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MULTIPLE POSITIVE SOLUTIONS FOR A CLASS OF FRACTIONAL SINGULAR BOUNDARY VALUE PROBLEMS

Abstract. For $(n-1, n]$ order singular fractional differential equations, conditions are established guaranteeing, respectively the existence of multiple positive solutions and the nonexistence of a positive solution of a class of boundary value problems.

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რეზიუმე. $(n-1, n]$ რიგის სინგულარული ფრაქციონალური დიფერენციალური განტოლებებისათვის დადგენილია პირობები, რომლებიც სათანადოდ უზრუნველყოფენ სასაზღვრო ამოცანათა ერთი კლასის ჯერადი დადებითი ამონახსნების არსებობასა და დადებითი ამონახსნის არარსებობას.

1. Introduction

The boundary value problem (BVP, for short), singular boundary value problem, and fractional order boundary value problem arise in a variety of differential applied mathematics and physics and hence, they have received much attention (see [1, 2, 6–12] and references therein). For example, in [1], Qiu and Bai considered the existence of positive solutions to BVP in the nonlinear fractional differential equation

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{cases}$$

where $2 < \alpha \leq 3$, and $f : (0,1] \times [0, +\infty) \to [0, +\infty)$ with $\lim\limits_{t \to 0+} f(t, u) = +\infty$ is continuous, that is, $f(t, u)$ may be singular at $t = 0$. They obtained the existence of at least one positive solution by using Krasnoselskii's fixed point theorem and nonlinear alternative of Leray–Schauder type in a cone.

In [14], Kaufmann obtained the existence and nonexistence of positive solutions to the nonlinear fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & t \in (0, \tau), \\ I^{\gamma} u(0^+) = 0, \quad I^{\beta} u(\tau) = 0, \end{cases}$$

where $\tau \in (0, T]$, $1 - \alpha < \gamma \leq 2 - \alpha$, $2 - \alpha < \beta < 0$, D_{0+}^{α} is the Riemann–Liouville differential operator of order α, $f \in C([0, T] \times \mathbb{R})$ is nonnegative.

In this paper, we consider the following singular fractional boundary value problem of the form

$$\begin{aligned} {}^C D_{0+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, &\ \ 0 < t < 1, \\ u^{(j)}(0) = 0, &\ \ 0 \leq j \leq n-1, \ \ j \neq 2, \\ u''(1) = 0, & \end{aligned} \tag{1.1}$$

where $n - 1 < \alpha \leq n$, $n \geq 4$, ${}^C D_{0+}^{\alpha}$ are the Caputo's fractional derivatives and $f : (0,1) \times (0, +\infty) \to [0, +\infty)$ is continuous, that is, $f(t, u)$ may be singular at $t = 0, 1$ and $u = 0$. When constructing a special cone and using approximation method and fixed point index theory, we have obtain the existence of multiple positive solutions and nonexistence for BVP (1.1).

The main features of the paper are as follows. Firstly, the degree of singularity in [1] is lower than that of the present paper (for details, please see our examples). Here, $f(t, u)$ may be singular not only at $t = 0, 1$, but also at $u = 0$. Secondly, the results we obtained are the existence of multiple positive solutions and nonexistence of positive solutions, while [1] just obtained the existence of at least one positive solution. Finally, BVP (1.1) is more general and extensive than that in [1].

The paper is organized as follows. Section 2 contains some definitions and lemmas. Moreover, the Green's function and its properties are derived. In Section 3, by constructing a special cone and using approximation method and fixed point index theory, the existence of multiple positive solutions and nonexistence result are established. Finally, in Section 4, two examples are worked out to demonstrate our main results.

2. Preliminaries

For convenience of the reader, we present some necessary definitions from fractional calculus theory (see [3, 5]).

Definition 2.1. The fractional (arbitrary) order integral of the function $h \in L^1([a,b])$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,\mathrm{d}s,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2. For a function h given on the interval $[a,b]$, the αth Caputo fractional-order derivative of h, is defined by

$$({}^C D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)\,ds.$$

Here, n is the smallest integer greater than or equal to α.

Lemma 2.3. *Let $\alpha > 0$. Then the differential equation*

$${}^C D_{0+}^\alpha u(t) = 0$$

has solutions $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$ for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, where n is the smallest integer greater than or equal to α.

Lemma 2.4. *Assume that $u \in C(0,1) \cap L^1[0,1]$ with a derivative of order n that belongs to $C(0,1) \cap L^1[0,1]$. Then*

$$I_{0+}^\alpha {}^C D_{0+}^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, where n is the smallest integer greater than or equal to α.

Lemma 2.5. *The relation*

$$I_{0+}^\alpha I_{0+}^\beta \varphi = I_{0+}^{\alpha+\beta} \varphi$$

is valid in the following case:

$$\operatorname{Re}\beta > 0, \quad \operatorname{Re}(\alpha+\beta) > 0, \quad \varphi \in L^1[a,b].$$

In the rest of this paper, we suppose $\alpha \in (n-1, n]$, $n \ge 4$.

Lemma 2.6. *Given $g \in C[0,1]$, the unique solution of*

$$\begin{aligned} {}^C D^{\alpha}_{0+} u(t) + g(t) = 0,\ \ 0 < t < 1,& \\ u^{(j)}(0) = 0,\ \ 0 \le j \le n-1,\ \ j \ne 2,& \\ u''(1) = 0& \end{aligned} \tag{2.1}$$

is

$$u(t) = \int_0^1 G(t,s) g(s)\,\mathrm{d}s, \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(\alpha-1)(\alpha-2)}{2}\, t^2 (1-s)^{\alpha-3} - (t-s)^{\alpha-1}, & s \le t, \\ \dfrac{(\alpha-1)(\alpha-2)}{2}\, t^2 (1-s)^{\alpha-3}, & t \le s. \end{cases} \tag{2.3}$$

Proof. Let $u \in C[0,1]$ be a solution of (2.1). By Lemma 2.3,

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, g(s)\,\mathrm{d}s.$$

From $u^{(j)}(0) = 0, 0 \le j \le n-1, j \ne 2, u''(1) = 0$, we have $c_0 = c_1 = c_3 = \cdots = c_{n-1} = 0$ and

$$c_2 = \frac{(\alpha-1)(\alpha-2)}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-3} g(s)\,\mathrm{d}s.$$

Then

$$\begin{aligned} u(t) &= c_2 t^2 - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\,\mathrm{d}s = \\ &= \frac{1}{\Gamma(\alpha)} \Bigg(\int_0^t \Big(\frac{(\alpha-1)(\alpha-2)}{2}\, t^2 (1-s)^{\alpha-3} - (t-s)^{\alpha-1} \Big) g(s)\,\mathrm{d}s + \\ &\qquad + \int_t^1 \frac{(\alpha-1)(\alpha-2)}{2}\, t^2 (1-s)^{\alpha-3} g(s)\,\mathrm{d}s \Bigg) = \\ &= \int_0^1 G(t,s) g(s)\,\mathrm{d}s. \end{aligned}$$

The proof is completed. □

Lemma 2.6 indicates that the solution of the BVP (1.1) coincides with the fixed point of the operator T defined as

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s))\,\mathrm{d}s, \quad \forall\, u \in C[0,1]. \tag{2.4}$$

Lemma 2.7. *The function* $G(t,s)$ *defined by* (2.3) *has the following properties:*

(i) $G(t,s) > 0, \ \ \forall\, t,s \in [0,1].$ (2.5)

(ii) $G(t,s) \le H(s) \le \dfrac{(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)}\,,$ (2.6)

where

$$H(s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(\alpha-1)(\alpha-2)}{2}\, s^2(1-s)^{\alpha-3} - (1-s)^{\alpha-1}, & s \le t, \\ \dfrac{(\alpha-1)(\alpha-2)}{2}\, s^2(1-s)^{\alpha-3}, & t \le s, \end{cases} \tag{2.7}$$

(iii) $G(t,s) \ge t^2 G(\tau,s), \ \ \forall\, t,s,\tau \in [0,1].$ (2.8)

Proof. First, since $\alpha \in (n-1,n]$ and $n \ge 4$, it is easy to see

$$\frac{(\alpha-1)(\alpha-2)}{2} > 1.$$

Furthermore, for $s,t \in [0,1]$,

$$\frac{(\alpha-1)(\alpha-2)}{2}\, t^2(1-s)^{\alpha-3} > t^2(1-s)^{\alpha-3} \ge$$
$$\ge (t-s)^2(t-s)^{\alpha-3} = (t-s)^{\alpha-1}.$$

Obviously, we can get (2.5).

Next, for the given $s \in (0,1)$, we can find that $G(t,s)$ is increasing with respect to t. For $t \le s$,

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\, \frac{(\alpha-1)(\alpha-2)}{2}\, t^2(1-s)^{\alpha-3} \le$$
$$\le \frac{1}{\Gamma(\alpha)}\, \frac{(\alpha-1)(\alpha-2)}{2}\, s^2(1-s)^{\alpha-3}$$

and for $t \ge s$,

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \Big(\frac{(\alpha-1)(\alpha-2)}{2}\, t^2(1-s)^{\alpha-3} - (t-s)^{\alpha-1} \Big),$$
$$G_t(t,s) = \frac{1}{\Gamma(\alpha)} \Big((\alpha-1)(\alpha-2)t(1-s)^{\alpha-3} - (\alpha-1)(t-s)^{\alpha-2} \Big) =$$
$$= \frac{1}{\Gamma(\alpha-1)} \Big((\alpha-2)t(1-s)^{\alpha-3} - (t-s)^{\alpha-2} \Big) \ge$$
$$\ge \frac{1}{\Gamma(\alpha-1)} \Big((t-s)(t-s)^{\alpha-3} - (t-s)^{\alpha-2} \Big) = 0.$$

Hence, we have

$$G(t,s) \le G(1,s) = \frac{1}{\Gamma(\alpha)}\Big(\frac{(\alpha-1)(\alpha-2)}{2}\,(1-s)^{\alpha-3} - (1-s)^{\alpha-1}\Big) = $$
$$= \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha)}\,\Big(\frac{(\alpha-1)(\alpha-2)}{2} - (1-s)^2\Big).$$

By the definition of $H(s)$, we know

$$H(s) \le \frac{1}{2\Gamma(\alpha)}\,(\alpha-1)(\alpha-2)(1-s)^{\alpha-3} = \frac{(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)}\,,$$

which means that (2.6) holds.

Finally, for $t \le s$, we have

$$\frac{G(t,s)}{H(s)} = \frac{\frac{1}{\Gamma(\alpha)}\,\frac{(\alpha-1)(\alpha-2)}{2}\,t^2(1-s)^{\alpha-3}}{\frac{1}{\Gamma(\alpha)}\,\frac{(\alpha-1)(\alpha-2)}{2}\,s^2(1-s)^{\alpha-3}} = \frac{t^2}{s^2} \ge t^2;$$

for $t \ge s$,

$$\frac{G(t,s)}{H(s)} = \frac{\frac{1}{\Gamma(\alpha)}\,\big(\frac{(\alpha-1)(\alpha-2)}{2}\,t^2(1-s)^{\alpha-3} - (t-s)^{\alpha-1}\big)}{\frac{1}{\Gamma(\alpha)}\,\big(\frac{(\alpha-1)(\alpha-2)}{2}(1-s)^{\alpha-3} - (1-s)^{\alpha-1}\big)} =$$
$$= \frac{1}{\frac{(\alpha-1)(\alpha-2)}{2} - (1-s)^2}\,\Big(\frac{(\alpha-1)(\alpha-2)}{2}\,t^2 - \frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-3}}\Big).$$

Since $s \le t \le 1$, $s \ge ts$ and $t-s \le t-ts$, we can get $(t-s)^{\alpha-3} \le (1-s)^{\alpha-3}$, $(t-s)^2 \le (t-ts)^2$. Thus,

$$\frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-3}} = \frac{(t-s)^2(t-s)^{\alpha-3}}{(1-s)^{\alpha-3}} \le \frac{(t-ts)^2(1-s)^{\alpha-3}}{(1-s)^{\alpha-3}} = t^2(1-s)^2.$$

Therefore,

$$\frac{G(t,s)}{H(s)} \ge \frac{1}{\frac{(\alpha-1)(\alpha-2)}{2} - (1-s)^2}\,\Big(\frac{(\alpha-1)(\alpha-2)}{2}t^2 - t^2(1-s)^2\Big) = t^2,$$

which implies that (iii) holds. The proof is completed. □

Lemma 2.8. *Let P be a cone of the real Banach space E, Ω be a bounded open set of E, $\theta \in \Omega$, $A : P \cap \overline{\Omega} \to P$ be completely continuous.*

(i) *If $x \ne \mu Ax$ for $x \in P \cap \partial\Omega$ and $\mu \in [0,1]$, then $i(A, P \cap \Omega, P) = 1$.*

(ii) *If $\inf_{x \in P\cap\partial\Omega} \|Ax\| > 0$ and $Ax \ne \mu x$ for $x \in P \cap \partial\Omega$ and $\mu \in (0,1]$, then $i(A, P \cap \Omega, P) = 0$.*

Let $J = [0,1]$. The basic space used in this paper is $E = C[J,\mathbb{R}]$. It is well known that E is a Banach space with norm $\|u\| = \max_{t\in J} |u(t)|$ $(\forall\, u \in E)$. From Lemma 2.7, it is easy to see that

$$Q := \Big\{u \in C[J,\mathbb{R}^+] :\ u(t) \ge t^2 u(s),\ \forall\, t,\ s \in J\Big\} \tag{2.9}$$

is a cone of E. Moreover, by (2.9), we have for all $u \in Q$,

$$u(t) \geq t^2 \|u\|, \quad \forall\, t \in J. \tag{2.10}$$

A function u is said to be a solution of BVP (1.1) if u satisfies (1.1). In addition, if $u(t) > 0$ for $t \in (0,1)$, then u is said to be a positive solution of BVP (1.1). Obviously, if $u \in Q \setminus \{\theta\}$ is a solution of BVP (1.1), then u is a positive solution of BVP (1.1), where θ denotes the zero element of the Banach space E.

3. Main Results

For convenience, we list the following assumptions.

(H1) $f \in C[(0,1) \times (0,+\infty), \mathbb{R}^+]$ and for every pair of positive numbers R and r with $R > r > 0$,

$$\int_0^1 (1-s)^{\alpha-3} f_{r,R}(s)\, ds < +\infty,$$

where $f_{r,R}(s) := \max\{f(s,u) :\ u \in [rs^2, R]\}$ for all $s \in (0,1)$.

(H2) For every $R > 0$, there exists $\psi_R \in C[J, \mathbb{R}^+]$ ($\psi_R \neq \theta$) such that $f(t,u) \geq \psi_R(t)$ for $t \in (0,1)$ and $u \in (0,R]$.

(H3) There exists an interval $[a,b] \subset (0,1)$ such that $\lim\limits_{u \to +\infty} f(s,u)/u = +\infty$ uniformly with respect to $s \in [a,b]$.

We remark that (H2) allows $f(t,u)$ being singular at $t = 0, 1$, and $u = 0$. Assumption (H3) shows that f is superlinear in u. The following theorem is our main results of this paper.

Theorem 3.1. *Assume* (H1)–(H3) *are satisfied. Then there exist positive numbers* λ^* *and* λ^{**} *with* $\lambda^* < \lambda^{**}$ *such that BVP* (1.1) *has at least two positive solutions for* $\lambda \in (0, \lambda^*)$ *and no solution for* $\lambda > \lambda^{**}$.

To overcome difficulties arising from singularity, we first consider the approximate problem

$$\begin{aligned} {}^C D^{\alpha}_{0+} u(t) + \lambda f_n(t, u(t)) &= 0, \quad 0 < t < 1, \\ u^{(j)}(0) &= 0, \quad 0 \leq j \leq n-1, \quad j \neq 2, \\ u''(1) &= 0, \end{aligned} \tag{3.1}$$

where $f_n(t,u) =: f\big(t, \max\{\frac{1}{n}, u\}\big)$, $n \in \mathbb{N}$. Define an operator A_n^{λ} on Q by

$$(A_n^{\lambda} u)(t) := \lambda \int_0^1 G(t,s) f_n(s, u(s))\, ds, \tag{3.2}$$

where $G(t,s)$ is defined by (2.3).

Obviously, $u = A_n^{\lambda} u$ is the corresponding integral equation of (3.1). Therefore, $u \in E$ is a solution of (3.1) if $u \in E$ is a fixed point of A_n^{λ}.

Furthermore, u is a positive solution of (3.1) if $u \in Q \setminus \{\theta\}$ is a fixed point of A_n^λ.

By (3.2), it is easy to see that A_n^λ is well defined on Q for each $n \in \mathbb{N}$ if the condition (H1) holds. For the sake of proving our main results we first prove some lemmas.

Lemma 3.2. *Under the condition* (H1)*,* $A_n^\lambda : Q \to Q$ *is completely continuous.*

Proof. First, we show that $A_n^\lambda Q \subset Q$ for each $n \in \mathbb{N}$ and $\lambda > 0$. From Lemma 2.7, it follows that

$$(A_n^\lambda u)(t) = \lambda \int_0^1 G(t,s) f_n(s,u(s))\,ds \geq$$

$$\geq t^2 \lambda \int_0^1 G(\tau,s) f_n(s,u(s))\,ds = t^2 (A_n^\lambda u)(\tau), \ \ \forall\, t,\tau \in J, \ \ u \in Q.$$

Therefore, $A_n^\lambda Q \subset Q$ for each $n \in \mathbb{N}$ and $\lambda > 0$.

Next, by standard methods and Ascoli–Arzela theorem one can prove that $A_n^\lambda : Q \to Q$ is completely continuous. So it is omitted. □

Lemma 3.3. *Suppose the conditions* (H1) *and* (H2) *hold. Then for each* $r > 0$ *there exists a positive number* $\lambda(r)$ *such that*

$$i(A_n^\lambda, Q_r, Q) = 1$$

for $\lambda \in (0, \lambda(r))$ *and* n *sufficiently large, where* $Q_r = \{u \in Q : \ \|u\| < r\}$.

Proof. For each $r > 0$ and $n > \frac{1}{r}$, let

$$\lambda(r) := r \left[\frac{1}{2\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} f_{r,r}(s)\,ds \right]^{-1}.$$

We assert $\|A_n^\lambda u\| < \|u\|$ for each $\lambda \in (0, \lambda(r))$ and $u \in \partial Q_r$. In fact, using (2.10) and

$$G(t,s) \leq \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} \ \text{ for } \ t,s \in J,$$

one can obtain

$$\|A_n^\lambda u\| \leq \lambda \int_0^1 \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_n(s,u(s))\,ds =$$

$$= \lambda \frac{1}{2\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} f_{r,r}(s)\,ds < r =$$

$$= \|u\| \ \text{ for } \lambda \in (0, \lambda(r)) \text{ and } u \in \partial Q_r.$$

Therefore, by Lemma 2.8, we have $i(A_n^\lambda, Q_r, Q) = 1$ for $\lambda \in (0, \lambda(r))$. □

Lemma 3.4. *Suppose the conditions* (H1) *and* (H2) *hold. Then for any given* $\lambda \in (0, \lambda(r))$ *there exists* $r' \in (0, r)$ *such that*

$$i(A_n^\lambda, Q_{r'}, Q) = 0$$

for n *sufficiently large, where* r *and* $\lambda(r)$ *are the same as in Lemma* 3.3.

Proof. Choose a positive number r' with

$$r' < \min\left\{r, \lambda \max_{t\in J} \int_0^1 G(t,s)\psi_r(s)\,ds\right\},$$

where $\psi_r(s)$ is defined as in (H2). Now, we claim that

$$A_n^\lambda u \neq \mu u, \quad \forall\, u \in \partial Q_{r'}, \quad \mu \in (0,1], \tag{3.3}$$

for $n > 1/r'$. Suppose, on the contrary, that there exist $u_0 \in \partial Q_{r'}$ and $\mu_0 \in (0,1]$ such that $A_n^\lambda u_0 = \mu_0 u_0$, namely,

$$u_0(t) \geq (A_n^\lambda u_0)(t) = \lambda \int_0^1 G(t,s) f_n(s, u_0(s))\,ds, \quad \forall t \in J.$$

Notice that $|u_0(s)| \leq r' < r$ and $n > \frac{1}{r'}$ imply $f_n(s, u_0(s)) \geq \psi_r(s)$ for $s \in (0,1)$. Therefore,

$$u_0(t) \geq (A_n^\lambda u_0)(t) \geq \lambda \int_0^1 G(t,s)\psi_r(s)\,ds,$$

that is,

$$r' \geq \lambda \max_{t\in J} \int_0^1 G(t,s)\psi_r(s)\,ds,$$

which is in contradiction with the selection of r'. This means that (3.3) holds. Thus, by Lemma 2.8, we have $i(A_n^\lambda, Q_{r'}, Q) = 0$ for $n > \frac{1}{r'}$. □

Lemma 3.5. *Suppose the condition* (H3) *holds. Then for every* $\lambda \in (0, \lambda(r))$*, there exists* $R > r$ *such that*

$$i(A_n^\lambda, Q_R, Q) = 0$$

for all $n \in \mathbb{N}$*, where* $\lambda(r)$ *is the same as in Lemma* 3.3.

Proof. By (H3) we know that there exists $R' > \max\{r, 1\}$ such that

$$\frac{f(t,u)}{u} > L := \left[a^2\left(\lambda \min_{t\in[a,b]} \int_a^b G(t,s)\,ds\right)\right]^{-1} \text{ for } u > R'. \tag{3.4}$$

Let $R := 1 + \frac{R'}{a^2}$. Then for $u \in \partial Q_R$, by (2.10) we have $u(t) \geq a^2\|u\| > R'$ as $t \in [a,b]$. Now we show that

$$A_n^\lambda u \neq \mu u \text{ for } u \in \partial Q_R \text{ and } \mu \in (0,1]. \tag{3.5}$$

Suppose, on the contrary, that there exist $u_0 \in \partial Q_R$ and $\mu_0 \in (0,1]$ such that $A_n^\lambda u_0 = \mu_0 u_0$, that is,

$$u_0(t) \geq (A_n^\lambda u_0)(t) = \lambda \int_0^1 G(t,s) f_n(s, u_0(s))\, ds, \quad \forall\, t \in J.$$

Furthermore,

$$u_0(t) \geq (A_n^\lambda u_0)(t) > \lambda \Big(\int_a^b G(t,s) \cdot L u_0(s)\, ds \Big) >$$

$$> \Big(\lambda \min_{t \in [a,b]} \int_a^b G(t,s)\, ds \Big) L a^2 R = R$$

for $t \in [a,b]$. That is in contradiction with $\|u_0\| = R$, which means that (3.5) holds. Therefore, by Lemma 2.8, we have $i(A_n^\lambda, Q_R, Q) = 0$ for $n \in \mathbb{N}$. □

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. For each $r > 0$, by Lemmas 3.3–3.5, there exist three positive numbers $\lambda(r), r'$, and R with $r' < r < R$ such that

$$i(A_n^\lambda, Q_{r'}, Q) = 0, \quad i(A_n^\lambda, Q_r, Q) = 1, \quad i(A_n^\lambda, Q_R, Q) = 0 \tag{3.6}$$

for n sufficiently large. Without loss of generality, suppose (3.6) holds for $n \geq n_0$. By virtue of the excision property of the fixed point index, we get

$$i\big(A_n^\lambda, Q_r \setminus \overline{Q_{r'}}, Q\big) = 1, \quad i\big(A_n^\lambda, Q_R \setminus \overline{Q_r}, Q\big) = -1$$

for $n \geq n_0$. Therefore, using the solution property of the fixed point index, there exist $u_n \in Q_r \setminus \overline{Q_{r'}}$ and $v_n \in Q_R \setminus \overline{Q_r}$ satisfying $A_n^\lambda u_n = u_n$ and $A_n^\lambda v_n = v_n$ as $n \geq n_0$. By the proof of Lemma 3.3, we know that there is no positive fixed point on ∂Q_r. Thus, $u_n \neq v_n$. Moreover, from (2.10) it follows that

$$r' t^2 \leq u_n(t) < r \text{ and } r t^2 < v_n(t) \leq R \text{ for } t \in J. \tag{3.7}$$

Further, we show that $\{u_n(t)\}_{n \geq n_0}$ are equicontinuous on J. To see this, we need to prove only that $\lim\limits_{t \to 0+} u_n(t) = 0$ uniformly with respect to $n \in \{n_0, n_0+1, n_0+2, \dots\}$ and $\{u_n(t)\}_{n \geq n_0}$ are equicontinuous on any subinterval of $(0,1]$. We first claim that $\lim\limits_{t \to 0+} u_n(t) = 0$ uniformly with respect to $n \in \{n_0, n_0+1, n_0+2, \dots\}$.

For arbitrary $\varepsilon > 0$, by (H1), there exists $\overline{\delta} > 0$ such that

$$\lambda \int_0^{\overline{\delta}} \frac{1}{2\Gamma(\alpha-2)} (1-s)^{\alpha-3} f_{r',r}(s)\, ds \leq \frac{\varepsilon}{3}\,. \tag{3.8}$$

Choose $\delta \in (0, \overline{\delta})$ sufficiently small such that

$$\lambda \delta^2 \int_0^1 \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds < \frac{\varepsilon}{3}\,. \tag{3.9}$$

Therefore, by (2.6), (3.8) and (3.9), we know for $t \in (0, \delta)$ and $\forall n \geq n_0$ that

$$\begin{aligned}
u_n(t) = \lambda \int_0^1 & G(t,s) f_n(s, u_n(s))\,ds \leq \\
&\leq \lambda \int_0^t \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds + \\
&\quad + \lambda\Big(\int_t^{\overline{\delta}} + \int_{\overline{\delta}}^1\Big) \frac{t^2}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds \leq \\
&\leq 2\lambda \int_0^{\overline{\delta}} \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds + \\
&\quad + \lambda t^2 \int_{\overline{\delta}}^1 \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds \leq \\
&\leq 2\lambda \int_0^{\overline{\delta}} \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds + \\
&\quad + \lambda \delta^2 \int_0^1 \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds \leq \\
&\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This implies that $\lim\limits_{t\to 0+} u_n(t) = 0$ uniformly with respect to $n \in \{n_0, n_0+1, n_0+2, \dots\}$.

Now we are in a position to show that $\{u_n(t)\}_{n\geq n_0}$ are equicontinuous on any subinterval $[a,b]$ of $(0,1]$. Notice that

$$u_n(t) = \lambda \int_0^1 G(t,s) f_n(s, u_n(s))\,ds, \quad \forall\, t \in (0,1].$$

Thus, for $t \in [a,b]$, we have

$$|u_n'(t)| = \lambda\Big|\int_0^1 G_t(t,s)f_n(s,u_n(s))ds\Big| \leq$$

$$\leq \frac{\lambda}{\Gamma(\alpha)}\Big(\int_0^t \Big|(\alpha-1)(\alpha-2)t(1-s)^{\alpha-3} - (\alpha-1)(t-s)^{\alpha-2}\Big| f_{r',r}(s)\,ds +$$

$$+ \int_t^1 \Big|(\alpha-1)(\alpha-2)t(1-s)^{\alpha-3}\Big| f_{r',r}(s)\,ds\Big) \leq$$

$$\leq \frac{\lambda(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_0^1 t(1-s)^{\alpha-3}f_{r',r}(s)\,ds \leq$$

$$\leq \frac{\lambda}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}f_{r',r}(s)\,ds < +\infty,$$

which implies that $\{u_n(t)\}_{n\geq n_0}$ are equicontinuous on $[a,b]$. Similarly as above, we can get that $\{v_n(t)\}_{n\geq n_0}$ are equicontinuous on $[0,1]$.

Then, the Ascoli–Arzela theorem guarantees the existence of $u, v \in Q\setminus\{\theta\}$ and two subsequences $\{u_{n_i}\}$ of $\{u_n\}$ and $\{v_{n_i}\}$ of $\{v_n\}$ such that $\lim_{i\to+\infty} u_{n_i}(t) = u(t)$ and $\lim_{i\to+\infty} v_{n_i}(t) = v(t)$ both uniformly with respect to $t\in J$. Moreover, by (H1), (3.7), and Lebesgue dominated convergence theorem, we obtain

$$u(t) = \lambda\int_0^1 G(t,s)f(s,u(s))\,ds, \quad v(t) = \lambda\int_0^1 G(t,s)f(s,v(s))\,ds, \ \ \forall\, t\in J$$

with $r' \leq \|u\| \leq r \leq \|v\| \leq R$. On the other hand, similarly to the proof of Lemma 3.3, it is easy to see $\|u\| < r < \|v\|$.

Choose $r = 1$. From the above we know that there exists $\lambda(1) > 0$ such that for each $\lambda\in(0,\lambda(1))$, BVP (1.1) has at least two positive solutions u_λ and v_λ with $0 < \|u_\lambda\| < 1 < \|v_\lambda\|$. Let

$$\lambda^* := \sup\{\overline{\lambda} > 0 :\ (1.1) \text{ have at least two positive solutions as } \lambda\in(0,\overline{\lambda})\}.$$

So, we get the existence of λ^* satisfying that BVP (1.1) has multiple positive solutions as $\lambda\in(0,\lambda^*)$.

Now we are in a position to prove the existence of λ^{**}. As above, we still choose $r = 1$ and corresponding $\lambda(1)$, R, r'. Here we show that BVP (1.1) has no positive solution for λ sufficiently large.

First suppose $\lambda \geq \lambda^*$. If BVP (1.1) has a positive solution u for some $\lambda\geq\lambda^*$, then by the corresponding integral equation

$$u(t) = \lambda\int_0^1 G(t,s)f(s,u(s))\,ds \tag{3.10}$$

and a process similar to the proof of Lemmas 3.4 and 3.5 (replacing λ in (3.4) with $\lambda(1)$), we obtain $r' < \|u\| < R$. This together with the condition (H2) and (3.10) guarantees that $u(t) \geq \lambda \int_0^1 G(t,s)\psi_R(s)ds$, that is, $R > \|u\| \geq \lambda \cdot \max_{t\in J} \int_0^1 G(t,s)\psi_R(s)\,ds$, which implies $\lambda < \big(\max_{t\in J} \int_0^1 G(t,s)\psi_R(s)\,ds\big)^{-1} R$. Therefore, we have obtained the existence of λ^{**}. The proof of Theorem 3.1 is complete. □

If $f(t,u)$ is not singular at $u = 0$, we have the following result, under the hypothesis

(H4) $f \in C[(0,1) \times [0,+\infty), \mathbb{R}^+]$ is nondecreasing with respect to u and for every positive number R,

$$\int_0^1 (1-s)^{\alpha-3} f_{0,R}(s)\,ds < +\infty,$$

where $f_{0,R}(s) = \max\{f(s,u) :\ u \in [0,R]\}$ for all $s \in (0,1)$.

Theorem 3.6. *Assume that the conditions* (H2)–(H4) *hold. Then there exist two positive numbers* λ^* *and* λ^{***} *with* $\lambda^* \leq \lambda^{***}$ *such that*

(i) *BVP* (1.1) *has at least two positive solutions for* $\lambda \in (0,\lambda^*)$*;*
(ii) *BVP* (1.1) *has at least one positive solution for* $\lambda \in (0,\lambda^{***}]$*;*
(iii) *BVP* (1.1) *has no solutions for* $\lambda > \lambda^{***}$*.*

Proof. Notice that the condition (H4) implies (H1). Therefore, the existence of λ^* can be obtained just as in Theorem 3.1. Now we claim that

$$\lambda^{***} := \sup\Big\{\lambda \in \mathbb{R}^+ : (1.1) \text{ has at least one positive solution}\Big\} \tag{3.11}$$

is required. First, from the proof of Theorem 3.1, we know that $\lambda^{***} \leq \lambda^{**}$. In the following we prove that (1.1) with $\lambda = \lambda^{***}$ has a positive solution $u^* \in Q$.

By (3.11), there exist two sequences $\{\lambda_n\}$ and $\{u_n\} \subset Q \setminus \{\theta\}$ such that $\{u_n\}$ is a positive solution of BVP (1.1) with $\lambda = \lambda_n$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_n \to \lambda^{***}$. Without loss of generality, suppose $\lambda_n \geq \lambda^*/2$ for each $n \in \mathbb{N}$. Similarly to the proof of Lemmas 3.4, 3.5 and Theorem 3.1, we can find that there exist two positive numbers r_1 and R_1 satisfying $r_1 \leq \|u_n\| \leq R_1$ for each $n \in \mathbb{N}$, and $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ which convergences to a function $u^* \in \overline{Q}_{R_1} \setminus Q_{r_1}$ uniformly as $t \in J$. Notice that

$$u_{n_k}(t) = \lambda_{n_k} \int_0^1 G(t,s) f(s,u_{n_k}(s))\,ds, \quad \forall\, t \in J.$$

Letting $k \to +\infty$, by the condition (H4) and Lebesgue dominated convergence theorem, we get

$$u^*(t) = \lambda^{***} \int_0^1 G(t,s) f(s, u^*(s))\, ds, \quad \forall\, t \in J.$$

This implies that $u^*(t)$ is a positive solution of BVP (1.1) with $\lambda = \lambda^{***}$.

Now we are in a position to prove that BVP (1.1) has at least one positive solution $u_\lambda(t)$ for each $\lambda \in (0, \lambda^{***})$. Notice that for $\lambda \in (0, \lambda^{***})$,

$$\begin{aligned} {}^C D_{0+}^{\alpha} u^*(t) &= \lambda^{***} f(t, u^*(t)) \geq \lambda f(t, u^*(t)), \quad t \in (0,1), \\ u^{*(j)}(0) &= 0, \quad 0 \leq j \leq n-1, \quad j \neq 2, \\ (u^*)''(1) &= 0. \end{aligned} \tag{3.12}$$

This implies that $u^*(t)$ is an upper solution of BVP (1.1). On the other hand, $u(t) \equiv 0$ is a lower solution for BVP (1.1). Applying [4, p. 244, Theorem 2.1], one can obtain that BVP (1.1) has at least one positive solution $u_\lambda(t) \in [0, u^*(t)]$ $(t \in J)$ for each $\lambda \in (0, \lambda^{***})$. □

4. Examples

Example 4.1. Consider the fractional singular boundary value problem

$$\begin{aligned} {}^C D_{0+}^{7/2} u(t) + \lambda \left[\frac{1}{\sqrt{t(1-t)}} \left(u^{-1/6} + u^2 \sin^2 t \right) \right] = 0, \quad t \in (0,1), \\ u(0) = u'(0) = u''(1) = u'''(0) = 0. \end{aligned} \tag{4.1}$$

Then there exist positive numbers λ^* and λ^{**} with $\lambda^* < \lambda^{**}$ such that BVP (4.1) has at least two positive solutions for $\lambda \in (0, \lambda^*)$ and no solution for $\lambda > \lambda^{**}$.

Proof. BVP (4.1) can be regarded as a BVP of the form (1.1), where $\alpha = \frac{7}{2}$, and

$$f(t,u) = \frac{1}{\sqrt{t(1-t)}} \left(u^{-1/6} + u^2 \sin^2 t \right).$$

We prove that $f(t,u)$ satisfies the conditions (H1)–(H3). For each pair of positive numbers R and r with $R > r > 0$, we know

$$f_{r,R}(t) \leq \frac{1}{\sqrt{t(1-t)}} \left((rt^2)^{-1/6} + R^2 \right).$$

Then

$$\int_0^1 (1-t)^{1/2} f_{r,R}(t)\, dt \leq \int_0^1 \frac{1}{\sqrt{t}} \left((rt^2)^{-1/6} + R^2 \right) dt < +\infty.$$

This means that the condition (H1) is satisfied. To see that (H2) holds, we notice that for each $R > 0$, one can choose $\psi_R(t) = R^{-1/6}/\sqrt{t(1-t)}$, which satisfies $\psi_R \neq \theta$ and $f(t,u) \geq \psi_R(t)$ for $t \in (0,1)$ and $u \in (0,R]$. Finally, it is easy to see that (H3) is satisfied since we can choose any subinterval of $[a,b] \subset (0,1)$ satisfying $\lim\limits_{u\to+\infty} f(s,u)/u = +\infty$ uniformly with respect to $s \in [a,b]$. By Theorem 3.1, the conclusion follows. □

Analogously, using Theorem 3.6, we can prove that the following statement holds.

Example 4.2. Consider the fractional singular boundary value problem

$$\begin{aligned} {}^C D^{\alpha}_{0+} u(t) &= \lambda t^{-1/2}(1-t)^{3-\alpha}(1+e^u+u^2\sin t), \quad t\in(0,1),\\ u^{(j)}(0) &= 0, \quad 0\le j\le n-1, \quad j\ne 2, \\ u''(1) &= 0. \end{aligned} \tag{4.2}$$

where $\alpha \in (n-1,n]$, $n \geq 4$. Then there exist two positive numbers λ^* and λ^{***} with $\lambda^* \leq \lambda^{***}$ such that:

(i) BVP (4.2) has at least two positive solutions for $\lambda \in (0,\lambda^*)$;
(ii) BVP (4.2) has at least one positive solution for $\lambda \in (0,\lambda^{***}]$;
(iii) BVP (4.2) has no solution for $\lambda > \lambda^{***}$.

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