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# ON NONLOCAL PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS FOR SINGULAR ORDINARY DIFFERENTIAL EQUATIONS

*Dedicated to the blessed memory of my dear teacher,*
*Professor Levan Magnaradze*

**Abstract.** For higher order singular ordinary differential equations, sufficient conditions for the solvability and unique solvability of nonlinear nonlocal boundary value problems are established.

**რეზიუმე.** მაღალი რიგის სინგულარული ჩვეულებრივი დიფერენციალური განტოლებებისათვის დადგენილია არაწრფივ არალოკალურ სასაზღვრო ამოცანათა ამოხსნადობისა და ცალსახად ამოხსნადობის საკმარისი პირობები.

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Let $n \geq 2$ be an arbitrary natural number, $-\infty < a < b < \infty$, and let $f : ]a, b[ \times \mathbb{R}^n \to \mathbb{R}$ be a measurable in the first and continuous in the last $n$ arguments function. The differential equation

$$u^{(n)} = f(t, u, \dots, u^{(n-1)}) \tag{1}$$

is said to be singular if the function $f$ with respect to the first argument is nonintegrable on $[a, b]$, having singularities at one or at several points of that segment. For the singular equation (1), the two-point boundary value problems and the multi-point problems of Valée-Poisson and Cauchy–Nicoletti type are thoroughly investigated (see [2]–[7], [11], [12], [20], [21] and the references therein).

As for the problems with nonlocal conditions, i.e., with the conditions connecting the values of an unknown solution and those of their derivatives at different points of the segment $[a, b]$, they are well-studied for the second order equations (see, e.g., [9], [10], [13], [14], [16]–[19]). Some nonlocal problems are studied for higher order linear singular differential equations as well (see [1], [15]). However, nonlocal problems with nonlinear boundary conditions both for linear and nonlinear singular differential equations remain still little studied. The present paper is devoted to the study of one of such problems. More precisely, for the equation (1) we consider the boundary value problem

$$u^{(i-1)}(a)=0 \;\; (i=1,\dots,n-1), \quad \varphi(u)=0, \tag{2}$$

where $\varphi$ is some, generally speaking, nonlinear functional, defined in the space of $(n-1)$-times continuously differentiable functions, satisfying the initial conditions

$$u^{(i-1)}(a)=0 \;\; (i=1,\dots,n-1). \tag{$2_0$}$$

Throughout the paper, the use will be made of the following notation.

$\mathbb{R}=\,]-\infty,+\infty[\,$, $\mathbb{R}_+=[0,+\infty[\,$.

$\mathbb{R}^n$ is the $n$-dimensional real Euclidean space,

$$D=\Big\{(x_1,\dots,x_n)\in\mathbb{R}^n:\; x_kx_1>0\;(k=1,\dots,n)\Big\}.$$

$L$ is the space of Lebesgue integrable on $[a,b]$ real functions.

$C_0^{n-1}$ is the Banach space of $(n-1)$-times continuously differentiable functions $u:[a,b]\to\mathbb{R}$, satisfying the initial conditions $(2_0)$, with the norm

$$\|u\|_{C_0^{n-1}}=\max\Big\{\big|u^{(n-1)}(t)\big|:\; a\le t\le b\Big\}.$$

$M_0^{n-1}=\Big\{u\in C_0^{n-1}:\; u^{(n-1)}(t)\neq 0 \text{ for } a\le t\le b\Big\}.$

$\widetilde{C}^{n-1}$ is the space of $(n-1)$-times continuously differentiable functions $u:[a,b]\to\mathbb{R}$ with an absolutely continuous $(n-1)$-th derivative.

Everywhere in the sequel, it is assumed that $\varphi: C_0^{n-1}\to\mathbb{R}$ is a continuous functional, bounded on every bounded set of the space $C_0^{n-1}$.

Along with $(1),(2)$, we consider the boundary value problem

$$u^{(n)}=(1-\lambda)\sum_{k=1}^{n}p_k(t)u^{(k-1)}+\lambda f(t,u,\dots,u^{(n-1)}), \tag{3}$$

$$u^{(i-1)}(a)=0 \;\; (i=1,\dots,n-1), \quad (1-\lambda)u^{(n-1)}(a)+\lambda\varphi(u)=0, \tag{4}$$

and the initial problem

$$u^{(n)}(t)\ge -\sum_{k=1}^{n}g_k(t)u^{(k-1)}(t), \tag{5}$$

$$u^{(i-1)}(a)=0 \;\; (i=1,\dots,n-1), \quad u^{(n-1)}(a)=1, \tag{6}$$

where $\lambda$ is a parameter, and $p_k : ]a,b[ \to \mathbb{R}$ $(k = 1,\dots,n)$ and $g_k : ]a,b[ \to \mathbb{R}_+$ $(k = 1,\dots,n)$ are measurable functions such that

$$\int_a^b (t-a)^{n-k} |p_k(t)|\,dt < +\infty \quad (k = 1,\dots,n), \tag{7}$$

$$\int_a^b (t-a)^{n-k} g_k(t)\,dt < +\infty \quad (k = 1,\dots,n). \tag{8}$$

A function $u \in \widetilde{C}^{n-1}$ is said to be a **solution of the differential equation** (3) (**of the differential inequality** (5)) if it almost everywhere on $]a,b[$ satisfies that equation (that inequality). If, moreover, this function satisfies the boundary conditions (4) (the initial conditions (6)), then it is called a **solution of the problem** (3), (4) (**of the problem** (5), (6)).

**Theorem 1** (The Principle of a Priori Boundedness)**.** *Let for every $\rho \in \mathbb{R}_+$ the function*

$$f_\rho(t) = \max\Big\{ \big|f(t,x_1,\dots,x_n)\big| : \ |x_k| \le \rho(t-a)^{n-k} \ (k = 1,\dots,n) \Big\}$$

*be integrable on $[a,b]$. Let, moreover, there exist measurable functions $p_k : ]a,b[ \to \mathbb{R}$ $(k = 1,\dots,n)$, satisfying the conditions* (7)*, and a positive number $\rho_0$ such that for any $\lambda \in ]0,1[$ an arbitrary solution of the problem* (3)*,* (4) *admits the estimate*

$$\|u\|_{C_0^{n-1}} \le \rho_0.$$

*Then the problem* (1)*,* (2) *has at least one solution.*

This theorem is proved on the basis of Theorem 1 appearing in [8].

Relying on the above-formulated theorem, we prove theorems below which contain effective sufficient conditions for the solvability of the problem (1), (2).

Introduce the following definition.

**Definition.** We say that the vector function $(g_1,\dots,g_n)$ with the measurable components $g_k : ]a,b[ \to \mathbb{R}_+$ $(k = 1,\dots,n)$ belongs to the set $V$ if the conditions (8) hold and an arbitrary solution of the problem (5), (6) satisfies the inequality

$$u^{(n-1)}(t) > 0 \ \text{ for } \ a \le t \le b.$$

**Theorem 2.** *Let on the set $]a,b[ \times D$ the inequality*

$$f(t,x_1,\dots,x_n)\operatorname{sgn}(x_1) \ge -\sum_{k=1}^n g_k(t)|x_k| - h(t) \tag{9}$$

*be fulfilled, and let on the set $]a,b[ \times \mathbb{R}^n$ the inequality*

$$\big|f(t,x_1,\dots,x_n)\big| \le \sum_{k=1}^n h_k(t)|x_k| + h(t) \tag{10}$$

*hold, where*

$$(g_1, \dots, g_n) \in V, \tag{11}$$

$h \in L$ *and* $h_k : ]a, b[ \to \mathbb{R}_+$ $(k = 1, \dots, n)$ *are measurable functions such that*

$$\int_a^b (t-a)^{n-k} h_k(t)\, dt < +\infty \quad (k = 1, \dots, n). \tag{12}$$

*If, moreover,*

$$\varphi(u) u^{(n-1)}(a) > 0 \ \ for \ \ u \in M_0^{n-1}, \tag{13}$$

*then the problem* (1), (2) *has at least one solution.*

**Corollary 1.** *Let on the sets* $]a, b[ \times D$ *and* $]a, b[ \times \mathbb{R}^n$ *be satisfied respectively the inequalities* (9) *and* (10), *where* $h \in L$ *and* $g_k : ]a, b[ \to \mathbb{R}_+$ *and* $h_k : ]a, b[ \to \mathbb{R}_+$ $(k = 1, \dots, n)$ *are measurable functions. Let, moreover,*

$$\sum_{k=1}^{n} \frac{1}{(n-k)!} \int_a^b (t-a)^{n-k} g_k(t)\, dt \le 1 \tag{14}$$

*and the conditions* (12) *and* (13) *hold. Then the problem* (1), (2) *has at least one solution.*

**Corollary 2.** *Let on the set* $]a, b[ \times D$ *the inequality*

$$f(t, x_1, \dots, x_n) \operatorname{sgn}(x_1) \ge -\sum_{k=1}^{n-1} \frac{\ell_k |x_k|}{(t-a)^{n-k-1}} - \ell |x_n| - h(t)$$

*be fulfilled and on the set* $]a, b[ \times \mathbb{R}^n$ *the inequality* (10) *hold, where* $h \in L$, $\ell_1, \dots, \ell_{n-1}$, $\ell$ *are nonnegative numbers such that*

$$\int_0^{+\infty} \frac{ds}{\ell_0 + \ell s + s^2} > b - a, \quad \ell_0 = \sum_{k=1}^{n-1} \frac{\ell_k}{(n-k-1)!}\,, \tag{15}$$

*and* $h_k : ]a, b[ \to \mathbb{R}_+$ $(k = 1, \dots, n)$ *are measurable functions satisfying the conditions* (12). *If, moreover, the functional* $\varphi$ *satisfies the condition* (13), *then the problem* (1), (2) *has at least one solution.*

As an example, we consider the boundary value problem

$$u^{(n)} = \sum_{k=1}^{n} p_k(t, u, \dots, u^{(n-1)}) u^{(k-1)} + p_0(t, u, \dots, u^{(n-1)}), \tag{16}$$

$$\begin{gathered} u^{(i-1)}(a) = 0 \ \ (i = 1, \dots, n-1), \\ \sum_{k=1}^{n} \bigg( \int_a^b (s-a)^{-\alpha_k} u^{(k-1)}(s)\, d\beta_k(s) \bigg)^{m_k} = 0. \end{gathered} \tag{17}$$

Here, $p_k : ]a,b[ \times \mathbb{R}^n \to \mathbb{R}$ $(k = 0,\dots,n)$ are functions, measurable in the first and continuous in the last $n$ arguments,

$$\alpha_k \in [0, n-k] \ (k = 1,\dots,n), \text{ every } m_k \text{ is an odd number;} \qquad (18)$$

$$\beta_k : [a,b] \to \mathbb{R} \ (k = 1,\dots,n) \text{ are nondecreasing functions,} \qquad (19)$$

and

$$\lim_{s\to a} \sum_{k=1}^{n} \big(\beta_k(b) - \beta_k(s)\big) > 0. \qquad (20)$$

Moreover, by the values of the functions $s \to (s-a)^{-\alpha_k} u^{(k-1)}(s)$ $(k = 1,\dots,n)$ at the point $a$ are meant their limits as $s \to a$.

From Corollaries 1 and 2 it follows

**Corollary 3.** *Let the conditions* (18)–(20) *be fulfilled and the functions* $p_k$ $(k = 0,\dots,n)$ *on the set* $]a,b[ \times \mathbb{R}^n$ *satisfy the inequalities*

$$-g_k(t) \le p_k(t, x_1,\dots,x_n) \le h_k(t) \ \ (k = 1,\dots,n),$$

$$\big|p_0(t, x_1,\dots,x_n)\big| \le h(t),$$

*where* $g_k$ *and* $h_k : ]a,b[ \to \mathbb{R}_+$ $(k = 1,\dots,n)$ *are measurable functions and* $h \in L$. *Let, moreover, the functions* $h_k$ $(k = 1,\dots,n)$ *satisfy the conditions* (12), *and as for the functions* $g_k$ $(k = 1,\dots,n)$, *they either satisfy the condition* (14), *or*

$$g_k(t) \equiv \ell_k (t-a)^{k+1-n} \ \ (k = 1,\dots,n-1), \quad g_n(t) \equiv \ell, \qquad (21)$$

*where* $\ell_1,\dots,\ell_{n-1}$, $\ell$ *are nonnegative numbers, satisfying the inequality* (15). *Then the problem* (16), (17) *has at least one solution.*

Theorem 3 below and its corollaries deal with the case, where the function $f$ for an arbitrary $t \in ]a,b[$ in the last $n$ arguments has continuous in $\mathbb{R}^n$ partial derivatives $\frac{\partial f_k(t,x_1,\dots,x_n)}{\partial x_k}$ $(k = 1,\dots,n)$.

**Theorem 3.** *Let on the set* $]a,b[ \times \mathbb{R}^n$ *the inequalities*

$$-g_k(t) \le \frac{\partial f(t, x_1,\dots,x_n)}{\partial x_k} \le h_k(t) \ \ (k = 1,\dots,n) \qquad (22)$$

*be fulfilled, where* $g_k$ *and* $h_k : ]a,b[ \to \mathbb{R}_+$ $(k = 1,\dots,n)$ *are measurable functions, satisfying the conditions* (11) *and* (12). *If, moreover,*

$$\int_a^b \big|f(t, 0,\dots,0)\big|\, dt < +\infty, \qquad (23)$$

$$\varphi(0)=0 \ \text{ and } \ \big(\varphi(u+v)-\varphi(u)\big)v^{(n-1)}(a)>0 \ \text{ for } \ u\in C_0^{n-1}, \ v\in M_0^{n-1}, \qquad (24)$$

*then the problem* (1), (2) *has one and only one solution.*

**Corollary 4.** *Let the conditions* (22)–(24) *be fulfilled, where* $g_k$ *and* $h_k : ]a,b[ \to \mathbb{R}_+$ $(k = 1,\dots,n)$ *are measurable functions. Let, moreover, the functions* $h_k$ $(k = 1,\dots,n)$ *satisfy the conditions* (12), *and the functions* $g_k$

$(k = 1, \dots, n)$ *either satisfy the inequality* (14)*, or admit the representations* (21)*, where* $\ell_1, \dots, \ell_{n-1}$*,* $\ell$ *are nonnegative numbers satisfying the condition* (15)*. Then the problem* (1), (2) *has one and only one solution.*

Finally, as an example, we consider the linear differential equation

$$u^{(n)} = \sum_{k=1}^{n} p_k(t) u^{(k-1)} + p_0(t) \tag{25}$$

with the nonlinear boundary conditions (17), where $p_k : ]a, b[ \to \mathbb{R}$ $(k = 0, \dots, n)$ are measurable functions such that

$$\int_a^b (t-a)^{n-k} |p_k(t)| \, dt < +\infty \quad (k = 1, \dots, n), \qquad \int_a^b |p_0(t)| \, dt < +\infty. \tag{26}$$

From Corollary 4, for the problem (25), (17) we have

**Corollary 5.** *Let the conditions* (18)–(20) *and* (26) *be fulfilled. Let, moreover, either*

$$\sum_{k=1}^{n} \frac{1}{(n-k)!} \int_a^b (t-a)^{n-k} \big( |p_k(t)| - p_k(t) \big) \, dt \le 2,$$

*or there exist nonnegative constants* $\ell_1, \dots, \ell_{n-1}$*,* $\ell$*, satisfying the condition* (15)*, such that on* $]a, b[$ *the inequalities*

$$p_k(t) \ge -\ell_k (t-a)^{k+1-n} \quad (k = 1, \dots, n-1), \quad p_n(t) \ge -\ell$$

*are fulfilled. Then the problem* (25), (17) *has one and only one solution.*


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