j,m=1,2;\ \ i,k=1,\dots,n),$$

and $\frac{1}{\mu}+\frac{2}{\nu}=1$. *Then the problem* (1.1), (1.2) *has one and only one solution.*

Remark 1.1. The $3n\times 3n$-matrix $\mathcal{H}$, appearing in Corollary 1.1 can be replaced by the $n\times n$-matrix

$$\bigg(\max\Big\{\sum_{j=0}^2\lambda_{kmij}\|h_{ik}\|_{\mu,S_m(\alpha_k)}:\ m=0,1,2\Big\}\bigg)_{i,k=1}^n.$$

By Remark 1.1, Corollary 1.1 has the following form for $h_{il}(t)\equiv h_{il}=$ *const* $(i,l=1,\dots,n)$, $\alpha_i(t)\equiv\alpha(t)$ $(i=1,\dots,n)$, $\beta_i(t)\equiv\beta(t)$ $(i=1,\dots,n)$ and $\mu=+\infty$.

Corollary 1.2. *Let the components of the matrix-function* $A=(a_{il})_{i,l=1}^n\in\mathrm{BV}_{loc}(]a,b[,\mathbb{R}^{n\times n})$ *satisfy the conditions*

$$\big(s_0(a_{ii})(t)-s_0(a_{ii})(s)\big)\operatorname{sgn}\Big(n_0+\frac{1}{2}-i\Big)\leq h_{ii}\big(s_0(\beta)(t)-s_0(\beta)(s)\big)-$$
$$-\big(s_0(\alpha)(t)-s_0(\alpha)(s)\big)\ \ for\ \ a<s<t<b\ \ (i=1,\dots,n),$$

$$(-1)^j\big(\big|1+(-1)^j d_j a_{ii}(t)\big|-1\big)\operatorname{sgn}\Big(n_0+\frac{1}{2}-i\Big)\leq h_{ii}d_j\beta(t)-d_j\alpha(t)\big)$$
$$\text{for } t\in\,]a,b] \ \ (j=1,2;\ i=1,\dots,n_0)$$
$$\text{and for } t\in[a,b[\ \ (j=1,2;\ i=n_0+1,\dots,n),$$
$$\big|s_0(a_{il})(t)-s_0(a_{il})(s)\big|\leq h_{il}\big(s_0(\beta)(t)-s_0(\beta)(s)\big)$$
$$\text{for } a<s<t<b \ \ (i\neq l;\ \ i,l=1,\dots,n)$$

and

$$|d_j a_{il}(t)|\leq h_{il}d_j\beta(t) \ \ \text{for } t\in[a,b] \ \ (i\neq l;\ i,l=1,\dots,n)$$

hold, where $\alpha:[a,b]\to\mathbb{R}$ *is a nondecreasing function satisfying the conditions* (1.11) *and* (1.12), β *is a function nondecreasing on* $[a,b]$ *and having not more than a finite number of points of discontinuity,* $h_{ii}\in\mathbb{R}$, $h_{il}\in\mathbb{R}_+$ $(i\neq l;\, i,l=1,\dots,n)$. *Let, moreover,*

$$\rho_0\, r(\mathcal{H})<1,$$

where

$$\mathcal{H}=(h_{ik})_{i,k=1}^n,\quad \rho_0=\max\bigg\{\sum_{j=0}^{2}\lambda_{mj}:m=0,1,2\bigg\},$$
$$\lambda_{00}=\frac{2}{\pi}\big(s_0(\beta)(b)-s_0(\beta)(a)\big),$$
$$\lambda_{0j}=\lambda_{j0}=\big(s_0(\beta)(b)-s_0(\alpha)(a)\big)^{\frac{1}{2}}\big(s_j(\beta)(b)-s_j(\beta)(a)\big)^{\frac{1}{2}}\quad (j=1,2),$$
$$\lambda_{mj}=\frac{1}{2}\big(\mu_{\alpha m}\nu_{\alpha m\alpha j}\big)^{\frac{1}{2}}\sin^{-1}\frac{\pi}{4n_{\alpha m}+2}\quad (m,j=1,2).$$

Then the problem (1.1), (1.2) *has one and only one solution.*

Theorem 1.2. *Let the components of the matrix-function* $A=(a_{il})_{i,l=1}^n\in\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$ *satisfy the conditions* (1.6)–(1.9), *where* $c_{il}(t)\equiv h_{il}\beta_i(t)+\beta_{il}(t)$ $(i,l=1,\dots,n)$,

$$d_2\beta_i(a)\leq 0 \ \text{ and } \ 0\leq d_1\beta_i(t)<|\eta_i|^{-1} \ \text{ for } \ a<t\leq b \ \ (i=1,\dots,n_0), \tag{1.17}$$
$$d_1\beta_i(b)\leq 0 \ \text{ and } \ 0\leq d_2\beta_i(t)<|\eta_i|^{-1} \ \text{ for } \ a\leq t<b \ \ (i=n_0+1,\dots,n), \tag{1.18}$$

where $\alpha_i:\,]a,b]\to\mathbb{R}$ $(i=1,\dots,n_0)$ *and* $\alpha_i:[a,b[\,\to\mathbb{R}$ $(i=n_0+1,\dots,n)$ *are nondecreasing functions satisfying the conditions* (1.11) *and* (1.12), $h_{ii}<0$, $h_{il}\geq 0$, $\eta_i<0$ $(i\neq l;\, i,l=1,\dots,n)$, β_{ii} $(i=1,\dots,n)$ *are the functions nondecreasing on* $[a,b]$; $\beta_{il},\beta_i\in\mathrm{BV}([a,b],\mathbb{R})$ $(i\neq l; i,l=1,\dots,n)$ *are the functions nondecreasing on the interval* $]a,b]$ *for* $i\in\{1,\dots,n_0\}$ *and on the interval* $[a,b[$ *for* $i\in\{n_0+1,\dots,n\}$. *Let, moreover, the condition* (1.16) *hold, where* $\mathcal{H}=(\xi_{il})_{i,l=1}^n$,

$$\xi_{ii}=\eta_i,\ \ \xi_{il}=\frac{h_{il}}{|h_{ii}|}\ \ (i\neq l;\ i,l=1,\dots,n),$$
$$\eta_i=V\big(\mathcal{A}(\zeta_i,a_i)\big)(b)-V\big(\mathcal{A}(\zeta_i,a_i)\big)(a+)\ \ \text{for } i\in\{1,\dots,n_0\},$$

$$\eta_i = V\big(\mathcal{A}(\zeta_i, a_i)\big)(b-) - V\big(\mathcal{A}(\zeta_i, a_i)\big)(a) \ \ \textit{for} \ \ i \in \{n_0+1,\dots,n\};$$

$$\zeta_i(t) \equiv \sum_{k=l}^{n} \beta_{il}(t) \ \ (i=1,\dots,n),$$

$$a_i(t) \equiv \big(\beta_i(t) - \beta_i(a+)\big)h_{ii} \ \ \textit{for} \ \ a < t \le b \ \ (i=1,\dots,n_0),$$

$$a_i(t) \equiv \big(\beta_i(b-) - \beta_i(t)\big)h_{ii} \ \ \textit{for} \ \ a \le t < b \ \ (i=n_0+1,\dots,n).$$

Then the problem (1.1), (1.2) *has one and only one solution.*

Remark 1.2. If

$$\eta_i < 1 \ \ (i=1,\dots,n), \tag{1.19}$$

then, in Theorem 1.2, we can assume that

$$\xi_{ii} = 0, \ \ \xi_{il} = \frac{h_{il}}{(1-\eta_i)|h_{ii}|} \ \ (i \ne l;\ i,l=1,\dots,n). \tag{1.20}$$

Theorem 1.3. *Let the matrix-function $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ be such that the functions c_{il} $(i \ne l;\ i,l=1,\dots,n)$ are nondecreasing on $[a,b]$ and the problem* (1.5), (1.2) *has a nontrivial nonnegative solution, i.e., the condition* (1.10) *is violated. Let, moreover, $\alpha_i :]a,b] \to \mathbb{R}$ $(i=1,\dots,n_0)$ and $\alpha_i : [a,b[\to \mathbb{R}$ $(i=n_0+1,\dots,n)$ be nondecreasing functions satisfying the conditions* (1.11), (1.12) *and*

$$1 + (-1)^j \operatorname{sgn}\Big(n_0 + \frac{1}{2} - i\Big) d_j\big(c_{ii}(t) - \alpha_i(t)\big) > 0$$
$$\textit{for} \ \ t \in]a,b] \ \ (j=1,2;\ i=1,\dots,n_0)$$
$$\textit{and for} \ \ t \in [a,b[\ \ (j=1,2;\ i=n_0+1,\dots,n). \tag{1.21}$$

Then there exist a matrix-function $A = (a_{il})_{i,l=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$, a vector-function $f = (f_l)_{l=1}^n \in \mathrm{BV}([a,b],R^n)$ and nondecreasing functions $\widetilde{\alpha}_i :]a,b] \to \mathbb{R}$ $(i=1,\dots,n_0)$ and $\widetilde{\alpha}_i : [a,b[\to \mathbb{R}$ $(i=n_0+1,\dots,n)$ such that the conditions (1.6)–(1.12) *and*

$$\widetilde{\alpha}_i(t) - \widetilde{\alpha}_i(s) \le \alpha_i(t) - \alpha_i(s)$$
$$\textit{for} \ \ a < t < s \le b \ \ \textit{and for} \ \ a \le t < s < b \ \ (i=n_0+1,\dots,n) \tag{1.22}$$

are fulfilled, but the problem (1.1), (1.2) *is unsolvable. In addition, if the matrix-function $C = (c_{il})_{i,l=1}^n$ is such that*

$$\det\left(\big(\delta_{il} + (-1)^j \varepsilon_l d_j c_{il}(t) \operatorname{sgn}\Big(n_0 + \frac{1}{2} - i\Big)_{i,l=1}^{n}\right) \ne 0$$
$$\textit{for} \ \ t \in [a,b];\ \ \varepsilon_1,\dots,\varepsilon_n \in [a,b] \ \ (j=1,2), \tag{1.23}$$

then the matrix-function $A = (a_{il})_{i,l=1}^n$ satisfies the condition (1.3).

Remark 1.3. The condition (1.23) holds, for example, if either

$$\sum_{l=1}^{n} |d_j c_{il}(t)| < 1 \ \ \text{for} \ \ t \in [a,b] \ \ (j=1,2;\ \ i=1,\dots,n), \tag{1.24}$$

$$\sum_{l=1,\, l\neq i}^{n} |d_j c_{il}(t)| < 1 + (-1)^j \operatorname{sgn}\Big(n_0 + \frac{1}{2} - i\Big) d_j c_{ii}(t)$$
$$\text{for } \; t \in [a,b] \;\; (j=1,2; \;\; i=1,\dots,n) \tag{1.25}$$

or

$$\sum_{l=1,\, l\neq i}^{n} |d_j c_{li}(t)| < 1 + (-1)^j \operatorname{sgn}\Big(n_0 + \frac{1}{2} - i\Big) d_j c_{ii}(t)$$
$$\text{for } \;\; t \in [a,b] \;\; (j=1,2; \;\; i=1,\dots,n). \tag{1.26}$$

2. Auxiliary Propositions

Lemma 2.1. *Let* $t_0 \in [a,b]$, α *and* $q \in \mathrm{BV}_{loc}([a,t_0[\,,\mathbb{R}^n) \cap \mathrm{BV}_{loc}(]t_0,b],\mathbb{R}^n)$ *be such that*

$$1 + (-1)^j \operatorname{sgn}(t-t_0) d_j \alpha(t) > 0 \;\; \textit{for} \;\; t \in [a,b] \;\; (j=1,2). \tag{2.1}$$

Let, moreover, $x \in \mathrm{BV}_{loc}([a,t_0[\,,\mathbb{R}^n) \cap \mathrm{BV}_{loc}(]t_0,b],\mathbb{R}^n)$ *be a solution of the linear generalized differential inequality*

$$\operatorname{sgn}(t-t_0) dx(t) \le x(t) d\alpha(t) + dq(t) \tag{2.2}$$

on the intervals $[a,t_0[$ *and* $]t_0,b]$, *satisfying the inequalities*

$$x(t_0+) \le y(t_0+) \;\; \textit{and} \;\; x(t_0-) \le y(t_0-), \tag{2.3}$$

where $y \in \mathrm{BV}_{loc}([a,t_0[\,,\mathbb{R}^n) \cap \mathrm{BV}_{loc}(]t_0,b],\mathbb{R}^n)$ *is a solution of the general differential equality*

$$\operatorname{sgn}(t-t_0) dy(t) = y(t) d\alpha(t) + dq(t). \tag{2.4}$$

Then

$$x(t) \le y(t) \;\; \textit{for} \;\; t \in [a,t_0[\cup]t_0,b]. \tag{2.5}$$

Proof of Lemma 2.1. Assume $t_0 < b$ and consider the closed interval $[t_0 + \varepsilon, b]$, where ε is an arbitrary sufficiently small positive number.

By (2.1), the Cauchy problem

$$d\gamma(t) = \gamma(t) d\alpha(t), \;\; \gamma(s) = 1$$

has the unique solution γ_s for every $s \in [t_0+\varepsilon, b]$ and, by (1.4), this is positive, i.e.,

$$\gamma_s(t) > 0 \;\; \text{for} \;\; t \in [t_0+\varepsilon, b]. \tag{2.6}$$

According to the variation-of-constant formula (see [29, Corollary III.2.14]), from (2.4) we have

$$y(t) = q(t) - q(s) +$$
$$+ \gamma(t)\Big\{\gamma^{-1}(s) y(s) - \int_s^t \big(q(\tau) - q(s)\big) d\gamma^{-1}(\tau)\Big\} \;\; \text{for} \;\; s,t \in [t_0+\varepsilon, b], \tag{2.7}$$

where $\gamma(t) \equiv \gamma_{t_0+\varepsilon}(t)$.

From (2.2), we have

$$dx(t) \le x(t)d\alpha(t) + d\big(q(t) - q_\varepsilon(t)\big) \ \text{ for } \ t \in [t_0+\varepsilon, b]$$

and, therefore,

$$x(t) = q(t) - q(t_0+\varepsilon) - q_\varepsilon(t) + q_\varepsilon(t_0+\varepsilon) + \gamma(t)\Big\{\gamma^{-1}(t_0+\varepsilon)x(t_0+\varepsilon) -$$
$$- \int\limits_{t_0+\varepsilon}^{t} \big(q(\tau) - q(t_0+\varepsilon) - q_\varepsilon(\tau) + q_\varepsilon(t_0+\varepsilon)\big)\, d\gamma^{-1}(\tau)\Big\} \ \text{ for } \ t \in [t_0+\varepsilon, b],$$

where

$$q_\varepsilon(t) = -x(t) + x(t_0+\varepsilon) + q(t) - q(t_0+\varepsilon) + \int\limits_{t_0+\varepsilon}^{t} x(\tau)d\alpha(\tau)$$
$$\text{for } \ t \in [t_0+\varepsilon, b].$$

Hence, by (2.7), we get

$$x(t) = y(t) + \gamma(t)\gamma^{-1}(t_0+\varepsilon)\big(x(t_0+\varepsilon) - y(t_0+\varepsilon)\big) + \\ + g_\varepsilon(t) \ \text{ for } \ t \in [t_0+\varepsilon, b], \tag{2.8}$$

where

$$g_\varepsilon(t) = -q_\varepsilon(t) + q_\varepsilon(t_0+\varepsilon) + \gamma(t) \int\limits_{t_0+\varepsilon}^{t} \big(q_\varepsilon(\tau) - q_\varepsilon(t_0+\varepsilon)\big)d\gamma^{-1}(\tau)$$
$$\text{for } \ t \in [t_0+\varepsilon, b].$$

Using the formula of integration-by-parts (see [29, Theorem I.4.33]), we find

$$g_\varepsilon(t) = -\gamma(t)\bigg(\int\limits_{t_0+\varepsilon}^{t} \gamma^{-1}(\tau)\, ds_0(q_\varepsilon)(\tau) +$$
$$+ \sum_{t_0+\varepsilon<\tau\le t} \gamma^{-1}(\tau-)d_1q_\varepsilon(\tau) + \sum_{t_0+\varepsilon\le\tau<t} \gamma^{-1}(\tau+)d_2q_\varepsilon(\tau)\bigg) \ \text{ for } \ t \in [t_0+\varepsilon, b]. \tag{2.9}$$

According to (2.6) and (2.9), we have

$$g_\varepsilon(t) \le 0 \ \text{ for } \ t \in [t_0+\varepsilon, b],$$

since by the definition of a solution of the generalized differential inequality (2.2) the function q_ε is nondecreasing on the interval $]t_0, b]$. By the equality $\gamma(t_0+\varepsilon) = 1$, from this and (2.8) we get

$$x(t) \le y(t) + \gamma(t)\big(x(t_0+\varepsilon) - y(t_0+\varepsilon)\big) \ \text{ for } \ t \in [t_0+\varepsilon, b].$$

Passing to the limit as $\varepsilon \to 0$ in the latter inequality and taking into account (2.3) and (2.6), we conclude

$$x(t) \le y(t) \ \text{ for } \ t \in]t_0, b].$$

Analogously we can show the validity of the inequality (2.5) for $t \in [a, t_0[$. The lemma is proved. □

The following lemma makes more precise the ones (see Lemma 6.5) in [10].

Lemma 2.2. *Let $t_1, \dots, t_n \in [a,b]$, $l_i : \mathrm{BV}_v([a,b], \mathbb{R}^n_+) \to \mathbb{R}_+$ $(i = 1, \dots, n)$ be linear bounded functionals, and $C_{kj} = (c_{kjil})_{i,l=1}^{n_k,n_j} \in \mathrm{BV}([a,b], \mathbb{R}^{n_k \times n_j})$ $(k, j = 1, 2)$ be such that the system*

$$\begin{gathered}
\operatorname{sgn}(t-t_i)dx_i(t) \le \sum_{l=1}^{n_1} x_l(t)dc_{11il}(t) + \sum_{l=1}^{n_2} x_{n_1+l}(t)dc_{12il}(t) \\
\text{for } \ t \in [a,b], \ \ t \ne t_i \ \ (i = 1, \dots, n_1), \\
(-1)^j d_j x_i(t_i) \le \sum_{l=1}^{n_1} x_{1l}(t_i)d_j c_{11il}(t_i) + \sum_{l=1}^{n_2} x_{n_1+l}(t_i)d_j c_{12il}(t_{1i}) \\
(j = 1, 2; \ \ i = 1, \dots, n_1), \\
dx_{n_1+i}(t) = \sum_{l=1}^{n_1} x_l(t)dc_{21il}(t) + \sum_{l=1}^{n_2} x_{n_1+l}(t)dc_{22il}(t) \\
\text{for } \ t \in [a,b] \ \ (i = 1, \dots, n_2),
\end{gathered} \tag{2.10}$$

has a nontrivial nonnegative solution under the condition

$$\begin{gathered}
x_i(t_i) \le l_i(x_1, \dots, x_n) \ \text{ for } \ i \in N_n, \\
x_i(t_i) = l_i(x_1, \dots, x_n) \ \text{ for } \ i \in \{1, \dots, n\} \setminus N_n,
\end{gathered} \tag{2.11}$$

where n_1 and n_2 $(n_1 + n_2 = n)$ are some nonnegative integers, and N_n is some subset of the set $\{1, \dots, n\}$. Let, moreover, the functions $\alpha_1, \dots, \alpha_{n_1} \in \mathrm{BV}([a,b], \mathbb{R}^n)$ be such that

$$d_j \alpha_i(t) \ge 0 \ \text{ for } \ t \in [a,b] \ \ (j = 1, 2; \ \ i = 1, \dots, n_1) \tag{2.12}$$

and

$$1 + (-1)^j \operatorname{sgn}(t - t_i) d_j \big(c_{11ii}(t) - \alpha_i(t)\big) > 0 \\
\text{for } \ t \in [a,b] \ \ (j = 1, 2; \ \ i = 1, \dots, n_1). \tag{2.13}$$

Then there exist matrix-functions $\widetilde{C}_{k1} = (\widetilde{c}_{k1il})_{i,l=1}^{n_k,n_1} \in \mathrm{BV}([a,b], \mathbb{R}^{n_k \times n_1})$ $(k = 1, 2)$, functions $\widetilde{\alpha}_i \in \mathrm{BV}([a,b], \mathbb{R}^n)$ $(i = 1, \dots, n_1)$, linear bounded functionals $\widetilde{l}_i : \mathrm{BV}_v([a,b], \mathbb{R}^n) \to \mathbb{R}$ $(i = 1, \dots, n)$ and numbers $c_{0i} \in \mathbb{R}$ $(i = 1, \dots, n)$ such that

$$\begin{gathered}
s_0(\widetilde{c}_{11ii})(t) - s_0(\widetilde{c}_{11ii})(s) \le \\
\le \big(s_0(c_{11ii} - \widetilde{\alpha}_i)(t) - s_0(c_{11ii} - \widetilde{\alpha}_i)(s)\big) \operatorname{sgn}(t - s) \\
\text{for } \ (t - s)(s - t_i) > 0, \ \ s, t \in [a,b] \ \ (i = 1, \dots, n_1),
\end{gathered} \tag{2.14}$$

$$(-1)^{j+m}\big(\big|1+(-1)^j d_j\widetilde{c}_{11ii}(t)\big|-1\big)\le d_j\big(c_{ii}(t)-\widetilde{\alpha}_i(t)\big)$$
$$\text{for } (-1)^m(t-t_i)>0 \ \ (j,m=1,2; i=1,\dots,n_1); \tag{2.15}$$
$$|s_0(\widetilde{c}_{21il})(t)-s_0(\widetilde{c}_{21il})(s)|\le$$
$$\le\bigvee_s^t(s_0(c_{21il})) \ \text{ for } a\le s\le t\le b \ \ (i=1,\dots,n_2,\ l=1,\dots,n_1), \tag{2.16}$$
$$\big|d_j\widetilde{c}_{21il}(t)\big|\le\big|d_j\widetilde{c}_{21il}(t)\big| \ \text{ for } t\in[a,b] \ \ (i=1,\dots,n_2,\ l=1,\dots,n_1), \tag{2.17}$$
$$0\le d_j\widetilde{\alpha}_i\le d_j\alpha_i(t) \ \text{ for } t\in[a,b] \ \ (j=1,2;\ \ i=1,\dots,n_1), \tag{2.18}$$

and the system

$$dx(t)=d\widetilde{A}(t)\cdot x(t) \tag{2.19}$$

under the n-condition

$$x_i(t_i)=\widetilde{l}_i(x_1,\dots,x_n)+c_{0i} \ \ (i=1,\dots,n) \tag{2.20}$$

is unsolvable, where

$$\widetilde{A}(t)\equiv\begin{pmatrix}\widetilde{C}_{11}(t), & C_{12}(t)\\ \widetilde{C}_{21}(t), & C_{22}(t)\end{pmatrix}. \tag{2.21}$$

Proof. Let $x=(x_i)_{i=1}^n$ be the nonnegative solution of the problem (2.10), (2.11). Let, moreover, $\varphi_i\in\mathrm{BV}([a,b],\mathbb{R})$ $(i=1,\dots,n_1)$ be the functions defined by

$$\varphi_i(t)\equiv\Bigg(\sum_{l=1}^{n_1}\int_{t_i}^t x_l(\tau)\,dc_{11\,il}(\tau)+$$
$$+\sum_{l=1}^{n_2}\int_{t_i}^t x_{n_1+l}(\tau)\,dc_{12\,il}(\tau)-\int_{t_i}^t x_i(\tau)db_i(\tau)\Bigg)\operatorname{sgn}(t-t_i) \ \ (i=1,\dots,n_1),$$

where $b_i(t)\equiv c_{11ii}-\alpha_i(t)$.

By the condition (2.13), it is evident that the Cauchy problem

$$dy(t)=y(t)\,d\widetilde{b}_i(t)+d\varphi_i(t), \tag{2.22}$$
$$y(t_i)=x_i(t_i), \tag{2.23}$$

where $\widetilde{b}_i(t)\equiv b_i(t)\operatorname{sgn}(t-t_i)$, has a unique solution y_i for every $i\in\{1,\dots,n_1\}$.

In addition, by (2.10) it is easy to verify that the function

$$z_i(t)\equiv x_i(t)-y_i(t)$$

satisfies the conditions of Lemma 2.1 and the problem

$$du(t)=u(t)\,d\widetilde{b}_i(t), \ \ u(t_i)=0$$

has only the trivial solution for every $i\in\{1,\dots,n_1\}$.

According to this lemma, we have

$$x_i(t) \le y_i(t) \ \text{ for } \ t \in [a,b] \ \ (i = 1,\dots,n_1)$$

and therefore

$$x_i(t) = \eta_i(t) y_i(t) \ \text{ for } \ t \in [a,b] \ \ (i = 1,\dots,n_1),$$

where for every $i \in \{1,\dots,n\}$, $\eta_i(t) = x_i(t)/y_i(t)$ if $t \in [a,b]$ is such that $y_i(t) \neq 0$, and $\eta_i(t) = 1$ if $t \in [a,b]$ is such that $y_i(t) = 0$.

It is evident that

$$0 \le \eta_i(t) \le 1 \ \text{ for } \ t \in [a,b] \ \text{ and } \ \eta_i(t_i) = 1 \ \ (i = 1,\dots,n). \tag{2.24}$$

Moreover, for every $i \in \{1,\dots,n\}$, $\eta_i : [a,b] \to [0,1]$ is the function bounded and measurable with respect to every measure along with x_i and y_i are integrable functions.

Hence there exist the integrals appearing in the notation

$$\begin{gathered}\widetilde{c}_{11\,ii}(t) \equiv \big(c_{11ii}(t) - \widetilde{\alpha}_i(t)\big)\,\text{sign}(t - t_i) \ \ (i = 1,\dots,n_1),\\ \widetilde{c}_{11\,il}(t) \equiv \text{sgn}(t - t_i) \int\limits_{t_i}^{t} \eta_l(\tau)\, dc_{11\,il}(\tau) \ \ (i \neq l; \ \ i,l = 1,\dots,n_1)\end{gathered} \tag{2.25}$$

and

$$\widetilde{c}_{21\,il}(t) \equiv \int\limits_{t_i}^{t} \eta_l(\tau)\, dc_{21\,il}(\tau) \ \ (i = 1,\dots,n_2; \ \ l = 1,\dots,n_1), \tag{2.26}$$

where

$$\widetilde{\alpha}_i(t) \equiv \int\limits_{t_i}^{t} \big(1 - \eta_i(\tau)\big)\, d\alpha_i(\tau) \ \ (i = 1,\dots,n_1). \tag{2.27}$$

Due to (2.11) and (2.22)–(2.24), the vector-function $z(t) = (z_i(t))_{i=1}^n$, $z_i(t) = y_i(t)$ $(i = 1,\dots,n_1)$, $z_{n_1+i}(t) = x_{n_1+i}(t)$ $(i = 1,\dots,n_2)$, is a non-trivial nonnegative solution of the problem

$$dz(t) = d\widetilde{A}(t) \cdot z(t), \tag{2.28}$$

$$z_i(t_i) = \widetilde{l}_i(z_1,\dots,z_n) \ \ (i = 1,\dots,n), \tag{2.29}$$

where the matrix-function $\widetilde{A}$ is defined by (2.21), (2.25)–(2.27); $\widetilde{l}_i$: $\text{BV}_v([a,b],\mathbb{R}^n) \to \mathbb{R}$ $(i = 1,\dots,n)$ are linear bounded functionals defined by

$$\begin{gathered}\widetilde{l}_i(z_1,\dots,z_{n_1},z_{n_1+1},\dots,z_n) =\\ = \delta_i l_i(\eta_1 z_1,\dots,\eta_{n_1} z_{n_1}, z_{n_1+1},\dots,z_n) \ \text{ for } \ (z_l)_{l=1}^n \in \text{BV}_v([a,b],\mathbb{R}^n),\end{gathered} \tag{2.30}$$

and $\delta_i \in [0,1]$ $(i = 1,\dots,n)$, $\delta_i = 1$ for $i \in \{1,\dots,n\} \setminus N_n$, are some numbers.

On the other hand, by Remark 1.2 from [9], there exist numbers $c_{0i} \in \mathbb{R}$ $(i = 1,\dots,n)$ such that the problem (2.19), (2.20) is not solvable, where the matrix-function $\widetilde{A}(t)$ and the linear functionals $\widetilde{l}_i$ $(i = 1, \dots, n)$ are defined as above.

Let us show the estimates (2.14)–(2.18). To this end, we use the following formulas obtained from Theorem I.4.12 and Lemma I.4.23 given in [29]. Let the functions $g \in \mathrm{BV}([a, b], \mathbb{R})$ and $f : [a, b] \to \mathbb{R}$ be such that the integral $\varphi(t) = \int\limits_a^t f(\tau)dg(\tau)$ exists for $t \in [a, b]$. Then the equalities

$$s_0(\varphi)(t) \equiv \int\limits_a^t f(\tau)ds_0(g)(\tau), \quad d_j\varphi(t) \equiv f(t)d_jg(t) \ \ (j = 1, 2) \tag{2.31}$$

hold.

Using (2.31), from (2.24)–(2.26) we get the estimates (2.14), (2.16) and (2.17). Moreover, by (2.12), (2.24) and (2.31), the estimate (2.18) holds. As for the estimate (2.15), it holds by general inequality $a-|b| \leq (a-b)\operatorname{sgn} a$ for the cases $t > t_i$, $j = 1$ $(i = 1, \dots, n_1)$ and $t < t_i$, $j = 2$ $(i = 1, \dots, n_1)$, and follows from (2.13) by using (2.18) for the cases $t > t_i$, $j = 2$ $(i = 1, \dots, n_1)$ and $t < t_i$, $j = 1$ $(i = 1, \dots, n_1)$. The lemma is proved. □

Remark 2.1. In Lemma 2.2, if the functions α_i and c_{21kl} are nondecreasing for some $i \in \{1, \dots, n_1\}$ and $k \in \{1, \dots, n_2\}$, $l \in \{1, \dots, n_1\}$, then the functions $\widetilde{\alpha}_i$ and $\widetilde{c}_{21kl}$, respectively, are nondecreasing as well, and

$$\widetilde{\alpha}_i(t) - \widetilde{\alpha}_i(s) \leq \alpha_i(t) - \alpha_i(s) \ \text{ and } \ \widetilde{c}_{21kl}(t) - \widetilde{c}_{21kl}(s) \leq c_{21kl}(t) - c_{21kl}(s) \quad \text{for } \ a \leq s < t \leq b.$$

The statement of Remark 2.1 follows from (2.26) and (2.27) with regard for (2.24).

3. Proofs of the Main Results

Proof of Theorem 1.1. Let us assume

$$t_{*k} = a + \frac{1}{k} \ \text{ and } \ t^*_k = b - \frac{1}{k} \ \ (k = 1, 2, \dots);$$

$$a_{ilk}(t) = \begin{cases} c_{il}(t) - c_{il}(t_{*k}-) + a_{il}(t_{*k}-) \ \text{ for } \ a \leq t < t_{*k}, \\ a_{il}(t) \ \text{ for } \ t_{*k} \leq t \leq t^*_k, \\ c_{il}(t) - c_{il}(t^*_k+) + a_{il}(t^*_k+) \ \text{ for } \ t^*_k < t \leq b \\ \qquad (i, l = 1, \dots, n; \ k = 1, 2, \dots) \end{cases} \tag{3.1}$$

and

$$A_k(t) \equiv (a_{ilk}(t))_{i,l=1}^n \ \ (k = 1, 2, \dots).$$

It is evident that $A_k \in \mathrm{BV}([a, b], \mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$.

For every natural k, consider the system

$$dx(t) = dA_k(t) \cdot x(t) + df(t) \ \text{ for } \ t \in [a, b]. \tag{3.2}$$

We show that the problem $(3.2), (1.2)$ has the unique solution. By Theorem 1.1 from [9] (see also [28]), for this it suffices to verify that the corresponding homogeneous system

$$dx(t) = dA_k(t) \cdot x(t) \ \text{ for } \ t \in [a,b] \tag{3.2_0}$$

has only the trivial solution under the condition (1.2).

Let us show that the problem $(3.2_0), (1.2)$ has only the trivial solution.

Indeed, if $x = (x_i)_{i=1}^n$ is an arbitrary solution of this problem, then due to Lemma 6.1 from [10], with regard for the conditions (1.6)–(1.9), the vector-function x satisfies the system (1.5) of generalized differential inequalities. But, by the condition (1.10), this system has only the trivial solution under the condition (1.2). Thus $x_i(t) \equiv 0$ $(i = 1, \dots, n)$.

We put

$$t_i = a \ \text{ for } \ i \in \{1, \dots, n_0\} \ \text{ and } \ t_i = b \ \text{ for } \ i \in \{n_0 + 1, \dots, n\}. \tag{3.3}$$

Let now k be an arbitrary fixed natural number, and $x_k = (x_{ik})_{i=1}^n$ be the unique solution of the problem $(3.2), (1.2)$. Then by the conditions (1.6)–(1.9) and the equalities (3.1) and (3.2), using Lemma 2.2 from [8] (or Lemma 6.1 from [10]), we find that the vector-function $x_k = (x_{ik})_{i=1}^n$ satisfies the system

$$\operatorname{sgn}(t - t_i) d|x_{ik}(t)| \le \sum_{l=1}^n |x_{lk}(t)| \, dc_{il}(t) + \operatorname{sgn}[x_{ik}(t)(t - t_i)] df_i(t)$$
$$\text{for } \ t \in [a,b], \ \ t \ne t_i \ \ (i = 1, \dots, n),$$
$$(-1)^j d_j |x_{ik}(t_i)| \le \sum_{l=1}^n |x_{lk}(t_i)| \, d_j c_{il}(t_i) + (-1)^j \operatorname{sgn}[x_{ik}(t_i)] df_i(t_i)$$
$$(j = 1, 2; \ \ i = 1, \dots, n),$$

where $t_1, \dots, t_n$ are defined by (3.3). From this, we have

$$\operatorname{sgn}(t - t_i) d|x_{ik}(t)| \le \sum_{l=1}^n |x_{lk}(t)| \, dc_{il}(t) + dv(f_i)(t)$$
$$\text{for } \ t \in [a,b], \ \ t \ne t_i \ \ (i = 1, \dots, n),$$
$$(-1)^j d_j |x_{ik}(t_i)| \le \sum_{l=1}^n |x_{lk}(t_i)| \, d_j c_{il}(t_i) + d_j v(f_i)(t_i) \ \ (j = 1, 2; \ i = 1, \dots, n).$$

Therefore, due to Lemma 2.4 from [8], there exists a number $\rho_0 > 0$ independent of k such that

$$\|x_{ik}\|_s \le \rho_0 \ \ (i = 1, \dots, n; \ \ k = 1, 2, \dots). \tag{3.4}$$

Let for every natural k, $t_{ik} = a + \frac{1}{k}$ and $\Delta_{ik} =]t_{ik}, b]$ for $i \in \{1, \dots, n_0\}$, and $t_{ik} = b - \frac{1}{k}$ and $\Delta_{ik} = [a, t_{ik}[$ for $i \in \{n_0 + 1, \dots, n\}$. Then, as above, using Lemma 2.2 from [8] and the estimate (3.4), we conclude that there exists a sufficiently large natural number k_0 such that for every $k \in \{k_0 + 1, k_0 + 2, \dots\}$, the vector-function $x_k = (x_{ik})_{i=1}^n$ satisfies the inequalities

$$\begin{gathered} \operatorname{sgn}(t - t_{ik}) d|x_{ik}(t)| \le -|x_{ik}(t)| d\alpha_i(t) + dq_i(t) \\ \text{for } t \in \Delta_{ik} \ \ (i = 1, \dots, n), \\ (-1)^j d_j |x_{ik}(t_{ik})| \le -|x_{ik}(t_{ik})| d_j \alpha_i(t_{ik}) + d_j q_i(t_{ik}) \\ (j = 1, 2; \ \ i = 1, \dots, n), \end{gathered} \tag{3.5}$$

where

$$q_i(t) \equiv \rho_0 \Big(\bigvee_{t_{ik}}^{t} (c_{ii}) + \sum_{l=1, l \ne i}^{n} \big(c_{il}(t) - c_{il}(t_{ik}) \big) \Big) \operatorname{sgn}(t - t_{ik}) + \\ + v(f_i)(t) - v(f_i)(t_{ik}) \ \ (i = 1, \dots, n).$$

Let $i \in \{1, \dots, n_0\}$ and $k \in \{k_0 + 1, k_0 + 2, \dots\}$. Consider the Cauchy problem

$$d\gamma(t) = -\,\gamma(t)\, d\alpha_i(t), \quad \gamma(t_{ik}) = 1.$$

Due to the condition (1.11), this problem has the unique solution γ_{ik} on the interval $\Delta_{ik\delta} = [t_{ik}, a + \delta]$ for sufficiently small $\delta > 0$. Then $\gamma_{ik}(t) = \gamma_{\beta_i}(t, t_{ik})$ for $t \in \Delta_{ik\delta}$, where the function γ_{α_i} is defined according to (1.4). Moreover, this function is positive and nonincreasing on the interval $t \in \Delta_{ik\delta}$. In addition, we can assume without loss of generality that the conditions of Lemma 2.1 are fulfilled on this interval. Therefore, according to this lemma, (3.5) and the variation-of-constant formula mentioned above, we have the estimate

$$|x_{ik}(t)| \le q_i(t) - q_i(t_{ik}) + \\ + \gamma_{ik}(t) \Big\{ \rho_0 - \int_{t_{ik}}^{t} \big(q_i(\tau) - q_i(t_{ik}) d\gamma_{ik}^{-1}(\tau) \Big\} \ \text{ for } \ t \in \Delta_{ik\delta}. \tag{3.6}$$

Taking into account the first equality of the condition (1.12) and the fact that the function q_i is nondecreasing on $\Delta_{ik\delta}$, from (3.6) we get

$$\lim_{t \to a+} \sup \Big\{ |x_{ik}(t)| : \ k = k_0 + 1, k_0 + 2, \dots \Big\} = 0 \ \ (i = 1, \dots, n_0). \tag{3.7}$$

Analogously, using the second parts of the conditions (1.11) and (1.12), as above we show that

$$\lim_{t \to b-} \sup \Big\{ |x_{ik}(t)| : \ k = k_0 + 1, k_0 + 2, \dots \Big\} = 0 \ \ (i = n_0 + 1, \dots, n). \tag{3.8}$$

Without loss of generality, we can assume that the natural number k_0 is such that $a < t_{1k_0} < t_{2k_0} < b$. Consider the sequence x_k $(k = k_0 + 1, k_0 + 2, \dots)$. Then by (3.1), (3.4) and the definition of the solution of the system

(3.2), we have

$$\|x_k(t)-x_k(s)\| \le \|f(t)-f(s)\| + \left\| \int\limits_s^t dA_k(\tau)\cdot(x_k(\tau)-x_k(s)) \right\| \le$$
$$\le \|f(t)-f(s)\| + \rho_0 \bigvee_s^t (A_{k_0}) \ \text{ for } \ t_{1k_0} \le s < t \le t_{2k_0},$$

since $A_k(t) = A_{k_0}(t) = A(t)$ for $t \in [t_{1k_0}, t_{2k_0}]$ $(k = k_0+1, k_0+2, \dots)$. Hence there exists a positive number ρ_{k_0} such that

$$\bigvee_{t_{1k_0}}^{t_{2k_0}} (x_k) \le \rho_{k_0} \ \ (k = k_0+1, k_0+2, \dots).$$

Consequently, in view of Helly's choose theorem, without loss of generality we can assume that the sequence x_k $(k = k_0+1, k_0+2, \dots)$ converges to some function $x_0 = (x_{io})_{i=1}^n \in \mathrm{BV}([t_{1k_0}, t_{2k_0} b], \mathbb{R}^n)$. If we continue this process, then in a standard way we can assume without loss of generality that

$$\lim_{k\to\infty} x_k(t) = x_0(t) \ \text{ for } \ t \in]a,b[\,, \tag{3.9}$$

where $x_0 = (x_{io})_{i=1}^n \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$.

Let now $[a_0, b_0] \subset]a,b[$ be an arbitrary closed interval. Then

$$\|x_k(t)-x_k(s)\| \le l_k + \|g_(t) - g_(s)\|$$
$$\text{for } \ a_0 \le s < t \le b_0 \ \ (k = k_0+1, k_0+2, \dots),$$

where

$$g(t) = f(t) + \int\limits_{a_0}^t dA_{k_0}(\tau)\cdot x_0(\tau), \quad l_k = \left\| \int\limits_{a_0}^{b_0} dV(A_{k_0})(\tau)\cdot |x_k(\tau) - x_0(\tau)| \right\|.$$

On the other hand, due to (3.9) and the Lebesgue theorem, we have $l_k \to 0$ as $k \to \infty$. Therefore, according to Lemma 2.3 from [7],

$$\lim_{k\to\infty} x_k(t) = x_0(t) \ \text{ uniformly on } \ [a_0, b_0].$$

Moreover, by (3.7), the sequences $\{x_{ik}\}_{k=1}^\infty (i = 1, \dots, n_0)$ converge uniformly on the interval $]a, t_0]$, and by (3.8), the sequences $\{x_{ik}\}_{k=1}^\infty$ $(i = n_0+1, \dots, n)$ converge uniformly on the interval $[t_0, b[$ for every $t_0 \in]a,b[$. Therefore, there exist one-sided limits $x_{i0}(a+)$ $(i = 1, \dots, n_0)$ and $x_{i0}(b-)$ $(i = n_0+1, \dots, n)$ and, in addition, they are equal to zero. Thus, due to (3.1) and (3.2), we have established that $x_0 \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ is a solution of the problem (1.1), (1.2).

Let us show that the problem (1.1), (1.2) has only one solution. Let x and y be two arbitrary solutions of the problem. Then the function

$z(t) \equiv x(t) - y(t)$, $z(t) \equiv (z_i(t))_{i=1}^n$, will be a solution of the homogeneous problem

$$dz(t) = dA(t) \cdot z(t),$$
$$z_i(a+) = 0 \;\; (i = 1, \dots, n_0), \quad z_i(b-) = 0 \;\; (i = n_0 + 1, \dots, n).$$

From this, by (1.6)–(1.9), z is a solution of the system of differential inequalities (1.5) under the condition (1.2). Thus, due to the condition (1.10), we conclude that $z(t) \equiv 0$. The theorem is proved. □

Proof of Corollary 1.1. The proof of this corollary slightly differs from that of Lemma 2.6 given in [3]. We give the main aspect of this proof for completeness.

It suffices to show that the problem (1.5), (1.2) has only the trivial nonnegative solution.

Let $(x_i)_{i=1}^n$ be an arbitrary nonnegative solution of the problem (1.5), (1.2). Let $i \in \{1, \dots, n_0\}$ be fixed, and ε be an arbitrary sufficiently small positive number. Then by (1.13)–(1.15) and Hölder's inequality, we have

$$|x_i(t)| \le |x_i(a+\varepsilon)| + \sum_{\sigma=0}^{2} \sum_{k=0}^{n} \left(\|h_{ik}\|_{\mu, s_\sigma(\beta_k)} \left| \int_{a+\varepsilon}^{t} |x_k(\tau)|^{\frac{\nu}{2}} \, ds_\sigma(\beta_k)(\tau) \right|^{\frac{2}{\nu}} \right) \quad \text{for } t \in \,]a, b].$$

This, in view of Minkowski's inequality, implies

$$\|x_i\|_{\nu, s_j(\beta_i)} \le |x_i(a+\varepsilon)| \big(s_j(\beta_i)(b) - s_j(\beta_i)(a) \big)^{\frac{1}{\nu}} + \sum_{\sigma=0}^{2} \sum_{k=0}^{n} \|h_{ik}\|_{\mu, s_\sigma(\beta_k)} \times$$
$$\times \left(\int_a^b \left| \int_{a+\varepsilon}^{t} |x_k(\tau)|^{\frac{\nu}{2}} ds_\sigma(\beta_k)(\tau) \right|^2 ds_j(\beta_i)(t) \right)^{\frac{1}{\nu}} \quad (j = 0, 1, 2). \qquad (3.10)$$

On the other hand, by virtue of Hölder's inequality, in case $\sigma^2 + j^2 + (i-k)^2 > 0$, $j = 0$, and by the generalized Wirtinger's inequalities (see Lemma 2.5 from [3]), in the other case we have

$$\lim_{\varepsilon \to 0} \left(\int_a^b \left| \int_{a+\varepsilon}^{t} |x_k(\tau)|^{\frac{\nu}{2}} \, ds_\sigma(\beta_k)(\tau) \right|^2 ds_j(\beta_i)(t) \right)^{\frac{1}{\nu}} \le$$
$$\le \lambda_{k\sigma ij} \left(\int_{a+}^{b} |x_k(\tau)|^{\nu} \, ds_\sigma(\beta_k)(\tau) \right)^{\frac{1}{\nu}} \quad (j, \sigma = 0, 1, 2; \;\; k = 1, \dots, n).$$

By this, (1.2) and (3.10), we get

$$\|x_i\|_{\nu,s_j(\beta_i)} \leq$$

$$\leq \sum_{\sigma=0}^{2}\sum_{k=0}^{n}\lambda_{k\sigma ij}\|h_{ik}\|_{\mu,s_\sigma(\beta_k)}\|x_k\|_{\nu,s_\sigma(\beta_k)} \;\; (j=0,1,2;\; i=1,\dots,n_0). \tag{3.11}$$

Analogously, we show that the estimate (3.11) is valid for $i \in \{n_0 + 1,\dots,n\}$, as well.

Therefore,

$$(I_{3n} - \mathcal{H})r \leq 0, \tag{3.12}$$

where $r \in \mathbb{R}^{3n}$ is the vector with the components

$$r_{i+nj} = \|x_i\|_{\nu,s_j(\beta_i)} \;\; (j = 0,1,2;\;\; i = 1,\dots,n).$$

From (3.12), due to (1.2) and (1.16), we find that $r = 0$ and $x_i(t) \equiv 0$ $(i = 1,\dots,n)$. Consequently, the problem (1.5), (1.2) has no nontrivial nonnegative solution. The corollary is proved. □

Proof of Theorem 1.2. It suffices to show that the problem (1.5), (1.2), where $c_{il}(t) = h_{il}\beta_i(t) + \beta_{il}(t)$ $(i,l = 1,\dots,n)$, has only the trivial nonnegative solution.

Let $(x_i)_{i=1}^{n}$ be an arbitrary nonnegative solution of the problem (1.5), (1.2). Let $i \in \{1,\dots,n_0\}$ be fixed. Then from (1.5), we have

$$dx_i(t) \leq x_i(t)da_i(t) + dg_i(t) \;\; \text{for} \;\; t \in]a,b], \tag{3.13}$$

where

$$g_i(t) = g_{1i}(t) + g_{2i}(t),$$

$$g_{1i}(t) = \sum_{l=1,l\neq i}^{n} r_l h_{il}\big(\beta_i(t) - \beta_i(a+)\big) \;\; \text{and} \;\; g_{2i}(t) = \sum_{l=1}^{n} r_l\big(\beta_{il}(t) - \beta_{il}(a+)\big)$$

and

$$r_l = \sup\big\{\|x_l(t)\| :\; t \in]a,b]\big\} \;\; (l = 1,\dots,n).$$

Hence the function x_i satisfies the inequality (2.2) for $t_0 = a$, $\alpha(t) \equiv a_i(t)$ and $q(t) \equiv g_i(t)$. Moreover, by (1.17), the condition (2.1) is fulfilled. Therefore, according to Lemma 2.1, we find

$$x_i(t) \leq y_i(t) \;\; \text{for} \;\; a < t \leq b, \tag{3.14}$$

where y_i is the solution of the Cauchy problem of the equation

$$dy(t) = y(t)da_i(t) + dg_i(t), \;\; y(a+) = 0.$$

Due to the variation-of-constant formula mentioned above, we have

$$y_i(t) = g_i(t) - \lambda_i(t)\int_{a+}^{t} g_i(\tau)d\lambda_i^{-1}(\tau) \;\; \text{for} \;\; t \in]a,b], \tag{3.15}$$

where λ_i is the solution of the Cauchy problem

$$d\lambda(t) = \lambda(t)da_i(t), \ \ \lambda(a+) = 1.$$

From (3.15), using the formula of integration-by-parts (see [29, Theorem I.4.33]), we conclude

$$y_i(t) = \lambda_i(t)\psi_i(t), \tag{3.16}$$

where

$$\psi_i(t) = \int\limits_{a+}^{t} \lambda_i^{-1}(\tau)dg_i(\tau) - \sum_{a<\tau<t} d_1 g_i(t)d_1\lambda_i(\tau) + \sum_{a<\tau<t} d_2 g_i(t)d_2\lambda_i(\tau) \quad \text{for } \ a < t \leq b.$$

Moreover, by the equalities

$$d_j\lambda_i^{-1}(t) = -\lambda_i^{-1}(t)\cdot\big(1+(-1)^j d_j a_i(t)\big)^{-1} d_j a_i(t) \ \ (j=1,2),$$

we have

$$\psi_i(t) = \psi_{1i}(t) + \psi_{2i}(t) \ \text{ for } \ a < t \leq b,$$

where

$$\psi_{ji}(t) = \int\limits_{a+}^{t} \lambda_i^{-1}(\tau)\, d\mathcal{A}(g_{ji}, a_i)(\tau) \ \text{ for } \ a < t \leq b \ \ (j=1,2).$$

Then by the equality $d\lambda_i^{-1}(t) = -\lambda_i^{-1}(t)d\mathcal{A}(a_i.a_i)(t)$ (see Lemma 2.1 from [11]) and the definition of the operator $\mathcal{A}$, we get

$$\psi_{1i}(t) = \sum_{l=1, l\neq i}^{n} r_l h_{il} \int\limits_{a+}^{t} \lambda_i^{-1}(\tau)\, d\mathcal{A}(a_i, a_i)(\tau) = \sum_{l=1, l\neq i}^{n} r_l \frac{h_{il}}{|h_{ii}|}\big(\lambda_i^{-1}(t) - 1\big)$$

and

$$\begin{aligned}\psi_{2i}(t) &= r_i \int\limits_{a+}^{t} \lambda_i^{-1}(\tau)d\mathcal{A}(\zeta_i, a_i)(\tau) \leq \\ &\leq r_i\lambda_i^{-1}(t)\Big(V\big(\mathcal{A}(\zeta_i, a_i)\big)(t) - V\big(\mathcal{A}(\zeta_i, a_i)\big)(a+)\Big) \leq \\ &\leq r_i\eta_i\lambda_i^{-1}(t) \ \text{ for } \ a < t \leq b.\end{aligned}$$

Hence, in view of (3.14) and (3.16), we find

$$r_i \leq \eta_i r_i + \sum_{l=1, l\neq i}^{n} r_l \frac{h_{il}}{|h_{ii}|} \tag{3.17}$$

for $i \in \{1, \dots, n_0\}$.

Analogously, we show the validity of the estimate (3.17) for $i \in \{n_0 + 1, \dots, n\}$, too.

Thus the constant vector $r=(r_i)_{i=1}^n$ satisfies the system of inequalities

$$(I-\mathcal{H})r\leq 0. \tag{3.18}$$

Therefore, according to the condition (1.16), we have $r=0$ and $x_i(t)\equiv 0$ $(i=1,\dots,n)$. The theorem is proved. □

Let us show Remark 1.2. Due to the condition (1.19), it is evident that (3.17) implies that the constant vector r, appearing in the proof of Theorem 1.2, satisfies the system (3.18), where the constant matrix $\mathcal{H}=(\xi_{il})_{i,l=1}^n$ is defined by (1.20). Therefore, by (1.16), we obtain $x_i(t)\equiv 0$ $(i=1,\dots,n)$ as in the proof of Theorem 1.2.

Proof of Theorem 1.3. Let the vector-function $x^*=(x_i^*)_{i=1}^n$ be the nontrivial nonnegative solution of the system (1.5) under the condition (1.2). Obviously, it will be a solution of the system (2.10), (2.11), where $C_{11}(t)\equiv C(T)$, $C_{12}(t)\equiv O_{n_1\times n_2}$, $C_{21}(t)\equiv O_{n_2\times n_1}$, $C_{22}(t)\equiv O_{n_2\times n_2}$, $t_i=a$ and $l_i(x_1,\dots,x_n)\equiv -d_2x_i(a)$ for $i\in\{1,\dots,n_0\}$, $t_i=b$ and $l_i(x_1,\dots,x_n)\equiv d_1x_i(b)$ for $i\in\{n_0+1,\dots,n\}$, and $N_n=\varnothing$. In addition, the condition (1.21) of Theorem 1.3 is equivalent to the condition (2.13) of Lemma 2.2. Therefore, according to Lemma 2.2 and Remark 2.1, there exist a matrix-function $\widetilde{A}\in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ and nondecreasing functions $\widetilde{\alpha}_i:[a,b]\to\mathbb{R}$ $(i=1,\dots,n)$ satisfying the conditions (2.14))–(2.18) of Lemma 2.2 and the condition (1.22), and a constant vector $c=(c_i)_{i=1}^n\in\mathbb{R}^n$ such that the system

$$dz(t)=d\widetilde{A}(t)\cdot z(t)$$

under the condition

$$z_i(t_i)=l_i(z_1,\dots,z_n)+c_i\ \ (i=1,\dots,n)$$

is unsolvable, where $z(t)=(z_i(t))_{i=1}^n$ and, due to the equalities (2.30), we have $\widetilde{l}_i(z_1,\dots,z_n)\equiv l_i(z_1,\dots,z_n)$. Consequently, using the mapping $x_i(t)=z_i(t)+c_i$ $(i=1,\dots,n)$ and definitions of the functionals l_i $(i=1,\dots,n)$, it is not difficult to see that the problem (1.1), (1.2) is not solvable as well, where $A(t)\equiv\widetilde{A}(t)$ and $f(t)\equiv\widetilde{A}(t)\cdot c$. Moreover, it is evident that in this case the conditions (1.6)–(1.9) coincide with the conditions (2.14)–(2.17), respectively. From the conditions (2.18) (or (1.22)) and (1.11) it follows that the functions $\widetilde{\alpha}_i$ $(i=1,\dots,n)$ satisfy the condition (1.22) as well. Therefore there exists the sufficiently small $\delta>0$ such that

$$1+(-1)^jd_j\widetilde{\beta}_i(t)>0\ \text{ for }\ t\in\,]a,a+\delta[\ \ (i=1,\dots,n_0)$$
$$\text{or }\ t\in\,]b-\delta,b[\ \ (i=n_0+1,\dots,n), \tag{3.19}$$

where $\widetilde{\beta}_i(t)\equiv\widetilde{\alpha}_i(t)\operatorname{sgn}(n_0+\frac{1}{2}-i)$.

Let us show that the condition (1.12) is valid. Let $i\in\{1,\dots,n_0\}$ be fixed and let a natural number k_0 be such that $a+\frac{1}{k}<a+\delta$ for $k>k_0$. Then, by the condition (3.19), there exists the nonnegative function $\gamma_{\widetilde{\beta}_i}(t)$ $(t\in\,]a,a+\delta[)$, since the corresponding Cauchy problem is uniquely solvable.

Let $t \in]a, a+\delta[$ and $k > k_0$ be such that $a + \frac{1}{k} < t$. Then, by definition of the solution, we have

$$\gamma_{\widetilde{\beta}_i}(t) = 1 + \int\limits_{a+\frac{1}{k}}^{t} \gamma_{\widetilde{\beta}_i}(\tau)\, d\widetilde{\beta}_i(\tau) \leq$$

$$\leq 1 + \int\limits_{a+\frac{1}{k}}^{t} \gamma_{\widetilde{\beta}_i}(\tau)\, d\alpha_i(\tau) + \int\limits_{a+\frac{1}{k}}^{t} \gamma_{\widetilde{\beta}_i}(\tau)\, d\big(\widetilde{\alpha}_i(\tau) - \alpha_i(\tau)\big).$$

Consequently, the function $\gamma_{\widetilde{\beta}_i}$ is a solution of the problem

$$\operatorname{sgn}(t - t_{ik}) d\gamma_(t) \leq \gamma_(t) d\widetilde{\beta}_i(t) \ \text{ for } \ t \in]t_{ik}, a+\delta[\,, \quad \gamma(t_{ik}) = 1,$$

where $t_{ik} = a + \frac{1}{k}$. On the other hand, the function $\gamma_{\widetilde{\beta}_i}$ is the unique solution of the problem

$$\operatorname{sgn}(t - t_{ik}) d\gamma_(t) = \gamma_(t) d\beta_i(t) \ \text{ for } \ t \in]t_{ik}, a+\delta[\,, \quad \gamma(t_{ik}) = 1.$$

Therefore, due to Lemma 2.1, we have

$$\gamma_{\widetilde{\beta}_i}(t) \leq \gamma_{\beta_i}(t) \ \text{ for } \ t \in]t_{ik}, a+\delta[\,.$$

From this, by (1.12) it follows that the function $\gamma_{\widetilde{\beta}_i}$ satisfies the first equality of the condition (1.12).

Analogously we show the second equality of the condition (1.12).

Let now the condition (1.23) hold. By definition of the matrix-function $A(t) \equiv \widetilde{A}(t)$ (see (2.21), (2.25)–(2.27)), we get

$$d_j A(t) = \left(\eta_i(t) d_j c_{il}(t) \operatorname{sgn}\left(n_0 + \frac{1}{2} - i\right)\right)_{i,l=1}^{n} \ \text{ for } \ t \in [a,b] \ \ (j = 1,2).$$

From this, by (1.23), it follows that the condition (1.3) holds. Thus the theorem is proved. □

Consider now Remark 1.3. The first case is evident. Indeed, by definition of the matrix-function $A = (a_{il})_{i,l=1}^{n}$, we have

$$d_j a_{il}(t) = \eta_l(t) d_j c_{il}(t) \operatorname{sgn}\left(n_0 + \frac{1}{2} - i\right) \ \text{ for } \ t \in [a,b] \ \ (j = 1,2;\ i,l = 1,\dots,n)$$

and

$$|d_j a_{il}(t)| \leq |d_j c_{il}(t)| \ \text{ for } \ t \in [a,b] \ \ (j = 1,2; \ \ i,l = 1,\dots,n).$$

Taking this into account, by (1.24), we have

$$\sum_{l=1}^{n} |d_j a_{il}(t)| < 1 \ \text{ for } \ t \in [a,b] \ \ (j = 1,2; \ \ i = 1,\dots,n).$$

Hence the condition (1.23) holds.

Let now the condition (1.25) be valid. Then we have

$$\sum_{l=1,\,l\neq i}^{n}\left|\varepsilon_i d_j c_{il}(t)\,\mathrm{sgn}\left(n_0+\frac{1}{2}-i\right)\right|\leq$$

$$\leq \varepsilon_i+(-1)^j\varepsilon_i d_j c_{ii}(t)\,\mathrm{sgn}\left(n_0+\frac{1}{2}-i\right)\leq 1+(-1)^j\varepsilon_i d_j c_{ii}(t)\,\mathrm{sgn}\left(n_0+\frac{1}{2}-i\right)$$

$$\text{for } t\in[a,b]\ \ (j=1,2;\ \ i=1,\dots,n). \tag{3.20}$$

Therefore, by Hadamard's theorem (see [14, p. 382]), the condition (1.23) holds. Remark 1.3 is proved analogously to the conditions (1.26).

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Author's addresses:

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 University St., Tbilisi 0186, Georgia;

2. Sukhumi State University, 12 Politkovskaia St., Tbilisi 0186, Georgia.

E-mail: ashord@rmi.ge