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GENERALIZED REGULARLY VARYING SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

Dedicated to 80th birthday anniversary of Professor Kusano Takaŝi

Abstract. The sharp sufficient conditions of the existence of generalized regularly varying solutions (in the sense of Karamata) of differential equations of the type

$$\big(p(t)\varphi(x'(t))\big)' \pm \sum_{i=1}^{n} \Big[q_i(t)\varphi\big(x(g_i(t))\big) + r_i(t)\varphi\big(x(h_i(t))\big)\Big] = 0$$

are established. Here, $p, q_i, r_i : [a,\infty) \to (0,\infty)$ are continuous functions, $g_i, h_i : [a,+\infty) \to R$ are continuous and increasing functions such that $g_i(t) < t$, $h_i(t) > t$ for $t \geq a$, $\lim\limits_{t\to\infty} g_i(t) = \infty$ and $\varphi(\xi) \equiv |\xi|^\alpha \operatorname{sgn} \xi$, $\alpha > 0$.

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$$\big(p(t)\varphi(x'(t))\big)' \pm \sum_{i=1}^{n} \Big[q_i(t)\varphi\big(x(g_i(t))\big) + r_i(t)\varphi\big(x(h_i(t))\big)\Big] = 0$$

სახის დიფერენციალური განტოლებებისათვის, სადაც $p, q_i, r_i : [a,\infty) \to (0,\infty)$ უწყვეტი ფუნქციებია, $g_i, h_i : [a,+\infty) \to R$ ისეთი უწყვეტი არაზრდადი ფუნქციებია, რომ $g_i(t) < t$, $h_i(t) > t$, როცა $t \geq a$, $\lim\limits_{t\to\infty} g_i(t) = \infty$, ხოლო $\varphi(\xi) \equiv |\xi|^\alpha \operatorname{sgn} \xi$, $\alpha > 0$.

1. Introduction

The equation to be studied in this paper is

$$\big(p(t)\varphi(x'(t))\big)' \pm \sum_{i=1}^{n} \Big[q_i(t)\varphi\big(x(g_i(t))\big) + r_i(t)\varphi\big(x(h_i(t))\big)\Big] = 0 \qquad (\mathrm{A}_\pm)$$

$$\big(\,\varphi(\xi) = |\xi|^\alpha \operatorname{sgn}\xi, \;\; \alpha > 0, \;\; \xi \in \mathbb{R}\,\big),$$

where $p, q_i, r_i : [a,\infty) \to (0,\infty)$ are continuous functions, g_i, h_i are continuous and increasing functions with $g_i(t) < t$, $h_i(t) > t$ and $\lim_{t\to\infty} g_i(t) = \infty$ for $i = 1, 2, \dots, n$. In what follows we always assume that the function $p(t)$ satisfies

$$\int_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \infty. \qquad (1.1)$$

It is shown in the monograph [8] that the class of regularly varying functions in the sense of Karamata is a well-suited framework for the asymptotic analysis of nonoscillatory solutions of the second order linear differential equation of the form

$$x''(t) = q(t)x(t), \;\; q(t) > 0.$$

The study of asymptotic analysis of nonoscillatory solutions of functional differential equations with deviating arguments in the framework of regularly varying functions (called Karamata functions) was first attempted by Kusano and Marić [5], [6]. They established a sharp condition for the existence of a slowly varying solution of the second order functional differential equation with retarded argument of the form

$$x''(t) = q(t)x(g(t)), \qquad (1.2)$$

and the following functional differential equation of the form

$$x''(t) \pm \big[q(t)x(g(t)) + r(t)x(h(t))\big] = 0, \qquad (1.3)$$

where $q, r : [a,\infty) \to (0,\infty)$ are continuous functions, g, h are continuous and increasing with $g(t) < t$, $h(t) > t$ for $t \geqq a$, $\lim_{t\to\infty} g(t) = \infty$.

It is well known that there is the qualitative similarity between linear differential equations and half-linear differential equations (see the book Došlý and Řehák [2]). Therefore, in our previous papers [4], [7] we proved how useful the regularly varying functions were for the study of nonoscillation and asymptotic analysis of the half-linear differential equation involving nonlinear Sturm–Liouville type differential operator of the form

$$\big(p(t)\varphi(x'(t))\big)' \pm f(t)\varphi(x(t)) = 0, \;\; p(t) > 0, \qquad (\mathrm{B}_\pm)$$

and the half-linear functional differential equation with both retarded and advanced arguments of the form

$$\big(\varphi(x'(t))\big)' \pm \Big[q(t)\varphi\big(x(g(t))\big) + r(t)\varphi\big(x(h(t))\big)\Big] = 0, \qquad (1.4)$$

where $f : [a,\infty) \to (0,\infty)$ is a continuous function, p, g, h are just as in the above equations.

Theorem A (J. Jaroš, T. Kusano and T. Tanigawa [4]). *Suppose that* (1.1) *holds. The equations* ($\mathrm{B}_\pm$) *have a normalized slowly varying solution with respect to* $P(t)$ *and a normalized regularly varying solution of index* 1 *with respect to* $P(t)$ *if and only if*

$$\lim_{t\to\infty} P(t)^\alpha \int_t^\infty f(s)\,ds = 0, \tag{1.5}$$

where the function $P(t)$ *is defined by*

$$P(t) = \int_a^t \frac{ds}{p(s)^{\frac{1}{\alpha}}}\,. \tag{1.6}$$

Theorem B (J. Manojlović and T. Tanigawa [7]). *Suppose that*

$$\lim_{t\to\infty} \frac{g(t)}{t} = 1 \ \text{ and } \ \lim_{t\to\infty} \frac{h(t)}{t} = 1$$

hold. Then the equations (1.4) *have a slowly varying solution and a regularly varying solution of index* 1 *if and only if*

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)\,ds = \lim_{t\to\infty} t^\alpha \int_t^\infty r(s)\,ds = 0.$$

The objective of this paper is to establish a sharp condition of the existence of a normalized slowly varying solution with respect to $P(t)$ and a normalized regularly varying solution of index 1 with respect to $P(t)$ of the equation ($\mathrm{A}_\pm$). Our main result is the following

Theorem 1.1. *Suppose that*

$$\lim_{t\to\infty} \frac{P(g_i(t))}{P(t)} = 1 \ \textit{ for } \ i = 1,2,\dots,n \tag{1.7}$$

and

$$\lim_{t\to\infty} \frac{P(h_i(t))}{P(t)} = 1 \ \textit{ for } \ i = 1,2,\dots,n \tag{1.8}$$

hold. The equation ($\mathrm{A}_\pm$) *possesses a normalized slowly varying solution with respect to* $P(t)$ *and a normalized regularly varying solution of index* 1 *with respect to* $P(t)$ *if and only if*

$$\lim_{t\to\infty} P(t)^\alpha \int_t^\infty q_i(s)\,ds = \lim_{t\to\infty} P(t)^\alpha \int_t^\infty r_i(s)\,ds = 0 \ \textit{ for } \ i = 1,2,\dots,n. \tag{1.9}$$

This paper is organized as follows. In Section 2 we briefly recall the definitions and properties of the slowly varying and regularly varying functions of index ρ with respect to $P(t)$ which are called the generalized regularly varying functions introduced by Jaroš and Kusano [3]. Explicit expressions for the normalized slowly varying solution with respect to $P(t)$ and the normalized regularly varying solution of index 1 with respect to $P(t)$ of the equations ($B_{\pm}$) obtained in [4] do not meet our need for application to the functional differential equations ($A_{\pm}$), and thus we present a modified proof of Theorem A in Section 3. The proof of Theorem 1.1 which is based on Theorems A and B will be presented in Section 4. Some examples illustrating our result will also be presented in Section 5.

2. Definitions and Properties of the Generalized Regularly Varying Functions

For the reader's convenience we first state the definitions and some basic properties of the regularly varying functions and then refer to the generalized regularly varying functions. The generalized regularly varying functions are introduced for the first time by Jaroš and Kusano [3] in order to gain useful information about an asymptotic behavior of nonoscillatory solutions for the self-adjoint differential equations of the form

$$\big(p(t)x'(t)\big)' + f(t)x(t) = 0.$$

The definitions and properties of regularly varying functions:

Definition 2.1. A measurable function $f : [a,\infty) \to (0,\infty)$ is said to be a regularly varying of index ρ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \ \text{ for any } \ \lambda > 0, \ \ \rho \in \mathbb{R}.$$

Proposition 2.1 (Representation Theorem)**.** *A measurable function $f : [a,\infty) \to (0,\infty)$ is regularly varying of index ρ if and only if it can be written in the form*

$$f(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\delta(s)}{s}\, ds \right\}, \ \ t \geqq t_0,$$

for some $t_0 > a$, where $c(t)$ and $\delta(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \ \ \text{and} \ \ \lim_{t\to\infty} \delta(t) = \rho.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. The symbol SV is used to denote $\mathrm{RV}(0)$ and a member of $\mathrm{SV} = \mathrm{RV}(0)$ is referred to as a slowly varying function. If $f(t) \in \mathrm{RV}(\rho)$, then $f(t) = t^{\rho} L(t)$ for some $L(t) \in \mathrm{SV}$. Therefore, the class of slowly varying functions is of fundamental importance in the theory of regular variation. In addition to the functions tending to positive constants as $t \to \infty$, the following functions

$$\prod_{i=1}^{N} (\log_i t)^{m_i} \ (m_i \in \mathbb{R}), \quad \exp\left\{ \prod_{i=1}^{N} (\log_i t)^{n_i} \right\} \ (0 < n_i < 1), \quad \exp\left\{ \frac{\log t}{\log_2 t} \right\},$$

where $\log_1 t = \log t$ and $\log_k t = \log \log_{k-1} t$ for $k = 2, 3, \ldots, N$, also belong to the set of slowly varying functions.

Proposition 2.2. *Let $L(t)$ be any slowly varying function. Then, for any $\gamma > 0$,*

$$\lim_{t\to\infty} t^{\gamma} L(t) = \infty \ \ \text{and} \ \ \lim_{t\to\infty} t^{-\gamma} L(t) = 0.$$

Proposition 2.3 (Karamata's integration theorem)**.** *Let $L(t) \in$ SV. Then*

(i) *if $\gamma > -1$,*

$$\int_a^t s^{\gamma} L(s)\, ds \ \sim \ \frac{t^{\gamma+1}}{\gamma+1} L(t), \quad \text{as } t \to \infty;$$

(ii) *if $\gamma < -1$,*

$$\int_t^{\infty} s^{\gamma} L(s)\, ds \ \sim \ -\frac{t^{\gamma+1}}{\gamma+1} L(t), \quad \text{as } t \to \infty.$$

Here and hereafter the notation $\varphi(t) \sim \psi(t)$ as $t \to \infty$ is used to mean the asymptotic equivalence of $\varphi(t)$ and $\psi(t)$: $\lim_{t\to\infty} \psi(t)/\varphi(t) = 1$.

For an excellent explanation of the theory of regularly varying functions the reader is referred to the book [1].

The definitions and properties of generalized regularly varying functions:

Definition 2.2. A measurable function $f : [a, \infty) \to (0, \infty)$ is said to be slowly varying with respect to $P(t)$ if the function $f \circ P(t)^{-1}$ is slowly varying in the sense of Karamata, where the function $P(t)$ is defined by (1.6) and $P(t)^{-1}$ denotes the inverse function of $P(t)$. The totality of slowly varying functions with respect to $P(t)$ is denoted by SV_P.

Definition 2.3. A measurable function $g : [a, \infty) \to (0, \infty)$ is said to be regularly varying function of index ρ with respect to $P(t)$ if the function $g \circ P(t)^{-1}$ is regularly varying of index ρ in the sense of Karamata. The set of all regularly varying functions of index ρ with respect to $P(t)$ is denoted by $\mathrm{RV}_P(\rho)$.

Of fundamental importance is the following representation theorem for the generalized slowly and regularly varying functions, which is an immediate consequence of Proposition 2.1.

Proposition 2.4.

(i) *A function $f(t)$ is slowly varying with respect to $P(t)$ if and only if it can be expressed in the form*

$$f(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\delta(s)}{p(s)^{\frac{1}{\alpha}} P(s)}\, ds \right\}, \quad t \geqq t_0 \tag{2.1}$$

for some $t_0 > a$, where $c(t)$ and $\delta(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0, \infty) \quad \textit{and} \quad \lim_{t\to\infty} \delta(t) = 0.$$

(ii) *A function $g(t)$ is regularly varying of index ρ with respect to $P(t)$ if and only if it has the representation*

$$g(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\delta(s)}{p(s)^{\frac{1}{\alpha}} P(s)}\, ds \right\}, \quad t \geqq t_0 \tag{2.2}$$

for some $t_0 > a$, where $c(t)$ and $\delta(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0, \infty) \quad \textit{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

If the function $c(t)$ in (2.1) (or (2.2)) is identically a constant on $[t_0, \infty)$, then the function $f(t)$ (or $g(t)$) is called normalized slowly varying (or normalized regularly varying of index ρ) with respect to $P(t)$. The totality of such functions is denoted by n-SV_P (or n-RV_P).

It is easy to see that if $g(t) \in \mathrm{RV}_P(\rho)$ (n-$\mathrm{RV}_P(\rho)$), then $g(t) = P(t)^\rho f(t)$ for some $f(t) \in \mathrm{SV}_P$ (or n-SV_P).

Proposition 2.5. *Let $f(t) \in \mathrm{SV}_P$. Then, for any $\gamma > 0$,*

$$\lim_{t\to\infty} P(t)^\gamma f(t) = \infty \quad \textit{and} \quad \lim_{t\to\infty} P(t)^{-\gamma} f(t) = 0. \tag{2.3}$$

The Karamata's integration theorem is generalized in the following manner.

Proposition 2.6 (The generalized Karamata's integration theorem)**.** *Let $f(t) \in$ n-SV_P. Then*

(i) *If $\gamma > -1$,*

$$\int_{t_0}^{t} \frac{P(s)^\gamma}{p(s)^{\frac{1}{\alpha}}} f(s)\, ds \sim \frac{P(t)^{\gamma+1}}{\gamma+1} f(t) \quad \textit{as} \quad t \to \infty; \tag{2.4}$$

(ii) *If $\gamma < -1$, $\int_{t_0}^{\infty} P(t)^\gamma f(t)/p(t)^{\frac{1}{\alpha}} dt < \infty$ and*

$$\int_{t}^{\infty} \frac{P(s)^\gamma}{p(s)^{\frac{1}{\alpha}}} f(s)\, ds \sim -\frac{P(t)^{\gamma+1}}{\gamma+1} f(t) \quad \textit{as} \quad t \to \infty. \tag{2.5}$$

3. The Existence of Generalized Regularly Varying Solution of Self-Adjoint Differential Equation without Deviating Arguments

Theorem 3.1. *Put* $F(t) = P(t)^{\alpha} \int_t^{\infty} f(s)\,ds$, $\widehat{F}(t) = \sup_{s \geqq t} F(s)$,

$$F_+(t,w) = |1 + F(t) - w|^{1+\frac{1}{\alpha}} + \Big(1 + \frac{1}{\alpha}\Big) w - 1, \tag{3.1}$$

and

$$F_-(t,w) = 1 + \Big(1 + \frac{1}{\alpha}\Big) w - |1 + F(t) - w|^{1+\frac{1}{\alpha}}. \tag{3.2}$$

(i) *The equation* (B_+) *possesses a* n-SV_P *solution* $x(t)$ *having the expression*

$$x(t) = \exp\bigg\{ \int_{t_0}^{t} \Big(\frac{v(s) + F(s)}{p(s)P(s)^{\alpha}} \Big)^{\frac{1}{\alpha}} ds \bigg\}, \quad t \geqq t_0 \tag{3.3}$$

for some $t_0 > a$, *in which* $v(t)$ *satisfies*

$$v(t) = \alpha P(t)^{\alpha} \int_t^{\infty} \frac{(v(s) + F(s))^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}}\, ds, \quad t \geqq t_0 \tag{3.4}$$

and

$$0 \leqq v(t) \leqq \widehat{F}(t_0) \ \text{ for } \ t \geqq t_0 \tag{3.5}$$

if and only if (1.5) *holds.*

(ii) *The equation* (B_+) *possesses a* n-$\mathrm{RV}_P(1)$ *solution* $x(t)$ *having the expression*

$$x(t) = \exp\bigg\{ \int_{t_1}^{t} \Big(\frac{1 + F(s) - w(s)}{p(s)P(s)^{\alpha}} \Big)^{\frac{1}{\alpha}} ds \bigg\}, \quad t \geqq t_1 \tag{3.6}$$

for some $t_1 > a$, *in which* $w(t)$ *satisfies*

$$w(t) = \frac{\alpha}{P(t)} \int_t^{\infty} F_+(s, w(s))\, ds, \quad t \geqq t_1 \tag{3.7}$$

and

$$0 \leqq w(t) \leqq \sqrt{\widehat{F}(t_1)} \ \text{ for } \ t \geqq t_1 \tag{3.8}$$

if and only if (1.5) *holds.*

(iii) *The equation* (B_-) *possesses a* n-SV_P *solution* $x(t)$ *having the expression*

$$x(t) = \exp\bigg\{ \int_{t_0}^{t} \Big(\frac{v(s) - F(s)}{p(s)P(s)^{\alpha}} \Big)^{\frac{1}{\alpha}*} ds \bigg\}, \quad t \geqq t_0 \tag{3.9}$$

for some $t_0 > a$*, in which* $v(t)$ *satisfies*

$$v(t) = \alpha P(t)^\alpha \int_t^\infty \frac{|v(s) - F(s)|^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}}\, ds, \quad t \geqq t_0 \tag{3.10}$$

and (3.5) *if and only if* (1.5) *holds. Here, the meaning of the asterisk notation is defined by* $\xi^{\gamma *} = |\xi|^\gamma \operatorname{sgn} \xi$, $\gamma > 0$, $\xi \in \mathbb{R}$.

(iv) *The equation* (B_-) *possesses a* n-$RV_P(1)$ *solution* $x(t)$ *having the expression*

$$x(t) = \exp\left\{ \int_{t_1}^t \Big(\frac{1 - F(s) + w(s)}{p(s)P(s)^\alpha} \Big)^{\frac{1}{\alpha}} ds \right\}, \quad t \geqq t_1 \tag{3.11}$$

for some $t_1 > a$*, in which* $w(t)$ *satisfies*

$$w(t) = \frac{\alpha}{P(t)} \int_t^\infty F_-(s, w(s))\, ds, \quad t \geqq t_1 \tag{3.12}$$

and (3.8) *if and only if* (1.5) *holds.*

Our purpose in this section is to give a proof of the above Theorem 3.1. The following lemma will be needed for our purpose.

Lemma 3.1.

(i) *If* $x(t)$*, a nonoscillatory solution of* ($B_\pm$)*, is not zero on* $[a, \infty)$*, then the function* $u(t) = p(t)\varphi(x'(t)/x(t))$ *satisfies the generalized Riccati equation*

$$u'(t) + \alpha \frac{|u(t)|^{1+\frac{1}{\alpha}}}{p(t)^{\frac{1}{\alpha}}} \pm f(t) = 0, \quad t \geqq a. \tag{$C_\pm$}$$

(ii) *If* $u(t)$ *is a solution of* ($C_\pm$)*, then the function*

$$x(t) = \exp\left\{ \int_a^t \Big(\frac{u(s)}{p(s)} \Big)^{\frac{1}{\alpha}*} ds \right\}$$

is a nonoscillatory solution of ($B_\pm$) *on* $[a, \infty)$.

Proof of Theorem 3.1. Since the idea of the proof of Theorem 3.1 for the equation (B_-) is similar to the way of proving the equation (B_+), we restrict our attention to the proof for equation (B_+).

(The "only if" part): Let $x(t)$ be a positive solution of (B_+) belonging to n-SV_P or n-$RV_P(1)$, respectively. Then, by the representation theorem,

$$x(t) = \exp\left\{ \int_{t_0}^t \frac{\delta(s)}{p(s)^{\frac{1}{\alpha}} P(s)}\, ds \right\}, \quad t \geqq t_0,$$

for some $t_0 > a$, where $\lim_{t\to\infty} \delta(t) = 0$ or 1 according as $x(t) \in$ n-SV$_P$ or $x(t) \in$ n-RV$_P(1)$. Since the function

$$u(t) = p(t)\varphi\Big(\frac{x'(t)}{x(t)}\Big) = \varphi\Big(\frac{\delta(t)}{P(t)}\Big)$$

satisfies the generalized Riccati equation (C_+) and $u(t) \to 0$ as $t \to \infty$, we obtain

$$u(t) = \alpha \int_t^\infty \frac{|P(s)^\alpha u(s)|^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}}\, ds + \int_t^\infty f(s)\, ds$$

or

$$P(t)^\alpha u(t) = \alpha P(t)^\alpha \int_t^\infty \frac{|P(s)^\alpha u(s)|^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}}\, ds + \\ + P(t)^\alpha \int_t^\infty f(s)\, ds, \quad t \geqq t_0. \tag{3.13}$$

Letting $t \to \infty$ in (3.13), we easily conclude that (1.5) holds in either case of $P(t)^\alpha u(t) \to 0$ or $P(t)^\alpha u(t) \to 1$ as $t \to \infty$.

(The "if" part) Suppose that (1.5) holds.

(The existence of a n-SV$_P$ solution of (B_+)): Choose $t_0 > \max\{a, 1\}$ so large that

$$\phi = (2\widehat{F}(t_0))^{\frac{1}{\alpha}} \max\Big\{2, 1 + \frac{1}{\alpha}\Big\} < 1, \tag{3.14}$$

and define the set of continuous functions V and the integral operators $\mathcal{F}$ by

$$V = \Big\{v \in C_0[t_0, \infty) : \;\; 0 \leqq v(t) \leqq \widehat{F}(t_0), \;\; t \geqq t_0\Big\} \tag{3.15}$$

and

$$\mathcal{F}v(t) = \alpha P(t)^\alpha \int_t^\infty \frac{(v(s) + F(s))^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}}\, ds, \;\; t \geqq t_0, \tag{3.16}$$

where $C_0[t_0, \infty)$ denotes the Banach space consisting of all continuous functions on $[t_0, \infty)$ and tend to 0 as $t \to \infty$ and equipped with the norm $\|v\|_0 = \sup_{t\geqq t_0} |v(t)|$. It can be verified that $\mathcal{F}$ is a contraction mapping on V. In fact, using (3.14), we see that $v \in V$ implies $\lim_{t\to\infty} \mathcal{F}v(t) = 0$ and

$$\mathcal{F}v(t) \leqq \alpha(2\widehat{F}(t_0))^{1+\frac{1}{\alpha}} P(t)^\alpha \int_t^\infty \frac{ds}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}} = (2\widehat{F}(t_0))^{1+\frac{1}{\alpha}} \leqq \widehat{F}(t_0),$$

and that $v_1, v_2 \in V$ implies

$$\Big| |v_1(t) + F(t)|^{1+\frac{1}{\alpha}} - \big|v_2(t) + F(t)\big|^{1+\frac{1}{\alpha}} \Big| \leqq$$
$$\leqq \Big(1 + \frac{1}{\alpha}\Big)(2\widehat{F}(t_0))^{\frac{1}{\alpha}} |v_1(t) - v_2(t)| \leqq \phi |v_1(t) - v_2(t)|, \quad t \geqq t_0,$$

which ensures that $\mathcal{F}$ is a contraction mapping. Therefore, there exists a unique element $v_0 \in V$ such that $v_0 = \mathcal{F} v_0$, that is,

$$v_0(t) = \alpha P(t)^{\alpha} \int_t^{\infty} \frac{(v_0(s) + F(s))^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}} \, ds, \quad t \geqq t_0.$$

Obviously, $v_0(t)$ satisfies the integral equation

$$\Big(\frac{v_0(t)}{P(t)^{\alpha}}\Big)' + \frac{(v(t) + F(t))^{1+\frac{1}{\alpha}}}{p(t)^{\frac{1}{\alpha}} P(t)^{\alpha+1}} = 0, \quad t \geqq t_0. \tag{3.17}$$

By virtue of the function $v_0(t)$ we define the function

$$x_0(t) = \exp\Big\{ \int_{t_0}^{t} \Big(\frac{v_0(s) + F(s)}{p(s) P(s)^{\alpha}}\Big)^{\frac{1}{\alpha}} ds \Big\}, \quad t \geqq t_0.$$

Since the function $u(t) = v_0(t) + F(t)/P(t)^{\alpha}$ satisfies the generalized Riccati equation (C_+) associated with (B_+) which is easily seen to be equivalent to (3.17), $x_0(t)$ is a solution of the differential equation (B_+).

(The existence of a n-$RV_P(1)$ solution of (B_+)): We will construct a n-$RV_P(1)$ solution of (B_+). Let us consider the function

$$x(t) = \exp\Big\{ \int_{t_1}^{t} \Big(\frac{1 + F(s) - w(s)}{p(s) P(s)^{\alpha}}\Big)^{\frac{1}{\alpha}*} ds \Big\}, \quad t \geqq t_1 \tag{3.18}$$

for some $t_1 > a$ to be determined later. According to (ii) of Lemma 3.1, the function $x(t)$ is a solution of (B_+) on $[t_1, \infty)$ if $w(t)$ is chosen in such way that $u(t) = 1 + F(t) - w(t)/P(t)^{\alpha}$ satisfies the generalized Riccati equation (C_+) on $[t_1, \infty)$. Then the differential equation for $w(t)$ is derived:

$$w'(t) - \frac{\alpha}{p(t)^{\frac{1}{\alpha}} P(t)} w(t) + \frac{\alpha}{p(t)^{\frac{1}{\alpha}} P(t)} \big[1 - |1 + F(t) - w(t)|^{1+\frac{1}{\alpha}}\big] = 0. \tag{3.19}$$

We rewrite (3.19) as

$$(P(t) w(t))' - \frac{\alpha}{p(t)^{\frac{1}{\alpha}}} F_+(t, w(t)) = 0, \tag{3.20}$$

where $F_+(t, w(t))$ is defined with (3.1). It is convenient to express $F_+(t, w)$ as

$$F_+(t, w) = G(t, w) + H(t, w) + k(t), \tag{3.21}$$

with $G(t, w)$, $H(t, w)$ and $k(t)$ defined, respectively, by

$$G(t, w) = |1 + F(t) - w|^{1+\frac{1}{\alpha}} + \Big(1 + \frac{1}{\alpha}\Big)(1 + F(t))^{\frac{1}{\alpha}} w - (1 + F(t))^{1+\frac{1}{\alpha}}, \tag{3.22}$$

$$H(t,w) = \Big(1+\frac{1}{\alpha}\Big)\Big\{1-(1+F(t))^{\frac{1}{\alpha}}\Big\}\,w, \tag{3.23}$$

and

$$k(t) = (1+F(t))^{1+\frac{1}{\alpha}} - 1. \tag{3.24}$$

Since $F(t) \to 0$ as $t \to \infty$ by hypothesis, we can choose $t_1 > \max\{a, 1\}$ such that

$$\Big(1+\frac{1}{\alpha}\Big)[K+L+\alpha]\sqrt{\widehat{F}(t_1)} \leqq 1, \tag{3.25}$$

where K and L are positive constants such that

$$\begin{aligned} K &= \Big(\frac{4}{3}\Big)^{1-\frac{1}{\alpha}} \ \text{ and } \ L = 1 \ \text{ if } \ \alpha > 1;\\ K &= \Big(\frac{3}{2}\Big)^{\frac{1}{\alpha}-1} \ \text{ and } \ L = \Big(\frac{5}{4}\Big)^{\frac{1}{\alpha}-1} \ \text{ if } \ \alpha \leqq 1. \end{aligned} \tag{3.26}$$

Noting that since $1+1/\alpha > 1$ and $K+L+\alpha \geqq 2$, we have in view of (3.25) that $\sqrt{\widehat{F}(t_1)} \leqq 1/2$ and $F(t) \leqq 1/4$ for all $t \geqq t_1$. It is easily shown that, using the mean value theorem and L'Hospital rule, the following inequalities hold for (3.22), (3.23) and (3.24):

$$\Big|\frac{\partial G(t,w)}{\partial w}\Big| \leqq \frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)K|w|, \tag{3.27}$$

$$\Big|\frac{\partial H(t,w)}{\partial w}\Big| \leqq \frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)L\,F(t), \tag{3.28}$$

$$|G(t,w)| \leqq \frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)L\,w^2, \tag{3.29}$$

$$|H(t,w)| \leqq \frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)L\,F(t)|w|, \tag{3.30}$$

and

$$|k(t)| \leqq \Big(1+\frac{1}{\alpha}\Big)F(t) \tag{3.31}$$

for $t \geqq t_1$ and for $|w| \leqq 1/4$.

Consider the set $W \subset C_0[t_1,\infty)$ defined by

$$W = \Big\{w \in C_0[t_1,\infty):\ |w(t)| \leqq \sqrt{\widehat{F}(t_1)}\,,\ \ t \geqq t_1\Big\} \tag{3.32}$$

and define the integral operator $\mathcal{G} : W \to C_0[t_1,\infty)$ by

$$\mathcal{G}w(t) = \frac{\alpha}{P(t)}\int_{t_1}^{t}\frac{F_+(s,w(s))}{p(s)^{\frac{1}{\alpha}}}\,ds, \ \ t \geqq t_1, \tag{3.33}$$

where $F_+(t,w)$ is given by (3.1). Then, it can be shown that $\mathcal{G}$ is a contraction mapping on W. In fact, if $w \in W$, then, by means of (3.29)–(3.31) and (3.25), we can see that

$$|\mathcal{G}w(t)| \leqq \frac{\alpha}{P(t)} \int_{t_1}^{t} \frac{1}{p(s)^{\frac{1}{\alpha}}} \Big[|G(s,w)| + |H(s,w)| + |k(s)|\Big]\, ds \leqq$$

$$\leqq \Big(1+\frac{1}{\alpha}\Big) \frac{1}{P(t)} \int_{t_1}^{t} \frac{1}{p(s)^{\frac{1}{\alpha}}} \Big[Lw(s)^2 + LF(s)|w(s)| + \alpha F(s)\Big]\, ds \leqq$$

$$\leqq \Big(1+\frac{1}{\alpha}\Big) \Big[L\widehat{F}(t_1) + L\widehat{F}(t_1)^{\frac{3}{2}} + \alpha \widehat{F}(t_1)\Big] =$$

$$= \Big(1+\frac{1}{\alpha}\Big) \widehat{F}(t_1) \Big[L + L\sqrt{\widehat{F}(t_1)} + \alpha\Big] \leqq$$

$$\leqq \sqrt{\widehat{F}(t_1)}\, \Big(1+\frac{1}{\alpha}\Big)[K+L+\alpha]\sqrt{\widehat{F}(t_1)} \leqq \sqrt{\widehat{F}(t_1)}\,, \quad t \geqq t_1.$$

Since $F_+(t,w(t)) \to 0$ as $t \to \infty$, we obtain $\lim\limits_{t\to\infty} \mathcal{G}w(t) = 0$. Thus, it follows that $\mathcal{G}w \in W$, and hence $\mathcal{G}$ maps W into itself. Moreover, if $w_1, w_2 \in W$, then, using (3.27) and (3.28), we obtain

$$\big|\mathcal{G}w_1(t) - \mathcal{G}w_2(t)\big| \leqq \frac{\alpha}{P(t)} \times$$

$$\times \int_{t_1}^{t} \frac{1}{p(s)^{\frac{1}{\alpha}}} \Big[\,\big|G(s,w_1(s)) - G(s,w_2(s))\big| + \big|H(s,w_1(s)) - H(s,w_2(s))\big|\,\Big]\, ds \leqq$$

$$\leqq \Big(1+\frac{1}{\alpha}\Big) \Big[K\sqrt{\widehat{F}(t_1)} + L\,\widehat{F}(t_1)\Big] \|w_1 - w_2\|_0 \leqq$$

$$\leqq \Big(1+\frac{1}{\alpha}\Big)[K+L]\sqrt{\widehat{F}(t_1)}\,\|w_1 - w_2\|_0,$$

which implies that

$$\big\|\mathcal{G}w_1 - \mathcal{G}w_2\big\|_0 \leqq \Big(1+\frac{1}{\alpha}\Big)[K+L]\sqrt{\widehat{F}(t_1)}\,\|w_1 - w_2\|_0.$$

In view of (3.25) this shows that $\mathcal{G}$ is a contraction mapping on W. Therefore, the contraction mapping principle ensures the existence of a unique fixed element $w_1 \in W$ such that $w_1 = \mathcal{G}w_1$, which is equivalent to the integral equation

$$w_1(t) = \frac{\alpha}{P(t)} \int_{t_1}^{t} \frac{F_+(s, w_1(s))}{p(s)^{\frac{1}{\alpha}}}\, ds, \quad t \geqq t_1. \tag{3.34}$$

Differentiation of (3.34) shows that $w_1(t)$ satisfies the differential equation (3.20), and substitution of this $w_1(t)$ into (3.6) gives rise to a solution $x(t)$ of the half-linear differential equation (B_+) defined on $[t_1, \infty)$. Furthermore, since $\lim\limits_{t\to\infty} w_1(t) = 0$, it follows from the representation theorem that

$x(t) \in \text{n-RV}_P(1)$. This completes the proof of Theorem 3.1 for the equation (B_+). □

Remark 3.1. Consider another half-linear differential equation

$$\big(p(t)\varphi(x'(t))\big)' + \widetilde{f}(t)\varphi(x(t)) = 0, \qquad (\widetilde{\mathrm{B}}_+)$$

where $\widetilde{f}(t)$ is a positive continuous function such that

$$\widetilde{f}(t) \geqq f(t), \quad t \geqq a$$

and

$$\lim_{t\to\infty} P(t)^\alpha \int_t^\infty \widetilde{f}(s)\,ds = 0.$$

We take $t_0 > \max\{a, 1\}$ so large that

$$(2\widetilde{F}(t_0))^{\frac{1}{\alpha}} \max\Big\{2, 1 + \frac{1}{\alpha}\Big\} < 1 \ \text{ where } \ \widetilde{F}(t) = P(t)^\alpha \int_t^\infty \widetilde{f}(s)\,ds.$$

Then, by means of Theorem 3.1, both $x_0(t)$ and $\widetilde{x}_0(t)$ are given, respectively, by (3.3) and

$$\widetilde{x}_0(t) = \exp\bigg\{\int_{t_0}^t \Big(\frac{\widetilde{v}_0(s) + \widetilde{F}(s)}{p(s)P(s)^\alpha}\Big)^{\frac{1}{\alpha}} ds\bigg\}, \quad t \geqq t_0,$$

where $\widetilde{v}_0(t)$ is a solution of the integral equation

$$\widetilde{v}_0(t) = \alpha\, P(t)^\alpha \int_t^\infty \frac{(\widetilde{v}_0(s) + \widetilde{F}(s))^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}}\,ds, \quad t \geqq t_0.$$

We here compare $x_0(t)$ with $\widetilde{x}_0(t)$. From the proof of Theorem 3.1, $v_0(t)$ and $\widetilde{v}_0(t)$ are the fixed points of the contraction mapping $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ given, respectively, by (3.16) and

$$\widetilde{\mathcal{F}}\widetilde{v}(t) = \alpha\, P(t)^\alpha \int_t^\infty \frac{(\widetilde{v}(s) + \widetilde{F}(s))^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} p(s)^{\alpha+1}}\,ds, \quad t \geqq t_0.$$

Noting that $v_0(t)$ and $\widetilde{v}_0(t)$ are the limit points of uniform convergence on $[t_0, \infty)$ of the sequences defined by

$$v_{n+1}(t) = \mathcal{F}v_n(t), \quad t \geqq t_0, \quad n = 1, 2, \dots, \quad v_1(t) = 0$$

and

$$\widetilde{v}_{n+1}(t) = \widetilde{\mathcal{F}}\widetilde{v}_n(t), \quad t \geqq t_0, \quad n = 1, 2, \dots, \quad \widetilde{v}_1(t) = 0.$$

We conclude that $\widetilde{v}_0(t) \geqq v_0(t)$, $t \geqq t_0$, which implies that $\widetilde{x}_0(t) \geqq x_0(t)$ for $t \geqq t_0$.

4. The Existence of Generalized Regularly Varying Solution of Self-Adjoint Functional Differential Equation with Deviating Arguments

In this section we first present the proof of Theorem 1.1 for equation (A_+) and then give the proof for the equation (A_-).

4.1. **The proof of Theorem 1.1 for the equation (A_+).** (The "only if" part) Suppose that there exists a positive solution $x_1(t) \in$ n-SV_P or $x_2(t) \in$ n-$RV_P(1)$ of (A_+). The equation (A_+) can be written as the half-linear differential equation without retarded and advanced arguments

$$\big(p(t)\varphi(x'(t))\big)' + \sum_{i=1}^{n} \big[q_{x,g_i}(t) + r_{x,h_i}(t)\big]\varphi(x(t)) = 0, \tag{4.1}$$

where

$$q_{x,g_i}(t) = q_i(t)\varphi\Big(\frac{x(g_i(t))}{x(t)}\Big) \text{ and } r_{x,h_i}(t) = r_i(t)\varphi\Big(\frac{x(h_i(t))}{x(t)}\Big), \quad i = 1, 2, \ldots, n. \tag{4.2}$$

Here, applying Theorem 3.1, we see that

$$\lim_{t\to\infty} P(t)^{\alpha} \int_t^{\infty} \sum_{i=1}^{n} \big[q_{x,g_i}(s) + r_{x,h_i}(s)\big]\, ds = 0$$

or

$$\lim_{t\to\infty} P(t)^{\alpha} \int_t^{\infty} \sum_{i=1}^{n} q_{x,g_i}(s)\, ds = \lim_{t\to\infty} P(t)^{\alpha} \int_t^{\infty} \sum_{i=1}^{n} r_{x,h_i}(s)\, ds = 0.$$

By the representation theorem, $x_j(t)$, $j = 1, 2$ can be expressed as

$$x_j(t) = \exp\Bigg\{\int_{t_0}^{t} \frac{\delta_j(s)}{p(s)^{\frac{1}{\alpha}} P(s)}\, ds\Bigg\}, \quad j = 1,\ 2$$

for some $t_0 > a$, where $\delta_j(t)$ satisfies

$$\lim_{t\to\infty} \delta_j(t) = \begin{cases} 0 & (j = 1) \\ 1 & (j = 2). \end{cases}$$

The solutions $x_j(t)$, $j = 1, 2$ satisfy

$$\frac{x_j(g_i(t))}{x_j(t)} = \exp\Bigg\{-\int_{g_i(t)}^{t} \frac{\delta_j(s)}{p(s)^{\frac{1}{\alpha}} P(s)}\, ds\Bigg\}, \quad t \geqq t_1,$$

and

$$\frac{x_j(h_i(t))}{x_j(t)} = \exp\Bigg\{\int_{t}^{h_i(t)} \frac{\delta_j(s)}{p(s)^{\frac{1}{\alpha}} P(s)}\, ds\Bigg\}, \quad t \geqq t_1,$$

respectively, where t_1 is such that $g_i(t_1) \geqq t_0$, $i = 1, 2, \ldots, n$. Then, using the properties of $\delta_j(t)$, (1.7) and (1.8), we see that

$$\int_{g_i(t)}^{t} \frac{|\delta_j(s)|}{p(s)^{\frac{1}{\alpha}} P(s)} \, ds \leqq \sup_{s \geqq g_i(t)} |\delta_j(s)| \cdot \log \frac{P(t)}{P(g_i(t))} \to 0 \text{ as } t \to \infty$$

and

$$\int_{t}^{h_i(t)} \frac{|\delta_j(s)|}{p(s)^{\frac{1}{\alpha}} P(s)} \, ds \leqq \sup_{s \geqq t} |\delta_j(s)| \cdot \log \frac{P(h_i(t))}{P(t)} \to 0 \text{ as } t \to \infty.$$

Thus, it follows that

$$\lim_{t\to\infty} \frac{x_j(g_i(t))}{x_j(t)} = \lim_{t\to\infty} \frac{x_j(h_i(t))}{x_j(t)} = 1, \quad i = 1, 2, \ldots, n, \quad j = 1, 2. \tag{4.3}$$

Consequently, from (4.3) we find that (1.9) holds.

(The "if" part)

(The existence of a n-SV$_P$ solution of (A$_+$)): Suppose that (1.9) is satisfied. Choose $t_0 > a$ so large that $t_* = \min_{i=1,2,\ldots,n} \left\{ \inf_{t \geqq t_0} g_i(t) \right\} > \max\{a, 1\}$,

$$\left\{ 2 \sum_{i=1}^{n} \left[\widehat{Q}_i(t_0) + 2^\alpha \widehat{R}_i(t_0) \right] \right\}^{\frac{1}{\alpha}} \max \left\{ 2, 1 + \frac{1}{\alpha} \right\} < 1 \tag{4.4}$$

and

$$\left(2 \sum_{i=1}^{n} \left[Q_i(t_0) + 2^\alpha R_i(t_0) \right] \right)^{\frac{1}{\alpha}} \log \frac{P(h_i(t))}{P(t)} \leqq \log 2, \quad t \geqq t_0, \tag{4.5}$$

where $Q_i(t)$, $R_i(t)$, $\widehat{Q}_i(t)$ and $\widehat{R}_i(t)$ for $i = 1, 2 \ldots, n$ are defined by

$$Q_i(t) = P(t)^\alpha \int_t^\infty q_i(s) \, ds, \quad \widehat{Q}_i(t) = \sup_{s \geqq t} Q_i(s) \tag{4.6}$$

and

$$R_i(t) = P(t)^\alpha \int_t^\infty r_i(s) \, ds, \quad \widehat{R}_i(t) = \sup_{s \geqq t} R_i(s). \tag{4.7}$$

Let Ξ denote the set of all positive continuous nondecreasing functions $\xi(t)$ on $[t_*, \infty)$ satisfying

$$\xi(t) = 1 \text{ for } t_* \leqq t \leqq t_0; \tag{4.8}$$

$$\xi(t) \leqq \exp \left\{ \int_{t_0}^{\infty} \left(\frac{v_0(s) + \sum_{i=1}^{n} [Q_i(s) + 2^\alpha R_i(s)]}{p(s) P(s)^\alpha} \right)^{\frac{1}{\alpha}} ds \right\} \text{ for } t \geqq t_0; \tag{4.9}$$

$$\frac{\xi(h_i(t))}{\xi(t)} \leqq 2 \text{ for } t \geqq t_0 \tag{4.10}$$

for $i = 1, 2, \dots, n$, where $v_0(t)$ satisfies the following integral equation:

$$v_0(t) = \alpha P(t)^\alpha \int_t^\infty \frac{\big(v_0(s) + \sum_{i=1}^{n} [Q_i(s) + 2^\alpha R_i(s)]\big)^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}} \, ds, \quad t \geqq t_0. \tag{4.11}$$

We note that the function

$$X_0(t) = \exp\left\{ \int_{t_0}^{t} \left(\frac{v_0(s) + \sum_{i=1}^{n} [Q_i(s) + 2^\alpha R_i(s)]}{p(s) P(s)^\alpha} \right)^{\frac{1}{\alpha}} ds \right\}, \quad t \geqq t_0 \tag{4.12}$$

is a solution of the half-linear differential equation

$$\big(p(t)\varphi(x'(t))\big)' + \sum_{i=1}^{n} \big[q_i(t) + 2^\alpha r_i(t)\big] \varphi(x(t)) = 0, \tag{4.13}$$

since the function

$$u(t) = \frac{v_0(t) + \sum_{i=1}^{n} [Q_i(t) + 2^\alpha R_i(t)]}{P(t)^\alpha} \tag{4.14}$$

satisfies the generalized Riccati equation

$$u'(t) + \alpha \frac{|u(t)|^{1+\frac{1}{\alpha}}}{p(t)^{\frac{1}{\alpha}}} + \sum_{i=1}^{n} \big[q_i(t) + 2^\alpha r_i(t)\big] = 0. \tag{4.15}$$

Since $v_0(t) + \sum_{i=1}^{n} [Q_i(t) + 2^\alpha R_i(t)] \to 0$ as $t \to \infty$, $X_0(t)$ is a normalized slowly varying function with respect to $P(t)$ by the representation theorem. It is obvious that Ξ is a nonvoid closed and convex subset of the locally convex space $C[t_0, \infty)$ of all continuous functions on $[t_0, \infty)$ equipped with the metric topology of uniform convergence on compact subintervals of $[t_0, \infty)$.

For any $\xi \in \Xi$, we define $q_{\xi,g_i}(t)$ and $r_{\xi,h_i}(t)$ by

$$q_{\xi,g_i}(t) = q_i(t)\varphi\Big(\frac{\xi(g_i(t))}{\xi(t)}\Big) \text{ and } r_{\xi,h_i}(t) = r_i(t)\varphi\Big(\frac{\xi(h_i(t))}{\xi(t)}\Big), \tag{4.16}$$

respectively. Taking into account (4.10), we have

$$\sum_{i=1}^{n} q_{\xi,g_i}(t) \leqq \sum_{i=1}^{n} q_i(t), \quad \sum_{i=1}^{n} r_{\xi.h_i}(t) \leqq 2^\alpha \sum_{i=1}^{n} r_i(t), \tag{4.17}$$

and accordingly,

$$\sum_{i=1}^{n} Q_{\xi,g_i}(t) \leqq \sum_{i=1}^{n} Q_i(t), \quad \sum_{i=1}^{n} R_{\xi,h_i}(t) \leqq 2^\alpha \sum_{i=1}^{n} R_i(t) \tag{4.18}$$

where $Q_{\xi,g_i}(t)$ and $R_{\xi,h_i}(t)$ are defined by

$$Q_{\xi,g_i}(t) = P(t)^\alpha \int_t^\infty q_{\xi,g_i}(s)\, ds, \quad R_{\xi,h_i}(t) = P(t)^\alpha \int_t^\infty r_{\xi,h_i}(s)\, ds. \tag{4.19}$$

Consequently, it follows from (4.4) that

$$\Big\{2\sum_{i=1}^{n}\Big[\widehat{Q}_{\xi,g_i}(t_0)+2^{\alpha}\widehat{R}_{\xi,h_i}(t_0)\Big]\Big\}^{\frac{1}{\alpha}}\max\Big\{2,1+\frac{1}{\alpha}\Big\}<1, \tag{4.20}$$

where $\widehat{Q}_{\xi,g_i}(t)=\sup_{s\geqq t}Q_{\xi,g_i}(t)$ and $\widehat{R}_{\xi,h_i}(t)=\sup_{s\geqq t}R_{\xi,h_i}(s)$. Thus, Theorem 3.1 implies that for any $\xi\in\Xi$ the half-linear differential equation

$$\big(p(t)\varphi(x'(t))\big)'+\sum_{i=1}^{n}\big[q_{x,g_i}(t)+r_{x,h_i}(t)\big]\varphi(x(t))=0 \tag{4.21}$$

has a n-SV$_P$ solution

$$X_\xi(t)=\exp\Bigg\{\int_{t_0}^{t}\Bigg(\frac{v_\xi(s)+\sum_{i=1}^{n}[Q_{\xi,g_i}(s)+R_{\xi,h_i}(s)]}{p(s)P(s)^{\alpha}}\Bigg)^{\frac{1}{\alpha}}ds\Bigg\},\quad t\geqq t_0, \tag{4.22}$$

where $v_\xi(t)$ is a solution of the integral equation

$$v_\xi(t)=\alpha P(t)^{\alpha}\int_{t}^{\infty}\frac{\big(v_\xi(s)+\sum_{i=1}^{n}[Q_{\xi,g_i}(s)+R_{\xi,h_i}(s)]\big)^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}}P(s)^{\alpha+1}}ds,\quad t\geqq t_0, \tag{4.23}$$

and satisfies

$$0\leqq v_\xi(t)\leqq\sum_{i=1}^{n}\big[\widehat{Q}_{\xi,g_i}(t_0)+\widehat{R}_{\xi,h_i}(t_0)\big]\leqq\sum_{i=1}^{n}\big[\widehat{Q}_i(t_0)+2^{\alpha}\widehat{R}_i(t_0)\big]\quad\text{for}\quad t\geqq t_0.$$

Let us now define the mapping Φ which assigns to each $\xi\in\Xi$ the function given by

$$\Phi\xi(t)=1\quad\text{for}\quad t_*\leqq t\leqq t_0,\quad \Phi\xi(t)=X_\xi(t)\quad\text{for}\quad t\geqq t_0. \tag{4.24}$$

To apply the Schauder–Tychonoff fixed point theorem to Φ we will show that Φ is a continuous mapping which sends Ξ into a relatively compact subset of Ξ.

(i) Φ maps Ξ into itself. Let $\xi\in\Xi$. Then

$$\begin{aligned}\Phi\xi(t)=X_\xi(t)&=\exp\Bigg\{\int_{t_0}^{t}\Bigg(\frac{v_\xi(s)+\sum_{i=1}^{n}[Q_{\xi,g_i}(s)+R_{\xi,h_i}(s)]}{p(s)P(s)^{\alpha}}\Bigg)^{\frac{1}{\alpha}}ds\Bigg\}\leqq\\ &\leqq\exp\Bigg\{\int_{t_0}^{t}\Bigg(\frac{v_\xi(s)+\sum_{i=1}^{n}[Q_i(s)+2^{\alpha}R_i(s)]}{p(s)P(s)^{\alpha}}\Bigg)^{\frac{1}{\alpha}}ds\Bigg\}\leqq\\ &\leqq\exp\Bigg\{\int_{t_0}^{t}\Bigg(\frac{v_0(s)+\sum_{i=1}^{n}[Q_i(s)+2^{\alpha}R_i(s)]}{p(s)P(s)^{\alpha}}\Bigg)^{\frac{1}{\alpha}}ds\Bigg\},\ t\geqq t_0,\end{aligned}$$

where we make use of the fact that $v_\xi(t) \leqq v_0(t)$, $t \geqq t_0$ for all $\xi \in \Xi$ (cf. Remark 3.1). Furthermore, since $v_\xi(t) \leqq \sum_{i=1}^{n} [\widehat{Q}_i(t_0) + 2^\alpha \widehat{R}_i(t_0)]$, using (4.5), we see that

$$\frac{\Phi(\xi(h_i(t)))}{\Phi(\xi(t))} = \exp\left\{ \int\limits_{t}^{h_i(t)} \left(\frac{v_\xi(s) + \sum_{i=1}^{n} [Q_{\xi,g_i}(s) + R_{\xi,h_i}(s)]}{p(s)P(s)^\alpha} \right)^{\frac{1}{\alpha}} ds \right\} \leqq$$

$$\leqq \exp\left\{ \left(2\sum_{i=1}^{n} \left[\widehat{Q}_i(t_0) + 2^\alpha \widehat{R}_i(t_0)\right] \right)^{\frac{1}{\alpha}} \int\limits_{t}^{h_i(t)} \frac{ds}{p(s)^{\frac{1}{\alpha}} P(s)} \right\} =$$

$$= \exp\left\{ \left(2\sum_{i=1}^{n} \left[\widehat{Q}_i(t_0) + 2^\alpha \widehat{R}_i(t_0)\right] \right)^{\frac{1}{\alpha}} \log \frac{P(h_i(t))}{P(t)} \right\} \leqq 2, \ t \geqq t_0.$$

This shows that $\Phi\xi \in \Xi$, that is, Φ is a self-map on Ξ.

(ii) $\Phi(\Xi)$ is relatively compact in $C[t_*, \infty)$. Since Φ maps Ξ into itself, that is, $\Phi(\Xi) \subset \Xi$, $\Phi(\Xi)$ is locally uniformly bounded on $[t_*, \infty)$, and since $\xi \in \Xi$ implies

$$0 \leqq \frac{d}{dt} \Phi\xi(t) = \frac{d}{dt} X_\xi(t) =$$

$$= \exp\left\{ \int\limits_{t_0}^{t} \left(\frac{v_\xi(s) + \sum_{i=1}^{n} [Q_{\xi,g_i}(s) + R_{\xi,h_i}(s)]}{p(s)P(s)^\alpha} \right)^{\frac{1}{\alpha}} ds \right\} \times$$

$$\times \left(\frac{v_\xi(t) + \sum_{i=1}^{n} [Q_{\xi,g_i}(t) + R_{\xi,h_i}(t)]}{p(t)P(t)^\alpha} \right)^{\frac{1}{\alpha}} \leqq$$

$$\leqq \exp\left\{ \left(\sum_{i=1}^{n} [\widehat{Q}_i(t_0) + 2^\alpha \widehat{R}_i(t_0)] \right)^{\frac{1}{\alpha}} \log \frac{P(t)}{P(t_0)} \right\} \times$$

$$\times \left(\frac{2\sum_{i=1}^{n} [\widehat{Q}_i(t_0) + 2^\alpha \widehat{R}_i(t_0)]}{p(t)P(t)^\alpha} \right)^{\frac{1}{\alpha}},$$

$\Phi(\Xi)$ is locally equi-continuous on $[t_*, \infty)$. From the Arzela–Ascoli lemma it then follows that $\Phi(\Xi)$ is relatively compact in $C[t_*, \infty)$.

(iii) Φ is a continuous mapping. Let $\{\xi_m(t)\}$ be a sequence of functions in Ξ converging to $\delta(t)$ uniformly on the compact subintervals of $[t_*, \infty)$. To prove the continuity of Φ, we have to prove that $\{\Phi\xi_m(t)\}$ converges to $\Phi\delta(t)$ uniformly on compact subintervals in $[t_*, \infty)$. Applying the mean value theorem, for $t \geqq t_*$ we obtain

$$\begin{aligned}
&\big|\Phi\xi_m(t)-\Phi\delta(t)\big| = |X_{\xi_m}(t)-X_\delta(t)| = \\
&= \Bigg|\exp\bigg\{\int_{t_0}^{t}\Big(\frac{v_{\xi_m}(s)+\sum\limits_{i=1}^{n}[Q_{\xi_m,g_i}(s)+R_{\xi_m,h_i}(s)]}{p(s)P(s)^\alpha}\Big)^{\frac{1}{\alpha}}ds\bigg\}- \\
&\quad - \exp\bigg\{\int_{t_0}^{t}\Big(\frac{v_\delta(s)+\sum\limits_{i=1}^{n}[Q_{\delta,g_i}(s)+R_{\delta,h_i}(s)]}{p(s)P(s)^\alpha}\Big)^{\frac{1}{\alpha}}ds\bigg\}\Bigg| \\
&\quad \leqq \exp\bigg\{\int_{t_0}^{t}\Big(\frac{v_0(s)+\sum\limits_{i=1}^{n}[\widehat{Q}_i(s)+2^\alpha\widehat{R}_i(s)]}{p(s)P(s)^\alpha}\Big)^{\frac{1}{\alpha}}ds\bigg\}\times \\
&\quad \times\int_{t_0}^{t}\frac{1}{p(s)^{\frac{1}{\alpha}}}\Bigg|\Big(\frac{v_{\xi_m}(s)+\sum\limits_{i=1}^{n}[Q_{\xi_m,g_i}(s)+R_{\xi_m,h_i}(s)]}{P(s)^\alpha}\Big)^{\frac{1}{\alpha}}- \\
&\qquad\qquad - \Big(\frac{v_\delta(s)+\sum\limits_{i=1}^{n}[Q_{\delta,g_i}(s)+R_{\delta,h_i}(s)]}{P(s)^\alpha}\Big)^{\frac{1}{\alpha}}\Bigg|ds.
\end{aligned}$$

By means of the inequality $|x^\lambda-y^\lambda| \leqq |x-y|^\lambda$ for $x,y\in\mathbb{R}^+$ and $0<\lambda<1$, we find that the integrand of the last integral in the previous inequality is bounded from above by the function

$$\begin{aligned}
&\Bigg|\Big(\frac{v_{\xi_m}(t)+\sum\limits_{i=1}^{n}[Q_{\xi_m,g_i}(t)+R_{\xi_m,h_i}(t)]}{P(t)^\alpha}\Big)^{\frac{1}{\alpha}}- \\
&\qquad - \Big(\frac{v_\delta(t)+\sum\limits_{i=1}^{n}[Q_{\delta,g_i}(t)+R_{\delta,h_i}(t)]}{P(t)^\alpha}\Big)^{\frac{1}{\alpha}}\Bigg| \leqq \\
&\leqq \Bigg(\frac{|v_{\xi_m}(t)-v_\delta(t)|+\sum\limits_{i=1}^{n}|Q_{\xi_m,g_i}(t)-Q_{\delta,g_i}(t)|+\sum\limits_{i=1}^{n}|R_{\xi_m,h_i}(t)-R_{\delta,h_i}(t)|}{P(t)^\alpha}\Bigg)^{\frac{1}{\alpha}} \\
&\qquad\qquad\qquad \text{if } \alpha>1.
\end{aligned}$$

Similarity, using the mean value theorem, we find that

$$\begin{aligned}
&\Bigg|\Big(\frac{v_{\xi_m}(t)+\sum\limits_{i=1}^{n}[Q_{\xi_m,g_i}(t)+R_{\xi_m,h_i}(t)]}{P(t)^\alpha}\Big)^{\frac{1}{\alpha}}- \\
&\qquad - \Big(\frac{v_\delta(t)+\sum\limits_{i=1}^{n}[Q_{\delta,g_i}(t)+R_{\delta,h_i}(t)]}{P(t)^\alpha}\Big)^{\frac{1}{\alpha}}\Bigg| \leqq
\end{aligned}$$

$$\leqq C_1 \frac{|v_{\xi_m}(t)-v_\delta(t)|+\sum\limits_{i=1}^n|Q_{\xi_m,g_i}(t)-Q_{\delta,g_i}(t)|+\sum\limits_{i=1}^n |R_{\xi_m,h_i}(t)-R_{\delta,h_i}(t)|}{P(t)^\alpha}$$
$$\text{if } \alpha \leqq 1,$$

where C_1 is a constant depending only on α, $\widehat{Q}_i(t_0)$ and $\widehat{R}_i(t_0)$. Accordingly, the continuity of Φ is guaranteed if we prove that the two sequences

$$\frac{|v_{\xi_m}(t)-v_\delta(t)|}{P(t)^\alpha}, \quad \frac{\sum\limits_{i=1}^n|Q_{\xi_m,g_i}(t)-Q_{\delta,g_i}(t)|+\sum\limits_{i=1}^n|R_{\xi_m,h_i}(t)-R_{\delta,h_i}(t)|}{P(t)^\alpha} \tag{4.25}$$

converge to 0 on any compact subinterval of $[t_*, \infty)$. In fact, it can be shown more strongly that they converge to 0 uniformly on $[t_*, \infty)$. The uniform convergence of the second sequence in (4.25) follows from the Lebesgue dominated convergence theorem applied to the inequality

$$\frac{\sum\limits_{i=1}^n |Q_{\xi_m,g_i}(t) - Q_{\delta,g_i}(t)| + \sum\limits_{i=1}^n |R_{\xi_m,h_i}(t) - R_{\delta,h_i}(t)|}{P(t)^\alpha} \leqq$$
$$\leqq \int\limits_t^\infty \Bigg[\sum_{i=1}^n q_i(s)\Big|\varphi\Big(\frac{\xi_m(g_i(s))}{\xi_m(s)}\Big) - \varphi\Big(\frac{\delta(g_i(s))}{\delta(s)}\Big)\Big| +$$
$$+ \sum_{i=1}^n r_i(s)\Big|\varphi\Big(\frac{\xi_m(h_i(s))}{\xi_m(s)}\Big) - \varphi\Big(\frac{\delta(h_i(s))}{\delta(s)}\Big)\Big|\Bigg]\, ds$$

for $t \geqq t_0$. To examine the first sequence in (4.25) we proceed as follows. Using (4.23) and the mean value theorem, we obtain

$$\frac{|v_{\xi_m}(t) - v_\delta(t)|}{P(t)^\alpha} \leqq \alpha \int\limits_t^\infty \frac{1}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}} \times$$
$$\times \Bigg|\Big(v_{\xi_m}(s) + \sum_{i=1}^n [Q_{\xi_m,g_i}(s) + R_{\xi_m,h_i}(s)]\Big)^{1+\frac{1}{\alpha}} -$$
$$- \Big(v_\delta(s) + \sum_{i=1}^n [Q_{\delta,g_i}(s) + R_{\delta,h_i}(s)]\Big)^{1+\frac{1}{\alpha}}\Bigg|\, ds.$$

Therefore, we have

$$\frac{|v_{\xi_m}(t) - v_\delta(t)|}{P(t)^\alpha} \leqq$$
$$\leqq \alpha\tau_1 \Bigg[\int\limits_t^\infty \frac{1}{p(s)^{\frac{1}{\alpha}}} \frac{|v_{\xi_m}(s) - v_\delta(s)|}{P(s)^{\alpha+1}}\, ds + \int\limits_t^\infty \frac{1}{p(s)^{\frac{1}{\alpha}}} \frac{S_{m,n}(s)}{P(s)^{\alpha+1}}\, ds\Bigg], \tag{4.26}$$

where τ_1 is a positive constant defined by

$$\tau_1 = \Big(1+\frac{1}{\alpha}\Big)\Big\{2\sum_{i=1}^{n}[\widehat{Q}_i(t_0)+2^\alpha \widehat{R}_i(t_0)]\Big\}^{\frac{1}{\alpha}} \tag{4.27}$$

and $S_{m,n}(t)$ is defined by

$$S_{m,n}(t) = \sum_{i=1}^{n}\Big[\big|Q_{\xi_m,g_i}(t)-Q_{\delta,g_i}(t)\big| + \big|R_{\xi_m,h_i}(t)-R_{\delta,h_i}(t)\big|\Big].$$

Note that $\tau_1 < 1$ by (4.27). Letting

$$Z_m(t) = \int_t^\infty \frac{|v_{\xi_m}(s)-v_\delta(s)|}{p(s)^{\frac{1}{\alpha}}P(s)^{\alpha+1}}\,ds, \tag{4.28}$$

we derive from (4.26) the following differential inequality for $z_m(t)$:

$$\big(P(t)^{\alpha\tau_1}Z_m(t)\big)' \geqq -\alpha\tau_1\frac{P(t)^{\alpha\tau_1-1}}{p(t)^{\frac{1}{\alpha}}}\int_t^\infty \frac{S_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}P(s)^{\alpha+1}}\,ds. \tag{4.29}$$

Noting that $P(t)^{\alpha\tau_1}Z_m(t) \to \infty$ and that the right-hand side of (4.29) is integrated over $[t,\infty)$, we obtain

$$Z_m(t) \leqq \frac{1}{P(t)^{\alpha\tau_1}}\int_t^\infty \frac{S_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}P(s)^{1+\alpha-\alpha\tau_1}}\,ds, \quad t \geqq t_0. \tag{4.30}$$

Combining (4.26) with (4.30), we have

$$\frac{|v_{\xi_m}(t)-v_\delta(t)|}{P(t)^\alpha} \leqq$$

$$\leqq \alpha\tau_1\Bigg[\frac{1}{P(t)^{\alpha\tau_1}}\int_t^\infty \frac{S_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}P(s)^{1+\alpha-\alpha\tau_1}}\,ds + \int_t^\infty \frac{S_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}P(s)^{\alpha+1}}\,ds\Bigg] \leqq$$

$$\leqq \frac{\alpha\tau_1}{P(t)^{\alpha\tau_1}}\int_t^\infty \frac{S_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}P(s)^{1+\alpha-\alpha\tau_1}}\,ds, \quad t \geqq t_0.$$

This shows that $|u_{\xi_m}(t)-v_\delta(t)|/P(t)^\alpha$ converges to 0 uniformly on $[t_*,\infty)$. We therefore conclude that the mapping Φ defined by (4.24) is continuous in the topology of $C[t_*,\infty)$. Thus, all the hypotheses of the Schauder–Tychonoff fixed point theorem are fulfilled, and hence there exists $\xi_0(t) \in \Xi$ satisfying the half-linear functional differential equation

$$\big(p(t)\varphi(\xi_0'(t))\big)' + \sum_{i=1}^{n}\big[q_{\xi_0,g_i}(t)+r_{\xi_0,h_i}(t)\big]\varphi(\xi_0(t)) = 0, \quad t \geqq t_0,$$

which is rewritten as

$$\big(p(t)\varphi(\xi_0'(t))\big)' + \sum_{i=1}^{n}\Big[q_i(t)\varphi\big(\xi_0(g_i(t))\big) + r_i(t)\varphi\big(\xi_0(h_i(t))\big)\Big] = 0, \quad t \geqq t_0.$$

This implies that the equation (A_+) has a n-SV$_P$ solution $\xi_0(t)$ existing on $[t_0,\infty)$.

(The existence of a n-RV$_P(1)$ solution of (A_+)): Next, we will be concerned with the construction of a n-RV$_P(1)$ solution of equation (A_+) under the condition (1.9). Choose $t_1 > a$ so large that $t_* = \min_{i=1,2,\dots,n} \big\{ \inf_{t \geqq t_1} g_i(t) \big\} > \max\{a,1\}$,

$$\Big(1+\frac{1}{\alpha}\Big)[K+L+\alpha]\sqrt{\sum_{i=1}^{n}\big[\widehat{Q}_i(t_1)+2^\alpha \widehat{R}_i(t_1)\big]} \leqq 1, \quad t \geqq t_1 \tag{4.31}$$

and

$$\bigg(\frac{3}{2}+\sum_{i=1}^{n}\big[\widehat{Q}_i(t_1)+2^\alpha \widehat{R}_i(t_1)\big]\bigg)^{\frac{1}{\alpha}} \log\frac{P(h_i(t))}{P(t)} \leqq \log 2, \quad t \geqq t_1. \tag{4.32}$$

Let $\mathbb{H}$ denote the set of all continuous nondecreasing functions $\eta(t)$ on $[t_*,\infty)$ satisfying

$$\eta(t) = 1 \ \text{ for } \ t_* \leqq t \leqq t_1; \tag{4.33}$$

$$1 \leqq \eta(t) \leqq \exp\left\{\int_{t_1}^{t}\left(\frac{\frac{3}{2}+\sum_{i=1}^{n}[Q_i(s)+2^\alpha R_i(s)]}{p(s)P(s)^\alpha}\right)^{\frac{1}{\alpha}} ds\right\} \ \text{ for } \ t \geqq t_1; \tag{4.34}$$

$$\frac{\eta(h_i(t))}{\eta(t)} \leqq 2 \ \text{ for } \ t \geqq t_1, \quad i=1,2,\dots,n. \tag{4.35}$$

For any $\eta \in \mathbb{H}$ we consider the differential equation

$$\big(p(t)\varphi(x'(t))\big)' + \sum_{i=1}^{n}[q_{\eta,g_i}(t)+r_{\eta,h_i}(t)]\varphi(x(t)) = 0, \quad t \geqq t_1, \tag{4.36}$$

where

$$q_{\eta,g_i}(t) = q_i(t)\varphi\Big(\frac{\eta(g_i(t))}{\eta(t)}\Big) \ \text{ and } \ r_{\eta,h_i}(t) = r_i(t)\varphi\Big(\frac{\eta(h_i(t))}{\eta(t)}\Big), \ i=1,2,\dots,n.$$

Since $\eta(g_i(t))/\eta(t) \leqq 1$ and $\eta(h_i(t))/\eta(t) \leqq 2$, we have

$$q_{\eta,g_i}(t) \leqq q_i(t) \ \text{ and } \ r_{\eta,h_i}(t) \leqq 2^\alpha r_i(t), \quad t \geqq t_1 \ \text{ for } \ i=1,2,\dots,n,$$

so that

$$Q_{\eta,g_i}(t) := P(t)^\alpha \int_t^\infty q_{\eta,g_i}(s)\,ds \leqq P(t)^\alpha \int_t^\infty q_i(s)\,ds = Q_i(t), \quad t \geqq t_1, \tag{4.37}$$

$$R_{\eta,h_i}(t) := P(t)^\alpha \int_t^\infty r_{\eta,h_i}(s)\,ds \leqq 2^\alpha P(t)^\alpha \int_t^\infty r_i(s)\,ds = 2^\alpha R_i(t), \quad t \geqq t_1. \tag{4.38}$$

Accordingly, from (4.31) we have

$$\Big(1+\frac{1}{\alpha}\Big)[K+L+\alpha]\sqrt{\sum_{i=1}^{n}\big[\widehat{Q}_{\eta,g_i}(t_1)+\widehat{R}_{\eta,h_i}(t_1)\big]}\leqq 1, \tag{4.39}$$

where $\widehat{Q}_{\eta,g_i}(t)=\sup\limits_{s\geqq t}Q_{\eta,g_i}(s)$ and $\widehat{R}_{\eta,h_i}(t)=\sup\limits_{s\geqq t}R_{\eta,h_i}(s)$. Moreover, we notice that from $K+L+\alpha>2$ and $1+\frac{1}{\alpha}>1$ follows

$$\sqrt{\sum_{i=1}^{n}\big[\widehat{Q}_{\eta,g_i}(t_1)+\widehat{R}_{\eta,h_i}(t_1)\big]}\leqq\frac{1}{2}\,. \tag{4.40}$$

This enables us to apply Theorem 3.1 and thus we conclude that half-linear differential equation (4.36) has a n-RV$_P(1)$ solution of the form

$$X_\eta(t)=\exp\left\{\int_{t_1}^{t}\left(\frac{1+\sum\limits_{i=1}^{n}[Q_{\eta,g_i}(s)+R_{\eta,h_i}(s)]-w_\eta(s)}{p(s)P(s)^\alpha}\right)^{\frac{1}{\alpha}}ds\right\},\quad t\geqq t_1, \tag{4.41}$$

where $w_\eta(t)$ is a solution of the integral equation

$$w_\eta(t)=\frac{\alpha}{P(t)}\int_{t}^{\infty}\frac{F_\eta(s,w_\eta(s))}{p(s)^{\frac{1}{\alpha}}}\,ds,\quad t\geqq t_1, \tag{4.42}$$

satisfying $|w_\eta(t)|\leqq\sqrt{\sum\limits_{i=1}^{n}[\widehat{Q}_{\eta,g_i}(t_1)+\widehat{R}_{\eta,h_i}(t_1)]}$ for $t\geqq t_1$. Furthermore, it follows from (4.40) that $|w_\eta(t)|\leqq 1/2$ for $t\geqq t_1$. Here $F_\eta(t,w_\eta(t))$ is

$$F_\eta(t,w_\eta)=\left|1+\sum_{i=1}^{n}\big[Q_{\eta,g_i}(t)+R_{\eta,h_i}(t)\big]-w_\eta\right|^{1+\frac{1}{\alpha}}+\Big(1+\frac{1}{\alpha}\Big)w_\eta-1,\quad t\geqq t_1.$$

Denote by Ψ the mapping which assigns to each $\eta\in\mathbb{H}$ the function $\Psi\eta(t)$ defined by

$$\Psi\eta(t)=1\ \text{ for }\ t_*\leqq t\leqq t_1,\quad \Psi\eta(t)=X_\eta(t)\ \text{ for }\ t\geqq t_1. \tag{4.43}$$

(i) Ψ is a self-map on $\mathbb{H}$. For any $\eta\in\mathbb{H}$ from (4.37) and (4.38), for $t\geqq t_1$ we find that

$$\left|1+\sum_{i=1}^{n}\big[Q_{\xi,g_i}(t)+R_{\xi,h_i}(t)\big]-w_\eta(t)\right|\leqq$$

$$\leqq 1+\sum_{i=1}^{n}\big[Q_i(t)+2^\alpha R_i(t)\big]+|w_\eta(t)|\leqq$$

$$\leqq\frac{3}{2}+\sum_{i=1}^{n}\big[Q_i(t)+2^\alpha R_i(t)\big],$$

or accordingly,

$$X_\eta(t) \leqq \exp\left\{ \int_{t_1}^{t} \left(\frac{\frac{3}{2} + \sum_{i=1}^{n}[Q_i(s) + 2^\alpha R_i(s)]}{p(s)P(s)^\alpha} \right)^{\frac{1}{\alpha}} ds \right\}, \quad t \geqq t_1.$$

Moreover, we have

$$\begin{aligned}
\frac{\Psi\eta(h_i(t))}{\Psi\eta(t)} &= \exp\left\{ \int_{t}^{h_i(t)} \frac{(1 + \sum_{i=1}^{n}[Q_{\eta,g_i}(s) + R_{\eta,h_i}(s)] - w_\eta(s))^{\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}}P(s)} ds \right\} \leqq \\
&\leqq \exp\left\{ \int_{t}^{h_i(t)} \frac{\left(\frac{3}{2} + \sum_{i=1}^{n}[Q_i(s) + 2^\alpha R_i(s)]\right)^{\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}}P(s)} ds \right\} \leqq \\
&\leqq \exp\left\{ \left(\frac{3}{2} + \sum_{i=1}^{n} \left[\widehat{Q}_i(s) + 2^\alpha \widehat{R}_i(s)\right] \right)^{\frac{1}{\alpha}} \int_{t}^{h_i(t)} \frac{ds}{p(s)^{\frac{1}{\alpha}}P(s)} \right\} \leqq \\
&\leqq \exp\left\{ \left(\frac{3}{2} + \sum_{i=1}^{n} \left[\widehat{Q}_i(s) + 2^\alpha \widehat{R}_i(s)\right] \right)^{\frac{1}{\alpha}} \log \frac{P(h_i(t))}{P(t)} \right\} \leqq 2, \quad t \geqq t_1.
\end{aligned}$$

(ii) $\Psi(\mathbb{H})$ is relatively compact in $C[t_*, \infty)$. This is a consequence of the inclusion $\Psi(\mathbb{H}) \subset \mathbb{H}$ and the following inequality holding for any $\eta \in \mathbb{H}$:

$$\begin{aligned}
0 \leqq \frac{d}{dt}\Psi\eta(t) &= \frac{d}{dt}X_\eta(t) = \\
&\left(\frac{1 + \sum_{i=1}^{n}[Q_{\eta,g_i}(t) + R_{\eta,h_i}(t)] - w_\eta(t)}{p(t)P(t)^\alpha} \right)^{\frac{1}{\alpha}} X_\eta(t) \leqq \\
&\leqq \left(\frac{\frac{3}{2} + \sum_{i=1}^{n}[\widehat{Q}_i(t_1) + 2^\alpha \widehat{R}_i(t_1)]}{p(t)P(t)^\alpha} \right)^{\frac{1}{\alpha}} \times \\
&\times \exp\left\{ \left(\frac{3}{2} + \sum_{i=1}^{n} \left[\widehat{Q}_i(t_1) + 2^\alpha \widehat{R}_i(t_1)\right] \right)^{\frac{1}{\alpha}} \log \frac{P(t)}{P(t_1)} \right\}.
\end{aligned}$$

(iii) Ψ is continuous in the topology of $C[t_*, \infty)$. Let $\{\eta_n\}$ be a sequence in $\mathbb{H}$ converging to $\theta \in \mathbb{H}$, which amounts to supposing that the sequence $\{\eta_n(t)\}$ converges to $\theta(t)$ uniformly on the compact subintervals of $[t_*, \infty)$. We will show that $\{\Psi\eta_n(t)\}$ converges to $\Psi\theta(t)$ uniformly on the compact subintervals $[t_*, \infty)$. In order to simplify notation, for arbitrary $\eta \in \mathbb{H}$ we denote

$$V_\eta(t) = \frac{1 + \sum_{i=1}^{n}[Q_{\eta,g_i}(t) + R_{\eta,h_i}(t)] - w_\eta(t)}{P(t)^\alpha}, \quad t \geqq t_1. \tag{4.44}$$

In view of (4.41), we have

$$\begin{aligned}
\big|\Psi\eta_m(t) - \Psi\theta(t)\big| &= \big|X_{\eta_m}(t) - X_\theta(t)\big| = \\
&= \left|\exp\left\{\int_{t_1}^{t}\Big(\frac{V_{\eta_m(s)}}{p(s)}\Big)^{\frac{1}{\alpha}}\,ds\right\} - \exp\left\{\int_{t_1}^{t}\Big(\frac{V_\theta(s)}{p(s)}\Big)^{\frac{1}{\alpha}}\,ds\right\}\right| \leqq \\
&\leqq \exp\left\{\int_{t_1}^{t}\left(\frac{\frac{3}{2}+\sum_{i=1}^{n}[Q_i(s)+2^\alpha R_i(s)]}{p(s)P(s)^\alpha}\right)^{\frac{1}{\alpha}}\,ds\right\}\times \\
&\qquad\times\int_{t_1}^{t}\frac{1}{p(s)^{\frac{1}{\alpha}}}\left|(V_{\eta_m}(s))^{\frac{1}{\alpha}} - (V_\theta(s))^{\frac{1}{\alpha}}\right|\,ds.
\end{aligned}$$

As in the previous part of the proof, we can verify that the integrand of the last integral is bounded from the above by

$$\left(\frac{\sum_{i=1}^{n}|Q_{\eta_m,g_i}(t)-Q_{\theta,g_i}(t)|+\sum_{i=1}^{n}|R_{\eta_m,h_i}(t)-R_{\theta,h_i}(t)|+|w_{\eta_m}(t)-w_\theta(t)|}{P(t)^\alpha}\right)^{\frac{1}{\alpha}} \quad \text{if } \alpha > 1,$$

$$C_2\,\frac{\sum_{i=1}^{n}|Q_{\eta_m,g_i}(t)-Q_{\theta,g_i}(t)|+\sum_{i=1}^{n}|R_{\eta_m,h_i}(t)-R_{\theta,h_i}(t)|+|w_{\eta_m}(t)-w_\theta(t)|}{P(t)^\alpha} \quad \text{if } \alpha \leqq 1,$$

where C_2 is a constant depending only on α, $\widehat{Q}_i(t_1)$ and $\widehat{R}_i(t_1)$. Accordingly, it suffices to prove the uniform convergence to 0 on the compact subintervals of the two sequences

$$\frac{|w_{\eta_m}(t) - w_\theta(t)|}{P(t)^\alpha} \quad \text{and} \quad \frac{\pi_{m,n}(t)}{P(t)^\alpha},$$

where

$$\pi_{m,n}(t) = \sum_{i=1}^{n}\big|Q_{\eta_m,g_i}(t) - Q_{\theta,g_i}(t)\big| + \sum_{i=1}^{n}\big|R_{\eta_m,h_i}(t) - R_{\theta,h_i}(t)\big|.$$

The uniform convergence of the sequence $\pi_{m,n}(t)/P(t)^\alpha$ is an immediate consequence of the Lebesgue dominated convergence theorem. Therefore, let us examine the sequence $|w_{\eta_m}(t) - w_\theta(t)|/P(t)^\alpha$. Applying the mean value theorem to $F_{\eta_m}(t, w_{\eta_m}(t))$ and $F_\theta(t, w_\theta(t))$ in $|w_{\eta_m}(t) - w_\theta(t)|/P(t)^\alpha$, we obtain for $t \geqq t_1$

$$\begin{aligned}\big|F_{\eta_m}(t,w_{\eta_m}(t)) - F_\theta(t,w_\theta(t))\big| &\leqq \Big(1+\frac{1}{\alpha}\Big)|w_{\eta_m}(t)-w_\theta(t)|+\\ &+\Bigg|\bigg|1+\sum_{i=1}^{n}\big[Q_{\eta_m,g_i}(t)+R_{\eta_m,h_i}(t)\big]-w_{\eta_m}(t)\bigg|^{1+\frac{1}{\alpha}}-\\ &-\bigg|1+\sum_{i=1}^{n}\big[Q_{\theta,g_i}(t)+R_{\theta,h_i}(t)\big]-w_\theta(t)\bigg|^{1+\frac{1}{\alpha}}\Bigg| \leqq\\ &\leqq \Big(1+\frac{1}{\alpha}\Big)(1+\tau_2)|w_{\eta_m}(t)-w_\theta(t)| + \Big(1+\frac{1}{\alpha}\Big)\tau_2\pi_{n,m}(t),\end{aligned}$$

where τ_2 is a positive constant depending only on α, $\widehat{Q}_i(t_1)$ and $\widehat{R}_i(t_1)$. Consequently, the sequence $|w_{\eta_m}(t)-w_\theta(t)|/P(t)^\alpha$ implies

$$\frac{|w_{\eta_m}(t)-w_\theta(t)|}{P(t)^\alpha} \leqq \frac{(\alpha+1)(1+\tau_2)}{P(t)^{\alpha+1}}\int_{t_1}^{t}\frac{|w_{\eta_m}(s)-w_\theta(s)|}{P(s)^\alpha}\,ds + \frac{(\alpha+1)\tau_2}{P(t)^{\alpha+1}}\int_{t_1}^{t}\frac{\pi_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}}\,ds, \quad t \geqq t_1. \tag{4.45}$$

Putting for simplicity

$$W_m(t) = \int_{t_1}^{t}\frac{|w_{\eta_m}(s)-w_\theta(s)|}{P(s)^\alpha}\,ds, \tag{4.46}$$

we transform (4.45) into

$$\big(P(t)^{-(\alpha+1)(1+\tau_2)}W_m(t)\big)' \leqq \frac{(\alpha+1)\tau_2}{p(t)^{\frac{1}{\alpha}}P(t)^{(\alpha+1)(1+\tau_2)+1}}\int_{t_1}^{t}\pi_{m,n}(s)\,ds, \quad t \geqq t_1,$$

which, after integration over $[t_1,t]$, yields

$$W_m(t) \leqq \frac{\tau_2}{1+\tau_2}\,P(t)^{(\alpha+1)(1+\tau_2)}\int_{t_1}^{t}\frac{\pi_{m,n}(s)}{P(s)^{(\alpha+1)(1+\tau_2)}}\,ds, \quad t \geqq t_1. \tag{4.47}$$

Combining (4.45) with (4.47), we have

$$\frac{|w_{\eta_m}(t)-w_\theta(t)|}{P(t)^\alpha} \leqq \frac{(\alpha+1)\tau_2}{P(t)^{-(\alpha+1)\tau_2}}\int_{t_1}^{t}\frac{\pi_{m,n}(s)}{P(s)^{(\alpha+1)(1+\tau_2)}}\,ds + \frac{(\alpha+1)\tau_2}{P(t)^{\alpha+1}}\int_{t_1}^{t}\frac{\pi_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}}\,ds, \quad t \geqq t_1.$$

This ensures the desired convergence of the sequence $|w_{\eta_m}(t)-w_\theta(t)|/P(t)^\alpha$, whence the continuity of the mapping Ψ has been assured. Thus, all the hypotheses of the Schauder–Tychonoff fixed point theorem are fulfilled, and so there exists $\eta_0 \in \mathbb{H}$ such that $\eta_0 = \Psi\eta_0$. Since $\eta_0(t) = X_{\eta_0}(t)$ for $t \geqq t_1$, $\eta_0(t)$ satisfies the differential equation

$$\big(p(t)\varphi(\eta_0'(t))\big)' + \sum_{i=1}^{n} \big[q_{\eta_0,g_i}(t) + r_{\eta_0,h_i}(t)\big]\varphi(\eta_0(t)) = 0, \quad t \geqq t_1$$

or

$$\big(p(t)\varphi(\eta_0'(t))\big)' + \sum_{i=1}^{n} \Big[q_i(t)\varphi\big(\eta_0(g_i(t))\big) + r_i(t)\varphi\big(\eta_0(h_i(t))\big)\Big] = 0, \quad t \geqq t_1.$$

Therefore, $\eta_0(t)$ is a desired n-RV$_P(1)$ solution of the functional differential equation (A$_+$) on $[t_1, \infty)$.

The proof of Theorem 1.1 for the equation (A$_-$).

(The existence of a n-SV$_P$ solution of (A$_-$)): Suppose that (1.9) holds. Choose $t_0 > a$ so large that $t_* = \min\limits_{i=1,2,\dots,n} \big\{\inf\limits_{t\geqq t_0} g_i(t)\big\} > \max\{a, 1\}$ and such that

$$\Big(2\sum_{i=1}^{n} \big[2^\alpha \widehat{Q}_i(t_0) + \widehat{R}_i(t_0)\big]\Big)^{\frac{1}{\alpha}} \max\Big\{2, 1 + \frac{1}{\alpha}\Big\} < 1, \tag{4.48}$$

$$\Big(2\sum_{i=1}^{n} \big[2^\alpha \widehat{Q}_i(t_0) + \widehat{R}_i(t_0)\big]\Big)^{\frac{1}{\alpha}} \log\frac{P(t)}{P(g_i(t))} < \log 2, \tag{4.49}$$

are satisfied for all $t \geqq t_0$, where $\widehat{Q}_i(t)$ and $\widehat{R}_i(t)$ are defined by (4.6) and (4.7).

Let $\mathbb{M}$ denote the set of all positive continuous nonincreasing functions $\mu(t)$ on $[t_*, \infty)$ with the properties

$$\mu(t) = 1 \ \text{ for } \ t_* \leqq t \leqq t_0; \tag{4.50}$$

$$\mu(t) \geqq \exp\left\{-\int_{t_0}^{t} \frac{\big(2\sum_{i=1}^{n}[2^\alpha\widehat{Q}_i(s) + \widehat{R}_i(s)]\big)^{\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}}P(s)}\,ds\right\} \ \text{ for } \ t \geqq t_0; \tag{4.51}$$

$$\frac{\mu(g_i(t))}{\mu(t)} \leqq 2 \ \text{ for } \ t \geqq t_0, \quad i = 1, 2, \dots, n. \tag{4.52}$$

We here consider the following differential equation:

$$\big(p(t)\varphi(x'(t))\big)' = \sum_{i=1}^{n} \big[q_{\mu,g_i}(t) + r_{\mu,h_i}(t)\big]\varphi(x(t)) \tag{4.53}$$

where, for arbitrary $\mu \in \mathbb{M}$, the functions $q_{\mu,g_i}(t)$ and $r_{\mu,h_i}(t)$ are defined by (4.2). In view of Theorem 3.1, for each $\mu \in \mathbb{M}$, the equation (4.53) has a n-SV$_P$ solution $X_\mu(t)$ having the representation

$$X_\mu(t) = \exp\left\{ \int_{t_0}^{t} \left(\frac{r_\mu(s) - \sum_{i=1}^{n} [Q_{\mu,g_i}(s) + R_{\mu,h_i}(s)]}{p(s)P(s)^\alpha} \right)^{\frac{1}{\alpha}*} ds \right\}, \quad t \geqq t_0, \quad (4.54)$$

where $r_\mu(t)$ is a solution of the integral equation

$$r_\mu(t) = \alpha P(t)^\alpha \int_t^\infty \frac{|r_\mu(s) - \sum_{i=1}^{n} [Q_{\mu,g_i}(s) + R_{\mu,h_i}(s)]|^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}} ds, \quad t \geqq t_0, \quad (4.55)$$

satisfying the inequality

$$0 \leqq r_\mu(t) \leqq \sum_{i=1}^{n} \left[\widehat{Q}_{\mu,g_i}(t_0) + \widehat{R}_{\mu,h_i}(t_0) \right] \leqq$$

$$\leqq \sum_{i=1}^{n} \left[2^\alpha \widehat{Q}_i(t_0) + \widehat{R}_i(t_0) \right], \quad t \geq t_0. \quad (4.56)$$

Here $Q_{\mu,g_i}(t)$ and $R_{\mu,h_i}(t)$ are defined by (4.19) and $\widehat{Q}_{\mu,g_i}(t) = \sup_{s \geqq t} Q_{\mu,g_i}(s)$ and $\widehat{R}_{\mu,h_i}(t) = \sup_{s \geqq t} R_{\mu,h_i}(s)$. Furthermore, using the decreasing nature of $\mu(t)$, we have

$$q_{\mu,g_i}(t) \leqq 2^\alpha q_i(t) \text{ and } r_{\mu,h_i}(t) \leqq r_i(t), \quad t \geqq t_0, \quad i = 1, 2, \ldots, n,$$

accordingly,

$$\sum_{i=1}^{n} \left[Q_{\mu,g_i}(t) + R_{\mu,h_i}(t) \right] \leqq \sum_{i=1}^{n} \left[2^\alpha Q_i(t) + R_i(t) \right] \leqq$$

$$\leqq \sum_{i=1}^{n} \left[2^\alpha \widehat{Q}_i(t_0) + \widehat{R}_i(t_0) \right], \quad t \geqq t_0. \quad (4.57)$$

Let us now define $\mathcal{H}$ to be the mapping which assigns to each $\mu \in \mathbb{M}$ the function $\mathcal{H}\mu$ given by

$$\mathcal{H}\mu(t) = 1 \text{ for } t_* \leqq t \leqq t_0, \quad \mathcal{H}\mu(t) = X_\mu(t) \text{ for } t \geqq t_0. \quad (4.58)$$

Proceeding as in the proof of the existence of n-SV$_P$ solution of (A$_+$), it can be proved that $\mathcal{H}$ maps $\mathbb{M}$ into a relatively compact subset of $\mathbb{M}$ with the help of the Schauder–Tychonoff fixed point theorem, so that there exists a $\mu_0 \in \mathbb{M}$ such that

$$\mu_0(t) = X_{\mu_0}(t) =$$

$$= \exp\left\{ \int_{t_0}^{t} \left(\frac{r_{\mu_0}(s) - \sum_{i=1}^{n} [Q_{\mu_0,g_i}(s) + R_{\mu_0,h_i}(s)]}{p(s)P(s)^\alpha} \right)^{\frac{1}{\alpha}*} ds \right\}, \quad t \geqq t_0.$$

This means that $\mu_0(t)$ is a solution satisfying the functional differential equation

$$\big(p(t)\varphi(\mu_0'(t))\big)' = \sum_{i=1}^{n} \big[q_{\mu_0,g_i}(t) + r_{\mu_0,h_i}(t)\big]\varphi(\mu_0(t)), \quad t \geqq t_0$$

or consequently,

$$\big(p(t)\varphi(\mu_0'(t))\big)' = \sum_{i=1}^{n} \Big[q_i(t)\varphi\big(\mu_0(g_i(t))\big) + r_i(t)\varphi\big(\mu_0(h_i(t))\big)\Big], \quad t \geqq t_0.$$

Therefore, we conclude that the equation (A_-) has a n-SV$_P$ solution.

(The existence of a n-RV$_P(1)$ solution of (A_-)): Suppose that (1.9) is satisfied. Choose $t_1 > a$ so large that $t_* = \min_{i=1,2,\ldots,n} \big\{ \inf_{t \geqq t_1} g_i(t) \big\} > \max\{a, 1\}$

$$\Big(1 + \frac{1}{\alpha}\Big)[\widetilde{K} + \widetilde{L} + \alpha]\sqrt{\sum_{i=1}^{n} \big[\widehat{Q}_i(t_1) + 2^\alpha \widehat{R}_i(t_1)\big]} \leqq 1 \tag{4.59}$$

and

$$\left\{1 + \sqrt{\sum_{i=1}^{n} [\widehat{Q}_i(t_1) + 2^\alpha R_i(t_1)]}\right\}^{\frac{1}{\alpha}} \log \frac{P(h_i(t))}{P(t)} \leqq \log 2, \tag{4.60}$$

where the functions $Q_i(t)$, $R_i(t)$, $\widehat{Q}_i(t)$ and $\widehat{R}_i(t)$ are defined by (4.6) and (4.7), while

$$\widetilde{K} = \begin{cases} \Big(\dfrac{4}{3}\Big)^{1-\frac{1}{\alpha}} & \text{if } \alpha > 1 \\ \Big(\dfrac{3}{2}\Big)^{\frac{1}{\alpha}-1} & \text{if } \alpha \leqq 1 \end{cases} \quad \text{and} \quad \widetilde{L} = \begin{cases} \Big(\dfrac{4}{3}\Big)^{1-\frac{1}{\alpha}} & \text{if } \alpha > 1 \\ 1 & \text{if } \alpha \leqq 1 \end{cases}. \tag{4.61}$$

Let $\mathbb{K}$ define the set of all positive continuous nondecreasing functions $\nu(t)$ on $[t_*, \infty)$ satisfying

$$\nu(t) = 1 \ \text{ for } \ t_* \leqq t \leqq t_1; \tag{4.62}$$

$$1 \leqq \nu(t) \leqq \exp\left\{ \int_{t_1}^{t} \Big(\frac{1 + \rho(s)}{p(s)P(s)^\alpha}\Big)^{\frac{1}{\alpha}} ds \right\} \ \text{ for } \ t \geqq t_1; \tag{4.63}$$

$$\frac{\nu(h_i(t))}{\nu(t)} \leqq 2 \ \text{ for } \ t \geqq t_1, \quad i = 1, 2, \ldots, n, \tag{4.64}$$

where $\rho(t)$ is a solution of the integral equation

$$\rho(t) = \Big(1 + \frac{1}{\alpha}\Big) \sum_{i=1}^{n} \big[\widehat{Q}_i(t_1) + 2^\alpha \widehat{R}_i(t_1)\big] \frac{1}{P(t)} \int_{t_1}^{t} \frac{[\widetilde{L}\rho(s) + \widetilde{L} + \alpha]}{p(s)^{\frac{1}{\alpha}}} ds. \tag{4.65}$$

In order to verify that $\rho(t)$ is a solution of (4.65), we now consider the integral operator $\mathcal{R}$ defined by

$$\mathcal{R}\rho(t) = \Big(1+\frac{1}{\alpha}\Big)\sum_{i=1}^{n}\big[\widehat{Q}_i(t_1)+2^{\alpha}\widehat{R}_i(t_1)\big]\times \\ \times\frac{1}{P(t)}\int_{t_1}^{t}\frac{1}{p(s)^{\frac{1}{\alpha}}}\big[\widetilde{L}\rho(s)+\widetilde{L}+\alpha\big]\,ds \qquad (4.66)$$

on the set

$$\mathbb{P} = \Bigg\{\rho\in C_0[t_1,\infty):\ 0\leqq\rho(t)\leqq\sqrt{\sum_{i=1}^{n}\big[\widehat{Q}_i(t_1)+2^{\alpha}\widehat{R}_i(t_1)\big]},\ \ t\geqq t_1\Bigg\}.$$

It is easy to see that $\mathcal{R}$ sends $\mathbb{P}$ into itself and satisfies

$$\big\|\mathcal{R}\rho_1-\mathcal{R}\rho_2\big\|\leqq\sum_{i=1}^{n}\big[\widehat{Q}_i(t_1)+2^{\alpha}\widehat{R}_i(t_1)\big]\Big(1+\frac{1}{\alpha}\Big)\widetilde{L}\|\rho_1-\rho_2\|_0,\ \ \rho_1,\rho_2\in\mathbb{R}.$$

Therefore, there exists a unique fixed point of $\mathcal{R}$ which solves the integral equation (4.65).

Consider a family of half-linear differential equations

$$\big(p(t)\varphi(x'(t))\big)' = \sum_{i=1}^{n}\big[q_{\nu,g_i}(t)+r_{\nu,h_i}(t)\big]\varphi(x(t)),\ \ t\geqq t_1, \qquad (4.67)$$

where, for any $\nu\in\mathbb{K}$, the functions $q_{\nu,g_i}(t)$ and $r_{\nu,h_i}(t)$ are defined by

$$q_{\nu,g_i}(t) = q_i(t)\varphi\Big(\frac{\nu(g_i(t))}{\nu(t)}\Big)\ \text{ and }\ r_{\nu,h_i}(t) = r_i(t)\varphi\Big(\frac{\nu(h_i(t))}{\nu(t)}\Big).$$

Then, we define $Q_{\nu,g_i}(t)$, $R_{\nu,h_i}(t)$ for every $\nu\in\mathbb{K}$ by (4.19) and $\widehat{Q}_{\nu,g_i}(t) = \sup_{s\geqq t}Q_{\nu,g_i}(t)$, $\widehat{R}_{\nu,h_i}(t) = \sup_{s\geqq t}R_{\nu,h_i}(t)$. It follows from Theorem 3.1 that for each $\nu\in\mathbb{K}$, the equation (4.67) has a n-RV$_P(1)$ solution $X_\nu(t)$ expressed in the form

$$X_\nu(t) = \exp\Bigg\{\int_{t_1}^{t}\Bigg(\frac{1-\sum\limits_{i=1}^{n}[Q_{\nu,g_i}(s)+R_{\nu,h_i}(s)]+w_\nu(s)}{p(s)P(s)^{\alpha}}\Bigg)^{\frac{1}{\alpha}}ds\Bigg\}, \qquad (4.68)$$
$$t\geqq t_1,$$

where $w_\nu(t)$ is a solution of the integral equation

$$w_\nu(t) = \frac{\alpha}{P(t)}\int_{t_1}^{t}\frac{\widetilde{F}_\nu(s,w_\nu(s))}{p(s)^{\frac{1}{\alpha}}}\,ds,\ \ t\geqq t_1 \qquad (4.69)$$

and

$$\widetilde{F}_\nu(t,w_\nu) = 1+\Big(1+\frac{1}{\alpha}\Big)w_\nu - \Big|1-\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]+w_\nu\Big|^{1+\frac{1}{\alpha}}. \quad (4.70)$$

We notice that for some fixed $\nu \in \mathbb{K}$, $w_\nu(t)$ is a fixed point of the contraction mapping $\mathcal{F}_\nu$ defined by

$$\mathcal{F}_\nu w_\nu(t) = \frac{\alpha}{P(t)}\int_{t_1}^{t}\frac{\widetilde{F}_\nu(s,w_\nu(s))}{p(s)^{\frac{1}{\alpha}}}\,ds, \quad t \geqq t_1, \quad (4.71)$$

which satisfies

$$|w_\nu(t)| \leqq \sqrt{\sum_{i=1}^{n}\big[\widehat{Q}_{\nu,g_i}(t_1)+\widehat{R}_{\nu,h_i}(t_1)\big]}, \quad t \geqq t_1. \quad (4.72)$$

Furthermore, using the increasing nature of $\nu(t)$, we obtain

$$q_{\nu,g_i}(t) \leqq q_i(t), \quad r_{\nu,h_i}(t) \leqq 2^\alpha r_i(t) \ \text{ for } \ t \geqq t_1, \quad \nu \in \mathbb{K},$$

or consequently,

$$\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big] \leqq$$
$$\leqq \sum_{i=1}^{n}\big[Q_i(t)+2^\alpha R_i(t)\big] \leqq \sum_{i=1}^{n}\big[\widehat{Q}_i(t_1)+2^\alpha \widehat{R}_i(t_1)\big], \quad t \geqq t_1. \quad (4.73)$$

We will show that for every $\nu \in \mathbb{K}$,

$$|w_\nu(t)| \leqq \rho(t), \quad t \geqq t_1. \quad (4.74)$$

To this end, it is convenient to express $\widetilde{F}_\nu(t,w_\nu)$ as

$$\widetilde{F}_\nu(t,w_\nu) = \widetilde{G}_\nu(t,w_\nu)+\widetilde{H}_\nu(t,w_\nu)+\widetilde{k}_\nu(t),$$

where $\widetilde{G}_\nu(t,w_\nu)$, $\widetilde{H}_\nu(t,w_\nu)$ and $\widetilde{k}_\nu(t)$ are defined, respectively, by

$$\widetilde{G}_\nu(t,w_\nu) = \Big(1-\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]\Big)^{1+\frac{1}{\alpha}}+$$
$$+\Big(1+\frac{1}{\alpha}\Big)\Big(1-\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]\Big)^{1+\frac{1}{\alpha}}w_\nu -$$
$$-\Big|1-\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]+w_\nu\Big|^{1+\frac{1}{\alpha}},$$

$$\widetilde{H}_\nu(t,w_\nu) = \Big(1+\frac{1}{\alpha}\Big)\Big\{1-\Big(1-\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]\Big)^{\frac{1}{\alpha}}\Big\}w_\nu,$$

and

$$\widetilde{k}_\nu(t) = 1 - \Big(1 - \sum_{i=1}^{n} \big[Q_{\nu,g_i}(t) + R_{\nu,h_i}(t)\big]\Big)^{1+\frac{1}{\alpha}}.$$

Using the mean value theorem, we find that for some $\theta \in (0,1)$ the inequalities hold:

$$\begin{aligned}\big|\widetilde{H}_\nu(t,w_\nu(t))\big| &\leqq \frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)\Big|1-(1-\theta)\Big(\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]\Big)\Big|^{\frac{1}{\alpha}-1}\times \\ &\qquad\times\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]|w_\nu(t)| \\ &\leqq \frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)\widetilde{L}\sum_{i=1}^{n}\big[\widehat{Q}_i(t_1)+2^\alpha\widehat{R}_i(t_1)\big]|w_\nu(t)| \qquad (4.75)\end{aligned}$$

and

$$\begin{aligned}&\big|\widetilde{k}_\nu(t)\big| \leqq \\ &\leqq \Big(1+\frac{1}{\alpha}\Big)\Big|1-(1-\theta)\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]\Big|^{\frac{1}{\alpha}}\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big] \leqq \\ &\leqq \Big(1+\frac{1}{\alpha}\Big)\sum_{i=1}^{n}\big[\widehat{Q}_i(t_1)+2^\alpha\widehat{R}_i(t_1)\big], \quad t \geqq t_1. \qquad (4.76)\end{aligned}$$

Moreover, by means of the mean value theorem and L'Hospital rule, it follows that

$$\big|\widetilde{G}_\nu(t,w_\nu(t))\big| \leqq \frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)\widetilde{L}w_\nu^2(t), \quad t \geqq t_1. \qquad (4.77)$$

Let $\nu \in \mathbb{K}$ be fixed. Recalling that ρ and w_ν are the fixed point of the contraction mappings $\mathcal{R}$ and $\mathcal{F}_\nu$ defined by (4.66) and (4.71), we see that ρ and w_ν are constructed, respectively, as the limits as $n \to \infty$ of the sequences $\{\rho_n = \mathcal{R}\rho_{n-1},\ n = 1,2,\ldots,$ with $\rho_0 = 0\}$ and $\{w_n = \mathcal{F}_\nu w_{n-1},\ n = 1,2,\ldots,n,$ with $w_0 = 0\}$. First we note that for $t \geqq t_1$,

$$\begin{aligned}|w_1(t)| &= \mathcal{F}_\nu w_0(t) = \\ &= \frac{\alpha}{P(t)}\int_{t_1}^{t}\frac{1}{p(s)^{\frac{1}{\alpha}}}\Big[1-\Big|1+\sum_{i=1}^{n}\big[Q_{\nu,g_i}(s)+R_{\nu,h_i}(s)\big]\Big|^{1+\frac{1}{\alpha}}\Big]\,ds \leqq \\ &\leqq \frac{\alpha+1}{P(t)}\int_{t_1}^{t}\frac{1}{p(s)^{\frac{1}{\alpha}}}\sum_{i=1}^{n}\big[Q_{\nu,g_i}(s)+R_{\nu,h_i}(s)\big]\,ds \leqq \\ &\leqq (\alpha+1)\sum_{i=1}^{n}\big[\widehat{Q}_i(t_1)(t)+2^\alpha\widehat{R}_i(t_1)\big] \leqq \\ &\leqq \big[\widehat{Q}_i(t_1)(t)+2^\alpha\widehat{R}_i(t_1)\big]\Big(1+\frac{1}{\alpha}\Big)[\widetilde{L}+\alpha] = \rho_1(t), \quad t \geqq t_1.\end{aligned}$$

Then, assuming that $|w_n(t)| \leqq \rho_n(t)$, $t \geqq t_1$, for some $n \in \mathbb{N}$ and using (4.75), (4.76) and (4.77), we have

$$|w_{n+1}(t)| = \mathcal{F}_\nu w_n(t) =$$

$$= \frac{\alpha}{P(t)} \int_{t_1}^{t} \frac{1}{p(s)^{\frac{1}{\alpha}}} \Big[\big|\widetilde{G}_\nu(s, w_n(s))\big| + \big|\widetilde{H}_\nu(s, w_n(s))\big| + \big|\widetilde{k}_\nu(s)\big| \Big] \, ds \leqq$$

$$\leqq \frac{\alpha+1}{P(t)} \int_{t_1}^{t} \frac{1}{p(s)^{\frac{1}{\alpha}}} \Bigg[\frac{\widetilde{L}}{\alpha} w_\nu^2(s) + \frac{\widetilde{L}}{\alpha} \sum_{i=1}^{n} \big[\widehat{Q}_i(t_1) + 2^\alpha \widehat{R}_i(t_1)\big] |w_n(s)| +$$

$$+ \sum_{i=1}^{n} \big[\widehat{Q}_i(t_1) + 2^\alpha \widehat{R}_i(t_1)\big] \Bigg] \, ds \leqq$$

$$\leqq \Big(1+\frac{1}{\alpha}\Big) \sum_{i=1}^{n} \big[\widehat{Q}_i(t_1)+2^\alpha \widehat{R}_i(t_1)\big] \, \frac{1}{P(t)} \int_{t_1}^{t} \frac{[\widetilde{L} + \widetilde{L}\rho_n(s) + \alpha]}{p(s)^{\frac{1}{\alpha}}} \, ds = \rho_{n+1}(t).$$

Therefore, inductive arguments ensure the validity of (4.74).

We define by $\mathcal{M}$ the mapping which assigns to each $\nu \in \mathbb{K}$ the function $\mathcal{H}\nu(t)$ defined by

$$\mathcal{M}\nu(t) = 1 \ \text{ for } \ t_* \leqq t \leqq t_1, \ \ \mathcal{M}\nu(t) = X_\nu(t) \ \text{ for } \ t \geqq t_1.$$

(i) $\mathcal{M}$ is a self-map on $\mathbb{K}$, since it readily follows from (4.60) and $0 \leqq \rho(t) \leqq \sqrt{\sum\limits_{i=1}^{n} [\widehat{Q}_i(t) + 2^\alpha \widehat{R}_i(t_1)]}$, $t \geqq t_1$ that

$$1 \leqq X_\nu(t) \leqq \exp\Bigg\{ \int_{t_1}^{t} \Big(\frac{1+\rho(s)}{p(s)P(s)^\alpha} \Big)^{\frac{1}{\alpha}} \Bigg\}, \ \ t \geqq t_1 \ \text{ for any } \ \nu \in \mathbb{K}$$

and

$$\frac{\mathcal{M}\nu(h_i(t))}{\mathcal{M}\nu(t)} =$$

$$= \exp\Bigg\{ \int_{t}^{h_i(t)} \Bigg(\frac{1 - \sum\limits_{i=1}^{n} [Q_{\nu,g_i}(s) + R_{\nu,h_i}(s)] + w_\nu(s)}{p(s)P(s)^\alpha} \Bigg)^{\frac{1}{\alpha}*} ds \Bigg\} \leqq$$

$$\leqq \exp\Bigg\{ \int_{t}^{h_i(t)} \Big(\frac{1+\rho(s)}{p(s)P(s)^\alpha} \Big)^{\frac{1}{\alpha}} ds \Bigg\} \leqq$$

$$\leqq \exp\Bigg\{ \Bigg(1 + \sqrt{\sum_{i=1}^{n} [\widehat{Q}_i(t_1) + 2^\alpha \widehat{R}_i(t_1)]}\Bigg)^{\frac{1}{\alpha}} \log \frac{P(h_i(t))}{P(t)} \Bigg\} \leqq 2, \ \ t \geqq t_1.$$

(ii) $\mathcal{M}(\mathbb{K})$ is relatively compact in $C[t_*,\infty)$. The inclusion $\mathcal{M}(\mathbb{K}) \subset \mathbb{K}$ shows that $\mathcal{M}(\mathbb{K})$ is locally uniformly bounded on $[t_*,\infty)$. Since

$$0 \leqq \frac{d}{dt}\mathcal{M}\nu(t) = \frac{d}{dt}X_\nu(t) = \\ = \left(\frac{1-\sum\limits_{i=1}^{n}[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)]+w_\nu(t)}{p(t)P(t)^\alpha}\right)^{\frac{1}{\alpha}} X_\nu(t) \leqq \\ \leqq \frac{\left\{1+\sqrt{\sum\limits_{i=1}^{n}[\widehat{Q}_i(t_1)+2^\alpha \widehat{R}_i(t_1)]}\right\}^{\frac{1}{\alpha}}}{p(t)^{\frac{1}{\alpha}}P(t)} \times \\ \times \exp\left\{\left(1+\sqrt{\sum_{i=1}^{n}[\widehat{Q}_i(t_1)+2^\alpha \widehat{R}_i(t_1)]}\right)^{\frac{1}{\alpha}} \log\frac{P(t)}{P(t_1)}\right\},$$

we conclude that $\mathcal{M}(\mathbb{K})$ is locally equi-continuous on $[t_*,\infty)$.
(iii) $\mathcal{M}$ is continuous in the topology of $C[t_*,\infty)$. Let $\{\nu_m(t)\}$ be a sequence in $\mathbb{K}$ converging to $\delta(t)$ uniformly on compact subintervals of $[t_*,\infty)$. We have to prove that $\{\mathcal{M}\nu_m(t)\}$ converges to $\mathcal{M}\delta(t)$ uniformly on any compact subintervals of $[t_*,\infty)$. In order to simplify the notation, for arbitrary $\nu \in \mathbb{K}$ we define

$$Z_\nu(t) = \frac{1-\sum\limits_{i=1}^{n}[q_{\nu,g_i}(t)+R_{\nu,h_i}(t)]+w_\nu(t)}{P(t)^\alpha}, \quad t \geqq t_1. \tag{4.78}$$

Then, using (4.68) and the mean value theorem, we get

$$\left|\mathcal{M}\nu_m(t)-\mathcal{M}\delta(t)\right| = \left|X_{\nu_m}(t)-X_\delta(t)\right| = \\ = \left|\exp\left\{\int_{t_1}^{t}\left(\frac{Z_{\nu_m}(s)}{p(s)}\right)^{\frac{1}{\alpha}}ds\right\} - \exp\left\{\int_{t_1}^{t}\left(\frac{Z_\delta(s)}{p(s)}\right)^{\frac{1}{\alpha}}ds\right\}\right| \leqq \\ \leqq \exp\left\{\int_{t_1}^{t}\left(\frac{1+\rho(s)}{p(s)P(s)^\alpha}\right)^{\frac{1}{\alpha}}ds\right\}\int_{t_1}^{t}\frac{|(Z_{\nu_m}(s))^{\frac{1}{\alpha}}-(Z_\delta(s))^{\frac{1}{\alpha}}|}{p(s)^{\frac{1}{\alpha}}}\,ds.$$

As in the proof of the existence of a n-$\mathrm{RV}_P(1)$ solution of the equation (A_+), we can show that the integrand of the last integral is bounded from above by the functions

$$\left(\frac{\sum\limits_{i=1}^{n}|Q_{\nu_m,g_i}(t)-Q_{\delta,g_i}(t)|+\sum\limits_{i=1}^{n}|R_{\nu_m,h_i}(t)-R_{\delta,h_i}(t)|+|w_{\nu_m}(t)-w_\delta(t)|}{P(t)^\alpha}\right)^{\frac{1}{\alpha}} \\ \text{if } \alpha > 1,$$

$$C_3 \frac{\sum\limits_{i=1}^{n} |Q_{\nu_m,g_i}(t) - Q_{\delta,g_i}(t)| + \sum\limits_{i=1}^{n} |R_{\nu_m,h_i}(t) - R_{\delta,h_i}(t)| + |w_{\nu_m}(t) - w_\delta(t)|}{P(t)^\alpha}$$

$$\text{if } \alpha \leqq 1,$$

where C_3 is a positive constant depending only on α and $\rho(t_1)$. Therefore, it suffices to prove the uniform convergence to 0 on the compact subintervals of the two sequences

$$\frac{|w_{\nu_m}(t) - w_\delta(t)|}{P(t)^\alpha} \text{ and}$$
$$\frac{\sum\limits_{i=1}^{n} |Q_{\nu_m,g_i}(t) - Q_{\delta,g_i}(t)| + \sum\limits_{i=1}^{n} |R_{\nu_m,h_i}(t) - R_{\delta,h_i}(t)|}{P(t)^\alpha} = \frac{\widetilde{S}_{n,m}(t)}{P(t)^\alpha}. \tag{4.79}$$

The second sequence in (4.79) can be dealt with exactly as in the case of n-RV$_P(1)$ solution of the equation (A$_+$). In order to prove the uniform convergence of the first sequence in (4.79), we consider $\widetilde{F}_{\nu_m}(t, w_{\nu_m})$ and $\widetilde{F}_\delta(t, w_\delta)$ defined by (4.70). Applying the mean value theorem, for $t \geqq t_1$ we get

$$\begin{aligned}
\big|\widetilde{F}_{\nu_m}(t, w_{\nu_m}(t)) - \widetilde{F}_\delta(t, w_\delta(t))\big| &\leqq \Big(1 + \frac{1}{\alpha}\Big)\big|w_{\nu_m}(t) - w_\delta(t)\big| + \\
&+ \bigg|\Big|1 - \sum_{i=1}^{n}\big[Q_{\nu_m,g_i}(t) + R_{\nu_m,h_i}(t)\big] + w_{\nu_m}(t)\Big|^{1+\frac{1}{\alpha}} - \\
&- \Big|1 - \sum_{i=1}^{n}\big[Q_{\delta,g_i}(t) + R_{\delta,h_i}(t)\big] + w_\delta(t)\Big|^{1+\frac{1}{\alpha}}\bigg| \leqq \\
\leqq \Big(1 + \frac{1}{\alpha}\Big)\big|w_{\nu_m}(t) - w_\delta(t)\big| &+ \Big(1 + \frac{1}{\alpha}\Big)\tau_3\Big\{\widetilde{S}_{m,n}(t) + \big|w_{\nu_m}(t) - w_\delta(t)\big|\Big\} = \\
&= \Big(1 + \frac{1}{\alpha}\Big)(1 + \tau_3)\big|w_{\nu_m}(t) - w_\delta(t)\big| + \Big(1 + \frac{1}{\alpha}\Big)\tau_3\widetilde{S}_{m,n}(t),
\end{aligned}$$

where τ_3 is a positive constant depending only on α, $\widehat{Q}_i(t_1)$ and $\widehat{R}_i(t_1)$. Consequently, the first sequence in (4.79) implies that

$$\begin{aligned}
\frac{|w_{\nu_m}(t) - w_\delta(t)|}{P(t)^\alpha} &\leqq \frac{(\alpha+1)(1+\tau_3)}{P(t)^{(\alpha+1)}} \int_{t_1}^{t} \frac{|w_{\nu_m}(s) - w_\delta(s)|}{p(s)^{\frac{1}{\alpha}}}\, ds + \\
&+ \frac{(\alpha+1)\tau_3}{P(t)^{\alpha+1}} \int_{t_1}^{t} \frac{\widetilde{S}_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}}\, ds, \quad t \geqq t_1.
\end{aligned} \tag{4.80}$$

Putting for brevity

$$\widetilde{W}_m(t) = \int_{t_1}^{t} \frac{|w_{\nu_m}(s) - w_\delta(s)|}{p(s)^{\frac{1}{\alpha}}}\, ds,$$

we derive the following differential inequality for $\widetilde{W}_m(t)$:

$$\big(P(t)^{-(\alpha+1)(1+\tau_3)}\widetilde{W}_m(t)\big)' \leqq$$
$$\leqq \frac{(\alpha+1)\tau_3}{p(t)^{\frac{1}{\alpha}} P(t)^{(\alpha+1)(\tau_3+1)+1}} \int_{t_1}^{t} \frac{\widetilde{S}_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}}\, ds, \quad t \geqq t_1. \qquad (4.81)$$

Integrating (4.81) from t_1 to t, we obtain

$$\widetilde{W}_m(t) \leqq \frac{\tau_3}{\tau_3+1}\, \frac{1}{P(t)^{(\alpha+1)(\tau_3+1)}} \int_{t_1}^{t} \frac{\widetilde{S}_{m,n}(s)}{p(s)^{\frac{1}{\alpha}} P(s)^{(\alpha+1)(\tau_3+1)}}\, ds, \quad t \geqq t_1. \qquad (4.82)$$

Using (4.80) and (4.82), we conclude that

$$\frac{|w_{\nu_m}(t) - w_\delta(t)|}{P(t)^\alpha} \leqq \frac{(\alpha+1)\tau_3}{P(t)^{(\alpha+1)(\tau_3+2)}} \int_{t_1}^{t} \frac{\widetilde{S}_{m,n}(s)}{p(s)^{\frac{1}{\alpha}} P(s)^{(\alpha+1)(\tau_3+1)}}\, ds +$$
$$+ \frac{(\alpha+1)\tau_3}{P(t)^{\alpha+1}} \int_{t_1}^{t} \frac{\widetilde{S}_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}}\, ds, \quad t \geqq t_1,$$

whence it follows that the sequence $|w_{\nu_m}(t) - w_\delta(t)|/P(t)^\alpha$ converges to 0 uniformly on $[t_1,\infty)$. Therefore, we have proved that the mapping $\mathcal{M}$ is continuous in the topology of $C[t_*,\infty)$. Thus, applying the Schauder–Tychonoff fixed point theorem, $\mathcal{M}$ has a fixed point ν_0 in $\mathbb{K}$. Since $\nu_0 = X_{\nu_0}(t)$ for $t \geqq t_1$, $\nu_0(t)$ satisfies the functional differential equation

$$\big(p(t)\varphi(\nu_0'(t))\big)' = \sum_{i=1}^{n} \Big[q_i(t)\varphi\big(\nu_0(g_i(t))\big) + r_i(t)\varphi\big(\nu_0(h_i(t))\big)\Big], \quad t \geqq t_1. \qquad (4.83)$$

It is obvious that $\nu_0(t)$ is a n-RV$_P(1)$ solution of the equation (A$_-$). This completes the proof of Theorem 1.1.

5. Examples

We here present four examples illustrating application of Theorem 1.1 to the functional differential equations of the type (A$_+$) and (A$_-$), respectively. We begin with two examples of the existence of n-SV$_P$ and n-RV$_P(1)$ solutions of the type (A$_+$) with the case $i = 1$.

Example 5.1. Consider the following functional differential equation with both retarded and advanced arguments

$$\big(e^{-\alpha t}\varphi(x'(t))\big)' + q(t)\varphi\Big(x\Big(t - \frac{1}{\log t}\Big)\Big) + r(t)\varphi\Big(x\Big(t + \frac{1}{\log t}\Big)\Big) = 0, \qquad (5.1)$$
$$t \geqq e,$$

where the functions $q(t)$ and $r(t)$ are given by

$$q(t) = \frac{\alpha}{2e^{\alpha t}\, t^{\alpha}} \Big(1 + \frac{\lambda}{\log t}\Big)^{\alpha - 1} \Big[1 - \frac{\lambda}{t \log t} + \frac{\lambda(\lambda - 1)}{t(\log t)^2} + \frac{\lambda}{\log t}\Big] \times \\ \times \Big(1 - \frac{1}{t \log t}\Big)^{-\alpha} \Big\{1 + \frac{\log(1 - \frac{1}{t \log t})}{\log t}\Big\}^{-\alpha\lambda}$$

and

$$r(t) = \frac{\alpha}{2e^{\alpha t} t^{\alpha}} \Big(1 + \frac{\lambda}{\log t}\Big)^{\alpha - 1} \Big[1 - \frac{\lambda}{t \log t} + \frac{\lambda(\lambda - 1)}{t(\log t)^2} + \frac{\lambda}{\log t}\Big] \times \\ \times \Big(1 + \frac{1}{t \log t}\Big)^{-\alpha} \Big\{1 + \frac{\log(1 + \frac{1}{t \log t})}{\log t}\Big\}^{-\alpha\lambda}$$

for λ being a positive constant. The function $p(t) = e^{-\alpha t}$ satisfies (1.1) and the function $P(t)$ given by (1.6) is $P(t) \sim e^t$. Moreover, the functions

$$g(t) = t - \frac{1}{\log t} \quad \text{and} \quad h(t) = t + \frac{1}{\log t} \tag{5.2}$$

satisfy the conditions (1.7) and (1.8). The condition (1.9) is satisfied for this equation, since

$$\int_t^{\infty} q(s)\, ds \sim \frac{\alpha}{2t^{\alpha} e^{\alpha t}} \quad \text{and} \quad \int_t^{\infty} h(s)\, ds \sim \frac{\alpha}{2t^{\alpha} e^{\alpha t}} \quad \text{as} \ \ t \to \infty.$$

Therefore, equation (5.1) has a n-SV$_{e^t}$ solution $x(t)$ by Theorem 1.1. One such solution is $x(t) = t(\log t)^{\lambda}$.

Example 5.2. Consider the following functional differential equation:

$$\big(t^{\alpha}\varphi(x'(t))\big)' + q(t)\varphi\big(x(te^{-\frac{1}{t}})\big) + r(t)\varphi\big(x(te^{\frac{1}{t}})\big) = 0, \ \ t \geqq e^e, \tag{5.3}$$

where the functions $q(t)$ and $r(t)$ are given by

$$q(t) = \frac{\alpha\mu}{2t(\log t)^{\alpha+1} \log_2 t} \Big(1 - \frac{\mu}{\log_2 t}\Big)^{\alpha - 1} \Big(1 - \frac{\mu + 1}{\log_2 t}\Big) \times \\ \times \Big(1 - \frac{1}{t \log t}\Big)^{-\alpha} \Big\{1 + \frac{\log(1 - \frac{1}{t \log t})}{\log_2 t}\Big\}^{\alpha\mu}$$

and

$$r(t) = \frac{\alpha\mu}{2t(\log t)^{\alpha+1} \log_2 t} \Big(1 - \frac{\mu}{\log_2 t}\Big)^{\alpha - 1} \Big(1 - \frac{\mu + 1}{\log_2 t}\Big) \times \\ \times \Big(1 + \frac{1}{t \log t}\Big)^{-\alpha} \Big\{1 + \frac{\log(1 + \frac{1}{t \log t})}{\log_2 t}\Big\}^{\alpha\mu},$$

respectively, and μ is a positive constant. The function $p(t) = t^{\alpha}$ satisfies (1.1) and the function $P(t)$ reduces to $P(t) \sim \log t$, while the functions

$g(t) = te^{-\frac{1}{t}}$ and $h(t) = te^{\frac{1}{t}}$ satisfy the conditions (1.7) and (1.8). Moreover, since

$$\int_t^\infty q(s)\,ds \sim \frac{\alpha\mu}{2t(\log t)^{\alpha+1}\log_2 t} \quad \text{and} \quad \int_t^\infty h(s)\,ds \sim \frac{\alpha\mu}{2t(\log t)^{\alpha+1}\log_2 t} \quad \text{as } t \to \infty,$$

the condition (1.9) is satisfied and thus, the equation (5.3) possesses a n-$\mathrm{RV}_{\log t}$ solution by Theorem 1.1. One such solution is $\log t/(\log_2 t)^\mu$.

Next, two examples illustrating application of Theorem 1.1 to the functional differential equation of the type (A_-) with the case $i = 1$ will be presented below.

Example 5.3. We consider the functional differential equation with both retarded and advanced arguments

$$\left(e^{-\alpha t}\varphi(x'(t))\right)' = q(t)\varphi\Big(x\Big(t-\frac{1}{\log t}\Big)\Big)+r(t)\varphi\Big(x\Big(t+\frac{1}{\log t}\Big)\Big), \quad t \geqq e, \tag{5.4}$$

where the functions $q(t)$ and $r(t)$ are given by

$$\begin{aligned}
q(t) = {} & \frac{\alpha}{2t^\alpha e^{\alpha t}}\Big(1-\frac{\lambda}{\log t}\Big)^{\alpha-1}\times \\
& \times\left[\Big(1+\frac{2}{t}\Big)\Big(1-\frac{\lambda}{\log t}\Big)+\frac{\lambda}{t\log t}\Big(1-\frac{\lambda}{\log t}\Big)+\frac{\lambda}{t(\log t)^2}\right]\times \\
& \times\Big(1-\frac{1}{t\log t}\Big)^{\alpha}\left\{1+\frac{\log(1-\frac{1}{t\log t})}{\log t}\right\}^{-\alpha\lambda}, \\
r(t) = {} & \frac{\alpha}{2t^\alpha e^{\alpha t}}\Big(1-\frac{\lambda}{\log t}\Big)^{\alpha-1}\times \\
& \times\left[\Big(1+\frac{2}{t}\Big)\Big(1-\frac{\lambda}{\log t}\Big)+\frac{\lambda}{t\log t}\Big(1-\frac{\lambda}{\log t}\Big)+\frac{\lambda}{t(\log t)^2}\right]\times \\
& \times\Big(1+\frac{1}{t\log t}\Big)^{\alpha}\left\{1+\frac{\log(1+\frac{1}{t\log t})}{\log t}\right\}^{-\alpha\lambda}
\end{aligned}$$

for λ being a positive constant. As in Example 5.1, it could be shown without difficulty that all conditions of Theorem 1.1 are satisfied, so that the equation (5.4) has a n-SV_{e^t} solution $x(t)$ by Theorem 1.1. One such solution is $(\log t)^\lambda/$t.

Example 5.4. Consider the following functional differential equation:

$$\left(t^\alpha\varphi(x'(t))\right)' = q(t)\varphi\big(x(te^{-\frac{1}{t}})\big)+r(t)\varphi\big(x(te^{\frac{1}{t}})\big), \quad t \geqq e^e, \tag{5.5}$$

where the functions $q(t)$ and $r(t)$ are given by

$$q(t) = \frac{\alpha\mu}{2t(\log t)^{\alpha+1}\log_2 t}\Big(1+\frac{\mu}{\log_2 t}\Big)^{\alpha-1}\Big(1+\frac{\mu-1}{\log_2 t}\Big)\times$$

$$\times \Big(1 - \frac{1}{t \log t}\Big)^{-\alpha} \Big\{1 + \frac{\log(1 - \frac{1}{t \log t})}{\log_2 t}\Big\}^{-\alpha\mu}$$

and

$$r(t) = \frac{\alpha\mu}{2t(\log t)^{\alpha+1} \log_2 t} \Big(1 + \frac{\mu}{\log_2 t}\Big)^{\alpha-1} \Big(1 + \frac{\mu - 1}{\log_2 t}\Big) \times$$
$$\times \Big(1 + \frac{1}{t \log t}\Big)^{-\alpha} \Big\{1 + \frac{\log(1 + \frac{1}{t \log t})}{\log_2 t}\Big\}^{-\alpha\mu},$$

respectively, and μ is a positive constant. As in Example 5.2, it can be verified that all conditions of Theorem 1.1 are satisfied. Therefore, the equation (5.5) possesses a n-$\mathrm{RV}_{\log t}$ solution $x(t)$. One such solution is $x(t) = \log t (\log_2 t)^{\mu}$.

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