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BOUNDED SOLUTIONS TO THE FIRST ORDER SCALAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Dedicated to the blessed memory of Professor Levan Magnaradze

Abstract. The question on the existence, uniqueness, and sign properties of a bounded on $[a, +\infty[$ solution to the boundary value problem for the first order functional differential equation is studied. The work is divided into two parts: in the first one, the linear problem is studied, the second one is devoted to the nonlinear problem.

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რეზიუმე. ნაშრომში შესწავლილია $[a, +\infty[$ ნახევარღერძზე მოცემული პირველი რიგის სკალარული ფუნქციონალურ დიფერენციალური განტოლებების შემოსაზღვრული ამონახსნის არსებობის, ერთადერთობის და ნიშან-მუდმივობის საკითხი. განხილულია როგორც წრფივი, ასევე არაწრფივი განტოლებები.

Introduction

In the present work, the problem on the existence, uniqueness, and sign properties of a bounded on $[a,+\infty[$ solution to the functional differential equation

$$u'(t) = F(u)(t) \tag{0.1}$$

is studied.

In Chapter 1 (Sections 1–5) we consider a linear equation, i.e., the equation

$$u'(t) = \ell(u)(t) + q(t), \tag{0.2}$$

with a "boundary" condition

$$\omega(u) = c.$$

Here $\ell : C_{loc}([a,+\infty[\,;R) \to L_{loc}([a,+\infty[\,;R)$ is a linear continuous operator, $\omega : C_{loc}([a,+\infty[\,;R) \to R$ is a linear bounded functional, $c \in R$, $q \in L_{loc}([a,+\infty[\,;R)$, and

$$\sup\left\{\Big|\int_a^t q(s)\,ds\Big| :\ t \geq a\right\} < +\infty.$$

Main results of this chapter are contained in Section 3. In Subsection 3.1, theorems on "Fredholmity" of above mentioned problem are presented (Theorems 3.1 and 3.2). Optimal (unimprovable) sufficient conditions of existence and uniqueness of a bounded solution to (0.2) satisfying one of the conditions

$$u(a) = c,$$
$$u(+\infty) = c,$$
$$u(a) - u(+\infty) = c$$

are presented in Subsection 3.2. Sign properties of those solutions are discussed in Subsection 3.3.

In Section 4, the results of Section 3 are concretized for a special case of the equation (0.2), for the equation with deviating arguments

$$u'(t) = \sum_{k=1}^{m}\Big(p_k(t)u(\tau_k(t)) - g_k(t)u(\mu_k(t))\Big) + q(t),$$

where $p_k, g_k \in L([a,+\infty[\,;R_+)$ and $\tau_k, \mu_k : [a,+\infty[\,\to [a,+\infty[$ are measurable functions.

Chapter 2 is devoted to the nonlinear equation (0.1) with a boundary condition

$$\omega(u) = h(u).$$

Here $h : C_{loc}([a,+\infty[\,;R) \to R$ is (in general) a nonlinear functional. In Section 7 we establish so–called principle of existence of a bounded solution, which is our main technical tool to investigate the nonlinear problem. Optimal sufficient conditions for existence and uniqueness of a bounded solution to (0.1) are established in Section 8. In Section 10 we concretize results obtained in Section 8 for the equation with deviating arguments of the form

$$u'(t) = \sum_{k=1}^{m} \Big(p_k(t)u(\tau_k(t)) - g_k(t)u(\mu_k(t)) \Big) + \\ + f\big(t, u(t), u(\nu_1(t)), \dots, u(\nu_n(t))\big),$$

where $p_k, g_k \in L([a,+\infty[\,;R_+)$, $\tau_k, \mu_k : [a,+\infty[\,\to [a,+\infty[$ are measurable functions, and $f : [a,+\infty[\,\times R^{n+1} \to R$ is a function satisfying local Carathéodory conditions.

Notation

N is a set of all natural numbers.

R is a set of all real numbers, $R_+ = [0, +\infty[$.

$C([a,b]; R)$ is a Banach space of continuous functions $u : [a,b] \to R$ with a norm $\|u\|_C = \max\{|u(t)| : a \le t \le b\}$.

$C([a,b]; R_+) = \{u \in C([a,b]; R) : u(t) \ge 0 \text{ for } t \in [a,b]\}$.

$C_{loc}([a, +\infty[; \mathcal{D})$, with $\mathcal{D} \subseteq R$, is a set of continuous functions $u : [a, +\infty[\to \mathcal{D}$ with a topology of uniform convergence on every compact subinterval of $[a, +\infty[$.

If $u \in C_{loc}([a, +\infty[; R)$, then

$$\|u\| = \sup\big\{|u(t)| : t \ge a\big\}.$$

$C_0([a, +\infty[; R)$ is a set of functions $u \in C_{loc}([a, +\infty[; R)$ (with a topology of uniform convergence on every compact subinterval of $[a, +\infty[$), for each of which there exists a finite limit

$$u(+\infty) \overset{def}{=} \lim_{t \to +\infty} u(t).$$

$\widetilde{C}([a,b]; \mathcal{D})$, where $\mathcal{D} \subseteq R$, is a set of absolutely continuous functions $u : [a,b] \to \mathcal{D}$.

$\widetilde{C}_{loc}([a, +\infty[; \mathcal{D})$, where $\mathcal{D} \subseteq R$, is a set of functions $u : [a, +\infty[\to \mathcal{D}$, absolutely continuous on every compact subinterval of $[a, +\infty[$.

$\widetilde{C}_0([a, +\infty[; \mathcal{D}) = \widetilde{C}_{loc}([a, +\infty[; \mathcal{D}) \cap C_0([a, +\infty[; R)$, where $\mathcal{D} \subseteq R$.

$L([a,b]; R)$ is a Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with a norm

$$\|p\|_L = \int_a^b |p(s)|\, ds.$$

$L([a,b]; R_+) = \big\{p \in L([a,b]; R) : p(t) \ge 0 \text{ for almost all } t \in [a,b]\big\}$.

$L([a,+\infty[\,;\mathcal{D})$, where $\mathcal{D} \subseteq R$, is a set of Lebesgue integrable functions $p : [a,+\infty[\,\to \mathcal{D}$.

$L_{loc}([a,+\infty[\,;\mathcal{D})$, where $\mathcal{D} \subseteq R$, is a set of locally Lebesgue integrable functions $p : [a,+\infty[\,\to \mathcal{D}$ with a topology of convergence in a mean on every compact subinterval of $[a,+\infty[\,$.

ch is a set of nontrivial linear bounded functionals $\omega : C_{loc}([a,+\infty[\,;R) \to R$.

$\mathcal{W}_0$ is a set of nontrivial linear bounded functionals $\omega : C_0([a,+\infty[\,;R) \to R$.

$\mathcal{H}$ is a set of continuous functionals $h : C_{loc}([a,+\infty[\,;R) \to R$ with the following property: for every $r > 0$ there exists $M_r \in R_+$ such that

$$|h(v)| \le M_r \quad \text{for} \quad \|v\| \le r.$$

$\mathcal{L}_{ab}$ is a set of linear bounded operators $\ell : C([a,b];R) \to L([a,b];R)$, for each of them there exists $\eta \in L([a,b];R_+)$ such that

$$|\ell(v)(t)| \le \eta(t)\|v\|_C \quad \text{for almost all} \quad t \in [a,b], \quad v \in C([a,b];R).$$

$\mathcal{P}_{ab}$ is a set of operators $\ell \in \mathcal{L}_{ab}$ transforming a set $C([a,b];R_+)$ into a set $L([a,b];R_+)$.

$\mathcal{M}_{ab}$ is a set of measurable functions $\tau : [a,b] \to [a,b]$.

$\mathcal{K}_{ab}$ is a set of continuous operators $F : C([a,b];R) \to L([a,b];R)$ satisfying Carathéodory conditions, i.e., for every $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$|F(v)(t)| \le q_r(t) \quad \text{for almost all} \quad t \in [a,b], \quad \|v\|_C \le r.$$

$\widetilde{\mathcal{P}}$ is a set of linear operators $\ell : C_{loc}([a,+\infty[\,;R) \to L_{loc}([a,+\infty[\,;R)$ which are continuous, transform a set $C_{loc}([a,+\infty[\,;R_+)$ into a set $L_{loc}([a,+\infty[\,;R_+)$, and such that $\ell(1) \in L([a,+\infty[\,;R_+)$.

$\widetilde{\mathcal{L}}$ is a set of linear operators $\ell : C_{loc}([a,+\infty[\,;R) \to L_{loc}([a,+\infty[\,;R)$ which are continuous and for which there exists an operator $\overline{\ell} \in \widetilde{\mathcal{P}}$ such that

$$|\ell(v)(t)| \le \overline{\ell}(|v|)(t) \quad \text{for almost all} \quad t \in [a,+\infty[\,, \quad v \in C_{loc}([a,+\infty[\,;R).$$

$\mathcal{M}$ is a set of locally measurable functions $\tau : [a,+\infty[\,\to [a,+\infty[\,$.

$\mathcal{K}$ is a set of continuous operators $F : C_{loc}([a,+\infty[\,;R) \to L_{loc}([a,+\infty[\,;R)$ satisfying local Carathéodory conditions, i.e., for every $r > 0$ there exists $q_r \in L_{loc}([a,+\infty[\,;R_+)$ such that

$$|F(v)(t)| \le q_r(t) \quad \text{for almost all} \quad t \in [a,+\infty[\,, \quad \|v\| \le r.$$

$K_{loc}\big([a,+\infty[\,\times A;B\big)$, where $A \subseteq R^n$ $(n \in N)$, $B \subseteq R$, is a set of functions $f : [a,+\infty[\,\times A \to B$ satisfying local Carathéodory conditions, i.e., for all $x \in A$, the function $f(\cdot,x) : [a,+\infty[\,\to B$ is measurable on every compact subinterval of $[a,+\infty[$, $f(t,\cdot): A \to B$ is a continuous function for almost all $t \in [a,+\infty[$, and for every $r > 0$ there exists $q_r \in L_{loc}([a,+\infty[;R_+)$ such that

$$|f(t,x)| \le q_r(t) \;\text{ for almost all }\; t \in [a,+\infty[,\;\; x \in A,\;\; \|x\| \le r.$$

$K\big([a,+\infty[\times A;B\big)$, where $A \subseteq R^n$ $(n \in N)$, $B \subseteq R$, is a set of functions $f: [a,+\infty[\times A \to B$ satisfying Carathéodory conditions, i.e., for all $x \in A$, the function $f(\cdot,x): [a,+\infty[\to B$ is measurable on every compact subinterval of $[a,+\infty[$, $f(t,\cdot): A \to B$ is a continuous function for almost all $t \in [a,+\infty[$, and for every $r > 0$ there exists $q_r \in L([a,+\infty[;R_+)$ such that

$$|f(t,x)| \le q_r(t) \;\text{ for almost all }\; t \in [a,+\infty[,\;\; x \in A,\;\; \|x\| \le r.$$

χ_{ab} is a characteristic function of the interval $[a,b]$, i.e.,

$$\chi_{ab}(t) = \begin{cases} 1 & \text{for } t \in [a,b], \\ 0 & \text{for } t \notin [a,b]. \end{cases}$$

$\theta_b : C_{loc}([a,+\infty[;R) \to C_0([a,+\infty[;R)$, where $b \in\,]a,+\infty[$, is an operator defined by

$$\theta_b(u)(t) = \chi_{ab}(t)u(t) + (1-\chi_{ab}(t))u(b) \;\text{ for }\; t \in [a,+\infty[.$$

If $q \in L_{loc}([a,+\infty[;R)$ and $b \in\,]a,+\infty[$, then

$$\widetilde{q}_b(t) = \chi_{ab}(t)q(t) \;\text{ for almost all }\; t \in [a,+\infty[.$$

If $\ell \in \widetilde{\mathcal{L}}$ and $b \in\,]a,+\infty[$, then

$$\widetilde{\ell}_b(u)(t) = \chi_{ab}(t)\ell(\theta_b(u))(t) \;\text{ for almost all }\; t \in [a,+\infty[.$$

If $F \in \mathcal{K}$ and $b \in\,]a,+\infty[$, then

$$\widetilde{F}_b(u)(t) = \chi_{ab}(t)F(\theta_b(u))(t) \;\text{ for almost all }\; t \in [a,+\infty[.$$

If $\omega \in \mathrm{ch}$, resp. $\omega \in \mathcal{W}_0$, and $b \in\,]a,+\infty[$, then

$$\widetilde{\omega}_b(u) = \omega(\theta_b(u)).$$

If $h \in \mathcal{H}$ and $b \in\,]a,+\infty[$, then

$$\widetilde{h}_b(u) = h(\theta_b(u)).$$

We will say that $\ell \in \mathcal{L}_{ab}$ is a t_0-Volterra operator, where $t_0 \in [a,b]$, if for arbitrary $a_1 \in [a,t_0]$, $b_1 \in [t_0,b]$, $a_1 \ne b_1$, and $v \in C([a,b];R)$, satisfying the equality

$$v(t) = 0 \;\text{ for }\; t \in [a_1,b_1],$$

we have

$$\ell(v)(t) = 0 \text{ for almost all } t \in [a_1, b_1].$$

We will say that $\ell \in \widetilde{\mathcal{L}}$ is an $a-$Volterra operator if for arbitrary $b \in [a, +\infty[$ and $v \in C_{loc}([a, +\infty[\,; R)$, satisfying the equality

$$v(t) = 0 \text{ for } t \in [a, b],$$

we have

$$\ell(v)(t) = 0 \text{ for almost all } t \in [a, b].$$

We will say that $\ell \in \widetilde{\mathcal{L}}$ is an anti–Volterra operator, if for arbitrary $b \in [a, +\infty[$ and $v \in C_{loc}([a, +\infty[\,; R)$, satisfying the equality

$$v(t) = 0 \text{ for } t \in [b, +\infty[\,,$$

we have

$$\ell(v)(t) = 0 \text{ for almost all } t \in [b, +\infty[\,.$$

If $x \in R$, then

$$[x]_+ = \frac{1}{2}\,(|x| + x), \quad [x]_- = \frac{1}{2}\,(|x| - x).$$

The uniform convergence in $[a, +\infty[$ is meant as a uniform convergence on every compact subinterval of $[a, +\infty[\,$.

The equalities and inequalities between measurable functions are understood almost everywhere in an appropriate interval.

CHAPTER 1

Linear Problem

1. Statement of the Problem

In this chapter, we will consider the problem on the existence, uniqueness, and nonnegativity of a bounded solution to the equation

$$u'(t) = \ell(u)(t) + q(t) \tag{1.1}$$

satisfying the condition

$$\omega(u) = c. \tag{1.2}$$

Here $\ell \in \widetilde{\mathcal{L}}$, $q \in L_{loc}([a,+\infty[\,;R)$, $\omega \in$ ch, resp. $\omega \in \mathcal{W}_0$, and $c \in R$.

By a solution to (1.1) we understand a function $u \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ satisfying the equality (1.1) almost everywhere in $[a,+\infty[$. By a solution to the problem (1.1), (1.2) we understand a solution to (1.1) which belongs to the domain of ω and satisfies (1.2).

Along with the problem (1.1), (1.2) we will consider the corresponding homogeneous problem

$$u'(t) = \ell(u)(t), \tag{1.1$_0$}$$

$$\omega(u) = 0. \tag{1.2$_0$}$$

Note that the particular cases of conditions (1.2), resp. (1.2$_0$), are the following conditions:

$$u(a) = c, \tag{1.3}$$

$$u(+\infty) = c, \tag{1.4}$$

$$u(a) - u(+\infty) = c, \tag{1.5}$$

resp.

$$u(a) = 0, \tag{1.3$_0$}$$

$$u(+\infty) = 0, \tag{1.4$_0$}$$

$$u(a) - u(+\infty) = 0. \tag{1.5$_0$}$$

Let $b > a$ be arbitrary but fixed. Define the operators $\varphi : C([a,b];R) \to C_0([a,+\infty[\,;R)$ and $\psi : L([a,+\infty[\,;R) \to L([a,b];R)$ by

$$\varphi(v)(t) \overset{def}{=} v(\nu(t)) \quad \text{for } t \in [a,+\infty[\,, \tag{1.6}$$

$$\psi(u)(t) \overset{def}{=} \frac{b-a}{(b-t)^2}\, u(\nu^{-1}(t)) \quad \text{for } t \in [a,b[\,, \tag{1.7}$$

where

$$\nu(t) = b - \frac{b-a}{1+t-a} \quad \text{for } t \in [a, +\infty[\,, \tag{1.8}$$

and ν^{-1} is the inverse function to ν, i.e.,

$$\nu^{-1}(t) = a + \frac{b-a}{b-t} - 1 \quad \text{for } t \in [a, b[\,.$$

Introduce the operator $\widehat{\ell} : C([a,b];R) \to L([a,b];R)$ by

$$\widehat{\ell}(v)(t) \stackrel{def}{=} \psi\big(\ell(\varphi(v))\big)(t) \quad \text{for } t \in [a,b]. \tag{1.9}$$

On account of the assumption $\ell \in \widetilde{\mathcal{L}}$ it is clear that the operator $\widehat{\ell}$ is well defined. Moreover, it can be easily verified that $\widehat{\ell}$ is a linear bounded operator. Analogously, the functional $\widehat{\omega} : C([a,b];R) \to R$, defined by

$$\widehat{\omega}(v) \stackrel{def}{=} \omega(\varphi(v)), \tag{1.10}$$

is a linear bounded functional (in both cases, when $\omega \in \mathrm{ch}$ and $\omega \in \mathcal{W}_0$).

Now assume that

$$q \in L([a, +\infty[\,; R), \tag{1.11}$$

$$\widehat{q} \stackrel{def}{=} \psi(q), \tag{1.12}$$

and consider the problem (on the interval $[a,b]$)

$$v'(t) = \widehat{\ell}(v)(t) + \widehat{q}(t), \quad \widehat{\omega}(v) = c \tag{1.13}$$

and the corresponding homogeneous problem

$$v'(t) = \widehat{\ell}(v)(t), \quad \widehat{\omega}(v) = 0. \tag{1.13$_0$}$$

By a direct calculation, it is easy to verify that if v is a solution* to the problem (1.13) then $u \stackrel{def}{=} \varphi(v)$ is a bounded solution to (1.1), (1.2), and vice versa, if u is a bounded solution to the problem (1.1), (1.2), then $u \in \widetilde{C}_0([a, +\infty[\,; R)$, and the function v defined by

$$v \stackrel{def}{=} \varphi^{-1}(u), \quad v(b) \stackrel{def}{=} u(+\infty)$$

is a solution to (1.13). Therefore, the following proposition holds:

Proposition 1.1. *Let* (1.11) *be fulfilled. Then the problem* (1.1), (1.2) *has a unique bounded solution u if and only if the problem* (1.13) *has a unique solution v. Moreover, $u \in C_0([a, +\infty[\,; R)$, $\varphi(v) \equiv u$, and $v(b) = u(+\infty)$.*

A particular case of Proposition 1.1 is the following

Proposition 1.2. *The only bounded solution of the problem* (1.1_0), (1.2_0) *is a trivial solution if and only if the problem* (1.13_0) *has only a trivial solution.*

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*Solutions are understood in the sense of Carathéodory, i.e., as absolutely continuous functions which satisfy the differential equality almost everywhere in $[a,b]$.

Furthermore, the following assertion is well-known from the general theory of boundary value problems for functional differential equations (see, e.g., [**1**, **2**, **10**, **15**, **19**]).

Proposition 1.3. *The problem* (1.13) *is uniquely solvable if and only if the corresponding homogeneous problem* (1.13_0) *has only a trivial solution.*

Consequently, from Propositions 1.1–1.3, it immediately follows

Proposition 1.4. *Let* (1.11) *be fulfilled. Then the problem* (1.1), (1.2) *has a unique bounded solution if and only if the only bounded solution of the homogeneous problem* $(1.1_0), (1.2_0)$ *is a trivial solution.*

Therefore, the question on the existence and uniqueness of a bounded solution to (1.1), (1.2) is equivalent (under the assumption (1.11)) to the question on the unique solvability of the homogeneous boundary value problem (1.13_0) (on a finite interval $[a, b]$).

Remark 1.1. It follows from the Riesz–Schauder theory that if the problem (1.13_0) has a nontrivial solution, then for every $c \in R$ there exists $\widehat{q} \in L([a, b]; R)$ such that the problem (1.13) has no solution.

Below we will study the problem (1.1), (1.2) under the less than (1.11) restricted condition, when

$$\sup\left\{\Big|\int\limits_a^t q(s)\,ds\Big| :\ t \geq a\right\} < +\infty \tag{1.14}$$

is fulfilled.

The chapter is organized as follows: Main results are presented in Section 3. First, the analogy of Proposition 1.4 is proved in Subsection 3.1 (see Theorems 3.1 and 3.2). In Subsection 3.2, sufficient conditions for the existence and uniqueness of a bounded solution to the equation (1.1) satisfying one of the conditions (1.3), (1.4), or (1.5) are established. The question on sign constant solutions to the problems (1.1), $(1.k)$ $(k = 3, 4, 5)$ is discussed in Subsection 3.3. In Section 4 we concretize results of Section 3 for particular cases of the equation (1.1) – for the equations with deviating arguments:

$$u'(t) = \sum_{k=1}^{m} p_k(t)u(\tau_k(t)) + q(t), \tag{1.15}$$

$$u'(t) = -\sum_{k=1}^{m} g_k(t)u(\mu_k(t)) + q(t), \tag{1.16}$$

and

$$u'(t) = \sum_{k=1}^{m} \Big(p_k(t)u(\tau_k(t)) - g_k(t)u(\mu_k(t))\Big) + q(t), \tag{1.17}$$

where $p_k, g_k \in L([a,+\infty[\,;R_+)$, $\tau_k, \mu_k \in \mathcal{M}$ $(k=1,\dots,m)$. Last section of the chapter – Section 5 – is devoted to the examples verifying the optimality of obtained results.

Auxiliary propositions contained in Section 2 play a crucial role in proving the main results. Namely Lemmas 2.1 and 2.2 (see p. 12 and p. 13) state that the unique bounded solution to (1.1), resp. (1.1_0), is a uniform limit of a suitable sequence $\{u_b\}_{b>a}$ of solutions to the problem

$$u'(t) = \widetilde{\ell}_b(u)(t) + \widetilde{q}_b(t), \tag{1.18}$$

$$\widetilde{\omega}_b(u) = c, \tag{1.19}$$

resp.

$$u'(t) = \widetilde{\ell}_b(u)(t), \tag{1.18_0}$$

$$\widetilde{\omega}_b(u) = 0 \tag{1.19_0}$$

(for the definition of $\widetilde{\ell}_b$, $\widetilde{\omega}_b$, and $\widetilde{q}_b$ see p. 7).

As it was mentioned above, we suppose that $\ell \in \widetilde{\mathcal{L}}$ and the condition (1.14) is fulfilled. If $\ell(v)(t) \overset{def}{=} p(t)v(t)$, i.e., if the equation (1.1) is of the form

$$u'(t) = p(t)u(t) + q(t), \tag{1.20}$$

the assumption $\ell \in \widetilde{\mathcal{L}}$ means that

$$p \in L([a,+\infty[\,;R). \tag{1.21}$$

The equation (1.20) is a suitable type of the equation (1.1) to verify that both conditions $\ell \in \widetilde{\mathcal{L}}$ (i.e., (1.21)) and (1.14) are essential for boundedness of its solutions.

2. Auxiliary Propositions

2.1. Lemma on Existence of a Bounded Solution.

Lemma 2.1. *Let the condition* (1.14) *be satisfied and let there exist* $\rho_0 > 0$ *and* $b_0 \in\,]a,+\infty[$ *, such that for every* $b \geq b_0$ *the equation* (1.18) *has a solution* u_b *satisfying the inequality*

$$\|u_b\| \leq \rho_0. \tag{2.1}$$

Then the equation (1.1) *has at least one bounded solution* u*. Moreover, there exists a sequence* $\{u_{b_n}\}_{n=1}^{+\infty} \subset \{u_b\}_{b\geq b_0}$ *such that*

$$\lim_{n\to+\infty} u_{b_n}(t) = u(t) \ \text{ uniformly in } \ [a,+\infty[\,. \tag{2.2}$$

Proof. In view of (2.1) and the condition $\ell \in \widetilde{\mathcal{L}}$, from (1.18) with $u = u_b$ we get

$$|u_b(t) - u_b(s)| \leq \rho_0 \int_s^t \overline{\ell}(1)(\xi)\,d\xi + \int_s^t |q(\xi)|\,d\xi \ \text{ for } \ a \leq s \leq t.$$

Hence, the set $\{u_b\}_{b\geq b_0}$ is uniformly bounded and equicontinuous on every compact subinterval of $[a,+\infty[$. Therefore, according to Arzelà–Ascoli lemma, there exist a sequence $\{u_{b_n}\}_{n=1}^{+\infty} \subset \{u_b\}_{b\geq b_0}$ and a function $u \in C_{loc}([a,+\infty[\,;R)$ such that $\lim\limits_{n\to+\infty} b_n = +\infty$ and (2.2) is fulfilled.

Obviously,

$$\theta_{b_n}(u_{b_n})(t) = u_{b_n}(t) \ \text{ for } \ t \geq a, \ \ n \in N,$$

and therefore the integration of (1.18) (with $u = u_{b_n}$) from a to t yields

$$u_{b_n}(t) = u_{b_n}(a) + \int\limits_a^t \ell(u_{b_n})(\xi)\,d\xi + \int\limits_a^t q(\xi)\,d\xi \ \text{ for } \ t \in [a,b_n], \ \ n \in N.$$

Consequently, with respect to (2.2) and the assumption $\ell \in \widetilde{\mathcal{L}}$,

$$u(t) = u(a) + \int\limits_a^t \ell(u)(\xi)\,d\xi + \int\limits_a^t q(\xi)d\xi \ \text{ for } \ t \geq a,$$

i.e., $u \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ and it is a solution to the equation (1.1). Moreover, from (2.1) and (2.2) we have $\|u\| \leq \rho_0$. □

2.2. Lemmas on A Priori Estimates.

Lemma 2.2. *Let the only bounded solution to the problem* $(1.1_0), (1.2_0)$ *be a trivial solution. Let, moreover, the condition* (1.14) *be fulfilled. Then there exist* $b_0 \in\,]a,+\infty[$ *and* $r_0 > 0$ *such that for every* $b \geq b_0$*, the problem* (1.18), (1.19) *has a unique solution* u_b*, and this solution admits the estimate*

$$\|u_b\| \leq r_0(|c| + q^*),$$

where

$$q^* = \sup\left\{ \Big| \int\limits_a^t q(s)\,ds \Big| : \ t \geq a \right\}. \tag{2.3}$$

To prove this lemma we need some auxiliary propositions.

Proposition 2.1. *Let the problem* (1.13_0) *have only a trivial solution. Then there exists* $r_0 > 0$ *such that for every* $\widehat{q} \in L([a,b];R)$ *and* $c \in R$*, the solution*† v *to the problem* (1.13) *admits the estimate*

$$\|v\|_C \leq r_0\left(|c| + \sup\left\{ \Big| \int\limits_a^t \widehat{q}\,(s)\,ds \Big| : \ t \in [a,b] \right\} \right). \tag{2.4}$$

Proof. Let

$$R \times L([a,b];R) = \big\{ (c,\widehat{q}\,) : \ c \in R, \ \widehat{q} \in L([a,b];R) \big\}$$

†The existence and uniqueness of such a solution is guaranteed by Proposition 1.3.

be a linear space with the norm

$$\|(c,\widehat{q}\,)\|_{R\times L} = |c| + \sup\left\{\Big|\int\limits_a^t \widehat{q}\,(s)\,ds\Big| :\ t\in[a,b]\right\},$$

and let Ω be an operator, which assigns to every $(c,\widehat{q}\,)\in R\times L([a,b];R)$ the solution v to the problem (1.13). According to Proposition 1.3, the operator Ω is defined correctly. Moreover, according to Theorem 1.4 in [**15**], $\Omega : R\times L([a,b];R)\to C([a,b];R)$ is a linear bounded operator (see also [**10**, Theorem 3.2]). Denote by r_0 the norm of Ω. Then, clearly, for any $(c,\widehat{q}\,)\in R\times L([a,b];R)$, the inequality

$$\|\Omega(c,\widehat{q}\,)\|_C \le r_0\|(c,\widehat{q}\,)\|_{R\times L}$$

holds. Consequently, the solution $v=\Omega(c,\widehat{q}\,)$ to the problem (1.13) admits the estimate (2.4). □

From Proposition 2.1 it immediately follows

Proposition 2.2. *Let $b\in\,]a,+\infty[$ be such that the problem* (1.18_0), (1.19_0) *has only a trivial solution. Then there exists $r_b>0$ such that for every $q\in L_{loc}([a,+\infty[\,;R)$ satisfying* (1.14) *and $c\in R$ the problem* (1.18), (1.19) *has a unique solution u_b, and this solution admits the estimate*

$$\|u_b\| \le r_b(|c|+q^*), \tag{2.5}$$

where q^* is defined by (2.3).

Proposition 2.3. *Let the only bounded solution to the problem* (1.1_0), (1.2_0) *be a trivial solution. Then there exists $b_0\in\,]a,+\infty[$ such that for every $b\ge b_0$, the problem* (1.18_0), (1.19_0) *has only a trivial solution.*

Proof. Assume on the contrary that there exists an increasing sequence $\{b_k\}_{k=1}^{+\infty}$, $\lim\limits_{k\to+\infty} b_k=+\infty$, such that, for every $k\in N$, the problem

$$u'(t)=\widetilde{\ell}_{b_k}(u)(t), \quad \widetilde{\omega}_{b_k}(u)=0 \tag{2.6}$$

has a nontrivial solution u_k. Obviously,

$$u_k\equiv\theta_{b_k}(u_k) \tag{2.7}$$

and, without loss of generality, we can assume that

$$\|u_k\|=1 \ \text{ for } \ k\in N. \tag{2.8}$$

Furthermore, according to (2.6)–(2.8), and the assumption $\ell\in\widetilde{\mathcal{L}}$, we have

$$|u_k(t)-u_k(s)| \le \int\limits_s^t |u_k'(\xi)|\,d\xi = \int\limits_s^t |\widetilde{\ell}_{b_k}(u_k)(\xi)|\,d\xi \le$$

$$\le \int\limits_s^t |\ell(u_k)(\xi)|\,d\xi \le \int\limits_s^t \overline{\ell}(1)(\xi)\,d\xi \ \text{ for } \ a\le s\le t, \ \ k\in N.$$

Consequently, the sequence of functions $\{u_k\}_{k=1}^{+\infty}$ is uniformly bounded and equicontinuous on every compact subinterval of $[a,+\infty[$. According to Arzelà–Ascoli lemma we can assume, without loss of generality, that there exists $u_0 \in C_{loc}([a,+\infty[\,;R)$ such that

$$\lim_{k\to+\infty} u_k(t) = u_0(t) \;\text{ uniformly in }\; [a,+\infty[\,. \tag{2.9}$$

Moreover, on account of (2.8) we have

$$\|u_0\| \le 1. \tag{2.10}$$

On the other hand, since u_k $(k\in N)$ are solutions to (2.6), we obtain

$$u_k(t) = u_k(a) + \int_a^t \widetilde{\ell}_{b_k}(u_k)(s)\,ds \;\text{ for }\; t\ge a, \;\; k\in N, \tag{2.11}$$

$$\widetilde{\omega}_{b_k}(u_k) = 0, \;\; k\in N. \tag{2.12}$$

From (2.11), in view of (2.9), we get

$$u_0(t) = u_0(a) + \int_a^t \ell(u_0)(s)\,ds \;\text{ for }\; t\ge a. \tag{2.13}$$

Thus $u_0 \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ and

$$|u_0(t)-u_0(s)| = \left|\int_s^t \ell(u_0)(\xi)\,d\xi\right| \le \|u_0\| \int_s^t \overline{\ell}(1)(\xi)\,d\xi \;\text{ for }\; a\le s\le t.$$

The last inequality (together with (2.10)) and the fact that $\overline{\ell}\in\widetilde{\mathcal{P}}$, ensures that there exists a finite limit $u_0(+\infty)$. Consequently, from (2.12), in view of (2.9), we obtain

$$\omega(u_0) = 0. \tag{2.14}$$

Now (2.13) and (2.14) imply that u_0 is a bounded solution to the problem $(1.1_0), (1.2_0)$. Therefore,

$$u_0 \equiv 0. \tag{2.15}$$

Since $\ell\in\widetilde{\mathcal{L}}$, we can choose $b_*\in\,]a,+\infty[$ such that

$$\int_{b_*}^{+\infty} \overline{\ell}(1)(s)\,ds \le \frac{1}{3}\,. \tag{2.16}$$

According to (2.9) and (2.15) there exists $k_0\in N$ such that

$$|u_k(t)| \le \frac{1}{3} \;\text{ for }\; t\in[a,b_*], \;\; k\ge k_0. \tag{2.17}$$

On the other hand, from (2.11), in view of (2.8), we have

$$|u_k(t)-u_k(b_*)| \le \int_{b_*}^t |\widetilde{\ell}_{b_k}(u_k)(s)|\,ds \le \int_{b_*}^{+\infty} \overline{\ell}(1)(s)\,ds \;\text{ for }\; t\ge b_*, \;\; k\in N.$$

Hence, on account of (2.16) and (2.17), we obtain

$$|u_k(t)| \le \frac{2}{3} \quad \text{for} \quad t \ge a, \quad k \ge k_0,$$

which contradicts (2.8). $\square$

Proof of Lemma 2.2. Since the only bounded solution to the problem $(1.1_0), (1.2_0)$ is a trivial solution, then according to Proposition 2.3 there exists $b_* \in \,]a, +\infty[$ such that for every $b \ge b_*$ the problem $(1.18_0), (1.19_0)$ has only a trivial solution. Consequently, according to Proposition 2.2 there exists $r_b > 0$ such that the solution u_b to (1.18), (1.19) admits the estimate (2.5).

If $|c|+q^* = 0$ then the conclusion of the lemma is evident. Let, therefore, $|c|+q^* \ne 0$ and assume on the contrary that there exists a sequence $\{b_n\}_{n=1}^{+\infty}$ such that $b_n \ge b_*$ for $n \in N$, $\lim\limits_{n\to+\infty} b_n = +\infty$, and

$$\|u_{b_n}\| > n(|c| + q^*) \quad \text{for} \quad n \in N. \tag{2.18}$$

Put

$$v_n(t) = \frac{u_{b_n}(t)}{\|u_{b_n}\|} \quad \text{for} \quad t \ge a, \quad n \in N. \tag{2.19}$$

Then

$$\|v_n\| = 1 \quad \text{for} \quad n \in N \tag{2.20}$$

and, in view of (1.18), (1.19), and (2.19), we have

$$v_n'(t) = \widetilde{\ell}_{b_n}(v_n)(t) + \frac{\widetilde{q}_{b_n}(t)}{\|u_{b_n}\|} \quad \text{for} \quad t \ge a, \quad n \in N, \tag{2.21}$$

$$\widetilde{\omega}_{b_n}(v_n) = \frac{c}{\|u_{b_n}\|} \quad \text{for} \quad n \in N. \tag{2.22}$$

Obviously, $\theta_{b_n}(v_n) \equiv v_n$. From (2.21), on account of (2.18), (2.20), and the assumption $\ell \in \widetilde{\mathcal{L}}$, we obtain

$$|v_n(t) - v_n(s)| \le \int_s^t |v_n'(\xi)|\, d\xi \le$$

$$\le \int_s^t \overline{\ell}(1)(\xi)\, d\xi + \frac{1}{|c| + q^*} \int_s^t |q(\xi)|\, d\xi \quad \text{for} \quad a \le s \le t, \quad n \in N. \tag{2.23}$$

From (2.20) and (2.23) it follows that the sequence of functions $\{v_n\}_{n=1}^{+\infty}$ is uniformly bounded and equicontinuous on every compact subinterval of $[a, +\infty[$. According to Arzelà–Ascoli lemma we can assume, without loss of generality, that there exists $v_0 \in C_{loc}([a, +\infty[\,; R)$ such that

$$\lim_{k\to+\infty} v_k(t) = v_0(t) \quad \text{uniformly in} \quad [a, +\infty[\,. \tag{2.24}$$

Moreover, on account of (2.20), we have

$$\|v_0\| \le 1. \tag{2.25}$$

On the other hand, the integration of (2.21) from a to t yields

$$v_n(t) = v_n(a) + \int\limits_a^t \widetilde{\ell}_{b_n}(v_n)(s)\,ds + \frac{1}{\|u_{b_n}\|}\int\limits_a^t \widetilde{q}_{b_n}(s)\,ds \ \text{ for } \ n \in N.$$

Hence, in view of (2.18) and (2.24), we get

$$v_0(t) = v_0(a) + \int\limits_a^t \ell(v_0)(s)\,ds \ \text{ for } \ t \geq a. \tag{2.26}$$

Thus $v_0 \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ and

$$|v_0(t) - v_0(s)| = \left|\int\limits_s^t \ell(v_0)(\xi)\,d\xi\right| \leq \|v_0\|\int\limits_s^t \overline{\ell}(1)(\xi)\,d\xi \ \text{ for } \ a \leq s \leq t.$$

The last inequality, together with (2.25) and the fact that $\overline{\ell} \in \widetilde{\mathcal{P}}$, ensures that there exists a finite limit $v_0(+\infty)$. Consequently, from (2.18), (2.22), and (2.24) we also obtain

$$\omega(v_0) = 0. \tag{2.27}$$

Now (2.26) and (2.27) imply that v_0 is a bounded solution to the problem $(1.1_0), (1.2_0)$. Therefore,

$$v_0 \equiv 0. \tag{2.28}$$

Since $\ell \in \widetilde{\mathcal{L}}$, we can choose $a_0 \in \,]a,+\infty[$ such that

$$\int\limits_{a_0}^{+\infty} \overline{\ell}(1)(s)\,ds \leq \frac{1}{5}\,. \tag{2.29}$$

According to (2.24) and (2.28) there exists $n_0 \in N$ such that

$$|v_n(t)| \leq \frac{1}{5} \ \text{ for } \ t \in [a, a_0], \ \ n \geq n_0. \tag{2.30}$$

On the other hand, from (2.21), in view of (2.18) and (2.20), we have

$$|v_n(t) - v_n(a_0)| \leq$$
$$\leq \int\limits_{a_0}^t |\widetilde{\ell}_{b_n}(v_n)(s)|\,ds + \frac{2q^*}{\|u_{b_n}\|} \leq \int\limits_{a_0}^{+\infty} \overline{\ell}(1)(s)\,ds + \frac{2}{n} \ \text{ for } \ t \geq a_0.$$

Hence, on account of (2.29) and (2.30), we obtain

$$|v_n(t)| \leq \frac{4}{5} \ \text{ for } \ t \geq a, \ \ n \geq \max\{n_0, 5\}$$

which contradicts (2.20). $\square$

2.3. Boundary Value Problems on Finite Interval. The following assertions are results from [**9**], formulated in a suitable for us form.

Lemma 2.3 ([**9**, Theorem 4.4, p. 83])**.** *Let* $\widehat{\ell} = \widehat{\ell}_0 - \widehat{\ell}_1$ *with* $\widehat{\ell}_0, \widehat{\ell}_1 \in \mathcal{P}_{ab}$, *and let there exist a function* $\widehat{\gamma} \in \widetilde{C}([a,b];\,]0,+\infty[\,)$ *such that*

$$\widehat{\gamma}'(t) \geq \widehat{\ell}_0(\widehat{\gamma})(t) + \widehat{\ell}_1(1)(t) \quad \text{for } t \in [a,b],$$
$$\widehat{\gamma}(b) - \widehat{\gamma}(a) < 3.$$

Then the problem

$$v'(t) = \widehat{\ell}(v)(t), \quad v(a) = 0 \tag{2.31}$$

has only a trivial solution.

Lemma 2.4 ([**9**, Theorem 4.2, p. 82])**.** *Let* $\widehat{\ell} = \widehat{\ell}_0 - \widehat{\ell}_1$ *with* $\widehat{\ell}_0, \widehat{\ell}_1 \in \mathcal{P}_{ab}$, *and let*

$$\int_a^b \widehat{\ell}_0(1)(s)\,ds < 1,$$
$$\int_a^b \widehat{\ell}_1(1)(s)\,ds < 1 + 2\sqrt{1 - \int_a^b \widehat{\ell}_0(1)(s)\,ds}\,.$$

Then the problem (2.31) *has only a trivial solution.*

Lemma 2.5 ([**9**, Theorem 4.10, p. 86])**.** *Let* $\widehat{\ell} = \widehat{\ell}_0 - \widehat{\ell}_1$ *with* $\widehat{\ell}_0, \widehat{\ell}_1 \in \mathcal{P}_{ab}$, *and let there exist a function* $\widehat{\gamma} \in \widetilde{C}([a,b];\,]0,+\infty[\,)$ *such that*

$$-\widehat{\gamma}'(t) \geq \widehat{\ell}_1(\widehat{\gamma})(t) + \widehat{\ell}_0(1)(t) \quad \text{for } t \in [a,b],$$
$$\widehat{\gamma}(a) - \widehat{\gamma}(b) < 3.$$

Then the problem

$$v'(t) = \widehat{\ell}(v)(t), \quad v(b) = 0 \tag{2.32}$$

has only a trivial solution.

Lemma 2.6 ([**9**, Theorem 4.8, p. 86])**.** *Let* $\widehat{\ell} = \widehat{\ell}_0 - \widehat{\ell}_1$ *with* $\widehat{\ell}_0, \widehat{\ell}_1 \in \mathcal{P}_{ab}$, *and let*

$$\int_a^b \widehat{\ell}_1(1)(s)\,ds < 1,$$
$$\int_a^b \widehat{\ell}_0(1)(s)\,ds < 1 + 2\sqrt{1 - \int_a^b \widehat{\ell}_1(1)(s)\,ds}\,.$$

Then the problem (2.32) *has only a trivial solution.*

Lemma 2.7 ([**9**, Theorem 4.1, p. 80])**.** *Let $\widehat{\ell}=\widehat{\ell}_0-\widehat{\ell}_1$ with $\widehat{\ell}_0,\widehat{\ell}_1\in\mathcal{P}_{ab}$, and let either*

$$\int_a^b \widehat{\ell}_0(1)(s)\,ds<1,$$

$$\frac{\int_a^b \widehat{\ell}_0(1)(s)\,ds}{1-\int_a^b \widehat{\ell}_0(1)(s)\,ds}<\int_a^b \widehat{\ell}_1(1)(s)\,ds<2+2\sqrt{1-\int_a^b \widehat{\ell}_0(1)(s)\,ds}\,,$$

or

$$\int_a^b \widehat{\ell}_1(1)(s)\,ds<1,$$

$$\frac{\int_a^b \widehat{\ell}_1(1)(s)\,ds}{1-\int_a^b \widehat{\ell}_1(1)(s)\,ds}<\int_a^b \widehat{\ell}_0(1)(s)\,ds<2+2\sqrt{1-\int_a^b \widehat{\ell}_1(1)(s)\,ds}\,.$$

Then the problem

$$v'(t)=\widehat{\ell}(v)(t),\quad v(a)-v(b)=0$$

has only a trivial solution.

Lemma 2.8 ([**9**, Theorem 2.1, p. 17])**.** *Let $\widehat{\ell}\in\mathcal{P}_{ab}$ and let there exist $\widehat{\gamma}\in\widetilde{C}([a,b];\,]0,+\infty[\,)$ such that*

$$\widehat{\gamma}'(t)\geq\widehat{\ell}(\widehat{\gamma})(t)\ \ \textit{for}\ \ t\in[a,b].$$

Then every function $v\in\widetilde{C}([a,b];R)$, satisfying

$$v'(t)\geq\widehat{\ell}(v)(t)\ \ \textit{for}\ \ t\in[a,b],\ \ v(a)\geq 0, \tag{2.33}$$

is nonnegative.

Lemma 2.9 ([**9**, Theorem 2.5, p. 22])**.** *Let $\widehat{\ell}=\widehat{\ell}_0-\widehat{\ell}_1$ with $\widehat{\ell}_0,\widehat{\ell}_1\in\mathcal{P}_{ab}$, and let $\widehat{\ell}_1$ be an $a-$Volterra operator. Let, moreover,*

$$\int_a^b \widehat{\ell}_0(1)(s)\,ds<1,\quad \int_a^b \widehat{\ell}_1(1)(s)\,ds\leq 1.$$

Then every function $v\in\widetilde{C}([a,b];R)$ satisfying (2.33) *is nonnegative.*

Remark 2.1. Note that under the conditions of Lemma 2.8 or Lemma 2.9, the problem

$$v'(t)=\widehat{\ell}(v)(t)+\widehat{q}(t),\quad v(a)=c$$

is uniquely solvable for any $\widehat{q} \in L([a,b];R)$ and $c \in R$. Moreover, the solution is nonnegative whenever $\widehat{q} \in L([a,b];R_+)$ and $c \in R_+$.

Indeed, let the assumptions of Lemma 2.8 or Lemma 2.9 be fulfilled and let v be a solution to (2.31). Then, obviously, $-v$ is also a solution to (2.31), and so both v and $-v$ are nonnegative functions. Therefore, $v \equiv 0$. Thus the assertion follows from Proposition 1.3 and Lemmas 2.8 and 2.9.

Lemma 2.10 ([**9**, Theorem 2.12, p. 26])**.** *Let $-\widehat{\ell} \in \mathcal{P}_{ab}$ and let there exist $\widehat{\gamma} \in \widetilde{C}([a,b];\,]0,+\infty[\,)$ such that*

$$\widehat{\gamma}'(t) \le \widehat{\ell}(\widehat{\gamma})(t) \ \ \textit{for} \ \ t \in [a,b].$$

Then every function $v \in \widetilde{C}([a,b];R)$ satisfying

$$v'(t) \le \widehat{\ell}(v)(t) \ \ \textit{for} \ \ t \in [a,b], \quad v(b) \ge 0 \tag{2.34}$$

is nonnegative.

Lemma 2.11 ([**9**, Theorem 2.16, p. 28])**.** *Let $\widehat{\ell} = \widehat{\ell}_0 - \widehat{\ell}_1$ with $\widehat{\ell}_0, \widehat{\ell}_1 \in \mathcal{P}_{ab}$, and let $\widehat{\ell}_0$ be a $b-$Volterra operator. Let, moreover,*

$$\int_a^b \widehat{\ell}_0(1)(s)\,ds \le 1, \quad \int_a^b \widehat{\ell}_1(1)(s)\,ds < 1.$$

Then every function $v \in \widetilde{C}([a,b];R)$ satisfying (2.34) *is nonnegative.*

Remark 2.2. Analogously to Remark 2.1 it can be shown that under the conditions of Lemma 2.10 or Lemma 2.11, the problem

$$v'(t) = \widehat{\ell}(v)(t) + \widehat{q}(t), \quad v(b) = c$$

is uniquely solvable for any $\widehat{q} \in L([a,b];R)$ and $c \in R$. Moreover, the solution is nonnegative whenever $-\widehat{q} \in L([a,b];R_+)$ and $c \in R_+$.

Lemma 2.12 ([**9**, Theorem 2.4, p. 21])**.** *Let $\widehat{\ell} = \widehat{\ell}_0 - \widehat{\ell}_1$ with $\widehat{\ell}_0, \widehat{\ell}_1 \in \mathcal{P}_{ab}$, and let*

$$\int_a^b \widehat{\ell}_0(1)(s)\,ds < 1, \quad \frac{\int_a^b \widehat{\ell}_0(1)(s)\,ds}{1 - \int_a^b \widehat{\ell}_0(1)(s)\,ds} < \int_a^b \widehat{\ell}_1(1)(s)\,ds \le 1.$$

Then every function $v \in \widetilde{C}([a,b];R)$ satisfying

$$v'(t) \ge \widehat{\ell}(v)(t) \ \ \textit{for} \ \ t \in [a,b], \quad v(a) - v(b) \ge 0 \tag{2.35}$$

is nonnegative.

Lemma 2.13 ([**9**, Theorem 2.15, p. 28])**.** *Let $\widehat{\ell} = \widehat{\ell}_0 - \widehat{\ell}_1$ with $\widehat{\ell}_0, \widehat{\ell}_1 \in \mathcal{P}_{ab}$, and let*

$$\int_a^b \widehat{\ell}_1(1)(s)\,ds < 1, \qquad \frac{\int_a^b \widehat{\ell}_1(1)(s)\,ds}{1 - \int_a^b \widehat{\ell}_1(1)(s)\,ds} < \int_a^b \widehat{\ell}_0(1)(s)\,ds \le 1.$$

Then every function $v \in \widetilde{C}([a,b];R)$ satisfying (2.35) *is nonpositive.*

Remark 2.3. Analogously to Remark 2.1 it can be shown that under the conditions of Lemma 2.12 or Lemma 2.13, the problem

$$v'(t) = \widehat{\ell}(v)(t) + \widehat{q}(t), \quad v(a) - v(b) = c$$

is uniquely solvable for any $\widehat{q} \in L([a,b];R)$ and $c \in R$. Moreover, the solution is nonnegative, resp. nonpositive, whenever $\widehat{q} \in L([a,b];R_+)$ and $c \in R_+$.

2.4. Nonnegative Solutions to a Certain Differential Inequality.

Lemma 2.14. *Let $\ell \in \widetilde{\mathcal{P}}$ and let there exist $\gamma \in \widetilde{C}_{loc}([a,+\infty[\,;\,]0,+\infty[\,)$ such that*

$$\gamma'(t) \ge \ell(\gamma)(t) \ \text{ for } \ t \ge a, \tag{2.36}$$

$$\lim_{t\to+\infty} \gamma(t) = +\infty. \tag{2.37}$$

Then every bounded function $u \in \widetilde{C}_{loc}([a,+\infty[\,;R)$, satisfying

$$u'(t) \ge \ell(u)(t) \ \text{ for } \ t \ge a, \quad u(a) \ge 0, \tag{2.38}$$

is nonnegative.

Proof. Assume on the contrary that there exist a bounded function $u \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ and $t_0 > a$ such that u satisfies (2.38) and

$$u(t_0) < 0. \tag{2.39}$$

Put

$$\lambda = \sup\left\{ -\frac{u(t)}{\gamma(t)} :\ t \ge a \right\}. \tag{2.40}$$

Obviously, by virtue of (2.39),

$$\lambda > 0. \tag{2.41}$$

Moreover, in view of (2.37) and the assumption that u is a bounded function, there exists $t_1 \ge a$ such that

$$\lambda = -\frac{u(t_1)}{\gamma(t_1)}\,. \tag{2.42}$$

Further, put

$$w(t) = \lambda\gamma(t) + u(t) \ \text{ for } \ t \ge a.$$

Then, on account of (2.36), (2.38), (2.40)–(2.42), we have

$$w'(t) \geq \ell(w)(t) \ \text{ for } \ t \geq a, \tag{2.43}$$

$$w(t) \geq 0 \ \text{ for } \ t \geq a, \quad w(a) > 0, \tag{2.44}$$

$$w(t_1) = 0. \tag{2.45}$$

Now, due to (2.44) and the assumption $\ell \in \widetilde{\mathcal{P}}$, from (2.43) we get $w'(t) \geq 0$ for $t \geq a$. Consequently, $w(t) > 0$ for $t \geq a$, which contradicts (2.45). $\square$

Remark 2.4. Under the conditions of Lemma 2.14, the only bounded solution to the problem $(1.1_0), (1.3_0)$ is a trivial solution. Indeed, let u be a bounded solution to (1.1_0), (1.3_0). Then, obviously, $-u$ is also a bounded solution to (1.1_0), (1.3_0), and both u and $-u$ are nonnegative functions. Therefore, $u \equiv 0$.

3. Main Results

3.1. Necessary and Sufficient Conditions.

Theorem 3.1. *Let* $\omega \in \mathrm{ch}$ *and the condition* (1.14) *be fulfilled. Then the problem* (1.1), (1.2) *has a unique bounded solution if and only if the only bounded solution to the problem* $(1.1_0), (1.2_0)$ *is a trivial solution.*

Proof. Evidently, if the problem (1.1), (1.2) has a unique bounded solution for arbitrary $c \in R$ and $q \in L_{loc}([a, +\infty[\,; R)$ satisfying (1.14), then the only bounded solution to the problem $(1.1_0), (1.2_0)$ is a trivial solution.

If the only bounded solution to the problem $(1.1_0), (1.2_0)$ is a trivial solution, then, according to Lemmas 2.1 and 2.2, the equation (1.1) has at least one bounded solution u and there exists a sequence of functions $\{u_{b_n}\}_{n=1}^{+\infty} \subset \widetilde{C}_0([a, +\infty[\,; R)$ such that $\omega(u_{b_n}) = c$ $(n \in N)$, and (2.2) holds. Consequently, since $\omega \in \mathrm{ch}$, we also have $\omega(u) = c$, i.e., u is a bounded solution to the problem (1.1), (1.2). The uniqueness of u is evident. $\square$

Theorem 3.2. *Let* $\omega \in \mathcal{W}_0$ *and the condition* (1.14) *be satisfied. Let, furthermore, the only bounded solution to the problem* $(1.1_0), (1.2_0)$ *be a trivial solution. Then the equation* (1.1) *has at least one bounded solution. If, moreover, there exists a finite limit*

$$\lim_{t \to +\infty} \int_a^t q(s)\, ds, \tag{3.1}$$

then the problem (1.1), (1.2) *has a unique bounded solution.*

Proof. If the assumptions of theorem are fulfilled, then, according to Lemmas 2.1 and 2.2, the equation (1.1) has at least one bounded solution u. Furthermore, there exist a sequence of functions $\{u_{b_n}\}_{n=1}^{+\infty} \subset \widetilde{C}_0([a, +\infty[\,; R)$ such that $\omega(u_{b_n}) = c$ $(n \in N)$, and (2.2) holds. Thus, if there exists a finite limit (3.1), there also exists a finite limit $u(+\infty)$. Consequently,

$u \in C_0([a,+\infty[\,;R)$ and since $\omega \in \mathcal{W}_0$, we also have $\omega(u) = c$, i.e., u is a bounded solution to the problem (1.1), (1.2). In this case, the uniqueness of u is evident. □

3.2. Bounded Solutions. In the first part of this subsection, there are formulated theorems dealing with the existence and uniqueness of a bounded solution to the problems (1.1), (1.k) ($k = 3, 4, 5$). The proofs of those theorems can be found in the second part of this subsection.

Theorem 3.3. *Let $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$, and let the condition* (1.14) *be satisfied. Let, moreover, there exist $\gamma \in \widetilde{C}_0([a,+\infty[\,;\,]0,+\infty[\,)$ such that*

$$\gamma'(t) \geq \ell_0(\gamma)(t) + \ell_1(1)(t) \quad \text{for } t \geq a, \tag{3.2}$$

$$\gamma(+\infty) - \gamma(a) < 3. \tag{3.3}$$

Then the problem (1.1), (1.3) *has a unique bounded solution.*

Remark 3.1. Theorem 3.3 is unimprovable in the sense that the strict inequality (3.3) cannot be replaced by the nonstrict one (see Example 5.1, p. 40).

Theorem 3.4. *Let $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$, and let the condition* (1.14) *be satisfied. Let, moreover,*

$$\int_a^{+\infty} \ell_0(1)(s)\,ds < 1, \tag{3.4}$$

$$\int_a^{+\infty} \ell_1(1)(s)\,ds < 1 + 2\sqrt{1 - \int_a^{+\infty} \ell_0(1)(s)\,ds}\,. \tag{3.5}$$

Then the problem (1.1), (1.3) *has a unique bounded solution.*

Remark 3.2. Denote by G_a the set of pairs $(x, y) \in R_+ \times R_+$ such that

$$x < 1, \quad y < 1 + 2\sqrt{1 - x}$$

(see Figure 3.1).

Theorem 3.4 states that if the condition (1.14) is fulfilled, $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$, and

$$\left(\int_a^{+\infty} \ell_0(1)(s)\,ds, \int_a^{+\infty} \ell_1(1)(s)\,ds \right) \in G_a,$$

then the problem (1.1), (1.3) has a unique bounded solution.

Below we will show (see On Remark 3.2, p. 40) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin G_a$ there exist $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$, $q \in L_{loc}([a,+\infty[\,;R)$, and $c \in R$

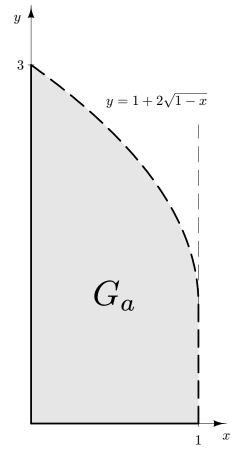

Figure 3.1

such that (1.14) holds,

$$\int_a^{+\infty} \ell_0(1)(s)\,ds = x_0, \qquad \int_a^{+\infty} \ell_1(1)(s)\,ds = y_0,$$

and the problem (1.1), (1.3) with $\ell = \ell_0 - \ell_1$ has no solution. In particular, the strict inequalities (3.4) and (3.5) cannot be replaced by the nonstrict ones.

Theorem 3.5. *Let $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$, and let the condition* (1.14) *be satisfied. Let, moreover, there exist $\gamma \in \widetilde{C}_0([a, +\infty[\,;\,]0, +\infty[\,)$ such that*

$$-\gamma'(t) \geq \ell_1(\gamma)(t) + \ell_0(1)(t) \ \ \text{for} \ \ t \geq a, \tag{3.6}$$

$$\gamma(+\infty) > 0, \tag{3.7}$$

$$\gamma(a) - \gamma(+\infty) < 3. \tag{3.8}$$

Then the equation (1.1) *has at least one bounded solution. If, moreover, there exists a finite limit* (3.1), *then the problem* (1.1), (1.4) *has a unique solution.*

Remark 3.3. Theorem 3.5 is unimprovable in the sense that neither one of the strict inequalities (3.7) and (3.8) can be replaced by the nonstrict one (see Example 5.2 on p. 41 and Example 5.3 on p. 42).

Theorem 3.6. *Let $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$, and let the condition* (1.14) *be satisfied. Let, moreover,*

$$\int_a^{+\infty} \ell_1(1)(s)\,ds < 1, \tag{3.9}$$

$$\int_a^{+\infty} \ell_0(1)(s)\,ds < 1 + 2\sqrt{1 - \int_a^{+\infty} \ell_1(1)(s)\,ds}\,. \tag{3.10}$$

Then the equation (1.1) *has at least one bounded solution. If, moreover, there exists a finite limit* (3.1)*, then the problem* (1.1), (1.4) *has a unique solution.*

Remark 3.4. Denote by $G_{+\infty}$ the set of pairs $(x, y) \in R_+ \times R_+$ such that

$$y < 1, \quad x < 1 + 2\sqrt{1 - y}$$

(see Figure 3.2).

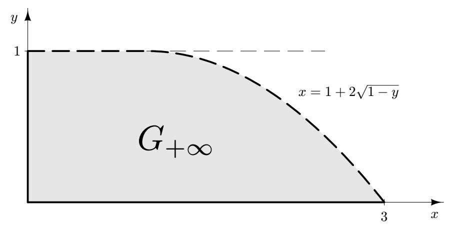

Figure 3.2

Theorem 3.6 states that if there exists a finite limit (3.1), $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$, and

$$\left(\int_a^{+\infty} \ell_0(1)(s)\,ds, \int_a^{+\infty} \ell_1(1)(s)\,ds \right) \in G_{+\infty},$$

then the problem (1.1), (1.4) has a unique bounded solution.

Below we will show (see On Remark 3.4, p. 43) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin G_{+\infty}$ there exist $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$, $q \in L_{loc}([a, +\infty[\,; R)$, and $c \in R$ such that there exists a finite limit (3.1),

$$\int_a^{+\infty} \ell_0(1)(s)\,ds = x_0, \qquad \int_a^{+\infty} \ell_1(1)(s)\,ds = y_0,$$

and the problem (1.1), (1.4) with $\ell = \ell_0 - \ell_1$ has no solution. In particular, the strict inequalities (3.9) and (3.10) cannot be replaced by the nonstrict ones.

Theorem 3.7. *Let $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$, and let the condition* (1.14) *be satisfied. Let, moreover, either*

$$\int_a^{+\infty} \ell_0(1)(s)\,ds < 1, \tag{3.11}$$

$$\frac{\int_a^{+\infty} \ell_0(1)(s)\,ds}{1 - \int_a^{+\infty} \ell_0(1)(s)\,ds} < \int_a^{+\infty} \ell_1(1)(s)\,ds < 2 + 2\sqrt{1 - \int_a^{+\infty} \ell_0(1)(s)\,ds}\,, \tag{3.12}$$

or

$$\int_a^{+\infty} \ell_1(1)(s)\,ds < 1, \tag{3.13}$$

$$\frac{\int_a^{+\infty} \ell_1(1)(s)\,ds}{1 - \int_a^{+\infty} \ell_1(1)(s)\,ds} < \int_a^{+\infty} \ell_0(1)(s)\,ds < 2 + 2\sqrt{1 - \int_a^{+\infty} \ell_1(1)(s)\,ds}\,. \tag{3.14}$$

Then the equation (1.1) *has at least one bounded solution. If, moreover, there exists a finite limit* (3.1)*, then the problem* (1.1), (1.5) *has a unique solution.*

Remark 3.5. Put

$$G_p^+ = \Big\{(x,y) \in R_+ \times R_+ :\ x < 1,\ \frac{x}{1-x} < y < 2 + 2\sqrt{1-x}\Big\},$$
$$G_p^- = \Big\{(x,y) \in R_+ \times R_+ :\ y < 1,\ \frac{y}{1-y} < x < 2 + 2\sqrt{1-y}\Big\}$$

(see Figure 3.3).

Theorem 3.7 states that if there exists a finite limit (3.1), $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$, and

$$\Big(\int_a^{+\infty} \ell_0(1)(s)\,ds,\ \int_a^{+\infty} \ell_1(1)(s)\,ds\Big) \in G_p^+ \cup G_p^-,$$

then the problem (1.1), (1.5) has a unique bounded solution.

Below we will show (see On Remark 3.5, p. 44) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin G_p^+ \cup G_p^-$ there exist $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$, $q \in L_{loc}([a,+\infty[\,;R)$, and $c \in R$ such that there exists a finite limit (3.1),

$$\int_a^{+\infty} \ell_0(1)(s)\,ds = x_0, \qquad \int_a^{+\infty} \ell_1(1)(s)\,ds = y_0,$$

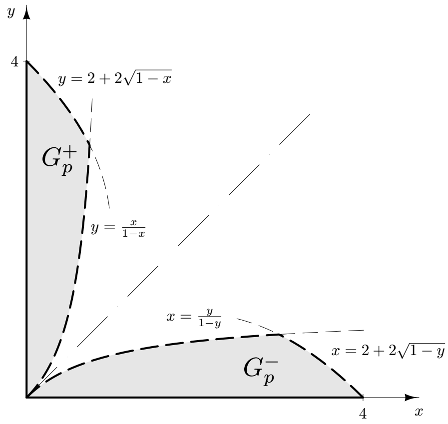

Figure 3.3

and the problem (1.1), (1.5) with $\ell = \ell_0 - \ell_1$ has no solution. In particular, neither of the strict inequalities in (3.11)–(3.14) can be replaced by the nonstrict one.

Remark 3.6. In Theorems 3.1–3.7, the condition (1.14) is essential and it cannot be omitted. Indeed, let $q \in L_{loc}([a, +\infty[\,; R)$, $p \in L([a, +\infty[\,; R_+)$ be such that

$$0 \neq \int_a^{+\infty} p(s)\, ds < 1,$$

and consider the equation

$$u'(t) = p(t)u(a) + q(t). \tag{3.15}$$

Put

$$\ell(v)(t) \stackrel{def}{=} p(t)v(a) \quad \text{for} \quad t \geq a.$$

Then the assumptions imposed on the operator ℓ (with $\ell_0 \equiv \ell$ and $\ell_1 \equiv 0$) in Theorems 3.3–3.7 are fulfilled.

On the other hand, every solution u to (3.15) is of the form

$$u(t) = c_0\left(1 + \int_a^t p(s)\, ds\right) + \int_a^t q(s)\, ds \quad \text{for} \quad t \geq a,$$

where $c_0 \in R$. However, u is bounded if and only if the condition (1.14) is fulfilled.

Remark 3.7. It is clear that if the problem (1.1), (1.4), resp. (1.1), (1.5), has a solution for some $c \in R$, then there exists a finite limit (3.1). Thus the condition (3.1) in Theorems 3.5–3.7 is also a necessary condition for the unique solvability of the mentioned problems.

Remark 3.8. Obviously, if ℓ is an $a-$Volterra operator and the assumptions of either Theorem 3.3 or Theorem 3.4 are fulfilled, then all solutions to the equation (1.1) are bounded.

In the proofs listed below, whenever $\ell \in \widetilde{\mathcal{L}}$, then the operator $\widehat{\ell}$ is defined by (1.9) with φ and ψ given by (1.6)–(1.8).

Proof of Theorem 3.3. Let $b \in \,]a, +\infty[$ and

$$\widehat{\gamma}(t) \stackrel{def}{=} \begin{cases} \varphi^{-1}(\gamma)(t) & \text{for } t \in [a, b[\,, \\ \gamma(+\infty) & \text{for } t = b, \end{cases} \tag{3.16}$$

where φ is given by (1.6) and (1.8). Then the assumptions of Lemma 2.3 are fulfilled. Consequently, according to Theorem 3.1 and Proposition 1.2, the theorem is valid. □

Proof of Theorem 3.4. All the assumptions of Lemma 2.4 are fulfilled. Consequently, according to Theorem 3.1 and Proposition 1.2, the theorem is valid. □

Proof of Theorem 3.5. Let $b \in \,]a, +\infty[$ and define function $\widehat{\gamma}$ by (3.16), where φ is given by (1.6) and (1.8). Then the assumptions of Lemma 2.5 are fulfilled. Consequently, according to Theorem 3.2 and Proposition 1.2, the theorem is valid. □

Proof of Theorem 3.6. All the assumptions of Lemma 2.6 are fulfilled. Consequently, according to Theorem 3.2 and Proposition 1.2, the theorem is valid. □

Proof of Theorem 3.7. All the assumptions of Lemma 2.7 are fulfilled. Consequently, according to Theorem 3.2 and Proposition 1.2, the theorem is valid. □

3.3. Sign Constant Solutions. As in the previous section, first we formulate theorems dealing with the existence and uniqueness of a sign constant bounded solution to the problems (1.1), (1.k) ($k = 3, 4, 5$). The proofs of those theorems can be found at the end of this subsection.

Theorem 3.8. *Let $\ell \in \widetilde{\mathcal{P}}$. Then for every $q \in L([a, +\infty[\,; R_+)$ and $c \in R_+$ the problem* (1.1), (1.3) *has a unique bounded solution and this solution is nonnegative if and only if there exists $\gamma \in \widetilde{C}_{loc}([a, +\infty[\,;]0, +\infty[\,)$ satisfying* (2.36).

Theorem 3.9. *Let $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$. Let, moreover, ℓ_1 be an $a-$Volterra operator and*

$$\int_a^{+\infty} \ell_0(1)(s)\, ds < 1, \qquad \int_a^{+\infty} \ell_1(1)(s)\, ds \leq 1. \tag{3.17}$$

Then for every $q \in L([a, +\infty[\,; R_+)$ and $c \in R_+$ the problem (1.1), (1.3) *has a unique bounded solution and this solution is nonnegative.*

Remark 3.9. Theorem 3.9 is unimprovable in the sense that the condition (3.17) can be replaced neither by the condition

$$\int_a^{+\infty} \ell_0(1)(s)\,ds \le 1+\varepsilon_0, \quad \int_a^{+\infty} \ell_1(1)(s)\,ds \le 1,$$

nor by the condition

$$\int_a^{+\infty} \ell_0(1)(s)\,ds < 1, \quad \int_a^{+\infty} \ell_1(1)(s)\,ds \le 1+\varepsilon,$$

no matter how small $\varepsilon_0 \ge 0$ and $\varepsilon > 0$ would be. More precisely, if the condition (3.17) is violated, then either the problem (1.1), (1.3) is not uniquely solvable for some $q \in L([a,+\infty[\,;R_+)$ and $c \in R_+$ (see On Remark 3.2, p. 40) or there exist $q \in L([a,+\infty[\,;R_+)$ and $c \in R_+$ such that the problem (1.1), (1.3) has a unique bounded solution, but this solution is not nonnegative (see Example 5.4 on p. 47 and Example 5.5 on p. 48).

Theorem 3.10. *Let* $-\ell \in \widetilde{\mathcal{P}}$. *Then for every* $-q \in L([a,+\infty[\,;R_+)$ *and* $c \in R_+$ *the problem* (1.1), (1.4) *has a unique solution and this solution is nonnegative if and only if there exists* $\gamma \in \widetilde{C}_0([a,+\infty[\,;\,]0,+\infty[\,)$ *satisfying* (3.7) *and*

$$\gamma'(t) \le \ell(\gamma)(t) \quad \text{for } t \ge a. \tag{3.18}$$

Theorem 3.11. *Let* $\ell = \ell_0 - \ell_1$ *with* $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$. *Let, moreover,* ℓ_0 *be an anti–Volterra operator and*

$$\int_a^{+\infty} \ell_0(1)(s)\,ds \le 1, \quad \int_a^{+\infty} \ell_1(1)(s)\,ds < 1. \tag{3.19}$$

Then for every $-q \in L([a,+\infty[\,;R_+)$ *and* $c \in R_+$ *the problem* (1.1), (1.4) *has a unique solution and this solution is nonnegative.*

Remark 3.10. Theorem 3.11 is unimprovable in the sense that the condition (3.19) can be replaced neither by the condition

$$\int_a^{+\infty} \ell_0(1)(s)\,ds \le 1+\varepsilon, \quad \int_a^{+\infty} \ell_1(1)(s)\,ds < 1,$$

nor by the condition

$$\int_a^{+\infty} \ell_0(1)(s)\,ds \le 1, \quad \int_a^{+\infty} \ell_1(1)(s)\,ds \le 1+\varepsilon_0,$$

no matter how small $\varepsilon > 0$ and $\varepsilon_0 \ge 0$ would be. More precisely, if the condition (3.19) is violated, then either the problem (1.1), (1.4) is not uniquely solvable for some $-q \in L([a,+\infty[\,;R_+)$ and $c \in R_+$ (see On Remark 3.4,

p. 43) or there exist $-q \in L([a,+\infty[\,;R_+)$ and $c \in R_+$ such that the problem (1.1), (1.4) has a unique solution, but this solution is not nonnegative (see Example 5.4 on p. 47 and Example 5.5 on p. 48).

Theorem 3.12. *Let $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$. Let, moreover, $k, j \in \{0,1\}$, $k \neq j$, and*

$$\int\limits_a^{+\infty} \ell_k(1)(s)\,ds < 1, \qquad \frac{\int\limits_a^{+\infty} \ell_k(1)(s)\,ds}{1 - \int\limits_a^{+\infty} \ell_k(1)(s)\,ds} < \int\limits_a^{+\infty} \ell_j(1)(s)\,ds \leq 1. \tag{3.20}$$

Then for every $q \in L([a,+\infty[\,;R_+)$ and $c \in R_+$ the problem (1.1), (1.5) *has a unique solution u, and this solution satisfies the inequality*

$$(-1)^k u(t) \geq 0 \ \ \textit{for} \ \ t \geq a.$$

Remark 3.11. Put

$$H^+ = \Big\{(x,y) \in R_+ \times R_+ :\ x < 1,\ \frac{x}{1-x} < y \leq 1\Big\},$$
$$H^- = \Big\{(x,y) \in R_+ \times R_+ :\ y < 1,\ \frac{y}{1-y} < x \leq 1\Big\}$$

(see Figures 3.4 and 3.5)

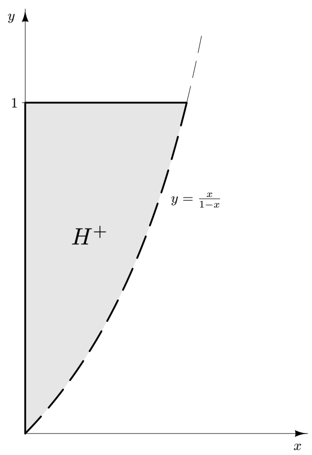

Figure 3.4

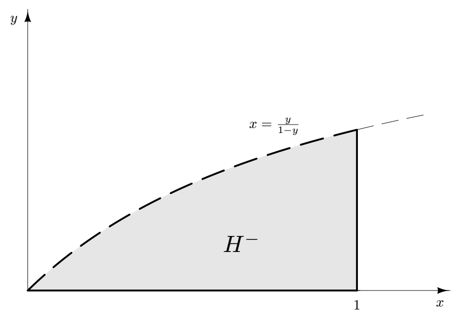

Figure 3.5

Theorem 3.12 states, that if $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$, and

$$\Big(\int\limits_a^{+\infty} \ell_0(1)(s)\,ds, \int\limits_a^{+\infty} \ell_1(1)(s)\,ds \Big) \in H^+,$$

resp.

$$\left(\int\limits_a^{+\infty} \ell_0(1)(s)\, ds, \int\limits_a^{+\infty} \ell_1(1)(s)\, ds \right) \in H^-,$$

then for every $q \in L([a,+\infty[\,;R_+)$ and $c \in R_+$, the problem $(1.1), (1.5)$ has a unique solution u, and this solution is nonnegative, resp. nonpositive.

Below we will show (see On Remark 3.11, p. 50) that for every $(x_0, y_0) \notin H^+$, resp. for every $(x_0, y_0) \notin H^-$, $x_0, y_0 \in R_+$, there exist $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$, $q \in L([a,+\infty[\,;R_+)$, and $c \in R_+$ such that

$$\int\limits_a^{+\infty} \ell_0(1)(s)\, ds = x_0, \quad \int\limits_a^{+\infty} \ell_1(1)(s)\, ds = y_0,$$

and the problem $(1.1), (1.5)$ with $\ell = \ell_0 - \ell_1$ has a solution, which is not nonnegative, resp. nonpositive. In particular, neither one of the strict inequalities in (3.20) can be replaced by the nonstrict one. Furthermore, the nonstrict inequality in (3.20) cannot be replaced by the inequality

$$\int\limits_a^{+\infty} \ell_j(1)(s)\, ds \leq 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

In the proofs listed below, whenever $\ell \in \widetilde{\mathcal{L}}$ and $\omega \in \mathrm{ch}$ ($\omega \in \mathcal{W}_0$), then the operators $\widehat{\ell}$ and $\widehat{\omega}$ are defined by (1.9) and (1.10), respectively, with φ and ψ given by (1.6)–(1.8).

Proof of Theorem 3.8. First suppose that for every $q \in L([a,+\infty[\,;R_+)$ and $c \in R_+$ the problem $(1.1), (1.3)$ has a unique bounded solution and this solution is nonnegative. Then, since $\ell \in \widetilde{\mathcal{P}}$, the problem

$$u'(t) = \ell(u)(t), \quad u(a) = 1$$

has a positive bounded solution u and we can put $\gamma \equiv u$.

Now suppose that there exists a function $\gamma \in \widetilde{C}_{loc}([a,+\infty[\,;\,]0,+\infty[\,)$ satisfying (2.36). Obviously, γ is nondecreasing. Therefore, there exists a finite or infinite limit

$$r \overset{def}{=} \lim_{t\to+\infty} \gamma(t).$$

If $r = +\infty$, then by virtue of Remark 2.4 (see p. 22), the only bounded solution to (1.1_0), (1.3_0) is a trivial solution. Now, according to Theorem 3.1 and Lemma 2.14, there is a unique bounded solution u_0 to the problem

$$u'(t) = \ell(u)(t), \quad u(a) = 1,$$

and u_0 is nonnegative. Taking into account the condition $\ell \in \widetilde{\mathcal{P}}$, we get that

$$0 < u_0(t) \;\text{ for }\; t \geq a, \quad \lim_{t\to+\infty} u_0(t) = u_0(+\infty) < +\infty.$$

Put

$$\alpha(t) = \begin{cases} \gamma(t) & \text{for } t \in [a, +\infty[\text{ if } r < +\infty, \\ u_0(t) & \text{for } t \in [a, +\infty[\text{ if } r = +\infty. \end{cases}$$

Obviously, $\alpha \in \widetilde{C}_0([a, +\infty[\,; \,]0, +\infty[\,)$ and

$$\alpha'(t) \geq \ell(\alpha)(t) \text{ for } t \in [a, +\infty[\,. \tag{3.21}$$

Let now $b \in \,]a, +\infty[$ and

$$\widehat{\gamma}(t) = \begin{cases} \varphi^{-1}(\alpha)(t) & \text{for } t \in [a, b[\,, \\ \alpha(+\infty) & \text{for } t = b, \end{cases}$$

where φ is given by (1.6) and (1.8). Then by virtue of (3.21), the assumptions of Lemma 2.8 hold. Consequently, according to Remark 2.1 and Proposition 1.1, the theorem is valid. $\square$

Proof of Theorem 3.9. Let $b \in \,]a, +\infty[\,$. Then all the assumptions of Lemma 2.9 are fulfilled. Consequently, according to Remark 2.1 and Proposition 1.1, the theorem is valid. $\square$

Proof of Theorem 3.10. First suppose that for every $c \in R_+$ and $-q \in L([a, +\infty[\,; R_+)$ the problem (1.1), (1.4) has a unique solution and this solution is nonnegative. Then, since $-\ell \in \widetilde{\mathcal{P}}$, the problem

$$u'(t) = \ell(u)(t), \quad u(+\infty) = 1$$

has a positive solution u and we can put $\gamma \equiv u$.

Now suppose that there exists a function $\gamma \in \widetilde{C}_0([a, +\infty[\,; \,]0, +\infty[\,)$ satisfying (3.7) and (3.18). Let $b \in \,]a, +\infty[\,$. Then the assumptions of Lemma 2.10 are fulfilled. Consequently, according to Remark 2.2 and Proposition 1.1, the theorem is valid. $\square$

Proof of Theorem 3.11. Let $b \in \,]a, +\infty[\,$. Then the assumptions of Lemma 2.11 are fulfilled. Consequently, according to Remark 2.2 and Proposition 1.1, the theorem is valid. $\square$

Proof of Theorem 3.12. Let $b \in \,]a, +\infty[\,$. If $k = 0$ (resp. $k = 1$), then the assumptions of Lemma 2.12 (resp. Lemma 2.13) are fulfilled. Consequently, according to Remark 2.3 and Proposition 1.1, the theorem is valid. $\square$

4. Equations with Deviating Arguments

In this section we will present some consequences of the main results from Section 3 for the equations with deviating arguments (1.15), (1.16), and (1.17), respectively.

In what follows we will use the notation

$$p_0(t) = \sum_{k=1}^{m} p_k(t), \quad g_0(t) = \sum_{k=1}^{m} g_k(t) \text{ for } t \geq a$$

4.1. Bounded Solutions.

Corollary 4.1. *Let the condition* (1.14) *be satisfied and let at least one of the following items be fulfilled:*

a) $\tau_k(t) \le t$ *for* $t \ge a$ $(k = 1, \dots, m)$,

$$\int_a^{+\infty} g_0(s) \exp\left(\int_s^{+\infty} p_0(\xi)\, d\xi \right) ds < 3; \tag{4.1}$$

b)

$$\int_a^{+\infty} p_0(s)\, ds < 1, \qquad \int_a^{+\infty} g_0(s)\, ds < 1 + 2\sqrt{1 - \int_a^{+\infty} p_0(s)\, ds}\,. \tag{4.2}$$

Then for every $c \in R$ *the problem* (1.17), (1.3) *has a unique bounded solution.*

Remark 4.1. Corollary 4.1 is unimprovable. More precisely, neither one of the strict inequalities in (4.1) and (4.2) can be replaced by the nonstrict one (see Example 5.1 on p. 40 and On Remark 3.2 on p. 40).

Remark 4.2. If $\tau_k(t) \le t$, $\mu_k(t) \le t$ for $t \ge a$ $(k = 1, \dots, m)$, (1.14) is satisfied, and at least one of the conditions a) and b) in Corollary 4.1 is fulfilled, then every solution of the equation (1.17) is bounded.

Corollary 4.2. *Let the condition* (1.14) *be satisfied and let at least one of the following items be fulfilled:*

a) $\mu_k(t) \ge t$ *for* $t \ge a$ $(k = 1, \dots, m)$,

$$\int_a^{+\infty} p_0(s) \exp\left(\int_a^{s} g_0(\xi)\, d\xi \right) ds < 3; \tag{4.3}$$

b)

$$\int_a^{+\infty} g_0(s)\, ds < 1, \qquad \int_a^{+\infty} p_0(s)\, ds < 1 + 2\sqrt{1 - \int_a^{+\infty} g_0(s)\, ds}\,. \tag{4.4}$$

Then the equation (1.17) *has at least one bounded solution. If, moreover, there exists a finite limit* (3.1), *then for every* $c \in R$ *the problem* (1.17), (1.4) *has a unique solution.*

Remark 4.3. Corollary 4.2 is unimprovable. More precisely, neither one of the strict inequalities in (4.3) and (4.4) can be replaced by the nonstrict one (see Example 5.3 on p. 42 and On Remark 3.4 on p. 43).

Corollary 4.3. *Let the condition* (1.14) *be satisfied and let either*

$$\int_a^{+\infty} p_0(s)\,ds < 1, \tag{4.5}$$

$$\frac{\int_a^{+\infty} p_0(s)\,ds}{1 - \int_a^{+\infty} p_0(s)\,ds} < \int_a^{+\infty} g_0(s)\,ds < 2 + 2\sqrt{1 - \int_a^{+\infty} p_0(s)\,ds}\,, \tag{4.6}$$

or

$$\int_a^{+\infty} g_0(s)\,ds < 1, \tag{4.7}$$

$$\frac{\int_a^{+\infty} g_0(s)\,ds}{1 - \int_a^{+\infty} g_0(s)\,ds} < \int_a^{+\infty} p_0(s)\,ds < 2 + 2\sqrt{1 - \int_a^{+\infty} g_0(s)\,ds}\,. \tag{4.8}$$

Then the equation (1.17) *has at least one bounded solution. If, moreover, there exists a finite limit* (3.1), *then for every* $c \in R$ *the problem* (1.17), (1.5) *has a unique solution.*

Remark 4.4. Corollary 4.3 is unimprovable in the sense that neither one of the strict inequalities in (4.5)–(4.8) can be replaced by the nonstrict one (see On Remark 3.5, p. 44).

4.2. Sign Constant Solutions.

Corollary 4.4. *Let the condition* (1.14) *be satisfied,* $\mu_k(t) \le t$ *for* $t \ge a$ $(k = 1, \dots, m)$, *and*

$$\int_a^{+\infty} p_0(s)\,ds < 1, \qquad \int_a^{+\infty} g_0(s)\,ds \le 1. \tag{4.9}$$

Then for every $c \in R$ *the problem* (1.17), (1.3) *has a unique bounded solution. If, moreover,* $q(t) \ge 0$ *for* $t \ge a$ *and* $c \ge 0$, *then this solution is nonnegative.*

Remark 4.5. Corollary 4.4 is unimprovable in the sense that the condition (4.9) can be replaced neither by the condition

$$\int_a^{+\infty} p_0(s)\,ds \le 1 + \varepsilon_0, \qquad \int_a^{+\infty} g_0(s)\,ds \le 1,$$

nor by the condition

$$\int_a^{+\infty} p_0(s)\,ds < 1, \quad \int_a^{+\infty} g_0(s)\,ds \le 1+\varepsilon,$$

no matter how small $\varepsilon_0 \ge 0$ and $\varepsilon > 0$ would be. More precisely, if the condition (4.9) is violated, then either the problem (1.17), (1.3) has no solution (see On Remark 3.2, p. 40) or this problem has a unique bounded solution which is not nonnegative (see Example 5.4 on p. 47 and Example 5.5 on p. 48).

Corollary 4.5. *Let the condition* (1.14) *be satisfied,* $\int_a^{\tau_k^*} p_0(\xi)\,d\xi \neq 0$ $(k=1,\dots,m)$*, and let there exist $x > 0$ such that*

$$\operatorname{ess\,sup}\left\{ \int_t^{\tau_k(t)} p_0(s)\,ds :\ t \ge a \right\} < \eta_k(x) \quad (k=1,\dots,m), \tag{4.10}$$

where

$$\eta_k(x) = \frac{1}{x}\ln\left(x + \frac{x}{\exp\big(x\int_a^{\tau_k^*} p_0(\xi)\,d\xi\big) - 1} \right),$$

$$\tau_k^* = \operatorname{ess\,sup}\big\{\tau_k(t) :\ t \ge a\big\}.$$

Then for every $c \in R$ the problem (1.15), (1.3) *has a unique bounded solution. If, moreover, $q(t) \ge 0$ for $t \ge a$ and $c \ge 0$, then this solution is nonnegative.*

Corollary 4.6. *Let the condition* (1.14) *be satisfied, $\tau_k(t) \ge t$ for $t \ge a$ $(k=1,\dots,m)$, and*

$$\int_a^{+\infty} p_0(s)\,ds \le 1, \quad \int_a^{+\infty} g_0(s)\,ds < 1. \tag{4.11}$$

Then the equation (1.17) *has at least one bounded solution. If, moreover, there exists a finite limit* (3.1)*, then for every $c \in R$ the problem* (1.17), (1.4) *has a unique solution. Furthermore, if $q(t) \le 0$ for $t \ge a$ and $c \ge 0$, then this solution is nonnegative.*

Remark 4.6. Corollary 4.6 is unimprovable in the sense that the condition (4.11) can be replaced neither by the condition

$$\int_a^{+\infty} p_0(s)\,ds \le 1+\varepsilon, \quad \int_a^{+\infty} g_0(s)\,ds < 1,$$

nor by the condition

$$\int_a^{+\infty} p_0(s)\,ds \le 1, \qquad \int_a^{+\infty} g_0(s)\,ds \le 1+\varepsilon_0,$$

no matter how small $\varepsilon > 0$ and $\varepsilon_0 \ge 0$ would be. More precisely, if the condition (4.11) is violated, then either the problem (1.17), (1.4) has no solution (see On Remark 3.4, p. 43) or this problem has a unique bounded solution which is not nonnegative (see Example 5.4 on p. 47 and Example 5.5 on p. 48).

Corollary 4.7. *Let the condition* (1.14) *be satisfied,* $\int_{\mu_k^*}^{+\infty} g_0(\xi)\,d\xi \ne 0$ $(k=1,\dots,m)$, *and let there exist* $x > 0$ *such that*

$$\operatorname{ess\,sup}\left\{ \int_{\mu_k(t)}^{t} g_0(s)\,ds : \ t \ge a \right\} < \vartheta_k(x) \quad (k=1,\dots,m),$$

where

$$\vartheta_k(x) = \frac{1}{x} \ln\left(x + \frac{x}{\exp\big(x \int_{\mu_k^*}^{+\infty} g_0(\xi)\,d\xi\big) - 1} \right),$$

$$\mu_k^* = \operatorname{ess\,inf}\big\{\mu_k(t) : \ t \ge a\big\}.$$

Then the equation (1.16) *has at least one bounded solution. If, moreover, there exists a finite limit* (3.1), *then for every* $c \in R$ *the problem* (1.16), (1.4) *has a unique solution. Furthermore, if* $q(t) \le 0$ *for* $t \ge a$ *and* $c \ge 0$, *then this solution is nonnegative.*

Corollary 4.8. *Let* $q \in L([a,+\infty[\,;R_+)$ *and*

$$\int_a^{+\infty} p_0(s)\,ds < 1, \qquad \frac{\int_a^{+\infty} p_0(s)\,ds}{1 - \int_a^{+\infty} p_0(s)\,ds} < \int_a^{+\infty} g_0(s)\,ds \le 1, \tag{4.12}$$

resp.

$$\int_a^{+\infty} g_0(s)\,ds < 1, \qquad \frac{\int_a^{+\infty} g_0(s)\,ds}{1 - \int_a^{+\infty} g_0(s)\,ds} < \int_a^{+\infty} p_0(s)\,ds \le 1. \tag{4.13}$$

Then for every $c \in R_+$ *the problem* (1.17), (1.5) *has a unique solution* u, *and this solution satisfies the inequality*

$$u(t) \ge 0 \ \text{ for } \ t \ge a,$$

resp.

$$u(t) \le 0 \ \ \textit{for} \ \ t \ge a.$$

Remark 4.7. Corollary 4.8 is unimprovable in the sense that neither one of the strict inequalities in (4.12) and (4.13) can be replaced by the nonstrict one. Furthermore, the nonstrict inequalities in (4.12) and (4.13) cannot be replaced by the inequalities

$$\int\limits_a^{+\infty} g_0(s)\,ds \le 1+\varepsilon, \quad \text{resp.} \quad \int\limits_a^{+\infty} p_0(s)\,ds \le 1+\varepsilon,$$

no matter how small $\varepsilon > 0$ would be (see On Remark 3.11, p. 50).

4.3. Proofs. Define operators ℓ_0 and ℓ_1 by

$$\begin{aligned} \ell_0(v)(t) &\stackrel{def}{=} \sum_{k=1}^{m} p_k(t)v(\tau_k(t)) \ \ \text{for} \ \ t \ge a, \\ \ell_1(v)(t) &\stackrel{def}{=} \sum_{k=1}^{m} g_k(t)v(\mu_k(t)) \ \ \text{for} \ \ t \ge a. \end{aligned} \tag{4.14}$$

Then Corollary 4.3 follows from Theorem 3.7, Corollary 4.4 follows from Theorems 3.4 and 3.9, Corollary 4.6 follows from Theorems 3.6 and 3.11, and Corollary 4.8 follows from Theorem 3.12.

Therefore we will prove only Corollaries 4.1, 4.2, 4.5, and 4.7.

Proof of Corollary 4.1. a) Choose $\varepsilon > 0$ such that

$$\varepsilon \exp\bigg(\int\limits_a^{+\infty} p_0(s)\,ds\bigg) + \int\limits_a^{+\infty} g_0(s)\exp\bigg(\int\limits_s^{+\infty} p_0(\xi)\,d\xi\bigg)\,ds \le 3,$$

and put

$$\gamma(t) = \exp\bigg(\int\limits_a^{t} p_0(s)\,ds\bigg)\bigg(\varepsilon + \int\limits_a^{t} g_0(s)\exp\bigg(-\int\limits_a^{s} p_0(\xi)\,d\xi\bigg)\,ds\bigg) \quad \text{for} \ \ t \ge a.$$

It can be easily verified that γ satisfies the inequalities (3.2) and (3.3) with ℓ_0 and ℓ_1 defined by (4.14). Consequently, the assumptions of Theorem 3.3 are fulfilled.

b) Obviously, the assumptions of Theorem 3.4 are satisfied, where ℓ_0 and ℓ_1 are defined by (4.14). □

Proof of Corollary 4.2. a) Choose $\varepsilon > 0$ such that

$$\varepsilon \exp\bigg(\int\limits_a^{+\infty} g_0(s)\,ds\bigg) + \int\limits_a^{+\infty} p_0(s)\exp\bigg(\int\limits_a^{s} g_0(\xi)\,d\xi\bigg)\,ds \le 3,$$

and put

$$\gamma(t) = \exp\bigg(\int\limits_t^{+\infty} g_0(s)\,ds\bigg)\bigg(\varepsilon + \int\limits_t^{+\infty} p_0(s)\exp\bigg(-\int\limits_s^{+\infty} g_0(\xi)\,d\xi\bigg)\,ds\bigg) \quad \text{for } t \geq a.$$

It can be easily verified that γ satisfies the inequalities (3.6), (3.7), and (3.8) with ℓ_0 and ℓ_1 defined by (4.14). Consequently, the assumptions of Theorem 3.5 are fulfilled.

b) Obviously, the assumptions of Theorem 3.6 are satisfied, where ℓ_0 and ℓ_1 are defined by (4.14). □

Proof of Corollary 4.5. Choose $\varepsilon > 0$ such that

$$\operatorname{ess\,sup}\bigg\{\int\limits_t^{\tau_k(t)} p_0(s)\,ds:\ t \geq a\bigg\} < \\ < \frac{1}{x}\ln\left(x + \frac{x(1-\varepsilon)}{\exp\big(x\int\limits_a^{\tau_k^*} p_0(\xi)\,d\xi\big) - (1-\varepsilon)}\right) \quad (k = 1,\dots,m),$$

and put

$$\gamma(t) = \frac{\exp\big(x\int\limits_a^t p_0(s)\,ds\big) - (1-\varepsilon)}{\exp\big(x\int\limits_a^{+\infty} p_0(s)\,ds\big) - 1} \quad \text{for } t \geq a.$$

It can be easily verified that γ satisfies the assumptions of Theorem 3.3 with ℓ_0 defined by (4.14) and $\ell_1 \equiv 0$, as well as the assumptions of Theorem 3.8 with

$$\ell(v)(t) \overset{def}{=} \sum_{k=1}^m p_k(t)v(\tau_k(t)) \quad \text{for } t \geq a.$$

This complete the proof. □

Proof of Corollary 4.7. Choose $\varepsilon > 0$ such that

$$\operatorname{ess\,sup}\bigg\{\int\limits_{\mu_k(t)}^{t} g_0(s)\,ds:\ t \geq a\bigg\} < \\ < \frac{1}{x}\ln\left(x + \frac{x(1-\varepsilon)}{\exp\big(x\int\limits_{\mu_k^*}^{+\infty} g_0(\xi)\,d\xi\big) - (1-\varepsilon)}\right) \quad (k = 1,\dots,m),$$

and put

$$\gamma(t) = \frac{\exp\Big(x \int\limits_t^{+\infty} g_0(s)\,ds\Big) - (1-\varepsilon)}{\exp\Big(x \int\limits_a^{+\infty} g_0(s)\,ds\Big) - 1} \quad \text{for } \ t \geq a.$$

It can be easily verified that γ satisfies the assumptions of Theorem 3.5 with ℓ_1 defined by (4.14) and $\ell_0 \equiv 0$, as well as the assumptions of Theorem 3.10 with

$$\ell(v)(t) \stackrel{def}{=} -\sum_{k=1}^{m} g_k(t)v(\mu_k(t)) \ \text{ for } \ t \geq a.$$

This complete the proof. □

5. Examples

Remark 5.1. Let functions $p, g \in L([a,b]; R_+)$ and $\tau, \mu \in \mathcal{M}_{ab}$ be such that on the segment $[a,b]$ the problem

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)), \quad u(a) = 0, \tag{5.1}$$

resp.

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)), \quad u(b) = 0, \tag{5.2}$$

resp.

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)), \quad u(a) - u(b) = 0, \tag{5.3}$$

has a nontrivial solution. Then, according to Remark 1.1 (see p. 11), there exist $q_0 \in L([a,b]; R)$ and $c \in R$ such that on the segment $[a,b]$ the problem

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + q_0(t), \quad u(a) = c, \tag{5.4}$$

resp.

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + q_0(t), \quad u(b) = c, \tag{5.5}$$

resp.

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + q_0(t), \quad u(a) - u(b) = c, \tag{5.6}$$

has no solution.

Further, if we put

$$\ell_0(v)(t) \stackrel{def}{=} \begin{cases} p(t)v(\tau(t)) & \text{for } \ t \in [a,b], \\ 0 & \text{for } \ t > b, \end{cases} \tag{5.7}$$

$$\ell_1(v)(t) \stackrel{def}{=} \begin{cases} g(t)v(\mu(t)) & \text{for } \ t \in [a,b], \\ 0 & \text{for } \ t > b, \end{cases} \tag{5.8}$$

$$q(t) \stackrel{def}{=} \begin{cases} q_0(t) & \text{for } \ t \in [a,b], \\ 0 & \text{for } \ t > b, \end{cases} \tag{5.9}$$

then, obviously, the condition (1.14) is satisfied, there exists a finite limit (3.1), and the problem (1.1), (1.3), resp. (1.1), (1.4), resp. (1.1), (1.5), with $\ell = \ell_0 - \ell_1$ has no solution.

Example 5.1. Let $t_0 \in \,]a,b[$ and choose $g \in L([a,b];R_+)$ such that

$$\int_a^{t_0} g(s)\,ds = 1, \qquad \int_{t_0}^b g(s)\,ds = 2.$$

Put

$$\mu(t) = \begin{cases} b & \text{for } \ t \in [a,t_0[\,, \\ t_0 & \text{for } \ t \in [t_0,b], \end{cases}$$

and

$$\gamma(t) = \begin{cases} 1 + \int\limits_a^t g(s)\,ds & \text{for } \ t \in [a,b], \\ 4 & \text{for } \ t > b. \end{cases}$$

Then γ satisfies all the assumptions of Theorem 3.3, where $\ell_0 \equiv 0$ and ℓ_1 is defined by (5.8), except of the condition (3.3), instead of which we have

$$\gamma(+\infty) - \gamma(a) = 3.$$

On the other hand, the problem (5.1) with $p \equiv 0$ has a nontrivial solution

$$u(t) = \begin{cases} \int\limits_a^t g(s)\,ds & \text{for } \ t \in [a,t_0[\,, \\ 1 - \int\limits_{t_0}^t g(s)\,ds & \text{for } \ t \in [t_0,b]. \end{cases}$$

Consequently, according to Remark 5.1 (see p. 39), we have shown that in Theorem 3.3, resp. in Corollary 4.1 a), the strict inequality (3.3), resp. the strict inequality (4.1), cannot be replaced by the nonstrict one.

On Remark 3.2. According to Remark 5.1 (see p. 39), for every $(x_0,y_0) \notin G_a$, it is sufficient to construct functions $p, g \in L([a,b];R_+)$ and $\tau, \mu \in \mathcal{M}_{ab}$ in such a way that

$$\int_a^b p(s)\,ds = x_0, \qquad \int_a^b g(s)\,ds = y_0 \tag{5.10}$$

and such that the problem (5.1) has a nontrivial solution.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0,y_0) \notin G_a$, then (x_0,y_0) belongs to one of the following sets:

$$G_a^1 = \Big\{(x,y) \in R_+ \times R_+ : \ x \geq 1\Big\},$$
$$G_a^2 = \Big\{(x,y) \in R_+ \times R_+ : \ x < 1, \ y \geq 1 + 2\sqrt{1-x}\,\Big\}.$$

Let $(x_0, y_0) \in G_a^1$, $t_0 \in [a,b]$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int_a^{t_0} p(s)\,ds = 1, \quad \int_{t_0}^{b} p(s)\,ds = x_0 - 1, \quad \int_a^{b} g(s)\,ds = y_0.$$

Put $\tau \equiv t_0$ and $\mu \equiv a$. Then (5.10) is satisfied and the problem (5.1) has a nontrivial solution

$$u(t) = \int_a^t p(s)\,ds \ \text{ for } \ t \in [a,b].$$

Let $(x_0, y_0) \in G_a^2$, $a < t_1 < t_2 < t_3 < b$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int_a^{t_1} p(s)\,ds = x_0, \quad \int_{t_1}^{b} p(s)\,ds = 0, \quad \int_a^{t_1} g(s)\,ds = \sqrt{1-x_0}\,,$$

$$\int_{t_1}^{t_2} g(s)\,ds = 1, \quad \int_{t_2}^{t_3} g(s)\,ds = y_0 - \big(1 + 2\sqrt{1-x_0}\,\big), \quad \int_{t_3}^{b} g(s)\,ds = \sqrt{1-x_0}\,.$$

Put $\tau \equiv t_1$ and

$$\mu(t) = \begin{cases} b & \text{for } \ t \in [a, t_1[\,, \\ t_1 & \text{for } \ t \in [t_1, t_2[\, \cup [t_3, b], \\ a & \text{for } \ t \in [t_2, t_3[\,. \end{cases}$$

Then (5.10) is satisfied and the problem (5.1) has a nontrivial solution

$$u(t) = \begin{cases} \displaystyle\int_a^t p(s)\,ds + \sqrt{1-x_0}\int_a^t g(s)\,ds & \text{for } \ t \in [a, t_1[\,, \\ \displaystyle 1 - \int_{t_1}^t g(s)\,ds & \text{for } \ t \in [t_1, t_2[\,, \\ 0 & \text{for } \ t \in [t_2, t_3[\,, \\ \displaystyle -\int_{t_3}^t g(s)\,ds & \text{for } \ t \in [t_3, b]. \end{cases}$$

Example 5.2. Let $c>0$, $-q \in L([a,+\infty[\,; R_+)$, and $g \in L([a,+\infty[\,; R_+)$ be such that

$$\int_a^{+\infty} g(s)\,ds = 1, \quad \int_t^{+\infty} g(s)\,ds > 0 \ \text{ for } \ t \geq a.$$

Put

$$\gamma(t) = \int_t^{+\infty} g(s)\,ds \quad \text{for } t \ge a.$$

Then $\gamma \in \widetilde{C}_0([a,+\infty[\,;\,]0,+\infty[\,)$, satisfies the differential inequality (3.18) with

$$\ell(v)(t) \stackrel{def}{=} -g(t)v(a) \quad \text{for } t \in [a,+\infty[\,,$$

and (3.6) with $\ell_1 \equiv -\ell$ and $\ell_0 \equiv 0$.

On the other hand,

$$\gamma(+\infty) = 0,$$

the condition (3.8) holds and every solution to (1.1) can be written in the form

$$u(t) = u_0\left(1 - \int_a^t g(s)\,ds\right) + \int_a^t q(s)\,ds \quad \text{for } t \ge a,$$

where $u_0 \in R$. Moreover,

$$u(t) \le u_0\left(1 - \int_a^t g(s)\,ds\right) = u_0\gamma(t) \quad \text{for } t \ge a.$$

Consequently,

$$u(+\infty) \le 0,$$

and so the problem (1.1), (1.4) has no solution.

Therefore, the condition (3.7) in Theorem 3.5 is essential and it cannot be omitted.

Example 5.3. Let $t_0 \in [a,b]$ and choose $p \in L([a,b];R_+)$ such that

$$\int_a^{t_0} p(s)\,ds = 2, \qquad \int_{t_0}^b p(s)\,ds = 1.$$

Put

$$\tau(t) = \begin{cases} t_0 & \text{for } [a,t_0[\,, \\ a & \text{for } [t_0,b], \end{cases}$$

and

$$\gamma(t) = \begin{cases} 1 + \displaystyle\int_t^b p(s)\,ds & \text{for } t \in [a,b], \\ 1 & \text{for } t > b. \end{cases}$$

Then γ satisfies all the assumptions of Theorem 3.5, where $\ell_1 \equiv 0$ and ℓ_0 is defined by (5.7), except of the condition (3.8), instead of which we have

$$\gamma(a) - \gamma(+\infty) = 3.$$

On the other hand, the problem (5.2) with $g \equiv 0$ has a nontrivial solution

$$u(t) = \begin{cases} 1 - \int\limits_t^{t_0} p(s)\,ds & \text{for } [a, t_0[\,, \\ \int\limits_t^{b} p(s)\,ds & \text{for } [t_0, b]. \end{cases}$$

Consequently, according to Remark 5.1 (see p. 39), we have shown that in Theorem 3.5, resp. in Corollary 4.2 a), the strict inequality (3.8), resp. (4.3) cannot be replaced by the nonstrict one.

On Remark 3.4. According to Remark 5.1 (see p. 39), for every $(x_0, y_0) \notin G_{+\infty}$, it is sufficient to costruct functions $p, g \in L([a, b]; R_+)$ and $\tau, \mu \in \mathcal{M}_{ab}$ in such a way that (5.10) holds and such that the problem (5.2) has a nontrivial solution.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin G_{+\infty}$, then (x_0, y_0) belongs to one of the following sets:

$$G^1_{+\infty} = \Big\{(x, y) \in R_+ \times R_+ :\ y \geq 1\Big\},$$
$$G^2_{+\infty} = \Big\{(x, y) \in R_+ \times R_+ :\ y < 1,\ x \geq 1 + 2\sqrt{1 - y}\,\Big\}.$$

Let $(x_0, y_0) \in G^1_{+\infty}$, $t_0 \in [a, b]$, and let $p, g \in L([a, b]; R_+)$ be such that

$$\int\limits_a^{t_0} g(s)\,ds = y_0 - 1, \quad \int\limits_{t_0}^{b} g(s)\,ds = 1, \quad \int\limits_a^{b} p(s)\,ds = x_0.$$

Put $\tau \equiv b$ and $\mu \equiv t_0$. Then (5.10) is satisfied and the problem (5.2) has a nontrivial solution

$$u(t) = \int\limits_t^{b} g(s)\,ds \ \text{ for } \ t \in [a, b].$$

Now suppose $(x_0, y_0) \in G^2_{+\infty}$, $a < t_1 < t_2 < t_3 < b$. Let $p, g \in L([a, b]; R_+)$ be such that

$$\int\limits_a^{t_3} g(s)\,ds = 0, \quad \int\limits_{t_3}^{b} g(s)\,ds = y_0, \quad \int\limits_a^{t_1} p(s)\,ds = \sqrt{1 - y_0}\,,$$

$$\int\limits_{t_1}^{t_2} p(s)\,ds = x_0 - \big(1 + 2\sqrt{1 - y_0}\,\big), \quad \int\limits_{t_2}^{t_3} p(s)\,ds = 1, \quad \int\limits_{t_3}^{b} p(s)\,ds = \sqrt{1 - y_0}\,.$$

Put $\mu \equiv t_3$ and

$$\tau(t) = \begin{cases} t_3 & \text{for } t \in [a, t_1[\, \cup [t_2, t_3[\,, \\ b & \text{for } t \in [t_1, t_2[\,, \\ a & \text{for } t \in [t_3, b]. \end{cases}$$

Then (5.10) is satisfied and the problem (5.2) has a nontrivial solution

$$u(t) = \begin{cases} -\int\limits_t^{t_1} p(s)\, ds & \text{for } t \in [a, t_1[\,, \\ 0 & \text{for } t \in [t_1, t_2[\,, \\ 1 - \int\limits_t^{t_3} p(s)\, ds & \text{for } t \in [t_2, t_3[\,, \\ \int\limits_t^b g(s)\, ds + \sqrt{1 - y_0} \int\limits_t^b p(s)\, ds & \text{for } t \in [t_3, b]. \end{cases}$$

On Remark 3.5. According to Remark 5.1 (see p. 39), for every $(x_0, y_0) \notin G_p^+ \cup G_p^-$, it is sufficient to costruct functions $p, g \in L([a, b]; R_+)$ and $\tau, \mu \in \mathcal{M}_{ab}$ in such a way that (5.10) holds and such that the problem (5.3) has a nontrivial solution.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin G_p^+ \cup G_p^-$, then (x_0, y_0) belongs at least to one of the following sets:

$$G_1 = \Big\{(x, y) \in R_+ \times R_+ : \ x \geq 1, \ y \geq 1\Big\},$$
$$G_2 = \Big\{(x, y) \in R_+ \times R_+ : \ x < 1, \ y \geq 2 + 2\sqrt{1 - x}\,\Big\},$$
$$G_3 = \Big\{(x, y) \in R_+ \times R_+ : \ y < 1, \ x \geq 2 + 2\sqrt{1 - y}\,\Big\},$$
$$G_4 = \left\{(x, y) \in R_+ \times R_+ : \ y < 1, \ y \leq x \leq \frac{y}{1 - y}\right\},$$
$$G_5 = \left\{(x, y) \in R_+ \times R_+ : \ 0 < x < 1, \ x \leq y \leq \frac{x}{1 - x}\right\}.$$

Let $(x_0, y_0) \in G_1$, $t_0 \in [a, b]$, and choose $p, g \in L([a, b]; R_+)$ such that

$$\int\limits_a^{t_0} p(s)\, ds = x_0 - 1, \quad \int\limits_{t_0}^b p(s)\, ds = 1, \quad \int\limits_a^{t_0} g(s)\, ds = 1, \quad \int\limits_{t_0}^b g(s)\, ds = y_0 - 1.$$

Put

$$\tau(t) = \begin{cases} t_0 & \text{for } t \in [a, t_0[\,, \\ b & \text{for } t \in [t_0, b], \end{cases} \qquad \mu(t) = \begin{cases} a & \text{for } t \in [a, t_0[\,, \\ t_0 & \text{for } t \in [t_0, b]. \end{cases}$$

Then (5.10) is satisfied and the problem (5.3) has a nontrivial solution

$$u(t) = \begin{cases} \int\limits_t^{t_0} g(s)\,ds & \text{for } t \in [a, t_0[\,, \\ \int\limits_{t_0}^{t} p(s)\,ds & \text{for } t \in [t_0, b]. \end{cases}$$

Let $(x_0, y_0) \in G_2$, $a < t_1 < t_2 < t_3 < t_4 < b$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int\limits_a^{t_1} p(s)\,ds = x_0, \quad \int\limits_{t_1}^{b} p(s)\,ds = 0, \quad \int\limits_a^{t_1} g(s)\,ds = \sqrt{1 - x_0}\,, \quad \int\limits_{t_1}^{t_2} g(s)\,ds = 1,$$

$$\int\limits_{t_2}^{t_3} g(s)\,ds = y_0 - \big(2 + 2\sqrt{1 - x_0}\,\big), \quad \int\limits_{t_3}^{t_4} g(s)\,ds = \sqrt{1 - x_0}\,, \quad \int\limits_{t_4}^{b} g(s)\,ds = 1.$$

Put $\tau \equiv t_1$ and

$$\mu(t) = \begin{cases} t_4 & \text{for } t \in [a, t_1[\, \cup [t_4, b], \\ t_1 & \text{for } t \in [t_1, t_2[\, \cup [t_3, t_4[\,, \\ a & \text{for } t \in [t_2, t_3[\,. \end{cases}$$

Then (5.10) is satisfied and the problem (5.3) has a nontrivial solution

$$u(t) = \begin{cases} \int\limits_a^{t} p(s)\,ds + \sqrt{1 - x_0} \int\limits_a^{t} g(s)\,ds & \text{for } t \in [a, t_1[\,, \\ 1 - \int\limits_{t_1}^{t} g(s)\,ds & \text{for } t \in [t_1, t_2[\,, \\ 0 & \text{for } t \in [t_2, t_3[\,, \\ -\int\limits_{t_3}^{t} g(s)\,ds & \text{for } t \in [t_3, t_4[\,, \\ -\sqrt{1 - x_0} \int\limits_t^{b} g(s)\,ds & \text{for } t \in [t_4, b]. \end{cases}$$

Let $(x_0, y_0) \in G_3$, $a < t_1 < t_2 < t_3 < t_4 < b$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int\limits_a^{t_4} g(s)\,ds = 0, \quad \int\limits_{t_4}^{b} g(s)\,ds = y_0, \quad \int\limits_a^{t_1} p(s)\,ds = 1, \quad \int\limits_{t_1}^{t_2} p(s)\,ds = \sqrt{1 - y_0}\,,$$

$$\int_{t_2}^{t_3} p(s)\,ds = x_0 - \big(2 + 2\sqrt{1-y_0}\,\big), \quad \int_{t_3}^{t_4} p(s)\,ds = 1, \quad \int_{t_4}^{b} p(s)\,ds = \sqrt{1-y_0}\,.$$

Put $\mu \equiv t_4$ and

$$\tau(t) = \begin{cases} t_4 & \text{for } t \in [t_1, t_2[\,\cup\,[t_3, t_4[\,, \\ b & \text{for } t \in [t_2, t_3[\,, \\ t_1 & \text{for } t \in [a, t_1[\,\cup\,[t_4, b]. \end{cases}$$

Then (5.10) is satisfied and the problem (5.3) has a nontrivial solution

$$u(t) = \begin{cases} -\sqrt{1-y_0}\displaystyle\int_a^t p(s)\,ds & \text{for } t \in [a, t_1[\,, \\ -\displaystyle\int_t^{t_2} p(s)\,ds & \text{for } t \in [t_1, t_2[\,, \\ 0 & \text{for } t \in [t_2, t_3[\,, \\ 1 - \displaystyle\int_t^{t_4} p(s)\,ds & \text{for } t \in [t_3, t_4[\,, \\ \displaystyle\int_t^b g(s)\,ds + \sqrt{1-y_0}\displaystyle\int_t^b p(s)\,ds & \text{for } t \in [t_4, b]. \end{cases}$$

Let $(x_0, y_0) \in G_4$. If $x_0 = 0$ and $y_0 = 0$, then we put $p \equiv 0$, $g \equiv 0$, $\tau \equiv a$, and $\mu \equiv a$. Consequently, (5.10) is satisfied and every nonzero constant function is a nontrivial solution to the problem (5.3).

Assume that $x_0 \neq 0$ and $y_0 \neq 0$. Let $a < t_1 < t_2 < b$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int_a^{t_1} g(s)\,ds = y_0, \quad \int_{t_1}^{b} g(s)\,ds = 0, \quad \int_a^{t_1} p(s)\,ds = 0,$$

$$\int_{t_1}^{t_2} p(s)\,ds = x_0 - \frac{x_0}{y_0} + 1, \quad \int_{t_2}^{b} p(s)\,ds = \frac{x_0}{y_0} - 1.$$

Put $\tau \equiv t_2$ and $\mu \equiv b$. Then (5.10) is satisfied and the problem (5.3) has a nontrivial solution

$$u(t) = \begin{cases} 1 - \displaystyle\int_a^t g(s)\,ds & \text{for } t \in [a, t_1[\,, \\ 1 - \dfrac{y_0}{x_0}\displaystyle\int_t^b p(s)\,ds & \text{for } t \in [t_1, b]. \end{cases}$$

Let $(x_0, y_0) \in G_5$, $a < t_1 < t_2 < b$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int_a^{t_2} p(s)\,ds = 0, \quad \int_{t_2}^{b} p(s)\,ds = x_0, \quad \int_a^{t_1} g(s)\,ds = \frac{y_0}{x_0} - 1,$$

$$\int_{t_1}^{t_2} g(s)\,ds = y_0 - \frac{y_0}{x_0} + 1, \quad \int_{t_2}^{b} g(s)\,ds = 0.$$

Put $\tau \equiv b$ and $\mu \equiv t_1$. Then (5.10) is satisfied and the problem (5.3) has a nontrivial solution

$$u(t) = \begin{cases} 1 - \dfrac{x_0}{y_0} \displaystyle\int_a^t g(s)\,ds & \text{for } t \in [a, t_2[\,, \\ 1 - \displaystyle\int_t^b p(s)\,ds & \text{for } t \in [t_2, b]. \end{cases}$$

Example 5.4. Let $b \in\,]a, +\infty[$, $\varepsilon > 0$, $\tau \equiv b$, $\mu \equiv a$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int_a^b p(s)\,ds = 1 + \varepsilon, \quad \int_a^b g(s)\,ds < 1.$$

Note that the problem (5.1), resp. (5.2), has only the trivial solution. Indeed, the integration of (5.1), resp. (5.2) from a to b yields

$$u(b) = (1+\varepsilon)u(b), \quad \text{resp.} \quad u(a) = u(a) \int_a^b g(s)\,ds,$$

whence we get $u(b) = 0$, resp. $u(a) = 0$. Consequently, $u'(t) = 0$ for $t \in [a,b]$, which together with $u(a) = 0$, resp. $u(b) = 0$, results in $u \equiv 0$.

Therefore, according to Proposition 1.3 (see p. 11), the problem

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)), \quad u(a) = 1, \tag{5.11}$$

resp.

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)), \quad u(b) = 1, \tag{5.12}$$

has a unique solution u.

On the other hand, the integration of (5.11), resp. (5.12), from a to b yields

$$u(b) - 1 = u(b)(1+\varepsilon) - \int_a^b g(s)\,ds,$$

resp.

$$1 - u(a) = (1+\varepsilon) - u(a) \int\limits_a^b g(s)\,ds,$$

whence we get

$$\varepsilon u(b) = \int\limits_a^b g(s)\,ds - 1 < 0, \quad \text{resp.} \quad u(a)\bigg(1 - \int\limits_a^b g(s)\,ds\bigg) = -\varepsilon < 0,$$

i.e., $u(b) < 0$, resp. $u(a) < 0$. Therefore, the problem $(1.1_0), (1.3)$, resp. $(1.1_0), (1.4)$, with $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 are defined by (5.7) and (5.8), and $c = 1$ has a unique solution

$$u_0(t) = \begin{cases} u(t) & \text{for } t \in [a,b], \\ u(b) & \text{for } t > b, \end{cases}$$

which assumes both positive and negative values.

Example 5.5. Let $b \in \,]a, +\infty[\,$, $\varepsilon > 0$, $\tau \equiv b$, $\mu \equiv a$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int\limits_a^b p(s)\,ds < 1, \quad \int\limits_a^b g(s)\,ds = 1 + \varepsilon.$$

Analogously to Example 5.4 one can verify that the problem (5.1), resp. (5.2), has only the trivial solution. Therefore, the problem (5.11), resp. (5.12), has a unique solution u.

On the other hand, the integration of (5.11), resp. (5.12), from a to b yields

$$u(b) - 1 = u(b) \int\limits_a^b p(s)\,ds - (1+\varepsilon),$$

resp.

$$1 - u(a) = \int\limits_a^b p(s)\,ds - u(a)(1+\varepsilon),$$

whence we get

$$u(b)\bigg(1 - \int\limits_a^b p(s)\,ds\bigg) = -\varepsilon < 0, \quad \text{resp.} \quad \varepsilon u(a) = \int\limits_a^b p(s)\,ds - 1 < 0,$$

i.e., $u(b) < 0$, resp. $u(a) < 0$. Therefore, the problem $(1.1_0), (1.3)$, resp. $(1.1_0), (1.4)$, with $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 are defined by (5.7) and (5.8), and $c = 1$ has a unique solution

$$u_0(t) = \begin{cases} u(t) & \text{for } t \in [a,b], \\ u(b) & \text{for } t > b, \end{cases}$$

which assumes both positive and negative values.

Examples 5.4 and 5.5 show that the condition (3.17) in Theorem 3.9, resp. the condition (4.9) in Corollary 4.4, can be replaced neither by the condition

$$\int_a^{+\infty} \ell_0(1)(s)\,ds \le 1+\varepsilon, \qquad \int_a^{+\infty} \ell_1(1)(s)\,ds \le 1,$$

resp.

$$\int_a^{+\infty} p_0(s)\,ds \le 1+\varepsilon, \qquad \int_a^{+\infty} g_0(s)\,ds \le 1,$$

nor by the condition

$$\int_a^{+\infty} \ell_0(1)(s)\,ds < 1, \qquad \int_a^{+\infty} \ell_1(1)(s)\,ds \le 1+\varepsilon,$$

resp.

$$\int_a^{+\infty} p_0(s)\,ds < 1, \qquad \int_a^{+\infty} g_0(s)\,ds \le 1+\varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

Further these examples also show that the condition (3.19) in Theorem 3.11, resp. the condition (4.11) in Corollary 4.6, can be replaced neither by the condition

$$\int_a^{+\infty} \ell_0(1)(s)\,ds \le 1+\varepsilon, \qquad \int_a^{+\infty} \ell_1(1)(s)\,ds < 1,$$

resp.

$$\int_a^{+\infty} p_0(s)\,ds \le 1+\varepsilon, \qquad \int_a^{+\infty} g_0(s)\,ds < 1,$$

nor by the condition

$$\int_a^{+\infty} \ell_0(1)(s)\,ds \le 1, \qquad \int_a^{+\infty} \ell_1(1)(s)\,ds \le 1+\varepsilon,$$

resp.

$$\int_a^{+\infty} p_0(s)\,ds \le 1, \qquad \int_a^{+\infty} g_0(s)\,ds \le 1+\varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

On Remark 3.11. In what follows, for every $(x_0, y_0) \not\in H^+$, resp. for every $(x_0, y_0) \not\in H^-$, $x_0, y_0 \in R_+$, the functions $p, g, q_0 \in L([a, b]; R_+)$ and $\tau, \mu \in \mathcal{M}$ are chosen in such a way that the equalities (5.10) are fulfilled and such that the problem (5.6) with $c = 0$ has a solution u, which is not nonnegative, resp. nonpositive. Consequently, if we define ℓ_0, ℓ_1, and q by (5.7)–(5.9), then the problem $(1.1), (1.5_0)$ with $\ell = \ell_0 - \ell_1$ has a solution

$$u_0(t) = \begin{cases} u(t) & \text{for } t \in [a, b], \\ u(b) & \text{for } t > b, \end{cases}$$

which is not nonnegative, resp. nonpositive.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \not\in H^+$, resp. $(x_0, y_0) \not\in H^-$, then (x_0, y_0) belongs to one of the following sets:

$$H_1^+ = \Big\{(x, y) \in R_+ \times R_+ : \ y > 1\Big\},$$
$$H_2^+ = \Big\{(x, y) \in R_+ \times R_+ : \ y \leq 1, \ \frac{y}{1+y} \leq x\Big\},$$

resp.

$$H_1^- = \Big\{(x, y) \in R_+ \times R_+ : \ x > 1\Big\},$$
$$H_2^- = \Big\{(x, y) \in R_+ \times R_+ : \ x \leq 1, \ \frac{x}{1+x} \leq y\Big\},$$

Let $(x_0, y_0) \in H_1^+$, $a < t_1 < t_2 < b$, and choose $p, g, q_0 \in L([a, b]; R_+)$ such that

$$\int_a^{t_2} p(s)\,ds = 0, \quad \int_{t_2}^{b} p(s)\,ds = x_0, \quad \int_a^{t_1} g(s)\,ds = y_0, \quad \int_{t_1}^{b} g(s)\,ds = 0,$$
$$\int_a^{t_1} q_0(s)\,ds = 0, \quad \int_{t_1}^{t_2} q_0(s)\,ds = y_0, \quad \int_{t_2}^{b} q_0(s)\,ds = x_0(y_0 - 1).$$

Put $\tau \equiv t_1$, $\mu \equiv a$. Then (5.10) is satisfied and the problem (5.6) has a solution

$$u(t) = \begin{cases} 1 - \displaystyle\int_a^t g(s)\,ds & \text{for } t \in [a, t_1[\,, \\ 1 - y_0 + \displaystyle\int_{t_1}^t q_0(s)\,ds & \text{for } t \in [t_1, t_2[\,, \\ 1 + (1 - y_0)\displaystyle\int_{t_2}^t p(s)\,ds + \int_{t_2}^t q_0(s)\,ds & \text{for } t \in [t_2, b] \end{cases}$$

with $u(t_1) = 1 - y_0 < 0$.

Let $(x_0, y_0) \in H_2^+$, $a < t_1 < t_2 < b$, and choose $p, g, q_0 \in L([a,b]; R_+)$ such that

$$\int_a^{t_1} p(s)\,ds = \frac{y_0}{1+y_0}, \quad \int_{t_1}^{t_2} p(s)\,ds = x_0 - \frac{y_0}{1+y_0}, \quad \int_{t_2}^{b} p(s)\,ds = 0,$$

$$\int_a^{t_2} g(s)\,ds = 0, \quad \int_{t_2}^{b} g(s)\,ds = y_0, \quad \int_a^{t_1} q_0(s)\,ds = 0,$$

$$\int_{t_1}^{t_2} q_0(s)\,ds = x_0 - \frac{y_0}{1+y_0}, \quad \int_{t_2}^{b} q_0(s)\,ds = 0.$$

Put $\mu \equiv a$,

$$\tau(t) = \begin{cases} t_1 & \text{for } t \in [a, t_2[\,, \\ a & \text{for } t \in [t_2, b]. \end{cases}$$

Then (5.10) is satisfied and the problem (5.6) has a solution

$$u(t) = \begin{cases} -1 - (1+y_0)\displaystyle\int_a^t p(s)\,ds & \text{for } t \in [a, t_1[\,, \\ -1 - y_0 - \displaystyle\int_{t_1}^t p(s)\,ds + \int_{t_1}^t q_0(s)\,ds & \text{for } t \in [t_1, t_2[\,, \\ -1 - y_0 + \displaystyle\int_{t_2}^t g(s)\,ds & \text{for } t \in [t_2, b] \end{cases}$$

with $u(a) = -1$.

Let $(x_0, y_0) \in H_1^-$, $a < t_1 < t_2 < b$, and choose $p, g, q_0 \in L([a,b]; R_+)$ such that

$$\int_a^{t_2} p(s)\,ds = 0, \quad \int_{t_2}^{b} p(s)\,ds = x_0, \quad \int_a^{t_1} g(s)\,ds = y_0, \quad \int_{t_1}^{b} g(s)\,ds = 0,$$

$$\int_a^{t_1} q_0(s)\,ds = y_0(x_0 - 1), \quad \int_{t_1}^{t_2} q_0(s)\,ds = x_0, \quad \int_{t_2}^{b} q_0(s)\,ds = 0.$$

Put $\tau \equiv t_2$, $\mu \equiv a$. Then (5.10) is satisfied and the problem (5.6) has a solution

$$u(t) = \begin{cases} -1 - (x_0 - 1) \int\limits_a^t g(s)\,ds + \int\limits_a^t q_0(s)\,ds & \text{for } t \in [a, t_1[\,, \\ -1 + \int\limits_{t_1}^t q_0(s)\,ds & \text{for } t \in [t_1, t_2[\,, \\ x_0 - 1 + \int\limits_{t_2}^t p(s)\,ds & \text{for } t \in [t_2, b] \end{cases}$$

with $u(t_2) = x_0 - 1 > 0$.

Let $(x_0, y_0) \in H_2^+$, $a < t_1 < t_2 < b$, and choose $p, g, q_0 \in L([a, b]; R_+)$ such that

$$\int\limits_a^{t_1} g(s)\,ds = 0, \quad \int\limits_{t_1}^{t_2} g(s)\,ds = y_0 - \frac{x_0}{1 + x_0}\,, \quad \int\limits_{t_2}^{b} g(s)\,ds = \frac{x_0}{1 + x_0},$$

$$\int\limits_a^{t_1} p(s)\,ds = x_0, \quad \int\limits_{t_1}^{b} p(s)\,ds = 0, \quad \int\limits_a^{t_1} q_0(s)\,ds = 0,$$

$$\int\limits_{t_1}^{t_2} q_0(s)\,ds = y_0 - \frac{x_0}{1 + x_0}, \quad \int\limits_{t_2}^{b} q_0(s)\,ds = 0.$$

Put $\tau \equiv a$,

$$\mu(t) = \begin{cases} a & \text{for } t \in [a, t_2[\,, \\ t_1 & \text{for } t \in [t_2, b]. \end{cases}$$

Then (5.10) is satisfied and the problem (5.6) has a solution

$$u(t) = \begin{cases} 1 + \int\limits_a^t p(s)\,ds & \text{for } t \in [a, t_1[\,, \\ 1 + x_0 - \int\limits_{t_1}^t g(s)\,ds + \int\limits_{t_1}^t q_0(s)\,ds & \text{for } t \in [t_1, t_2[\,, \\ 1 + x_0 - (1 + x_0) \int\limits_{t_2}^t g(s)\,ds & \text{for } t \in [t_2, b] \end{cases}$$

with $u(a) = 1$.

We have shown that in Theorem 3.12 and Corollary 4.8 neither one of the strict inequalities in (3.20), (4.12) and (4.13) can be replaced by the nonstrict one. Furthermore, the nonstrict inequalities in (3.20) and in (4.12), resp. (4.13), cannot be replaced by the inequalities

$$\int_a^{+\infty} \ell_j(1)(s)\,ds \leq 1+\varepsilon$$

and

$$\int_a^{+\infty} g_0(s)\,ds \leq 1+\varepsilon, \quad \text{resp.} \quad \int_a^{+\infty} p_0(s)\,ds \leq 1+\varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

CHAPTER 2

Nonlinear Problem

6. Statement of the Problem

In this chapter, we will consider the problem on the existence and uniqueness of a bounded solution to the equation

$$u'(t) = F(u)(t) \tag{6.1}$$

satisfying the condition

$$\omega(u) = h(u). \tag{6.2}$$

Here, $F \in \mathcal{K}$, $\omega \in$ ch, resp. $\omega \in \mathcal{W}_0$, and $h \in \mathcal{H}$.

By a solution to (6.1) we understand a function $u \in \widetilde{C}_{loc}([a, +\infty[\,; R)$ satisfying the equality (6.1) almost everywhere in $[a, +\infty[$. By a solution to the problem (6.1), (6.2) we understand a solution to (6.1) which belongs to the domain of ω and satisfies (6.2).

The particular cases of the condition (6.2) are:

$$u(a) = h(u), \tag{6.3}$$

$$u(+\infty) = h(u), \tag{6.4}$$

$$u(a) - u(+\infty) = h(u). \tag{6.5}$$

The chapter is organized as follows: Main results are presented in Section 8, where sufficient conditions for the existence and uniqueness of a bounded solution to the equation (6.1) satisfying one of the conditions (6.3), (6.4), or (6.5) are established. The proofs of the main results are contained in Section 9. In Section 10 we concretize results of Section 8 for particular cases of the equation (6.1) – for the equation with deviating arguments:

$$u'(t) = \sum_{k=1}^{m} \Big(p_k(t)u(\tau_k(t)) - g_k(t)u(\mu_k(t))\Big) + \\ + f\big(t, u(t), u(\nu_1(t)), \dots, u(\nu_n(t))\big), \tag{6.1$'$}$$

where $f \in K_{loc}([a, +\infty[\, \times R^{n+1}; R)$, $p_k, g_k \in L([a, +\infty[\,; R_+)$, $\tau_k, \mu_k \in \mathcal{M}$ $(k = 1, \dots, m)$, $\nu_j \in \mathcal{M}$ $(j = 1, \dots, n)$, $m, n \in N$. Last section of the chapter – Section 11 – is devoted to the examples verifying the optimality of obtained results.

The general principle of the existence of a bounded solution contained in Section 7 play a crucial role in proving the main results. Namely Lemmas 7.2 and 7.3 (see p. 60 and p. 60) state a bounded solution to (6.1) can be represented as a uniform limit of a suitable sequence $\{u_b\}_{b>a}$ of solutions to the problem

$$u'(t) = \widetilde{F}_b(u)(t), \tag{6.6}$$

$$\widetilde{\omega}_b(u) = \widetilde{h}_b(u) \tag{6.7}$$

(for the definition of $\widetilde{F}_b$, $\widetilde{\omega}_b$, and $\widetilde{h}_b$ see p. 7).

7. The Principle of Existence of Bounded Solutions

7.1. Main Results.

Theorem 7.1. *Let $\omega \in \mathrm{ch}$, and let there exist $\ell \in \widetilde{\mathcal{L}}$ such that the only bounded solution to the problem $(1.1_0), (1.2_0)$ is a trivial solution. Let, moreover, there exist a continuous function $c : R_+ \to R_+$ such that*

$$|h(v)| \le c(\|v\|) \ \ \text{for} \ \ v \in C_0([a,+\infty[\,;R), \tag{7.1}$$

and let on the set $\{v \in C_0([a,+\infty[\,;R) : \ |\omega(v)| \le c(\|v\|)\}$ the inequalities

$$\Big| \int\limits_a^t \big[F(v)(s) - \ell(v)(s)\big]\,ds \Big| \le \Big| \int\limits_a^t q(s,\|v\|)\,ds \Big| \ \ \text{for} \ \ t \ge a, \tag{7.2}$$

$$|F(v)(t) - \ell(v)(t)| \le \eta(t)\|v\| \ \ \text{for} \ \ t \ge a, \quad \|v\| \ge \rho_1 \tag{7.3}$$

be fulfilled, where $q \in K_{loc}([a,+\infty[\,\times R_+;R)$, $\eta \in L_{loc}([a,+\infty[\,;R_+)$, and $\rho_1 > 0$. Assume also, that the functions c and q satisfy

$$\sup\bigg\{ \Big| \int\limits_a^t q(s,x)\,ds \Big| : \ t \ge a \bigg\} < +\infty \ \ \text{for every} \ \ x \in R_+, \tag{7.4}$$

$$\lim_{x\to+\infty} \frac{1}{x}\bigg(c(x) + \sup\bigg\{ \Big| \int\limits_a^t q(s,x)\,ds \Big| : \ t \ge a \bigg\}\bigg) = 0. \tag{7.5}$$

Then the problem $(6.1), (6.2)$ has at least one bounded solution.

Theorem 7.2. *Let $\omega \in \mathcal{W}_0$, and let there exist $\ell \in \widetilde{\mathcal{L}}$ such that the only bounded solution to the problem $(1.1_0), (1.2_0)$ is a trivial solution. Let, moreover, there exist a continuous function $c : R_+ \to R_+$ such that the inequality (7.1) holds, and let on the set $\{v \in C_0([a,+\infty[\,;R) : |\omega(v)| \le c(\|v\|)\}$ the inequalities (7.2) and (7.3) be fulfilled, where $\rho_1 > 0$, $q \in K_{loc}([a,+\infty[\,\times R_+;R)$, and $\eta \in L_{loc}([a,+\infty[\,;R_+)$. Assume also, that the functions c and q satisfy (7.4) and (7.5). Then the equation (6.1) has at least one bounded solution. If, moreover, for every $v \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ with $\|v\| < +\infty$, there exists a finite limit*

$$\lim_{t\to+\infty} \int\limits_a^t F(v)(s)\,ds, \tag{7.6}$$

then the problem $(6.1), (6.2)$ has at least one bounded solution.

Remark 7.1. From Theorems 7.1 and 7.2, together with the results from Chapter 1, we immediately get several criteria guaranteeing the existence of a bounded solution to the equation (6.1), resp. to the problem (6.1), (6.2).

Definition 7.1. We will say that an operator $\ell \in \widetilde{\mathcal{L}}$ belongs to the set $\mathcal{A}(\omega)$, if there exists $r > 0$ such that for any $q^* \in L([a,+\infty[\,;R_+)$ and $c^* \in R_+$, every function $u \in \widetilde{C}_0([a,+\infty[\,;R)$ satisfying the inequalities

$$\omega(u)\operatorname{sgn} u(a) \le c^*, \tag{7.7}$$

$$\big[u'(t) - \ell(u)(t)\big]\operatorname{sgn} u(t) \le q^*(t) \ \text{ for } \ t \ge a, \tag{7.8}$$

admits the estimate

$$\|u\| \le r\bigg(c^* + \int_a^{+\infty} q^*(s)\,ds\bigg). \tag{7.9}$$

Definition 7.2. We will say that an operator $\ell \in \widetilde{\mathcal{L}}$ belongs to the set $\mathcal{B}(\omega)$, if there exists $r > 0$ such that for any $q^* \in L([a,+\infty[\,;R_+)$ and $c^* \in R_+$, every function $u \in \widetilde{C}_0([a,+\infty[\,;R)$ satisfying the inequalities

$$\omega(u)\operatorname{sgn} u(+\infty) \le c^*, \tag{7.10}$$

$$\big[u'(t) - \ell(u)(t)\big]\operatorname{sgn} u(t) \ge -q^*(t) \ \text{ for } \ t \ge a, \tag{7.11}$$

admits the estimate (7.9).

Theorem 7.3. *Let $\omega \in \mathrm{ch}$, and let there exist a continuous function $c : R_+ \to R_+$ such that*

$$h(v)\operatorname{sgn} v(a) \le c(\|v\|) \ \textit{ for } \ v \in C_0([a,+\infty[\,;R). \tag{7.12}$$

Let, moreover, there exist an operator $\ell \in \mathcal{A}(\omega)$ such that on the set $\{v \in C_0([a,+\infty[\,;R) :\ \omega(v)\operatorname{sgn} v(a) \le c(\|v\|)\}$ the inequality

$$\big[F(v)(t) - \ell(v)(t)\big]\operatorname{sgn} v(t) \le q(t,\|v\|) \ \textit{ for } \ t \ge a \tag{7.13}$$

is fulfilled, where $q \in K([a,+\infty[\,\times R_+;R_+)$. Assume also, that the functions c and q satisfy

$$\lim_{x\to+\infty} \frac{1}{x}\left(c(x) + \int_a^{+\infty} q(s,x)\,ds\right) = 0. \tag{7.14}$$

Then the problem (6.1), (6.2) *has at least one bounded solution.*

Theorem 7.4. *Let $\omega \in \mathcal{W}_0$, and let there exist a continuous function $c : R_+ \to R_+$ such that*

$$h(v)\operatorname{sgn} v(+\infty) \le c(\|v\|) \ \textit{ for } \ v \in C_0([a,+\infty[\,;R). \tag{7.15}$$

Let, moreover, there exist an operator $\ell \in \mathcal{B}(\omega)$ such that on the set $\{v \in C_0([a,+\infty[\,;R) :\ \omega(v)\operatorname{sgn} v(+\infty) \le c(\|v\|)\}$ the inequality

$$\big[F(v)(t) - \ell(v)(t)\big]\operatorname{sgn} v(t) \ge -q(t,\|v\|) \ \textit{ for } \ t \ge a \tag{7.16}$$

is fulfilled, where $q \in K([a,+\infty[\times R_+; R_+)$*. Assume also, that the functions* c *and* q *satisfy* (7.14)*. Then the equation* (6.1) *has at least one bounded solution. If, moreover, for every* $v \in \widetilde{C}_{loc}([a,+\infty[;R)$ *with* $\|v\| < +\infty$*, there exists a finite limit* (7.6)*, then the problem* (6.1), (6.2) *has at least one bounded solution.*

Theorem 7.5. *Let* $\omega \in \mathrm{ch}$*,* $F(0) \in L([a,+\infty[;R)$*,*

$$\big[h(v) - h(w)\big]\operatorname{sgn}(v(a) - w(a)) \le 0 \ \ \textit{for} \ \ v, w \in C_0([a,+\infty[;R), \tag{7.17}$$

and let there exist $\ell \in \mathcal{A}(\omega)$ *such that on the set* $\{v \in C_0([a,+\infty[;R) : \omega(v)\operatorname{sgn} v(a) \le |h(0)|\}$ *the inequality*

$$\big[F(v)(t) - F(w)(t) - \ell(v-w)(t)\big]\operatorname{sgn}(v(t) - w(t)) \le 0 \ \ \textit{for} \ \ t \ge a \tag{7.18}$$

is fulfilled. Then the problem (6.1), (6.2) *has a unique bounded solution.*

Theorem 7.6. *Let* $\omega \in \mathcal{W}_0$*,* $F(0) \in L([a,+\infty[;R)$*,*

$$\big[h(v) - h(w)\big]\operatorname{sgn}(v(+\infty) - w(+\infty)) \le 0 \ \ \textit{for} \ \ v, w \in C_0([a,+\infty[;R), \tag{7.19}$$

and let there exist $\ell \in \mathcal{B}(\omega)$ *such that on the set* $\{v \in C_0([a,+\infty[;R) : \omega(v)\operatorname{sgn} v(+\infty) \le |h(0)|\}$ *the inequality*

$$\big[F(v)(t) - F(w)(t) - \ell(v-w)(t)\big]\operatorname{sgn}(v(t) - w(t)) \ge 0 \ \ \textit{for} \ \ t \ge a \tag{7.20}$$

is fulfilled. Let, moreover, for every $v \in \widetilde{C}_{loc}([a,+\infty[;R)$ *with* $\|v\| < +\infty$*, there exist a finite limit* (7.6)*. Then the problem* (6.1), (6.2) *has a unique bounded solution.*

7.2. Auxiliary Propositions. To prove Theorems 7.1–7.6 we will need some auxiliary propositions established in this subsection.

In what follows, we will also consider on the interval $[a,b]$ the problem

$$v'(t) = \widehat{F}(v)(t), \tag{7.21}$$

$$\widehat{\omega}(v) = \widehat{h}(v), \tag{7.22}$$

where $\widehat{F} \in \mathcal{K}_{ab}$, $\widehat{\omega} : C([a,b];R) \to R$ is a linear bounded functional, and $\widehat{h} : C([a,b];R) \to R$ is a continuous functional satisfying that for every $r > 0$ there exists $M_r > 0$ such that

$$|\widehat{h}(v)| \le M_r \ \ \text{for} \ \ \|v\|_C \le r.$$

By a solution to the problem (7.21), (7.22) we understand a function $v \in \widetilde{C}([a,b];R)$, which satisfies the equality (7.21) almost everywhere in $[a,b]$ and the condition (7.22) is fulfilled.

Now let us formulate the result from [**16**, Theorem 1] in a suitable for us form.

Lemma 7.1. *Let there exist a positive number* ρ *and an operator* $\widehat{\ell} \in \mathcal{L}_{ab}$ *such that the homogeneous problem*

$$v'(t) = \widehat{\ell}(v)(t), \quad \widehat{\omega}(v) = 0 \tag{7.23}$$

has only a trivial solution, and let for every $\delta \in \,]0,1[$ *and for an arbitrary function* $v \in \widetilde{C}([a,b];R)$ *satisfying*

$$v'(t) = \widehat{\ell}(v)(t) + \delta\big[\widehat{F}(v)(t) - \widehat{\ell}(v)(t)\big] \ \text{ for } \ t \in [a,b], \tag{7.24}$$

$$\widehat{\omega}(v) = \delta\widehat{h}(v), \tag{7.25}$$

the estimate

$$\|v\|_C \le \rho \tag{7.26}$$

hold. Then the problem (7.21), (7.22) *has at least one solution* v, *which satisfies* (7.26).

Proof. Since $\widehat{\ell} \in \mathcal{L}_{ab}$ and $\widehat{F} \in \mathcal{K}_{ab}$, there exist $\eta, \eta_0 \in L([a,b];R_+)$ and $\alpha \in R_+$ such that

$$|\widehat{\ell}(y)(t)| \le \eta(t)\|y\|_C \ \text{ for } \ t \in [a,b], \quad y \in C([a,b];R),$$
$$|\widehat{F}(y)(t)| \le \eta_0(t) \ \text{ for } \ t \in [a,b], \quad \|y\|_C \le 2\rho,$$
$$|\widehat{h}(y)| \le \alpha \ \text{ for } \ \|y\|_C \le 2\rho.$$

Put

$$\gamma(t) \stackrel{def}{=} \eta_0(t) + 2\rho\eta(t) \ \text{ for } \ t \in [a,b],$$

$$\sigma(s) \stackrel{def}{=} \begin{cases} 1 & \text{for } \ 0 \le s \le \rho, \\ 2 - \dfrac{s}{\rho} & \text{for } \ \rho < s < 2\rho, \\ 0 & \text{for } \ 2\rho \le s, \end{cases} \tag{7.27}$$

$$\begin{aligned} q_0(y)(t) &\stackrel{def}{=} \sigma\big(\|y\|_C\big)\big[\widehat{F}(y)(t) - \widehat{\ell}(y)(t)\big] \ \text{ for } \ t \in [a,b], \\ c_0(y) &\stackrel{def}{=} \sigma\big(\|y\|_C\big)\widehat{h}(y). \end{aligned} \tag{7.28}$$

Then for every $y \in C([a,b];R)$ and almost all $t \in [a,b]$, the inequalities

$$|q_0(y)(t)| \le \gamma(t), \quad |c_0(y)| \le \alpha$$

hold.

For arbitrarily fixed $u \in C([a,b];R)$, let us consider the problem

$$v'(t) = \widehat{\ell}(v)(t) + q_0(u)(t), \quad \widehat{\omega}(v) = c_0(u). \tag{7.29}$$

According to Proposition 1.3 (see p. 11), the problem (7.29) has a unique solution v and, moreover, by virtue of Proposition 2.1 (see p. 13), there exists $\beta > 0$ such that

$$\|v\|_C \le \beta\big(|c_0(u)| + \|q_0(u)\|_L\big).$$

Therefore, for arbitrarily fixed $u \in C([a,b];R)$, the solution v to the problem (7.29) admits the estimates

$$\|v\|_C \le \rho_0, \quad |v'(t)| \le \gamma^*(t) \ \text{ for } \ t \in [a,b], \tag{7.30}$$

where $\rho_0 = \beta\big(\|\gamma\|_L + \alpha\big)$ and $\gamma^*(t) = \rho_0\eta(t) + \gamma(t)$ for $t \in [a,b]$.

Let $\Omega : C([a,b];R) \to C([a,b];R)$ be an operator which to every $u \in C([a,b];R)$ assigns the solution v to the problem (7.29). Due to Theorem 1.4 from [**15**], the operator Ω is continuous (see also [**10**, Theorem 3.2]). On the other hand, by virtue of (7.30), for every $u \in C([a,b];R)$ we have

$$\|\Omega(u)\|_C \le \rho_0, \quad \big|\Omega(u)(t) - \Omega(u)(s)\big| \le \left|\int\limits_s^t \gamma^*(\xi)\,d\xi\right| \text{ for } s,t \in [a,b].$$

Thus the operator Ω continuously maps the Banach space $C([a,b];R)$ into its relatively compact subset. Therefore, using the Schauder's principle, there exists $u \in C([a,b];R)$ such that

$$\Omega(u)(t) = u(t) \text{ for } t \in [a,b].$$

By the equalities (7.28), u is obviously a solution to the problem (7.24), (7.25) with

$$\delta = \sigma(\|u\|_C). \tag{7.31}$$

Now we will show that u admits the estimate (7.26). Suppose the contrary. Then either

$$\rho < \|u\|_C < 2\rho \tag{7.32}$$

or

$$\|u\|_C \ge 2\rho. \tag{7.33}$$

If we assume that the inequalities (7.32) are fulfilled, then, on account of (7.27) and (7.31), we have $\delta \in \,]0,1[\,$. However, by the assumptions of the lemma, in this case we have the estimate (7.26), which contradicts (7.32).

Suppose now that (7.33) is satisfied. Then by (7.27) and (7.31), we have $\delta = 0$. Hence u is a solution to the problem (7.23). But this is impossible because the problem (7.23) has only a trivial solution. Thus, above–obtained contradiction proves the validity of the estimate (7.26).

By virtue of (7.26), (7.27), and (7.31), it is clear that $\delta = 1$ and thus, u is a solution to the problem (7.21), (7.22). ☐

From Lemma 7.1 it immediately follows

Proposition 7.1. *Let* $b \in \,]a,+\infty[\,$, *and let there exist* $\rho > 0$*and an operator* $\ell \in \widetilde{\mathcal{L}}$ *such that the homogeneous problem* $(1.18_0), (1.19_0)$ *has only a trivial solution, and let for every* $\delta \in \,]0,1[$ *and for an arbitrary function* $u \in \widetilde{C}_0([a,+\infty[\,;R)$ *satisfying*

$$u'(t) = \widetilde{\ell}_b(u)(t) + \delta\big[\widetilde{F}_b(u)(t) - \widetilde{\ell}_b(u)(t)\big] \text{ for } t \ge a, \tag{7.34}$$

$$\widetilde{\omega}_b(u) = \delta \widetilde{h}_b(u), \tag{7.35}$$

the estimate

$$\|u\| \le \rho \tag{7.36}$$

hold. Then the problem (6.6), (6.7) *has at least one solution* u*, which satisfies* (7.36)*.*

Lemma 7.2. *Let there exist $\rho > 0$ and $b_0 \in \,]a, +\infty[$ such that for every $b \geq b_0$, the equation* (6.6) *has a solution u_b satisfying*

$$\|u_b\| \leq \rho. \tag{7.37}$$

Then the equation (6.1) *has at least one bounded solution u. Moreover, there exists a sequence $\{u_{b_n}\}_{n=1}^{+\infty} \subset \{u_b\}_{b \geq b_0}$ such that*

$$\lim_{n \to +\infty} u_{b_n}(t) = u_0(t) \quad \text{uniformly in } [a, +\infty[\,. \tag{7.38}$$

Proof. Since $F \in \mathcal{K}$, there exists $q \in L_{loc}([a, +\infty[\,; R_+)$ such that

$$|F(v)(t)| \leq q(t) \ \text{ for } \ t \geq a, \quad v \in C_{loc}([a, +\infty[\,; R), \quad \|v\| \leq \rho.$$

Therefore, since $\theta_b(u_b)(t) = u_b(t)$ for $t \geq a$, in view of (6.6) we have

$$|u_b(t) - u_b(s)| \leq \int\limits_s^t |\widetilde{F}_b(u_b)(\xi)| \, d\xi \leq$$

$$\leq \int\limits_s^t |F(u_b)(\xi)| \, d\xi \leq \int\limits_s^t q(\xi) \, d\xi \ \text{ for } \ a \leq s \leq t.$$

Consequently, the set of functions $\{u_b\}_{b \geq b_0}$ is uniformly bounded and equicontinuous on every compact subinterval of $[a, +\infty[\,$. According to Arzelà–Ascoli lemma, there exist a sequence $\{u_{b_n}\}_{n=1}^{+\infty} \subset \{u_b\}_{b \geq b_0}$ and a function $u \in C_{loc}([a, +\infty[\,; R)$ such that $\lim\limits_{n \to +\infty} b_n = +\infty$ and (7.38) is fulfilled. Obviously,

$$\theta_{b_n}(u_{b_n})(t) = u_{b_n}(t) \ \text{ for } \ t \geq a, \ \ n \in N,$$

and from (7.37) it follows that $\|u\| \leq \rho$. Moreover, the integration of (6.6) from a to t (with $u = u_{b_n}$) yields

$$u_{b_n}(t) = u_{b_n}(a) + \int\limits_a^t F(u_{b_n})(s) \, ds \ \text{ for } \ t \in [a, b_n], \ \ n \in N.$$

Consequently, with respect to (7.38) and the assumption $F \in \mathcal{K}$ we have

$$u(t) = u(a) + \int\limits_a^t F(u)(s) \, ds \ \text{ for } \ t \geq a,$$

i.e., $u \in \widetilde{C}_{loc}([a, +\infty[\,; R)$ and it is a bounded solution to the equation (6.1). ☐

Lemma 7.3. *Let $\ell \in \widetilde{\mathcal{L}}$ and let the only bounded solution to the problem* $(1.1_0), (1.2_0)$ *be a trivial solution. Let, moreover, there exist a continuous function $c : R_+ \to R_+$ such that the inequality* (7.1) *holds, and let on the set $\{v \in C_0([a, +\infty[\,; R) : \ |\omega(v)| \leq c(\|v\|)\}$ the inequalities* (7.2) *and* (7.3) *be fulfilled, where $q \in K_{loc}([a, +\infty[\, \times R_+; R)$, $\eta \in L_{loc}([a, +\infty[\,; R_+)$, and $\rho_1 > 0$. Assume also, that the functions c and q satisfy* (7.4) *and*

(7.5). *Then there exist* $\rho > 0$ *and* $b_0 \in \,]a, +\infty[$ *such that for every* $b \geq b_0$, *an arbitrary function* $u \in \widetilde{C}_0([a, +\infty[\,; R)$ *satisfying* (7.34) *and* (7.35) *with some* $\delta \in \,]0, 1[\,$, *admits the estimate* (7.36).

Proof. By virtue of Proposition 2.3 (see p. 14) there exists $b_* \in \,]a, +\infty[$ such that for every $b \geq b_*$ the problem $(1.18_0), (1.19_0)$ has only a trivial solution. We will show that there exist $\rho > 0$ and $b_0 \geq b_*$ such that for every $b \geq b_0$, an arbitrary function $u \in \widetilde{C}_0([a, +\infty[\,; R)$ satisfying (7.34) and (7.35) with some $\delta \in \,]0, 1[\,$, admits the estimate (7.36).

Assume on the contrary that for every $n \in N$ there exist $b_n \geq b_*$, $\delta_n \in \,]0, 1[\,$, and $u_{b_n} \in \widetilde{C}_0([a, +\infty[\,; R)$ satisfying (7.34) and (7.35) with $b = b_n$ and $\delta = \delta_n$, such that $\lim\limits_{n\to+\infty} b_n = +\infty$ and

$$\|u_{b_n}\| > n.$$

Obviously,

$$\lim_{n\to+\infty} \|u_{b_n}\| = +\infty. \tag{7.39}$$

Put

$$v_n(t) = \frac{u_{b_n}(t)}{\|u_{b_n}\|} \quad \text{for} \quad t \geq a, \quad n \in N. \tag{7.40}$$

Then

$$\|v_n\| = 1 \quad \text{for} \quad n \in N \tag{7.41}$$

and, in view of (7.34), (7.35), and (7.40), we have

$$v_n'(t) = \widetilde{\ell}_{b_n}(v_n)(t) + \frac{\delta_n}{\|u_{b_n}\|}\big[\widetilde{F}_{b_n}(u_{b_n})(t) - \widetilde{\ell}_{b_n}(u_{b_n})(t)\big] \quad \text{for} \quad t \geq a, \tag{7.42}$$

$$\widetilde{\omega}_{b_n}(v_n) = \frac{\delta_n}{\|u_{b_n}\|}\,\widetilde{h}_{b_n}(u_{b_n}). \tag{7.43}$$

Obviously, $\theta_{b_n}(v_n) \equiv v_n$ and $\theta_{b_n}(u_{b_n}) \equiv u_{b_n}$. Therefore, on account of (7.3), (7.39), (7.41), and the assumption $\ell \in \widetilde{\mathcal{L}}$, from (7.42) it follows that there exists $n_0 \in N$ such that

$$|v_n(t) - v_n(s)| \leq \int_s^t |v_n'(\xi)|\, d\xi \leq$$
$$\leq \int_s^t \overline{\ell}(1)(\xi)\, d\xi + \int_s^t \eta(\xi)\, d\xi \quad \text{for} \quad a \leq s \leq t, \quad n \geq n_0. \tag{7.44}$$

Therefore, the sequence $\{v_n\}_{n=n_0}^{+\infty}$ is uniformly bounded and equicontinuous on every compact subinterval of $[a, +\infty[\,$. According to Arzelà–Ascoli lemma we can assume, without loss of generality, that there exists $v_0 \in C_{loc}([a, +\infty[\,; R)$ such that

$$\lim_{n\to+\infty} v_n(t) = v_0(t) \quad \text{uniformly in} \quad [a, +\infty[\,. \tag{7.45}$$

Moreover, on account of (7.41), we have

$$\|v_0\| \le 1, \tag{7.46}$$

and the integration of (7.42) from a to t yields

$$v_n(t) = v_n(a) + \int_a^t \widetilde{\ell}_{b_n}(v_n)(s)\,ds + \\ + \frac{\delta_n}{\|u_{b_n}\|} \int_a^t \big[\widetilde{F}_{b_n}(u_{b_n})(s) - \widetilde{\ell}_{b_n}(u_{b_n})(s)\big]\,ds \quad \text{for} \quad n \in N. \tag{7.47}$$

On the other hand, by virtue of (7.1) and (7.2) we have

$$\frac{\delta_n}{\|u_{b_n}\|} \left| \int_a^t \big[\widetilde{F}_{b_n}(u_{b_n})(s) - \widetilde{\ell}_{b_n}(u_{b_n})(s)\big]\,ds \right| \le \\ \le \frac{1}{\|u_{b_n}\|} \sup\left\{ \left| \int_a^\xi q(s, \|u_{b_n}\|)\,ds \right| : \ \xi \ge a \right\} \quad \text{for} \quad t \ge a, \tag{7.48}$$

$$\frac{\delta_n}{\|u_{b_n}\|}\, \widetilde{h}_{b_n}(u_{b_n}) \le \frac{1}{\|u_{b_n}\|}\, c(\|u_{b_n}\|). \tag{7.49}$$

Now, from (7.47), in view of (7.5), (7.39), (7.45), and (7.48), we get

$$v_0(t) = v_0(a) + \int_a^t \ell(v_0)(s)\,ds \quad \text{for} \quad t \ge a. \tag{7.50}$$

Thus $v_0 \in \widetilde{C}_{loc}([a, +\infty[\,; R)$ and

$$|v_0(t) - v_0(s)| = \left| \int_s^t \ell(v_0)(\xi)\,d\xi \right| \le \|v_0\| \int_s^t \overline{\ell}(1)(\xi)\,d\xi \quad \text{for} \quad a \le s \le t.$$

The last inequality, together with (7.46) and the fact that $\overline{\ell} \in \widetilde{\mathcal{P}}$, ensures that there exists a finite limit $v_0(+\infty)$. Consequently, from (7.43), in view of (7.5), (7.39), (7.45), and (7.49) we also obtain

$$\omega(v_0) = 0. \tag{7.51}$$

Now (7.46), (7.50), and (7.51) imply that v_0 is a bounded solution to the problem $(1.1_0), (1.2_0)$. Therefore,

$$v_0 \equiv 0. \tag{7.52}$$

Since $\ell \in \widetilde{\mathcal{L}}$, we can choose $a_0 \in \,]a, +\infty[$ such that

$$\int_{a_0}^{+\infty} \overline{\ell}(1)(s)\,ds \le \frac{1}{5}\,. \tag{7.53}$$

According to (7.5), (7.39), (7.45), and (7.52) there exists $n_1 \in N$ such that

$$|v_n(t)| \le \frac{1}{5} \ \text{ for } \ t \in [a, a_0], \quad n \ge n_1, \tag{7.54}$$

$$\frac{1}{\|u_{b_n}\|} \sup\left\{ \Big| \int\limits_a^{\xi} q(s, \|u_{b_n}\|)\, ds \Big| :\ \xi \ge a \right\} \le \frac{1}{5} \ \text{ for } \ n \ge n_1. \tag{7.55}$$

On the other hand, from (7.47), in view of (7.41) and (7.48), we have

$$|v_n(t) - v_n(a_0)| \le$$
$$\le \int\limits_{a_0}^{+\infty} \bar{\ell}(1)(s)\, ds + \frac{2}{\|u_{b_n}\|} \sup\left\{ \Big| \int\limits_a^{\xi} q(s, \|u_{b_n}\|)\, ds \Big| :\ \xi \ge a \right\} \ \text{ for } \ t \ge a_0.$$

Hence, on account of (7.53)–(7.55), we obtain

$$|v_n(t)| \le \frac{4}{5} \ \text{ for } \ t \ge a, \ \ n \ge n_1$$

which contradicts (7.41). □

Lemma 7.4. *Let $\omega \in \mathrm{ch}$, $c \in R_+$, and let $v \in C_{loc}([a, +\infty[\,; R)$ be a function satisfying*

$$\omega(v) \operatorname{sgn} v(a) \le c.$$

Then there exists a sequence of functions $\{v_n\}_{n=1}^{+\infty} \subset C_0([a, +\infty[\,; R)$ such that

$$\omega(v_n) \operatorname{sgn} v_n(a) \le c \ \text{ for } \ n \in N \tag{7.56}$$

and

$$\lim_{n \to +\infty} v_n(t) = v(t) \ \text{ uniformly in } \ [a, +\infty[\,. \tag{7.57}$$

Proof. Let $\{b_k\}_{k=1}^{+\infty}$ be a sequence of numbers such that $b_k \ge a$ for $k \in N$, $\lim\limits_{k \to +\infty} b_k = +\infty$. If for every $n \in N$ there exists $k_n \in N$ such that $k_n \ge n$ and

$$\omega(\theta_{b_{k_n}}(v)) \operatorname{sgn} \theta_{b_{k_n}}(v)(a) \le c,$$

then we put

$$v_n(t) = \theta_{b_{k_n}}(v)(t) \ \text{ for } \ t \ge a, \ \ n \in N.$$

If there exists $k_0 \in N$ such that

$$\omega(\theta_{b_k}(v)) \operatorname{sgn} \theta_{b_k}(v)(a) > c \ \text{ for } \ k \ge k_0,$$

then for every $n \in N$ there exists $k_n \ge k_0$ such that

$$\omega(\theta_{b_{k_n}}(v)) \operatorname{sgn} \theta_{b_{k_n}}(v)(a) - \frac{1}{n} \le c,$$

because

$$\lim_{k \to +\infty} \omega(\theta_{b_k}(v)) \operatorname{sgn} \theta_{b_k}(v)(a) = \omega(v) \operatorname{sgn} v(a) \le c.$$

Put

$$\widetilde{v}_n(t) = \theta_{b_{k_n}}(v)(t) - \frac{\theta_{b_{k_0}}(v)(t)}{n\omega(\theta_{b_{k_0}}(v)) \operatorname{sgn} \theta_{b_{k_0}}(v)(a)} \quad \text{for } t \geq a, \ \ n \in N.$$

Obviously, there exists $n_0 \in N$ such that

$$\operatorname{sgn} \widetilde{v}_n(a) = \operatorname{sgn} v(a) \quad \text{for } n \geq n_0.$$

Thus we can put

$$v_n(t) = \widetilde{v}_{n+n_0}(t) \quad \text{for } t \geq a, \ \ n \in N.$$

Consequently, in both cases we have that (7.56) and (7.57) hold. □

Lemma 7.5. *Let $\omega \in$ ch, $c \in R_+$, and let there exist $\ell \in \widetilde{\mathcal{L}}$ such that on the set $\{v \in C_0([a,+\infty[\,;R) : \ \omega(v) \operatorname{sgn} v(a) \leq c\}$ the inequality* (7.18) *is fulfilled. Then on the set $\{v \in C_{loc}([a,+\infty[\,;R) : \ \omega(v) \operatorname{sgn} v(a) \leq c\}$ the inequality* (7.18) *holds, as well.*

Proof. Assume on the contrary that there exist $v_0, w_0 \in C_{loc}([a,+\infty[\,;R)$ satisfying $\omega(v_0) \operatorname{sgn} v_0(a) \leq c$ and $\omega(w_0) \operatorname{sgn} w_0(a) \leq c$ such that

$$\begin{aligned} \big[F(v_0)(t) - F(w_0)(t) - \ell(v_0 - w_0)(t)\big] \times \\ \times \operatorname{sgn}(v_0(t) - w_0(t)) > 0 \quad \text{for } t \in M, \end{aligned} \tag{7.58}$$

where $M \subseteq [a,+\infty[$ is a measurable set with $\operatorname{meas} M > 0$. According to Lemma 7.4 there exist sequences of functions $\{v_n\}_{n=1}^{+\infty}$, $\{w_n\}_{n=1}^{+\infty} \subset C_0([a,+\infty[\,;R)$ such that

$$\omega(v_n) \operatorname{sgn} v_n(a) \leq c, \quad \omega(w_n) \operatorname{sgn} w_n(a) \leq c \quad \text{for } n \in N, \tag{7.59}$$

and $\lim\limits_{n\to+\infty} v_n(t) = v_0(t)$, $\lim\limits_{n\to+\infty} w_n(t) = w_0(t)$ uniformly in $[a,+\infty[$. Put

$$M_n = [a, a+n] \cap M \quad \text{for } n \in N, \tag{7.60}$$

$$\begin{aligned} q_n(t) = \big[F(v_n)(t) - F(w_n)(t) - \ell(v_n - w_n)(t)\big] \times \\ \times \operatorname{sgn}(v_n(t) - w_n(t)) \quad \text{for } t \geq a, \ \ n \in N, \end{aligned} \tag{7.61}$$

$$\begin{aligned} q_0(t) = \big[F(v_0)(t) - F(w_0)(t) - \ell(v_0 - w_0)(t)\big] \times \\ \times \operatorname{sgn}(v_0(t) - w_0(t)) \quad \text{for } t \geq a. \end{aligned} \tag{7.62}$$

Obviously, there exists $n_0 \in N$ such that

$$\operatorname{meas} M_{n_0} > 0. \tag{7.63}$$

Furthermore, let

$$K_n = \left\{ t \in M_{n_0} : \ |v_0(t) - w_0(t)| \geq \frac{1}{n} \right\} \quad \text{for } n \in N.$$

Then, in view of (7.58), (7.60), and (7.63), there exists $n_1 \in N$ such that

$$\operatorname{meas} K_{n_1} > 0.$$

Put

$$\varepsilon \stackrel{def}{=} \int\limits_{K_{n_1}} q_0(s)\,ds. \tag{7.64}$$

Obviously, $\varepsilon > 0$ and there exists $n_2 \in N$ such that

$$\operatorname{sgn}(v_n(t) - w_n(t)) = \operatorname{sgn}(v_0(t) - w_0(t)) \ \text{ for } \ t \in K_{n_1}, \quad n \geq n_2.$$

Since $F \in \mathcal{K}$ and $\ell \in \widetilde{\mathcal{L}}$, there exists $n_3 \geq n_2$ such that

$$\int\limits_{K_{n_1}} |q_n(s) - q_0(s)|\,ds < \varepsilon \ \text{ for } \ n \geq n_3, \tag{7.65}$$

where ε is defined by (7.64). Moreover, because we assume that (7.18) is fulfilled on the set $\{v \in C_0([a,+\infty[\,;R) : \ \omega(v)\operatorname{sgn} v(a) \leq c\}$, in view of (7.59) and (7.61) we get

$$\int\limits_{K_{n_1}} q_n(s)\,ds \leq 0 \ \text{ for } \ n \in N. \tag{7.66}$$

On the other hand, on account of (7.65) and (7.66), we have

$$\int\limits_{K_{n_1}} q_0(s)\,ds \leq \int\limits_{K_{n_1}} q_{n_1}(s)\,ds + \int\limits_{K_{n_1}} |q_0(s) - q_{n_1}(s)|\,ds < \varepsilon,$$

which contradicts (7.64). □

7.3. Proofs of Theorems 7.1–7.6. *Proof of Theorem* 7.1. According to Lemmas 7.2 and 7.3, and Propositions 2.3 and 7.1, there exist a bounded solution u_0 to the equation (6.1) and a sequence of functions $\{u_n\}_{n=1}^{+\infty} \subset \widetilde{C}_0([a,+\infty[\,;R)$ such that

$$\omega(u_n) = h(u_n) \ \text{ for } \ n \in N, \tag{7.67}$$

and

$$\lim_{n\to+\infty} u_n(t) = u_0(t) \ \text{ uniformly in } \ [a,+\infty[\,. \tag{7.68}$$

Hence, in view of the fact that ω and h are continuous functionals on $C_{loc}([a,+\infty[\,;R)$, we get $\omega(u_0) = h(u_0)$, i.e., u_0 is a bounded solution to the problem (6.1), (6.2). □

Proof of Theorem 7.2. According to Lemmas 7.2 and 7.3, and Propositions 2.3 and 7.1, there exist a bounded solution u_0 to the equation (6.1) and a sequence of functions $\{u_n\}_{n=1}^{+\infty} \subset \widetilde{C}_0([a,+\infty[\,;R)$ such that (7.67) and (7.68) hold. If, moreover, there exists a finite limit (7.6) for every $v \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ with $\|v\| < +\infty$, then there exists a finite limit $u(+\infty)$, i.e., $u_0 \in \widetilde{C}_0([a,+\infty[\,;R)$. Hence, in view of the fact that ω and h are continuous functionals on $C_0([a,+\infty[\,;R)$, we get $\omega(u_0) = h(u_0)$, i.e., u_0 is a bounded solution to the problem (6.1), (6.2). □

Proof of Theorem 7.3. First note that due to the assumption $\ell \in \mathcal{A}(\omega)$, the only bounded solution to the problem $(1.1_0), (1.2_0)$ is a trivial solution. Therefore, according to Proposition 2.3 (see p. 14), there exists $b_0 \in]a, +\infty[$ such that for every $b \geq b_0$, the problem $(1.18_0), (1.19_0)$ has only a trivial solution.

Let r be the number appearing in Definition 7.1. According to (7.14), there exists $\rho > 0$ such that

$$\frac{1}{x}\left(c(x) + \int_a^{+\infty} q(s,x)\,ds\right) < \frac{1}{r} \quad \text{for} \quad x > \rho. \tag{7.69}$$

Let now $b \geq b_0$ be arbitrary but fixed and assume that a function $u \in \widetilde{C}_0([a, +\infty[\,; R)$ satisfies (7.34) and (7.35) with some $\delta \in\,]0, 1[$. Then, obviously, $\theta_b(u) \equiv u$, and, according to (7.12), u satisfies the inequality (7.7) with $c^* = c(\|u\|)$, i.e., $u \in \{v \in C_0([a, +\infty[\,; R) : \ \omega(v) \operatorname{sgn} v(a) \leq c(\|v\|)\}$. By virtue of (7.13) we have that u satisfies also the inequality (7.8) with $q^*(t) = q(t, \|u\|)$ for $t \geq a$. Hence, by the assumption $\ell \in \mathcal{A}(\omega)$ and the definition of the number ρ we get the estimate (7.36).

Since ρ depends neither on u nor on δ, it follows from Proposition 7.1 that the problem (6.6), (6.7) has at least one solution u. Obviously, this solution admits the estimate (7.36). Furthermore, since $b \geq b_0$ was chosen arbitrarily, according to Lemma 7.2, the equation (6.1) has a bounded solution u_0, and there exist a sequence of functions $\{u_n\}_{n=1}^{+\infty} \subset \widetilde{C}_0([a, +\infty[\,; R)$ such that (7.67) and (7.68) hold. Hence, since ω and h are continuous functionals on $C_{loc}([a, +\infty[\,; R)$, we get $\omega(u_0) = h(u_0)$. Therefore, u_0 is a bounded solution to the problem (6.1), (6.2). □

Proof of Theorem 7.4. First note that due to the assumption $\ell \in \mathcal{B}(\omega)$, the only bounded solution to the problem $(1.1_0), (1.2_0)$ is a trivial solution. Therefore, according to Proposition 2.3 (see p. 14), there exists $b_0 \in]a, +\infty[$ such that for every $b \geq b_0$, the problem $(1.18_0), (1.19_0)$ has only a trivial solution.

Let r be the number appearing in Definition 7.2. According to (7.14), there exists $\rho > 0$ such that (7.69) holds. Let now $b \geq b_0$ be arbitrary but fixed and assume that a function $u \in \widetilde{C}_0([a, +\infty[\,; R)$ satisfies (7.34) and (7.35) with some $\delta \in\,]0, 1[$. Then, obviously, $\theta_b(u) \equiv u$, and, according to (7.15), u satisfies the inequality (7.10) with $c^* = c(\|u\|)$, i.e., $u \in \{v \in C_0([a, +\infty[\,; R) : \ \omega(v) \operatorname{sgn} v(+\infty) \leq c(\|v\|)\}$. By virtue of (7.16) we have that u satisfies also the inequality (7.11) with $q^*(t) = q(t, \|u\|)$ for $t \geq a$. Hence, by the assumption $\ell \in \mathcal{B}(\omega)$ and the definition of the number ρ we get the estimate (7.36).

Since ρ depends neither on u nor on δ, it follows from Proposition 7.1 that the problem (6.6), (6.7) has at least one solution u. Obviously, this solution admits the estimate (7.36). Furthermore, since $b \geq b_0$ was chosen arbitrarily, according to Lemma 7.2, the equation (6.1) has a bounded solution u_0, and there exist a sequence of functions $\{u_n\}_{n=1}^{+\infty} \subset \widetilde{C}_0([a,+\infty[\,;R)$ such that (7.67) and (7.68) hold. If, moreover, there exists a finite limit (7.6) for every $v \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ with $\|v\| < +\infty$, then there exists a finite limit $u(+\infty)$, i.e., $u_0 \in \widetilde{C}_0([a,+\infty[\,;R)$. Hence, since ω and h are continuous functionals on $C_0([a,+\infty[\,;R)$, we get $\omega(u_0) = h(u_0)$. Therefore, u_0 is a bounded solution to the problem (6.1), (6.2). $\square$

Proof of Theorem 7.5. It follows from (7.17) that the condition (7.12) is fulfilled with $c \equiv |h(0)|$. By virtue of (7.18) we see that on the set $\{v \in C_0([a,+\infty[\,;R) :\ \omega(v)\operatorname{sgn} v(a) \leq |h(0)|\}$ the inequality (7.13) holds, where $q \equiv |F(0)|$. Consequently, the assumptions of Theorem 7.3 are fulfilled and so the problem (6.1), (6.2) has at least one bounded solution. It remains to show that the problem (6.1), (6.2) has at most one bounded solution.

Let u_1, u_2 be bounded solutions to the problem (6.1), (6.2), and let $b \in\,]a,+\infty[$ be arbitrary but fixed. Since $\theta_b(u_i) \in C_0([a,+\infty[\,;R)$ $(i=1,2)$, on account of (7.17) we get

$$\big[h(\theta_b(u_1)) - h(\theta_b(u_2))\big]\operatorname{sgn}(u_1(a)-u_2(a)) \leq 0,$$
$$h(\theta_b(u_i))\operatorname{sgn} u_i(a) \leq |h(0)| \ \text{ for } \ i=1,2.$$

Consequently, in view of (6.2), we have

$$\omega(u_i)\operatorname{sgn} u_i(a) = h(u_i)\operatorname{sgn} u_i(a) = \\ = \lim_{b\to+\infty} h(\theta_b(u_i))\operatorname{sgn} u_i(a) \leq |h(0)| \ \text{ for } \ i=1,2,$$

$$\big[h(u_1) - h(u_2)\big]\operatorname{sgn}(u_1(a)-u_2(a)) \leq 0. \tag{7.70}$$

Thus, according to Lemma 7.5 and (7.18), we obtain

$$\big[F(u_1)(t) - F(u_2)(t) - \ell(u_1-u_2)(t)\big]\times \\ \times \operatorname{sgn}(u_1(t)-u_2(t)) \leq 0 \ \text{ for } \ t \geq a. \tag{7.71}$$

Therefore, from (6.1) and (6.2), on account of (7.70) and (7.71) we obtain

$$\omega(u_1-u_2)\operatorname{sgn}(u_1(a)-u_2(a)) \leq 0, \tag{7.72}$$

$$\big[u_1'(t) - u_2'(t) - \ell(u_1-u_2)(t)\big]\operatorname{sgn}(u_1(t)-u_2(t)) \leq 0 \ \text{ for } \ t \geq a. \tag{7.73}$$

Furthermore, from (7.73) it follows that

$$|u_1(t)-u_2(t)|' \leq |\ell(u_1-u_2)(t)| \ \text{ for } \ t \geq a,$$

whence, on account of the assumption $\ell \in \widetilde{\mathcal{L}}$, we get

$$\big||w(t)| - |w(s)|\big| \leq \|w\| \int_s^t \overline{\ell}(1)(\xi)\,d\xi \ \text{ for } \ a \leq s \leq t,$$

where $w(t) = u_1(t) - u_2(t)$ for $t \geq a$. Consequently, $w \in \widetilde{C}_0([a,+\infty[\,;R)$, and by virtue of (7.72), (7.73), and the assumption $\ell \in \mathcal{A}(\omega)$, we have $\|w\| = 0$, i.e., $u_1 \equiv u_2$. $\square$

Proof of Theorem 7.6. It follows from (7.19) that the condition (7.15) is fulfilled with $c \equiv |h(0)|$. By virtue of (7.20) we see that on the set $\{v \in C_0([a,+\infty[\,;R) : \ \omega(v)\operatorname{sgn} v(+\infty) \le |h(0)|\}$ the inequality (7.16) holds, where $q \equiv |F(0)|$. Consequently, the assumptions of Theorem 7.4 are fulfilled and so the problem (6.1), (6.2) has at least one bounded solution. It remains to show that the problem (6.1), (6.2) has at most one bounded solution.

Let u_1, u_2 be bounded solutions to the problem (6.1), (6.2). Then, obviously, $u_1, u_2 \in \widetilde{C}_0([a,+\infty[\,;R)$. Consequently, by virtue of (7.19) and (7.20), we have

$$\omega(u_1 - u_2)\operatorname{sgn}(u_1(+\infty) - u_2(+\infty)) \le 0, \tag{7.74}$$

$$\big[u_1'(t) - u_2'(t) - \ell(u_1 - u_2)(t)\big]\operatorname{sgn}(u_1(t) - u_2(t)) \ge 0 \quad \text{for} \ \ t \ge a. \tag{7.75}$$

Now, since $u_1 - u_2 \in \widetilde{C}_0([a,+\infty[\,;R)$ and $\ell \in \mathcal{B}(\omega)$, by virtue of (7.74) and (7.75) we obtain $\|u_1 - u_2\| = 0$, i.e., $u_1 \equiv u_2$. □

8. Existence and Uniqueness of Bounded Solutions

In what follows, we will always assume that $q \in K([a,+\infty[\,\times R_+; R_+)$, $c : R_+ \to R_+$ is a continuous function, and

$$\lim_{x\to+\infty} \frac{1}{x}\left(c(x) + \int_a^{+\infty} q(s,x)\,ds\right) = 0.$$

Theorem 8.1. *Let the inequality* (7.12) *hold, and let there exist* $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ *such that on the set* $\{v \in C_0([a,+\infty[\,;R) : \ |v(a)| \le c(\|v\|)\}$ *the inequality*

$$\big[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)\big]\operatorname{sgn} v(t) \le q(t, \|v\|) \quad \textit{for} \ \ t \ge a \tag{8.1}$$

is fulfilled. If, moreover, there exists $\gamma \in \widetilde{C}_0([a,+\infty[\,;\,]0,+\infty[\,)$ *satisfying*

$$\gamma'(t) \ge \ell_0(\gamma)(t) + \ell_1(1)(t) \quad \textit{for} \ \ t \ge a, \tag{8.2}$$

$$\gamma(+\infty) - \gamma(a) < 2, \tag{8.3}$$

then the problem (6.1), (6.3) *has at least one bounded solution.*

Remark 8.1. Theorem 8.1 is unimprovable in the sense that the inequality (8.3) cannot be replaced by the nonstrict one (see On Remark 8.1, p. 89).

Theorem 8.2. *Let the inequality* (7.12) *hold, and let there exist* $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ *such that on the set* $\{v \in C_0([a,+\infty[\,;R) : \ |v(a)| \le c(\|v\|)\}$ *the inequality*

(8.1) *is fulfilled. If, moreover,*

$$\int_a^{+\infty} \ell_0(1)(s)\,ds < 1, \tag{8.4}$$

$$\int_a^{+\infty} \ell_1(1)(s)\,ds < 2\sqrt{1 - \int_a^{+\infty} \ell_0(1)(s)\,ds}\,, \tag{8.5}$$

then the problem (6.1), (6.3) *has at least one bounded solution.*

Remark 8.2. Denote by D_a the set of pairs $(x, y) \in R_+ \times R_+$ such that

$$x < 1, \quad y < 2\sqrt{1-x}$$

(see Figure 8.1).

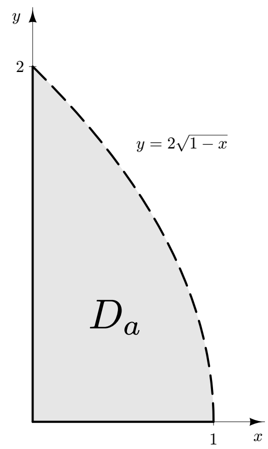

Figure 8.1

According to Theorem 8.2, if (7.12) holds, there exist $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ such that the inequality (8.1) is satisfied on the set $\{v \in C_0([a,+\infty[\,;R) : |v(a)| \le c(\|v\|)\}$, and

$$\left(\int_a^{+\infty} \ell_0(1)(s)\,ds, \int_a^{+\infty} \ell_1(1)(s)\,ds \right) \in D_a,$$

then the problem (6.1), (6.3) has at least one bounded solution. Below we will show (see On Remark 8.2, p. 90) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin D_a$ there exist $F \in \mathcal{K}$ and $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ such that (7.12) (with $h \equiv 0$) and (8.1) hold,

$$\int_a^{+\infty} \ell_0(1)(s)\,ds = x_0, \qquad \int_a^{+\infty} \ell_1(1)(s)\,ds = y_0,$$

and the problem (6.1), (6.3) (with $h \equiv 0$) has no solution. In particular, the strict inequalities (8.4) and (8.5) cannot be replaced by the nonstrict ones.

Theorem 8.3. *Let the inequality* (7.15) *hold, and let there exist* $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ *such that on the set* $\{v \in C_0([a,+\infty[\,;R) : \ |v(+\infty)| \leq c(\|v\|)\}$ *the inequality*

$$\big[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)\big] \operatorname{sgn} v(t) \geq -q(t, \|v\|) \quad \text{for } t \geq a \tag{8.6}$$

is fulfilled. If, moreover, there exists $\gamma \in \widetilde{C}_0([a,+\infty[\,;\,]0,+\infty[\,)$ *satisfying*

$$-\gamma'(t) \geq \ell_1(\gamma)(t) + \ell_0(1)(t) \quad \text{for } t \geq a, \tag{8.7}$$

$$\gamma(+\infty) > 0, \tag{8.8}$$

$$\gamma(a) - \gamma(+\infty) < 2, \tag{8.9}$$

then the equation (6.1) *has at least one bounded solution. If, moreover, for every* $v \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ *with* $\|v\| < +\infty$*, there exist a finite limit* (7.6)*, then the problem* (6.1)*,* (6.4) *has at least one bounded solution.*

Remark 8.3. The Example 5.2 (see p. 41) shows that the condition (8.8) is essential and it cannot be omitted even in the linear case. Furthermore, Theorem 8.1 is unimprovable in the sense that the inequality (8.9) cannot be replaced by the nonstrict one (see On Remark 8.3, p. 92).

Theorem 8.4. *Let the inequality* (7.15) *hold, and let there exist* $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ *such that on the set* $\{v \in C_0([a,+\infty[\,;R) : \ |v(+\infty)| \leq c(\|v\|)\}$ *the inequality* (8.6) *is fulfilled. If, moreover,*

$$\int_a^{+\infty} \ell_1(1)(s)\,ds < 1, \tag{8.10}$$

$$\int_a^{+\infty} \ell_0(1)(s)\,ds < 2\sqrt{1 - \int_a^{+\infty} \ell_1(1)(s)\,ds}\,, \tag{8.11}$$

then the equation (6.1) *has at least one bounded solution. If, moreover, for every* $v \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ *with* $\|v\| < +\infty$*, there exist a finite limit* (7.6)*, then the problem* (6.1)*,* (6.4) *has at least one bounded solution.*

Remark 8.4. Denote by $D_{+\infty}$ the set of pairs $(x,y) \in R_+ \times R_+$ such that

$$y < 1, \quad x < 2\sqrt{1-y}$$

(see Figure 8.2).

According to Theorem 8.4, if (7.15) holds, there exist $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ such that the inequality (8.6) is satisfied on the set $\{v \in C_0([a,+\infty[\,;R) : |v(+\infty)| \leq c(\|v\|)\}$, and

$$\bigg(\int_a^{+\infty} \ell_0(1)(s)\,ds, \int_a^{+\infty} \ell_1(1)(s)\,ds\bigg) \in D_{+\infty},$$

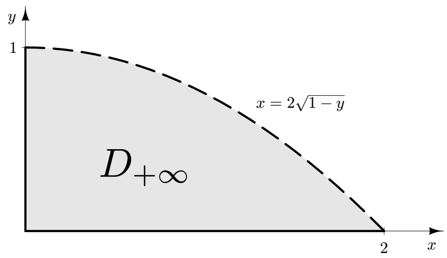

Figure 8.2

and for every $v \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ with $\|v\| < +\infty$, there exist a finite limit (7.6), then the problem (6.1), (6.4) has at least one bounded solution. Below we will show (see On Remark 8.4, p. 93) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin D_{+\infty}$, there exist $F \in \mathcal{K}$ and $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ such that (7.15) (with $h \equiv 0$) and (8.6) hold,

$$\int_a^{+\infty} \ell_0(1)(s)\,ds = x_0, \qquad \int_a^{+\infty} \ell_1(1)(s)\,ds = y_0,$$

and the problem (6.1), (6.4) (with $h \equiv 0$) has no solution. In particular, the strict inequalities (8.10) and (8.11) cannot be replaced by the nonstrict ones.

Theorem 8.5. *Let the inequality* (7.12) *hold, and let there exist* $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ *such that on the set* $\{v \in C_0([a,+\infty[\,;R) : \ (v(a) - v(+\infty))\operatorname{sgn} v(a) \le c(\|v\|)\}$ *the inequality* (8.1) *is fulfilled. If, moreover,*

$$\int_a^{+\infty} \ell_0(1)(s)\,ds < 1, \tag{8.12}$$

$$\frac{\int_a^{+\infty} \ell_0(1)(s)\,ds}{1 - \int_a^{+\infty} \ell_0(1)(s)\,ds} < \int_a^{+\infty} \ell_1(1)(s)\,ds < 2\sqrt{1 - \int_a^{+\infty} \ell_0(1)(s)\,ds}\,, \tag{8.13}$$

then the equation (6.1) *has at least one bounded solution. If, moreover, for every* $v \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ *with* $\|v\| < +\infty$, *there exist a finite limit* (7.6), *then the problem* (6.1), (6.5) *has at least one bounded solution.*

Remark 8.5. Denote by D_p^+ the set of pairs $(x,y) \in R_+ \times R_+$ such that

$$x < 1, \quad \frac{x}{1-x} < y < 2\sqrt{1-x}$$

(see Figure 8.3).

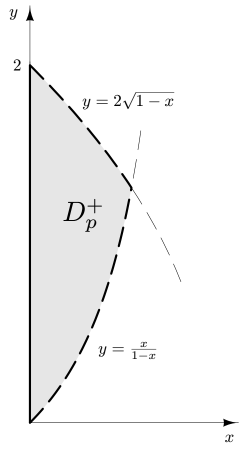

Figure 8.3

According to Theorem 8.5, if (7.12) holds, there exist $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ such that the inequality (8.1) is satisfied on the set $\{v \in C_0([a,+\infty[\,;R) :\ (v(a) - v(+\infty)) \operatorname{sgn} v(a) \le c(\|v\|)\}$, and

$$\Bigg(\int\limits_a^{+\infty} \ell_0(1)(s)\,ds, \int\limits_a^{+\infty} \ell_1(1)(s)\,ds \Bigg) \in D_p^+,$$

and for every $v \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ with $\|v\| < +\infty$, there exist a finite limit (7.6), then the problem (6.1), (6.5) has at least one bounded solution. Below we will show (see On Remark 8.5, p. 94) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin D_p^+$ there exist $F \in \mathcal{K}$ and $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ such that (7.12) (with $h \equiv 0$) and (8.1) hold,

$$\int\limits_a^{+\infty} \ell_0(1)(s)\,ds = x_0, \qquad \int\limits_a^{+\infty} \ell_1(1)(s)\,ds = y_0,$$

and the problem (6.1), (6.5) (with $h \equiv 0$) has no solution. In particular, the strict inequalities (8.12) and (8.13) cannot be replaced by the nonstrict ones.

Theorem 8.6. *Let the inequality* (7.15) *hold, and let there exist* $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ *such that on the set* $\{v \in C_0([a,+\infty[\,;R) :\ (v(a) - v(+\infty)) \operatorname{sgn} v(+\infty) \le$

$c(\|v\|)\}$ *the inequality* (8.6) *is fulfilled. If, moreover,*

$$\int_a^{+\infty} \ell_1(1)(s)\,ds < 1, \tag{8.14}$$

$$\frac{\int_a^{+\infty} \ell_1(1)(s)\,ds}{1 - \int_a^{+\infty} \ell_1(1)(s)\,ds} < \int_a^{+\infty} \ell_0(1)(s)\,ds < 2\sqrt{1 - \int_a^{+\infty} \ell_1(1)(s)\,ds}\,, \tag{8.15}$$

then the equation (6.1) *has at least one bounded solution. If, moreover, for every* $v \in \widetilde{C}_{loc}([a, +\infty[\,; R)$ *with* $\|v\| < +\infty$, *there exist a finite limit* (7.6), *then the problem* (6.1), (6.5) *has at least one bounded solution.*

Remark 8.6. Denote by D_p^- the set of pairs $(x, y) \in R_+ \times R_+$ such that

$$y < 1, \qquad \frac{y}{1-y} < x < 2\sqrt{1-y}$$

(see Figure 8.4).

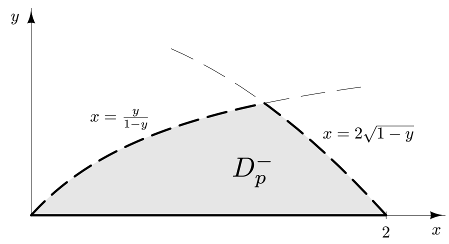

Figure 8.4.

According to Theorem 8.6, if (7.15) holds, there exist $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ such that the inequality (8.6) is satisfied on the set $\{v \in C_0([a, +\infty[\,; R) : \ (v(a) - v(+\infty)) \operatorname{sgn} v(+\infty) \le c(\|v\|)\}$, and

$$\left(\int_a^{+\infty} \ell_0(1)(s)\,ds, \int_a^{+\infty} \ell_1(1)(s)\,ds \right) \in D_p^-,$$

and for every $v \in \widetilde{C}_{loc}([a, +\infty[\,; R)$ with $\|v\| < +\infty$, there exist a finite limit (7.6), then the problem (6.1), (6.5) has at least one bounded solution. Below we will show (see On Remark 8.6, p. 97) that for every $x_0, y_0 \in R_+$, $(x_0, y_0) \notin D_p^-$, there exist $F \in \mathcal{K}$ and $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ such that (7.15) (with $h \equiv 0$) and (8.6) hold,

$$\int_a^{+\infty} \ell_0(1)(s)\,ds = x_0, \qquad \int_a^{+\infty} \ell_1(1)(s)\,ds = y_0,$$

and the problem (6.1), (6.5) (with $h \equiv 0$) has no solution. In particular, the strict inequalities (8.14) and (8.15) cannot be replaced by the nonstrict ones.

In Theorems 8.7–8.12, the conditions guaranteeing the existence of a unique bounded solution to the problems (6.1), (6.k) ($k = 3, 4, 5$) are established.

Theorem 8.7. *Let the inequality* (7.17) *hold,* $F(0) \in L([a, +\infty[\,; R)$, *and let there exist* $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ *such that on the set* $\{v \in C_0([a, +\infty[\,; R) : |v(a)| \leq |h(0)|\}$ *the inequality*

$$\big[F(v)(t) - F(w)(t) - \ell_0(v - w)(t) + \ell_1(v - w)(t)\big] \times \\ \times \operatorname{sgn}(v(t) - w(t)) \leq 0 \ \ for \ \ t \geq a \quad (8.16)$$

is fulfilled. If, moreover, there exists $\gamma \in \widetilde{C}_0([a, +\infty[\,;]0, +\infty[\,)$ *satisfying* (8.2) *and* (8.3)*, then the problem* (6.1), (6.3) *has a unique bounded solution.*

Remark 8.7. The example constructed in Section 11 (see On Remark 8.1, p. 89) also shows that the strict inequality (8.3) cannot be replaced by the nonstrict one.

Theorem 8.8. *Let the inequality* (7.17) *hold,* $F(0) \in L([a, +\infty[\,; R)$, *and let there exist* $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ *such that on the set* $\{v \in C_0([a, +\infty[\,; R) : |v(a)| \leq |h(0)|\}$ *the inequality* (8.16) *is fulfilled. If, moreover, the inequalities* (8.4) *and* (8.5) *hold, then the problem* (6.1), (6.3) *has a unique bounded solution.*

Remark 8.8. Examples constructed in Section 11 (see On Remark 8.2, p. 90) also show that neither one of the strict inequalities (8.4) and (8.5) can be replaced by the nonstrict one.

Theorem 8.9. *Let* (7.19) *hold,* $F(0) \in L([a, +\infty[\,; R)$, *and let there exist* $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ *such that on the set* $\{v \in C_0([a, +\infty[\,; R) : |v(+\infty)| \leq |h(0)|\}$ *the inequality*

$$\big[F(v)(t) - F(w)(t) - \ell_0(v - w)(t) + \ell_1(v - w)(t)\big] \times \\ \times \operatorname{sgn}(v(t) - w(t)) \geq 0 \ \ for \ \ t \geq a \quad (8.17)$$

is fulfilled. If, moreover, there exists $\gamma \in \widetilde{C}_0([a, +\infty[\,;]0, +\infty[\,)$ *satisfying* (8.7)–(8.9)*, and for every* $v \in \widetilde{C}_{loc}([a, +\infty[\,; R)$ *with* $\|v\| < +\infty$*, there exists a finite limit* (7.6)*, then the problem* (6.1), (6.4) *has a unique bounded solution.*

Remark 8.9. The Example 5.2 (see p. 41) shows that the condition (8.8) is essential and it cannot be omitted even in the linear case. Furthermore, the example constructed in Section 11 (see On Remark 8.3, p. 92) also shows that the strict inequality (8.9) cannot be replaced by the nonstrict one.

Theorem 8.10. *Let the inequality* (7.19) *hold,* $F(0) \in L([a,+\infty[\,;R)$, *and let there exist* $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ *such that on the set* $\{v \in C_0([a,+\infty[\,;R) : |v(+\infty)| \le |h(0)|\}$ *the inequality* (8.17) *is fulfilled. If, moreover, the inequalities* (8.10) *and* (8.11) *hold, and for every* $v \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ *with* $\|v\| < +\infty$*, there exists a finite limit* (7.6)*, then the problem* (6.1), (6.4) *has a unique bounded solution.*

Remark 8.10. The examples constructed in Section 11 (see On Remark 8.4, p. 93) also show that neither one of the strict inequalities (8.10) and (8.11) can be replaced by the nonstrict one.

Theorem 8.11. *Let the inequality* (7.17) *hold,* $F(0) \in L([a,+\infty[\,;R)$, *and let there exist* $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ *such that on the set* $\{v \in C_0([a,+\infty[\,;R) : (v(a)-v(+\infty))\operatorname{sgn} v(a) \le |h(0)|\}$ *the inequality* (8.16) *is fulfilled. If, moreover, the inequalities* (8.12) *and* (8.13) *hold, and for every* $v \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ *with* $\|v\| < +\infty$*, there exists a finite limit* (7.6)*, then the problem* (6.1), (6.5) *has a unique bounded solution.*

Remark 8.11. The examples constructed in Section 11 (see On Remark 8.5, p. 94) also show that neither one of the strict inequalities in (8.12) and (8.13) can be replaced by the nonstrict one.

Theorem 8.12. *Let the inequality* (7.19) *hold,* $F(0) \in L([a,+\infty[\,;R)$, *and let there exist* $\ell_0, \ell_1 \in \widetilde{\mathcal{P}}$ *such that on the set* $\{v \in C_0([a,+\infty[\,;R) : (v(a) - v(+\infty))\operatorname{sgn} v(+\infty) \le |h(0)|\}$ *the inequality* (8.17) *is fulfilled. If, moreover,* (8.14) *and* (8.15) *hold, and for every* $v \in \widetilde{C}_{loc}([a,+\infty[\,;R)$ *with* $\|v\| < +\infty$*, there exists a finite limit* (7.6)*, then the problem* (6.1), (6.5) *has a unique bounded solution.*

Remark 8.12. The examples constructed in Section 11 (see On Remark 8.6, p. 97) also show that neither one of the strict inequalities in (8.14) and (8.15) can be replaced by the nonstrict one.

9. Proofs of Theorems 8.1–8.12

Definition 9.1. Let $\widehat{\omega} : C([a,b];R) \to R$ be a linear bounded functional. We will say that an operator $\widehat{\ell} \in \mathcal{L}_{ab}$ belongs to the set $\mathcal{A}_{ab}(\widehat{\omega})$, if there exists $r > 0$ such that for every $q^* \in L([a,b];R_+)$ and $c^* \in R_+$, an arbitrary function $u \in \widetilde{C}([a,b];R)$ satisfying the inequalities

$$\widehat{\omega}(u)\operatorname{sgn} u(a) \le c^*, \tag{9.1}$$

$$\big[u'(t) - \widehat{\ell}(u)(t)\big]\operatorname{sgn} u(t) \le q^*(t) \quad \text{for} \quad t \in [a,b], \tag{9.2}$$

admits the estimate

$$\|u\|_C \le r\bigg(c^* + \int_a^b q^*(s)\,ds\bigg). \tag{9.3}$$

Definition 9.2. Let $\widehat{\omega} : C([a,b];R) \to R$ be a linear bounded functional. We will say that an operator $\widehat{\ell} \in \mathcal{L}_{ab}$ belongs to the set $\mathcal{B}_{ab}(\widehat{\omega})$, if there exists $r > 0$ such that for every $q^* \in L([a,b];R_+)$ and $c^* \in R_+$, an arbitrary function $u \in \widetilde{C}([a,b];R)$ satisfying the inequalities

$$\widehat{\omega}(u)\operatorname{sgn} u(b) \le c^*, \tag{9.4}$$

$$\big[u'(t) - \widehat{\ell}(u)(t)\big]\operatorname{sgn} u(t) \ge -q^*(t) \ \text{ for } \ t \in [a,b], \tag{9.5}$$

admits the estimate (9.3).

Remark 9.1. Let $b > a$, $\ell \in \widetilde{\mathcal{L}}$, $\omega \in \mathrm{ch}$, resp. $\omega \in \mathcal{W}_0$, and let the operators $\widehat{\ell}$ and $\widehat{\omega}$ be defined by (1.9) and (1.10), respectively, with φ and ψ given by (1.6)–(1.8). Then it is not difficult to verify that $\ell \in \mathcal{A}(\omega)$ ($\ell \in \mathcal{B}(\omega)$) if and only if $\widehat{\ell} \in \mathcal{A}_{ab}(\widehat{\omega})$ ($\widehat{\ell} \in \mathcal{B}_{ab}(\widehat{\omega})$).

To prove Theorems 8.1–8.12, we will need some auxiliary propositions. First we formulate several assertions from [**9**] in a suitable for us form.

Lemma 9.1 ([**9**, Lemma 12.4, p. 228])**.** *Let the operator $\widehat{\ell} \in \mathcal{L}_{ab}$ admit the representation $\widehat{\ell} = \widehat{\ell}_0 - \widehat{\ell}_1$, where $\widehat{\ell}_0, \widehat{\ell}_1 \in \mathcal{P}_{ab}$. If there exists a function $\widehat{\gamma} \in \widetilde{C}([a,b];\,]0,+\infty[\,)$ satisfying the inequalities*

$$\widehat{\gamma}'(t) \ge \widehat{\ell}_0(\widehat{\gamma})(t) + \widehat{\ell}_1(1)(t) \ \text{ for } \ t \in [a,b], \tag{9.6}$$

$$\widehat{\gamma}(b) - \widehat{\gamma}(a) < 2, \tag{9.7}$$

then $\widehat{\ell} \in \mathcal{A}_{ab}(\widehat{\omega})$, where $\widehat{\omega}(v) \equiv v(a)$.

Lemma 9.2. *Let the operator $\widehat{\ell} \in \mathcal{L}_{ab}$ admit the representation $\widehat{\ell} = \widehat{\ell}_0 - \widehat{\ell}_1$, where $\widehat{\ell}_0, \widehat{\ell}_1 \in \mathcal{P}_{ab}$. If there exists a function $\widehat{\gamma} \in \widetilde{C}([a,b];\,]0,+\infty[\,)$ satisfying the inequalities*

$$-\widehat{\gamma}'(t) \ge \widehat{\ell}_1(\widehat{\gamma})(t) + \widehat{\ell}_0(1)(t) \ \text{ for } \ t \in [a,b],$$

$$\widehat{\gamma}(a) - \widehat{\gamma}(b) < 2,$$

then $\widehat{\ell} \in \mathcal{B}_{ab}(\widehat{\omega})$, where $\widehat{\omega}(v) \equiv v(b)$.

Proof. The assertion follows from Remark 2.16 and Lemma 12.4 in [**9**] (see pp. 28 and 228 therein). $\square$

Lemma 9.3 ([**9**, Lemma 12.1, p. 211])**.** *Let the operator $\widehat{\ell} \in \mathcal{L}_{ab}$ admit the representation $\widehat{\ell} = \widehat{\ell}_0 - \widehat{\ell}_1$, where $\widehat{\ell}_0, \widehat{\ell}_1 \in \mathcal{P}_{ab}$. If*

$$\int_a^b \widehat{\ell}_0(1)(s)\,ds < 1, \tag{9.8}$$

$$\frac{\int_a^b \widehat{\ell}_0(1)(s)\,ds}{1 - \int_a^b \widehat{\ell}_0(1)(s)\,ds} < \int_a^b \widehat{\ell}_1(1)(s)\,ds < 2\sqrt{1 - \int_a^b \widehat{\ell}_0(1)(s)\,ds}\,, \tag{9.9}$$

then $\widehat{\ell} \in \mathcal{A}_{ab}(\widehat{\omega})$, *where* $\widehat{\omega}(v) \equiv v(a) - v(b)$.

Lemma 9.4 ([**9**, Lemma 12.2, p. 219])**.** *Let the operator* $\widehat{\ell} \in \mathcal{L}_{ab}$ *admit the representation* $\widehat{\ell} = \widehat{\ell}_0 - \widehat{\ell}_1$,*where* $\widehat{\ell}_0, \widehat{\ell}_1 \in \mathcal{P}_{ab}$. *If*

$$\int_a^b \widehat{\ell}_1(1)(s)\, ds < 1,$$

$$\frac{\int_a^b \widehat{\ell}_1(1)(s)\, ds}{1 - \int_a^b \widehat{\ell}_1(1)(s)\, ds} < \int_a^b \widehat{\ell}_0(1)(s)\, ds < 2\sqrt{1 - \int_a^b \widehat{\ell}_1(1)(s)\, ds}\,,$$

then $\widehat{\ell} \in \mathcal{B}_{ab}(\widehat{\omega})$, *where* $\widehat{\omega}(v) \equiv v(b) - v(a)$.

Lemma 9.5. *Let the operator* $\widehat{\ell} \in \mathcal{L}_{ab}$ *admit the representation* $\widehat{\ell} = \widehat{\ell}_0 - \widehat{\ell}_1$, *where* $\widehat{\ell}_0, \widehat{\ell}_1 \in \mathcal{P}_{ab}$. *If*

$$\int_a^b \widehat{\ell}_0(1)(s)\, ds < 1, \tag{9.10}$$

$$\int_a^b \widehat{\ell}_1(1)(s)\, ds < 2\sqrt{1 - \int_a^b \widehat{\ell}_0(1)(s)\, ds}\,, \tag{9.11}$$

then $\widehat{\ell} \in \mathcal{A}_{ab}(\widehat{\omega})$, *where* $\widehat{\omega}(v) \equiv v(a)$.

Proof. Let $q^* \in L([a,b]; R_+)$, $c \in R_+$, and $u \in \widetilde{C}([a,b]; R)$ satisfy (9.1), (9.2) with $\widehat{\omega}(u) = u(a)$. We will show that (9.3) holds with

$$r = \frac{1}{1 - \|\widehat{\ell}_0(1)\|_L} + \frac{1 + \|\widehat{\ell}_1(1)\|_L}{1 - \|\widehat{\ell}_0(1)\|_L - \frac{1}{4}\|\widehat{\ell}_1(1)\|_L^2}\,. \tag{9.12}$$

Obviously, u satisfies

$$u'(t) = \widehat{\ell}_0(u)(t) - \widehat{\ell}_1(u)(t) + \overline{q}(t) \quad \text{for} \quad t \in [a,b], \tag{9.13}$$

where

$$\overline{q}(t) = u'(t) - \widehat{\ell}(u)(t) \quad \text{for} \quad t \in [a,b]. \tag{9.14}$$

Obviously,

$$\overline{q}(t) \operatorname{sgn} u(t) \le q^*(t) \quad \text{for} \quad t \in [a,b], \tag{9.15}$$

and

$$|u(a)| \le c^*. \tag{9.16}$$

First suppose that u does not assume both positive and negative values (u is still nonnegative or still nonpositive). Put

$$\overline{M} = \max\big\{|u(t)| : \ t \in [a,b]\big\}, \tag{9.17}$$

and choose $t_0 \in [a,b]$ such that

$$|u(t_0)| = \overline{M}. \tag{9.18}$$

Obviously, $\overline{M} \geq 0$. Due to (9.15), (9.17), and the assumptions $\widehat{\ell}_0, \widehat{\ell}_1 \in \mathcal{P}_{ab}$, the equality (9.13) implies

$$|u(t)|' \leq \overline{M}\widehat{\ell}_0(1)(t) + q^*(t) \quad \text{for} \quad t \in [a,b]. \tag{9.19}$$

Now the integration of (9.19) from a to t_0, in view of (9.16) and (9.18), results in

$$\overline{M} \leq \overline{M} \int_a^{t_0} \widehat{\ell}_0(1)(s)\,ds + c^* + \int_a^{t_0} q^*(s)\,ds.$$

The last inequality, by virtue of (9.10), (9.17), and the assumptions $\widehat{\ell}_0 \in \mathcal{P}_{ab}$, $q^* \in L([a,b];R_+)$, implies

$$\|u\|_C \leq \left(1 - \int_a^b \widehat{\ell}_0(1)(s)\,ds\right)^{-1} \left(c^* + \int_a^b q^*(s)\,ds\right),$$

i.e., the estimate (9.3) holds with r defined by (9.12).

Now suppose that u assumes both positive and negative values. Put

$$M = \max\big\{u(t):\ t \in [a,b]\big\}, \quad m = -\min\big\{u(t):\ t \in [a,b]\big\}, \tag{9.20}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \quad u(t_m) = -m. \tag{9.21}$$

Obviously, $M > 0$, $m > 0$, and either

$$t_m < t_M \tag{9.22}$$

or

$$t_m > t_M. \tag{9.23}$$

First assume that (9.22) is fulfilled. It is clear that there exists $\alpha_2 \in\,]t_m, t_M[$ such that

$$u(t) > 0 \quad \text{for} \quad \alpha_2 < t \leq t_M, \quad u(\alpha_2) = 0. \tag{9.24}$$

Let

$$\alpha_1 = \inf\Big\{t \in [a, t_m]:\ u(s) < 0 \ \text{ for } t \leq s \leq t_m\Big\}. \tag{9.25}$$

Obviously,

$$u(t) < 0 \quad \text{for} \quad \alpha_1 < t \leq t_m \tag{9.26}$$

and

$$\text{if} \quad \alpha_1 > a, \quad \text{then} \quad u(\alpha_1) = 0. \tag{9.27}$$

The integration of (9.13) from α_1 to t_m and from α_2 to t_M, in view of (9.15), (9.20), (9.21), (9.24), and (9.26), yields

$$u(\alpha_1)+m\leq M\int\limits_{\alpha_1}^{t_m}\widehat{\ell}_1(1)(s)\,ds+m\int\limits_{\alpha_1}^{t_m}\widehat{\ell}_0(1)(s)\,ds+\int\limits_{\alpha_1}^{t_m}q^*(s)\,ds, \tag{9.28}$$

$$M\leq M\int\limits_{\alpha_2}^{t_M}\widehat{\ell}_0(1)(s)\,ds+m\int\limits_{\alpha_2}^{t_M}\widehat{\ell}_1(1)(s)\,ds+\int\limits_{\alpha_2}^{t_M}q^*(s)\,ds. \tag{9.29}$$

It follows from (9.28) and (9.29), on account of (9.16) and (9.27), that

$$\begin{gathered} m(1-C_1)\leq MA_1+c^*+\int\limits_a^b q^*(s)\,ds,\\ M(1-D_1)\leq mB_1+c^*+\int\limits_a^b q^*(s)\,ds, \end{gathered} \tag{9.30}$$

where

$$\begin{gathered} A_1=\int\limits_{\alpha_1}^{t_m}\widehat{\ell}_1(1)(s)\,ds,\quad B_1=\int\limits_{\alpha_2}^{t_M}\widehat{\ell}_1(1)(s)\,ds,\\ C_1=\int\limits_{\alpha_1}^{t_m}\widehat{\ell}_0(1)(s)\,ds,\quad D_1=\int\limits_{\alpha_2}^{t_M}\widehat{\ell}_0(1)(s)\,ds. \end{gathered}$$

Due to (9.10), $C_1<1$, $D_1<1$. Consequently, (9.30) implies

$$\begin{aligned} 0<m(1-C_1)(1-D_1)&\leq A_1\big(mB_1+c^*+\|q^*\|_L\big)+c^*+\|q^*\|_L\leq\\ &\leq mA_1B_1+\big(1+\|\widehat{\ell}_1(1)\|_L\big)\big(c^*+\|q^*\|_L\big),\\ 0<M(1-C_1)(1-D_1)&\leq B_1\big(MA_1+c^*+\|q^*\|_L\big)+c^*+\|q^*\|_L\leq\\ &\leq MA_1B_1+\big(1+\|\widehat{\ell}_1(1)\|_L\big)\big(c^*+\|q^*\|_L\big). \end{aligned} \tag{9.31}$$

Obviously,

$$\begin{gathered} (1-C_1)(1-D_1)\geq 1-(C_1+D_1)\geq 1-\|\widehat{\ell}_0(1)\|_L>0,\\ 4A_1B_1\leq (A_1+B_1)^2\leq \|\widehat{\ell}_1(1)\|_L^2. \end{gathered} \tag{9.32}$$

By (9.32) and (9.11), from (9.31) we get

$$\begin{aligned} m&\leq \frac{1+\|\widehat{\ell}_1(1)\|_L}{1-\|\widehat{\ell}_0(1)\|_L-\frac14\|\widehat{\ell}_1(1)\|_L^2}\,\big(c^*+\|q^*\|_L\big),\\ M&\leq \frac{1+\|\widehat{\ell}_1(1)\|_L}{1-\|\widehat{\ell}_0(1)\|_L-\frac14\|\widehat{\ell}_1(1)\|_L^2}\,\big(c^*+\|q^*\|_L\big). \end{aligned} \tag{9.33}$$

The inequalities (9.33), on account of (9.20) and (9.12), imply the estimate (9.3).

Now suppose that (9.23) is fulfilled. It is clear that there exists $\alpha_5 \in]t_M, t_m[$ such that

$$u(t) < 0 \ \text{ for } \ \alpha_5 < t \le t_m, \quad u(\alpha_5) = 0. \tag{9.34}$$

Let

$$\alpha_4 = \inf\Big\{t \in [a, t_M] : \ u(s) > 0 \ \text{ for } t \le s \le t_M\Big\}. \tag{9.35}$$

Obviously,

$$u(t) > 0 \ \text{ for } \ \alpha_4 < t \le t_M, \tag{9.36}$$

and

$$\text{if} \quad \alpha_4 > a, \quad \text{then} \quad u(\alpha_4) = 0. \tag{9.37}$$

The integration of (9.13) from α_4 to t_M and from α_5 to t_m, in view of (9.15), (9.20), (9.21), (9.34), and (9.36), yields

$$M - u(\alpha_4) \le M \int\limits_{\alpha_4}^{t_M} \widehat{\ell}_0(1)(s)\,ds + m \int\limits_{\alpha_4}^{t_M} \widehat{\ell}_1(1)(s)\,ds + \int\limits_{\alpha_4}^{t_M} q^*(s)\,ds, \tag{9.38}$$

$$m \le M \int\limits_{\alpha_5}^{t_m} \widehat{\ell}_1(1)(s)\,ds + m \int\limits_{\alpha_5}^{t_m} \widehat{\ell}_0(1)(s)\,ds + \int\limits_{\alpha_5}^{t_m} q^*(s)\,ds. \tag{9.39}$$

It follows from (9.38) and (9.39), on account (9.16) and (9.37), that

$$\begin{aligned} M(1 - C_2) &\le mA_2 + c^* + \int\limits_a^b q^*(s)\,ds, \\ m(1 - D_2) &\le MB_2 + c^* + \int\limits_a^b q^*(s)\,ds, \end{aligned} \tag{9.40}$$

where

$$A_2 = \int\limits_{\alpha_4}^{t_M} \widehat{\ell}_1(1)(s)\,ds, \quad B_2 = \int\limits_{\alpha_5}^{t_m} \widehat{\ell}_1(1)(s)\,ds,$$

$$C_2 = \int\limits_{\alpha_4}^{t_M} \widehat{\ell}_0(1)(s)\,ds, \quad D_2 = \int\limits_{\alpha_5}^{t_m} \widehat{\ell}_0(1)(s)\,ds.$$

Due to (9.10), $C_2 < 1$, $D_2 < 1$. Consequently, (9.40) implies

$$\begin{aligned} 0 < M(1 - C_2)(1 - D_2) &\le A_2\big(MB_2 + c^* + \|q^*\|_L\big) + c^* + \|q^*\|_L \le \\ &\le MA_1B_1 + \big(1 + \|\widehat{\ell}_1(1)\|_L\big)\big(c^* + \|q^*\|_L\big), \\ 0 < m(1 - C_2)(1 - D_2) &\le B_2\big(mA_2 + c^* + \|q^*\|_L\big) + c^* + \|q^*\|_L \le \\ &\le mA_2B_2 + \big(1 + \|\widehat{\ell}_1(1)\|_L\big)\big(c^* + \|q^*\|_L\big). \end{aligned} \tag{9.41}$$

Obviously,

$$\begin{gathered}(1-C_2)(1-D_2)\geq 1-(C_2+D_2)\geq 1-\|\widehat{\ell}_0(1)\|_L>0,\\ 4A_2B_2\leq (A_2+B_2)^2\leq \|\widehat{\ell}_1(1)\|_L^2.\end{gathered}\tag{9.42}$$

By (9.42) and (9.11), from (9.41) we get the inequalities (9.33), which, on account of (9.20) and (9.12), imply the estimate (9.3). □

Analogously one can prove the following assertion.

Lemma 9.6. *Let the operator* $\widehat{\ell}\in\mathcal{L}_{ab}$ *admit the representation* $\widehat{\ell}=\widehat{\ell}_0-\widehat{\ell}_1$, *where* $\widehat{\ell}_0,\widehat{\ell}_1\in\mathcal{P}_{ab}$. *If, moreover,*

$$\int\limits_a^b \widehat{\ell}_1(1)(s)\,ds<1,$$

$$\int\limits_a^b \widehat{\ell}_0(1)(s)\,ds<2\sqrt{1-\int\limits_a^b \widehat{\ell}_1(1)(s)\,ds}\,,$$

then $\widehat{\ell}\in\mathcal{B}_{ab}(\widehat{\omega})$, *where* $\widehat{\omega}(v)\equiv v(b)$.

In the proofs listed below, whenever $\ell\in\widetilde{\mathcal{L}}$, then the operator $\widehat{\ell}$ is defined by (1.9) with φ and ψ given by (1.6)–(1.8).

Proof of Theorem 8.1. Let $b\in\,]a,+\infty[$, and define function $\widehat{\gamma}$ by (3.16), where φ is given by (1.6) and (1.8). Then the assumptions of Lemma 9.1 are satisfied. Consequently, by virtue of Remark 9.1, we get $\ell_0-\ell_1\in\mathcal{A}(\omega)$ (with $\omega(v)\equiv v(a)$). Now the validity of theorem follows from Theorem 7.3. □

Proof of Theorem 8.2. All the assumptions of Lemma 9.5 are satisfied. Consequently, by virtue of Remark 9.1, we get $\ell_0-\ell_1\in\mathcal{A}(\omega)$ (with $\omega(v)\equiv v(a)$). Now the validity of theorem follows from Theorem 7.3. □

Proof of Theorem 8.3. Let $b\in\,]a,+\infty[$, and define function $\widehat{\gamma}$ by (3.16), where φ is given by (1.6) and (1.8). Then the assumptions of Lemma 9.2 are satisfied. Consequently, by virtue of Remark 9.1, we get $\ell_0-\ell_1\in\mathcal{B}(\omega)$ (with $\omega(v)\equiv v(+\infty)$). Now the validity of theorem follows from Theorem 7.4. □

Proof of Theorem 8.4. All the assumptions of Lemma 9.6 are satisfied. Consequently, by virtue of Remark 9.1, we get $\ell_0-\ell_1\in\mathcal{B}(\omega)$ (with $\omega(v)\equiv v(+\infty)$). Now the validity of theorem follows from Theorem 7.4. □

Proof of Theorem 8.5. All the assumptions of Lemma 9.3 are satisfied. Consequently, by virtue of Remark 9.1, we get $\ell_0-\ell_1\in\mathcal{A}(\omega)$ (with $\omega(v)\equiv v(a)-v(+\infty)$). Now the validity of theorem follows from Theorem 7.3. □

Proof of Theorem 8.6. All the assumptions of Lemma 9.4 are satisfied. Consequently, by virtue of Remark 9.1, we get $\ell_0-\ell_1\in\mathcal{B}(\omega)$ (with $\omega(v)\equiv v(+\infty)-v(a)$). Now the validity of theorem follows from Theorem 7.4. □

Theorems 8.7–8.12 can be proven analogously to Theorems 8.1–8.6, just Theorem 7.5 (Theorem 7.6) should to be used instead of Theorem 7.3 (Theorem 7.4).

10. Equations with Deviating Arguments

In this section we will establish some consequences of the main results from Section 8 for the equation with deviating arguments (6.1′).

In what follows we will use the notation

$$p_0(t) = \sum_{k=1}^{m} p_k(t), \quad g_0(t) = \sum_{k=1}^{m} g_k(t) \ \text{ for } \ t \geq a$$

and we will assume that $q \in K([a,+\infty[\times R_+; R_+)$ is nondecreasing in the second argument, $c : R_+ \to R_+$ is a continuous function, and

$$\lim_{x\to+\infty} \frac{1}{x}\left(c(x) + \int_a^{+\infty} q(s,x)\,ds\right) = 0.$$

10.1. Bounded Solutions.

Corollary 10.1. *Let the condition* (7.12) *be fulfilled and let on the set* R^{n+1} *the inequality*

$$f(t,x,x_1,\dots,x_n)\operatorname{sgn} x \leq q(t,|x|) \ \textit{ for } \ t \geq a \tag{10.1}$$

hold. Let, moreover, at least one of the following items be fulfilled:

a) $\tau_k(t) \leq t$ *for* $t \geq a$ $(k = 1,\dots,m)$,

$$\int_a^{+\infty} g_0(s)\exp\left(\int_s^{+\infty} p_0(\xi)\,d\xi\right) ds < 2; \tag{10.2}$$

b) $g_0 \equiv 0$, $\int_a^{\tau_k^*} p_0(\xi)\,d\xi \neq 0$ $(k = 1,\dots,m)$, *and let there exist* $y > 0$ *such that*

$$\operatorname{ess\,sup}\left\{\int_t^{\tau_k(t)} p_0(s)\,ds : \ t \geq a\right\} < \eta_k(y) \quad (k = 1,\dots,m), \tag{10.3}$$

where

$$\eta_k(y) = \frac{1}{y}\ln\left(y + \frac{y}{\exp\left(y\int_a^{\tau_k^*} p_0(\xi)\,d\xi\right) - 1}\right),$$

$$\tau_k^* = \operatorname{ess\,sup}\left\{\tau_k(t) : \ t \geq a\right\}.$$

Then the problem (6.1′), (6.3) *has at least one bounded solution.*

Remark 10.1. Corollary 10.1 is unimprovable in that sense that the strict inequality (10.2) cannot be replaced by the nonstrict one (see On Remark 8.1, p. 89).

Corollary 10.2. *Let the condition* (7.12) *be fulfilled and let on the set* R^{n+1} *the inequality* (10.1) *hold. If, moreover,*

$$\int_a^{+\infty} p_0(s)\,ds < 1, \qquad \int_a^{+\infty} g_0(s)\,ds < 2\sqrt{1-\int_a^{+\infty} p_0(s)\,ds}\,, \tag{10.4}$$

then the problem $(6.1'), (6.3)$ *has at least one bounded solution.*

Remark 10.2. Corollary 10.2 is unimprovable in that sense that neither one of the strict inequalities in (10.4) can be replaced by the nonstrict one (see On Remark 8.2, p. 90).

Corollary 10.3. *Let the condition* (7.15) *be fulfilled and let on the set* R^{n+1} *the inequality*

$$f(t,x,x_1,\dots,x_n)\operatorname{sgn} x \geq -q(t,|x|) \ \ \text{for} \ \ t \geq a \tag{10.5}$$

hold. Let, moreover, at least one of the following items be fulfilled:

a) $\mu_k(t) \geq t$ *for* $t \geq a$ $(k=1,\dots,m)$,

$$\int_a^{+\infty} p_0(s)\exp\bigg(\int_a^s g_0(\xi)\,d\xi\bigg)\,ds < 2; \tag{10.6}$$

b) $p_0 \equiv 0$, $\int\limits_{\mu_k^*}^{+\infty} g_0(\xi)\,d\xi \neq 0$ $(k=1,\dots,m)$, *and let there exist* $y>0$ *such that*

$$\operatorname{ess\,sup}\bigg\{\int_{\mu_k(t)}^t g_0(s)\,ds : \ t \geq a\bigg\} < \vartheta_k(y) \ \ (k=1,\dots,m),$$

where

$$\vartheta_k(y) = \frac{1}{y}\,\ln\left(y + \frac{y}{\exp\big(y\int\limits_{\mu_k^*}^{+\infty} g_0(\xi)\,d\xi\big)-1}\right),$$

$$\mu_k^* = \operatorname{ess\,inf}\big\{\mu_k(t) : \ t \geq a\big\}.$$

Then the equation $(6.1')$ *has at least one bounded solution. If, moreover,*

$$f \in K\big([a+\infty[\,\times R^{n+1}; R\big), \tag{10.7}$$

then the problem $(6.1'), (6.4)$ *has at least one bounded solution.*

Remark 10.3. Corollary 10.3 is unimprovable in that sense that the strict inequality (10.6) cannot be replaced by the nonstrict one (see On Remark 8.3, p. 92).

Corollary 10.4. *Let the condition* (7.15) *be fulfilled and let on the set* R^{n+1} *the inequality* (10.5) *hold. If, moreover,*

$$\int_a^{+\infty} g_0(s)\,ds < 1, \qquad \int_a^{+\infty} p_0(s)\,ds < 2\sqrt{1 - \int_a^{+\infty} g_0(s)\,ds}\,, \tag{10.8}$$

then the equation $(6.1')$ *has at least one bounded solution. If, moreover, the inclusion* (10.7) *holds, then the problem* $(6.1'), (6.4)$ *has at least one bounded solution.*

Remark 10.4. Corollary 10.4 is unimprovable in that sense that neither one of the strict inequalities in (10.8) can be replaced by the nonstrict one (see On Remark 8.4, p. 93).

Corollary 10.5. *Let the condition* (7.12) *be fulfilled and let on the set* R^{n+1} *the inequality* (10.1) *hold. If, moreover,*

$$\int_a^{+\infty} p_0(s)\,ds < 1, \tag{10.9}$$

$$\frac{\int_a^{+\infty} p_0(s)\,ds}{1 - \int_a^{+\infty} p_0(s)\,ds} < \int_a^{+\infty} g_0(s)\,ds < 2\sqrt{1 - \int_a^{+\infty} p_0(s)\,ds}\,, \tag{10.10}$$

then the equation $(6.1')$ *has at least one bounded solution. If, moreover, the inclusion* (10.7) *holds, then the problem* $(6.1'), (6.5)$ *has at least one bounded solution.*

Remark 10.5. Corollary 10.5 is unimprovable in that sense that neither one of the strict inequalities (10.9) and (10.10) can be replaced by the nonstrict one (see On Remark 8.5, p. 94).

Corollary 10.6. *Let the condition* (7.15) *be fulfilled and let on the set* R^{n+1} *the inequality* (10.5) *hold. If, moreover,*

$$\int_a^{+\infty} g_0(s)\,ds < 1, \tag{10.11}$$

$$\frac{\int_a^{+\infty} g_0(s)\,ds}{1 - \int_a^{+\infty} g_0(s)\,ds} < \int_a^{+\infty} p_0(s)\,ds < 2\sqrt{1 - \int_a^{+\infty} g_0(s)\,ds}\,, \tag{10.12}$$

then the equation $(6.1')$ *has at least one bounded solution. If, moreover, the inclusion* (10.7) *holds, then the problem* $(6.1'), (6.5)$ *has at least one bounded solution.*

Remark 10.6. Corollary 10.6 is unimprovable in that sense that neither one of the strict inequalities (10.11) and (10.12) can be replaced by the nonstrict one (see On Remark 8.6, p. 97).

In Corollaries 10.7–10.12, the conditions guaranteeing the existence of a unique bounded solution to the problems $(6.1'), (6.k)$ $(k = 3, 4, 5)$ are established.

Corollary 10.7. *Let* (7.17) *be fulfilled,* $f(\cdot, 0) \in L([a, +\infty[\,; R)$*, and let on the set* R^{n+1} *the inequality*

$$\big[f(t, x, x_1, \dots, x_n) - f(t, y, y_1, \dots, y_n)\big] \operatorname{sgn}(x - y) \le 0 \;\; \textit{for} \;\; t \ge a \quad (10.13)$$

hold. If, moreover, at least one of the conditions a) or b) in Corollary 10.1 is fulfilled, then the problem $(6.1'), (6.3)$ *has a unique bounded solution.*

Remark 10.7. Corollary 10.7 is unimprovable in that sense that the strict inequality (10.2) cannot be replaced by the nonstrict one (see On Remark 8.1, p. 89).

Corollary 10.8. *Let* (7.17) *be fulfilled,* $f(\cdot, 0) \in L([a, +\infty[\,; R)$*, and let on the set* R^{n+1} *the inequality* (10.13) *hold. If, moreover, the inequalities* (10.4) *hold, then the problem* $(6.1'), (6.3)$ *has a unique bounded solution.*

Remark 10.8. Corollary 10.8 is unimprovable in that sense that neither one of the strict inequalities in (10.4) can be replaced by the nonstrict one (see On Remark 8.2, p. 90).

Corollary 10.9. *Let the condition* (7.19) *be fulfilled, and let on the set* R^{n+1} *the inequality*

$$\big[f(t, x, x_1, \dots, x_n) - f(t, y, y_1, \dots, y_n)\big] \operatorname{sgn}(x - y) \ge 0 \;\; \textit{for} \;\; t \ge a \quad (10.14)$$

hold. If, moreover, at least one of the conditions a) or b) in Corollary 10.3 is fulfilled and the inclusion (10.7) *holds, then the problem* $(6.1'), (6.4)$ *has a unique bounded solution.*

Remark 10.9. Corollary 10.9 is unimprovable in that sense that the strict inequality (10.6) cannot be replaced by the nonstrict one (see On Remark 8.3, p. 92).

Corollary 10.10. *Let the condition* (7.19) *be fulfilled and let on the set* R^{n+1} *the inequality* (10.14) *hold. If, moreover, the inequalities* (10.8) *are satisfied and the inclusion* (10.7) *holds, then the problem* $(6.1'), (6.4)$ *has a unique bounded solution.*

Remark 10.10. Corollary 10.10 is unimprovable in that sense that neither one of the strict inequalities in (10.8) can be replaced by the nonstrict one (see On Remark 8.4, p. 93).

Corollary 10.11. *Let the condition* (7.17) *be fulfilled and let on the set* R^{n+1} *the inequality* (10.13) *hold. If, moreover, the inequalities* (10.9) *and* (10.10) *are satisfied and the inclusion* (10.7) *holds, then the problem* $(6.1'), (6.5)$ *has a unique bounded solution.*

Remark 10.11. Corollary 10.11 is unimprovable in that sense that neither one of the strict inequalities (10.9) and (10.10) can be replaced by the nonstrict one (see On Remark 8.5, p. 94).

Corollary 10.12. *Let the condition* (7.19) *be fulfilled and let on the set* R^{n+1} *the inequality* (10.14) *hold. If, moreover, the inequalities* (10.11) *and* (10.12) *are satisfied and the inclusion* (10.7) *holds, then the problem* $(6.1'), (6.5)$ *has a unique bounded solution.*

Remark 10.12. Corollary 10.12 is unimprovable in that sense that neither one of the strict inequalities (10.11) and (10.12) can be replaced by the nonstrict one (see On Remark 8.6, p. 97).

10.2. Proofs. *Proof of Corollary* 10.1. Put

$$F(v)(t) \overset{def}{=} \sum_{k=1}^{m} \Big(p_k(t)v(\tau_k(t)) - g_k(t)v(\mu_k(t)) \Big) + \\ + f(t, v(t), v(\nu_1(t)), \dots, v(\nu_n(t))) \ \text{ for } \ t \geq a, \tag{10.15}$$

$$\ell_0(v)(t) \overset{def}{=} \sum_{k=1}^{m} p_k(t)v(\tau_k(t)) \ \text{ for } \ t \geq a, \tag{10.16}$$

$$\ell_1(v)(t) \overset{def}{=} \sum_{k=1}^{m} g_k(t)v(\mu_k(t)) \ \text{ for } \ t \geq a. \tag{10.17}$$

Then, obviously, (10.1) implies (8.1).

a) Choose $\varepsilon > 0$ such that

$$\varepsilon \exp\left(\int_a^{+\infty} p_0(s)\, ds \right) + \int_a^{+\infty} g_0(s) \exp\left(\int_s^{+\infty} p_0(\xi)\, d\xi \right) ds < 2$$

and put

$$\gamma(t) = \exp\left(\int_a^t p_0(s)\, ds \right) \times \\ \times \left(\varepsilon + \int_a^t g_0(s) \exp\left(- \int_a^s p_0(\xi)\, d\xi \right) ds \right) \ \text{ for } \ t \geq a.$$

b) Choose $\varepsilon > 0$ such that

$$\operatorname{ess\,sup} \left\{ \int_t^{\tau_k(t)} p_0(s)\, ds : \ t \geq a \right\} < \\ < \frac{1}{y} \ln \left(y + \frac{y(1-\varepsilon)}{\exp\big(y \int_a^{\tau_k^*} p_0(\xi)\, d\xi\big) - (1-\varepsilon)} \right) \ (k = 1, \dots, m),$$

and put

$$\gamma(t) = \frac{\exp\big(y \int\limits_a^t p_0(s)\,ds\big) - (1-\varepsilon)}{\exp\big(y \int\limits_a^{+\infty} p_0(\xi)\,d\xi\big) - 1} \quad \text{for } \ t \geq a.$$

Then, in both cases a) and b), it can be easily verified that γ satisfies (8.2) and (8.3). Consequently, the assumptions of Theorem 8.1 are fulfilled. □

Proof of Corollary 10.2. Let the operators F, ℓ_0, and ℓ_1 be defined by (10.15)–(10.17). Then (10.1) implies (8.1) and the condition (10.4) results in (8.4) and (8.5). Consequently, the assumptions of Theorem 8.2 are fulfilled. □

Proof of Corollary 10.3. Let the operators F, ℓ_0, and ℓ_1 be defined by (10.15)–(10.17). Then, obviously, (10.5) and (10.7) imply (8.6) and (7.6), respectively.

a) Choose $\varepsilon > 0$ such that

$$\varepsilon \exp\bigg(\int\limits_a^{+\infty} g_0(s)\,ds\bigg) + \int\limits_a^{+\infty} p_0(s) \exp\bigg(\int\limits_a^{s} g_0(\xi)\,d\xi\bigg)\,ds < 2$$

and put

$$\gamma(t) = \exp\bigg(\int\limits_t^{+\infty} g_0(s)\,ds\bigg) \times$$
$$\times \bigg(\varepsilon + \int\limits_t^{+\infty} p_0(s) \exp\bigg(-\int\limits_s^{+\infty} g_0(\xi)\,d\xi\bigg)\,ds\bigg) \quad \text{for } \ t \geq a.$$

b) Choose $\varepsilon > 0$ such that

$$\operatorname{ess\,sup}\bigg\{\int\limits_{\mu_k(t)}^{t} g_0(s)\,ds : \ t \geq a\bigg\} <$$
$$< \frac{1}{y} \ln\left(y + \frac{y(1-\varepsilon)}{\exp\big(y \int\limits_{\mu_k^*}^{+\infty} g_0(\xi)\,d\xi\big) - (1-\varepsilon)}\right) \quad (k = 1, \dots, m),$$

and put

$$\gamma(t) = \frac{\exp\big(y \int\limits_t^{+\infty} g_0(s)\,ds\big) - (1-\varepsilon)}{\exp\big(y \int\limits_a^{+\infty} g_0(\xi)\,d\xi\big) - 1} \quad \text{for } \ t \geq a.$$

Then, in both cases a) and b)), it can be easily verified that γ satisfies (8.7)–(8.9). Consequently, the assumptions of Theorem 8.3 are fulfilled. □

Proof of Corollary 10.4. Let the operators F, ℓ_0, and ℓ_1 be defined by (10.15)–(10.17). Then (10.5) and (10.7) imply (8.6) and (7.6), respectively, and the condition (10.8) results in (8.10) and (8.11). Consequently, the assumptions of Theorem 8.4 are fulfilled. □

Proof of Corollary 10.5. Let the operators F, ℓ_0, and ℓ_1 be defined by (10.15)–(10.17). Then (10.1) and (10.7) imply (8.1) and (7.6), respectively, and the conditions (10.9) and (10.10) result in (8.12) and (8.13). Consequently, the assumptions of Theorem 8.5 are fulfilled. □

Proof of Corollary 10.6. Let the operators F, ℓ_0, and ℓ_1 be defined by (10.15)–(10.17). Then (10.5) and (10.7) imply (8.6) and (7.6), respectively, and the conditions (10.11) and (10.12) result in (8.14) and (8.15). Consequently, the assumptions of Theorem 8.6 are fulfilled. □

In a similar manner as in the proofs of Corollaries 10.1–10.6 one can show that Corollaries 10.7–10.12 follow from Theorems 8.7–8.12.

11. Comments

Remark 11.1. Let the functions $p, g \in L([a,b]; R_+)$, $\tau, \mu \in \mathcal{M}_{ab}$, and the operator $G \in \mathcal{K}_{ab}$ be such that the inequality

$$\big[G(u)(t) - G(v)(t)\big] \operatorname{sgn}(u(t) - v(t)) \le 0 \quad \text{for} \ \ t \in [a,b], \quad u, v \in C([a,b]; R),$$

resp.

$$\big[G(u)(t) - G(v)(t)\big] \operatorname{sgn}(u(t) - v(t)) \ge 0 \quad \text{for} \ \ t \in [a,b], \quad u, v \in C([a,b]; R),$$

is fulfilled, and such that on the segment $[a,b]$ the problem

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + G(u)(t), \quad u(a) = 0, \tag{11.1}$$

resp.

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + G(u)(t), \quad u(b) = 0, \tag{11.2}$$

resp.

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + G(u)(t), \quad u(a) - u(b) = 0, \tag{11.3}$$

has no solution. If we put, for $v \in C_{loc}([a, +\infty[\,; R)$,

$$\ell_0(v)(t) = \begin{cases} p(t)v(\tau(t)) & \text{for} \ \ t \in [a,b], \\ 0 & \text{for} \ \ t > b, \end{cases} \tag{11.4}$$

$$\ell_1(v)(t) = \begin{cases} g(t)v(\mu(t)) & \text{for} \ \ t \in [a,b], \\ 0 & \text{for} \ \ t > b, \end{cases} \tag{11.5}$$

$$F(v)(t) = \begin{cases} p(t)v(\tau(t)) - g(t)v(\mu(t)) + G(v_b)(t) & \text{for} \ \ t \in [a,b], \\ 0 & \text{for} \ \ t > b, \end{cases} \tag{11.6}$$

$$h(v) \equiv 0, \tag{11.7}$$

where v_b is a restriction of v to the segment $[a,b]$, then there exists a finite limit (7.6) for every $v \in \widetilde{C}_{loc}([a,+\infty[\,;R)$, the inequalities (7.12), (7.15), (7.17), (7.19), (8.1), (8.6), (8.16), and (8.17) are fulfilled, respectively, with $c \equiv 0$ and

$$q(t,x) = \begin{cases} |G(0)(t)| & \text{for } \ t \in [a,b], \ x \in R_+, \\ 0 & \text{for } \ t > b, \ x \in R_+, \end{cases} \tag{11.8}$$

and the problem (6.1), (6.3), resp. (6.1), (6.4), resp. (6.1), (6.5) has no solution.

On Remark 8.1. Let $a < t_1 < t_2 < b$, $p \equiv 0$, $\tau \in \mathcal{M}_{ab}$ be arbitrary,

$$\mu(t) = \begin{cases} b & \text{for } \ t \in [a,t_2[\,, \\ t_1 & \text{for } \ t \in [t_2,b], \end{cases}$$

and choose $g \in L([a,b];R_+)$ such that

$$\int_a^{t_1} g(s)\,ds = 1, \qquad \int_{t_1}^{t_2} g(s)\,ds = 0, \qquad \int_{t_2}^{b} g(s)\,ds = 1.$$

Further, let

$$G(v)(t) = \begin{cases} 0 & \text{for } \ t \in [a,t_1[\,, \\ -v(t)|v(t)| & \text{for } \ t \in [t_1,t_2[\,, \\ q_0(t) & \text{for } \ t \in [t_2,b], \end{cases}$$

where $q_0 \in L([a,b];R)$ is such that

$$\int_{t_2}^{b} q_0(s)\,ds \geq \frac{1}{t_2 - t_1}\,.$$

Put

$$\gamma(t) = \begin{cases} 1 + \displaystyle\int_a^t g(s)\,ds & \text{for } \ t \in [a,b], \\ 1 + \displaystyle\int_a^b g(s)\,ds & \text{for } \ t > b. \end{cases}$$

Then the assumptions of Theorems 8.1 and 8.7 are fulfilled (with ℓ_0, ℓ_1, F, h, and q defined by (11.4)–(11.8), $c \equiv 0$) except (8.3), instead of which we have

$$\gamma(+\infty) - \gamma(a) = 2.$$

Moreover, the problem (11.1) has no solution. Indeed, assume on the contrary that (11.1) has a solution u. Then from (11.1) we get

$$\begin{aligned} u(t_1) &= -u(b), \\ u(t_2) &= \frac{u(t_1)}{1+|u(t_1)|(t_2-t_1)}\,, \\ u(b) &= u(t_2) - u(t_1) + \int\limits_{t_2}^{b} q_0(s)\,ds. \end{aligned}$$

Hence we obtain

$$\int\limits_{t_2}^{b} q_0(s)\,ds = -\frac{u(t_1)}{1+|u(t_1)|(t_2-t_1)} \le \frac{|u(t_1)|}{1+|u(t_1)|(t_2-t_1)} < \frac{1}{t_2-t_1}\,,$$

a contradiction.

Consequently, according to Remark 11.1 (see p. 88), we have shown that the strict inequality (8.3) in Theorems 8.1 and 8.7 (the inequality (10.2) in Corollaries 10.1 and 10.7) cannot be replaced by the nonstrict one.

On Remark 8.2. According to Remark 11.1 (see p. 88), for every point $(x_0, y_0) \notin D_a$, it is sufficient to costruct functions $p, g \in L([a,b]; R_+)$, $\tau, \mu \in \mathcal{M}_{ab}$, and a suitable operator $G \in \mathcal{K}_{ab}$ in such a way that

$$\int\limits_a^b p(s)\,ds = x_0, \quad \int\limits_a^b g(s)\,ds = y_0 \tag{11.9}$$

and such that the problem (11.1) has no solution.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin D_a$, then (x_0, y_0) belongs to one of the following sets:

$$\begin{aligned} D_a^1 &= \Big\{(x,y) \in R_+ \times R_+ :\ x \ge 1\Big\}, \\ D_a^2 &= \Big\{(x,y) \in R_+ \times R_+ :\ x < 1,\ y \ge 2\sqrt{1-x}\,\Big\}. \end{aligned}$$

Let $(x_0, y_0) \in D_a^1$, $t_0 \in [a,b]$, and choose $p, g \in L([a,b]; R_+)$ and $q_0 \in L([a,b]; R)$ such that

$$\int\limits_a^{t_0} p(s)\,ds = 1, \quad \int\limits_{t_0}^{b} p(s)\,ds = x_0 - 1, \quad \int\limits_a^{b} g(s)\,ds = y_0, \quad \int\limits_a^{t_0} q_0(s)\,ds \ne 0.$$

Put $\tau \equiv t_0$, $\mu \equiv a$, $G \equiv q_0$. Then the problem (11.1) has no solution. Indeed, assuming that u is a solution to (11.1), the integration of (11.1) from a to t_0 yields

$$u(t_0) = u(a) + u(t_0) \int\limits_a^{t_0} p(s)\,ds - u(a) \int\limits_a^{t_0} g(s)\,ds + \int\limits_a^{t_0} q_0(s)\,ds,$$

which, together with the initial condition in (11.1) results in a contradiction

$$\int_a^{t_0} q_0(s)\,ds = 0.$$

Let $(x_0, y_0) \in D_a^2$, $a < t_1 < t_2 < b$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int_a^{t_2} p(s)\,ds = 0, \quad \int_{t_2}^{b} p(s)\,ds = x_0, \quad \int_a^{t_1} g(s)\,ds = \sqrt{1-x_0}\,,$$

$$\int_{t_1}^{t_2} g(s)\,ds = y_0 - 2\sqrt{1-x_0}\,, \quad \int_{t_2}^{b} g(s)\,ds = \sqrt{1-x_0}\,.$$

Put $\tau \equiv b$,

$$\mu(t) = \begin{cases} b & \text{for } \ t \in [a, t_1[\,, \\ a & \text{for } \ t \in [t_1, t_2[\,, \\ t_1 & \text{for } \ t \in [t_2, b], \end{cases}$$

and

$$G(v)(t) = \begin{cases} 0 & \text{for } \ t \in [a, t_1[\,, \\ -v(t)|v(t)| & \text{for } \ t \in [t_1, t_2[\,, \\ q_0(t) & \text{for } \ t \in [t_2, b], \end{cases}$$

where $q_0 \in L([a,b]; R)$ is such that

$$\int_{t_2}^{b} q_0(s)\,ds \geq \frac{1}{t_2 - t_1}\,.$$

Then the problem (11.1) has no solution. Indeed, assume on the contrary that (11.1) has a solution u. Then from (11.1) we get

$$\begin{aligned} u(t_1) &= -u(b)\sqrt{1-x_0}\,, \\ u(t_2) &= \frac{u(t_1)}{1 + |u(t_1)|(t_2 - t_1)}\,, \\ u(b) &= u(t_2) + u(b)x_0 - u(t_1)\sqrt{1-x_0} + \int_{t_2}^{b} q_0(s)\,ds. \end{aligned}$$

Hence we obtain

$$\int_{t_2}^{b} q_0(s)\,ds = -\frac{u(t_1)}{1 + |u(t_1)|(t_2 - t_1)} \leq \frac{|u(t_1)|}{1 + |u(t_1)|(t_2 - t_1)} < \frac{1}{t_2 - t_1}\,,$$

a contradiction.

Consequently, we have shown that the strict inequalities (8.4) and (8.5) in Theorems 8.2 and 8.8 (the inequalities (10.4) in Corollaries 10.2 and 10.8) cannot be replaced by the nonstrict ones.

On Remark 8.3. Let $a < t_1 < t_2 < b$, $g \equiv 0$, $\mu \in \mathcal{M}_{ab}$ be arbitrary,

$$\tau(t) = \begin{cases} t_2 & \text{for } t \in [a, t_1[\,, \\ a & \text{for } t \in [t_1, b], \end{cases}$$

and choose $p \in L([a,b]; R_+)$ such that

$$\int_a^{t_1} p(s)\,ds = 1, \quad \int_{t_1}^{t_2} p(s)\,ds = 0, \quad \int_{t_2}^{b} p(s)\,ds = 1.$$

Further, let

$$G(v)(t) = \begin{cases} q_0(t) & \text{for } t \in [a, t_1[\,, \\ v(t)|v(t)| & \text{for } t \in [t_1, t_2[\,, \\ 0 & \text{for } t \in [t_2, b], \end{cases}$$

where $q_0 \in L([a,b]; R)$ is such that

$$\int_a^{t_1} q_0(s)\,ds \geq \frac{1}{t_2 - t_1}\,.$$

Put

$$\gamma(t) = \begin{cases} 1 + \displaystyle\int_t^b p(s)\,ds & \text{for } t \in [a, b], \\ 1 & \text{for } t > b. \end{cases}$$

Then the assumptions of Theorems 8.3 and 8.9 are fulfilled (with ℓ_0, ℓ_1, F, h, and q defined by (11.4)–(11.8), $c \equiv 0$) except (8.9), instead of which we have

$$\gamma(a) - \gamma(+\infty) = 2.$$

Moreover, the problem (11.2) has no solution. Indeed, assume on the contrary that (11.2) has a solution u. Then from (11.2) we get

$$u(t_1) = u(a) + u(t_2) + \int_a^{t_1} q_0(s)\,ds,$$

$$u(t_1) = \frac{u(t_2)}{1 + |u(t_2)|(t_2 - t_1)}\,,$$

$$0 = u(t_2) + u(a).$$

Hence we obtain

$$\int_a^{t_1} q_0(s)\,ds = \frac{u(t_2)}{1 + |u(t_2)|(t_2 - t_1)} \leq \frac{|u(t_2)|}{1 + |u(t_2)|(t_2 - t_1)} < \frac{1}{t_2 - t_1}\,,$$

a contradiction.

Consequently, according to Remark 11.1 (see p. 88), we have shown that the strict inequality (8.9) in Theorems 8.3 and 8.9 (the inequality (10.6) in Corollaries 10.3 and 10.9) cannot be replaced by the nonstrict one.

On Remark 8.4. According to Remark 11.1 (see p. 88), for every point $(x_0, y_0) \notin D_{+\infty}$, it is sufficient to costruct functions $p, g \in L([a,b]; R_+)$, $\tau, \mu \in \mathcal{M}_{ab}$, and a suitable operator $G \in \mathcal{K}_{ab}$ in such a way that (11.9) holds and such that the problem (11.2) has no solution.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin D_{+\infty}$, then (x_0, y_0) belongs to one of the following sets:

$$D^1_{+\infty} = \Big\{(x,y) \in R_+ \times R_+ : \ y \geq 1\Big\},$$
$$D^2_{+\infty} = \Big\{(x,y) \in R_+ \times R_+ : \ y < 1, \ x \geq 2\sqrt{1-y}\,\Big\}.$$

Let $(x_0, y_0) \in D^1_{+\infty}$, $t_0 \in [a,b]$, and choose $p, g \in L([a,b]; R_+)$ and $q_0 \in L([a,b]; R)$ such that

$$\int_a^b p(s)\,ds = x_0, \quad \int_a^{t_0} g(s)\,ds = y_0 - 1, \quad \int_{t_0}^b g(s)\,ds = 1, \quad \int_{t_0}^b q_0(s)\,ds \neq 0.$$

Put $\tau \equiv b$, $\mu \equiv t_0$, $G \equiv q_0$. Then the problem (11.2) has no solution. Indeed, assuming that u is a solution to (11.2), the integration of (11.2) from t_0 to b yields

$$u(b) = u(t_0) + u(b) \int_{t_0}^b p(s)\,ds - u(t_0) \int_{t_0}^b g(s)\,ds + \int_{t_0}^b q_0(s)\,ds,$$

which, together with the initial condition in (11.2) results in a contradiction

$$\int_{t_0}^b q_0(s)\,ds = 0.$$

Let $(x_0, y_0) \in D^2_{+\infty}$, $a < t_1 < t_2 < b$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int_a^{t_1} p(s)\,ds = \sqrt{1-y_0}\,, \quad \int_{t_1}^{t_2} p(s)\,ds = x_0 - 2\sqrt{1-y_0}\,,$$
$$\int_{t_2}^b p(s)\,ds = \sqrt{1-y_0}\,, \quad \int_a^{t_1} g(s)\,ds = y_0, \quad \int_{t_1}^b g(s)\,ds = 0.$$

Put $\mu \equiv a$,

$$\tau(t) = \begin{cases} t_2 & \text{for } t \in [a, t_1[\,, \\ b & \text{for } t \in [t_1, t_2[\,, \\ a & \text{for } t \in [t_2, b], \end{cases}$$

and

$$G(v)(t) = \begin{cases} q_0(t) & \text{for } t \in [a, t_1[\,, \\ v(t)|v(t)| & \text{for } t \in [t_1, t_2[\,, \\ 0 & \text{for } t \in [t_2, b], \end{cases}$$

where $q_0 \in L([a, b]; R)$ is such that

$$\int_a^{t_1} q_0(s)\, ds \geq \frac{1}{t_2 - t_1}\,.$$

Then the problem (11.2) has no solution. Indeed, assume on the contrary that (11.2) has a solution u. Then from (11.2) we get

$$u(t_1) = u(a) + u(t_2)\sqrt{1 - y_0} - u(a)y_0 + \int_a^{t_1} q_0(s)\, ds,$$

$$u(t_1) = \frac{u(t_2)}{1 + |u(t_2)|(t_2 - t_1)}\,,$$

$$0 = u(t_2) + u(a)\sqrt{1 - y_0}\,.$$

Hence we obtain

$$\int_a^{t_1} q_0(s)\, ds = \frac{u(t_2)}{1 + |u(t_2)|(t_2 - t_1)} \leq \frac{|u(t_2)|}{1 + |u(t_2)|(t_2 - t_1)} < \frac{1}{t_2 - t_1}\,,$$

a contradiction.

Consequently, we have shown that the strict inequalities (8.10) and (8.11) in Theorems 8.4 and 8.10 (the inequalities (10.8) in Corollaries 10.4 and 10.10) cannot be replaced by the nonstrict ones.

On Remark 8.5. According to Remark 11.1 (see p. 88), for every point $(x_0, y_0) \notin D_p^+$, it is sufficient to costruct functions $p, g \in L([a, b]; R_+)$, $\tau, \mu \in \mathcal{M}_{ab}$, and a suitable operator $G \in \mathcal{K}_{ab}$ in such a way that (11.9) holds and such that the problem (11.3) has no solution.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin D_p^+$, then (x_0, y_0) belongs at least to one of the following sets:

$$D_1 = \Big\{(x, y) \in R_+ \times R_+ :\ x \geq 1\Big\},$$

$$D_2 = \Big\{(x, y) \in R_+ \times R_+ :\ x < 1,\ y \leq \frac{x}{1 - x}\Big\},$$

$$D_3 = \Big\{(x, y) \in R_+ \times R_+ :\ x < 1,\ y \geq 2\sqrt{1 - x}\,\Big\}.$$

Let $(x_0, y_0) \in D_1$, $a < t_1 < t_2 < b$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int\limits_a^{t_1} p(s)\,ds = 0, \quad \int\limits_{t_1}^{t^2} p(s)\,ds = x_0, \quad \int\limits_{t_2}^{b} p(s)\,ds = 0,$$

$$\int\limits_a^{t_1} g(s)\,ds = y_0, \quad \int\limits_{t_1}^{b} g(s)\,ds = 0.$$

Put $\tau \equiv b$, $\mu \equiv t_1$, and

$$z(t) = \begin{cases} 0 & \text{for } t \in [a, t_2[\,, \\ -\dfrac{x_0(1+y_0) - y_0}{(b-t_2)(1+y_0) + (x_0(1+y_0) - y_0)(b-t)} & \text{for } t \in [t_2, b]. \end{cases}$$

Then the problem

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + z(t)u(t), \quad u(a) - u(b) = 0 \tag{11.10}$$

has a nontrivial solution

$$u(t) = \begin{cases} 1 + y_0 - \int\limits_a^t g(s)\,ds & \text{for } t \in [a, t_1[\,, \\ 1 + (1+y_0)\int\limits_a^t p(s)\,ds & \text{for } t \in [t_1, t_2[\,, \\ 1 + y_0 + \dfrac{x_0(1+y_0) - y_0}{b - t_2}(b-t) & \text{for } t \in [t_2, b]. \end{cases}$$

According to Remark 1.1 (see p. 11) there exists $q_0 \in L([a,b]; R)$ such that the problem

$$\begin{gathered} u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + z(t)u(t) + q_0(t), \\ u(a) - u(b) = 0 \end{gathered} \tag{11.11}$$

has no solution. Now, if we put

$$G(v)(t) \overset{def}{=} z(t)v(t) + q_0(t) \quad \text{for } t \in [a,b], \tag{11.12}$$

then the problem (11.3) has no solution.

Let $(x_0, y_0) \in D_2$, $a < t_1 < t_2 < b$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int\limits_a^{t_1} p(s)\,ds = x_0, \quad \int\limits_{t_1}^{b} p(s)\,ds = x_0, \quad \int\limits_a^{t_1} g(s)\,ds = 0,$$

$$\int\limits_{t_1}^{t_2} g(s)\,ds = y_0, \quad \int\limits_{t_2}^{b} g(s)\,ds = 0.$$

Put $\tau \equiv t_1$, $\mu \equiv a$, and

$$z(t) = \begin{cases} 0 & \text{for } t \in [a, t_2[\,, \\ -\dfrac{x_0 - y_0(1-x_0)}{(b-t_2)(1-x_0) + (x_0 - y_0(1-x_0))(b-t)} & \text{for } t \in [t_2, b]. \end{cases}$$

Then the problem (11.10) has a nontrivial solution

$$u(t) = \begin{cases} 1 - x_0 + \displaystyle\int_a^t p(s)\,ds & \text{for } t \in [a, t_1[\,, \\ 1 - (1-x_0)\displaystyle\int_a^t g(s)\,ds & \text{for } t \in [t_1, t_2[\,, \\ 1 - x_0 + \dfrac{x_0 - y_0(1-x_0)}{b-t_2}(b-t) & \text{for } t \in [t_2, b]. \end{cases}$$

According to Remark 1.1 (see p. 11) there exists $q_0 \in L([a,b];R)$ such that the problem (11.11) has no solution. Now, if we define the operator G by (11.12), then the problem (11.3) has no solution.

Let $(x_0, y_0) \in D_3$, $a < t_1 < t_2 < t_3 < t_4 < b$, and choose $p, g \in L([a,b];R_+)$ such that

$$\int_a^{t_3} p(s)\,ds = 0, \quad \int_{t_3}^{t_4} p(s)\,ds = x_0, \quad \int_{t_4}^{b} p(s)\,ds = 0, \quad \int_a^{t_1} g(s)\,ds = 0,$$

$$\int_{t_1}^{t_2} g(s)\,ds = \sqrt{1-x_0}\,, \quad \int_{t_2}^{t_3} g(s)\,ds = 0, \quad \int_{t_3}^{t_4} g(s)\,ds = \sqrt{1-x_0}\,,$$

$$\int_{t_4}^{b} g(s)\,ds = y_0 - 2\sqrt{1-x_0}\,.$$

Put $\tau \equiv b$,

$$\mu(t) = \begin{cases} b & \text{for } t \in [a, t_2[\,, \\ t_2 & \text{for } t \in [t_2, t_4[\,, \\ t_1 & \text{for } t \in [t_4, b], \end{cases}$$

and

$$G(v)(t) = \begin{cases} -v(t)|v(t)| & \text{for } t \in [a, t_1[\,\cup [t_2, t_3[\,, \\ 0 & \text{for } t \in [t_1, t_2[\,\cup [t_3, t_4[\,, \\ q_0(t) & \text{for } t \in [t_4, b], \end{cases}$$

where $q_0 \in L([a,b];R)$ is such that

$$\int_{t_4}^{b} q_0(s)\,ds \geq \frac{y_0 - \sqrt{1-x_0}}{t_1 - a} + \frac{1}{t_3 - t_2}\,.$$

Then the problem (11.3) has no solution. Indeed, assume on the contrary that (11.3) has a solution u. Then from (11.3) we get

$$u(t_1) = \frac{u(a)}{1 + |u(a)|(t_1 - a)}\,, \tag{11.13}$$

$$u(t_2) = u(t_1) - u(b)\sqrt{1 - x_0}\,, \tag{11.14}$$

$$u(t_3) = \frac{u(t_2)}{1 + |u(t_2)|(t_3 - t_2)}\,, \tag{11.15}$$

$$u(t_4) = u(t_3) - u(t_2)\sqrt{1 - x_0} + u(b)x_0, \tag{11.16}$$

$$u(b) = u(t_4) - u(t_1)\big(y_0 - 2\sqrt{1 - x_0}\,\big) + \int\limits_{t_4}^{b} q_0(s)\,ds. \tag{11.17}$$

The equalities (11.14), (11.16), and (11.17) imply

$$\int\limits_{t_4}^{b} q_0(s)\,ds = u(t_1)\big(y_0 - \sqrt{1 - x_0}\,\big) - u(t_3).$$

Hence, by virtue of (11.13) and (11.15), we get

$$\begin{aligned}\int\limits_{t_4}^{b} q_0(s)\,ds &= \frac{(y_0 - \sqrt{1 - x_0}\,)u(a)}{1 + |u(a)|(t_1 - a)} - \frac{u(t_2)}{1 + |u(t_2)|(t_3 - t_2)} \leq \\ &\leq \frac{(y_0 - \sqrt{1 - x_0}\,)|u(a)|}{1 + |u(a)|(t_1 - a)} + \frac{|u(t_2)|}{1 + |u(t_2)|(t_3 - t_2)} < \\ &< \frac{y_0 - \sqrt{1 - x_0}}{t_1 - a} + \frac{1}{t_3 - t_2}\,,\end{aligned}$$

a contradiction.

Consequently, we have shown that the strict inequalities (8.12) and (8.13) in Theorems 8.5 and 8.11 (the inequalities (10.9) and (10.10) in Corollaries 10.5 and 10.11) cannot be replaced by the nonstrict ones.

On Remark 8.6. According to Remark 11.1 (see p. 88), for every point $(x_0, y_0) \notin D_p^-$, it is sufficient to costruct functions $p, g \in L([a, b]; R_+)$, $\tau, \mu \in \mathcal{M}_{ab}$, and a suitable operator $G \in \mathcal{K}_{ab}$ in such a way that (11.9) holds and such that the problem (11.3) has no solution.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0, y_0) \notin D_p^-$, then (x_0, y_0) belongs at least to one of the following sets:

$$D_4 = \Big\{(x, y) \in R_+ \times R_+ :\ y \geq 1\Big\},$$

$$D_5 = \left\{(x, y) \in R_+ \times R_+\ :\ y < 1,\ x \leq \frac{y}{1 - y}\right\},$$

$$D_6 = \Big\{(x, y) \in R_+ \times R_+\ :\ y < 1,\ x \geq 2\sqrt{1 - y}\,\Big\}.$$

Let $(x_0, y_0) \in D_4$, $a < t_1 < t_2 < b$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int_a^{t_2} p(s)\,ds = 0, \quad \int_{t_2}^{b} p(s)\,ds = x_0, \quad \int_a^{t_1} g(s)\,ds = 0,$$
$$\int_{t_1}^{t_2} g(s)\,ds = y_0, \quad \int_{t_2}^{b} g(s)\,ds = 0.$$

Put $\tau \equiv t_2$, $\mu \equiv a$, and

$$z(t) = \begin{cases} \dfrac{y_0(1+x_0) - x_0}{(t_1 - a)(1+x_0) + (y_0(1+x_0) - x_0)(t-a)} & \text{for } \ t \in [a, t_1[\,, \\ 0 & \text{for } \ t \in [t_1, b]. \end{cases}$$

Then the problem (11.10) has a nontrivial solution

$$u(t) = \begin{cases} 1 + x_0 + \dfrac{y_0(1+x_0) - x_0}{t_1 - a}\,(t-a) & \text{for } \ t \in [a, t_1[\,, \\ 1 + (1+x_0) \displaystyle\int_t^b g(s)\,ds & \text{for } \ t \in [t_1, t_2[\,, \\ 1 + x_0 - \displaystyle\int_t^b p(s)\,ds & \text{for } \ t \in [t_2, b]. \end{cases}$$

According to Remark 1.1 (see p. 11) there exists $q_0 \in L([a,b]; R)$ such that the problem (11.11) has no solution. Now, if we define the operator G by (11.12), then the problem (11.3) has no solution.

Let $(x_0, y_0) \in D_5$, $a < t_1 < t_2 < b$, and choose $p, g \in L([a,b]; R_+)$ such that

$$\int_a^{t_1} p(s)\,ds = x_0, \quad \int_{t_1}^{t_2} p(s)\,ds = x_0, \quad \int_{t_2}^{b} p(s)\,ds = 0,$$
$$\int_a^{t_2} g(s)\,ds = 0, \quad \int_{t_2}^{b} g(s)\,ds = y_0.$$

Put $\tau \equiv b$, $\mu \equiv t_2$, and

$$z(t) = \begin{cases} \dfrac{y_0 - x_0(1-y_0)}{(t_1 - a)(1-y_0) + (y_0 - x_0(1-y_0))(t-a)} & \text{for } \ t \in [a, t_1[\,, \\ 0 & \text{for } \ t \in [t_1, b]. \end{cases}$$

Then the problem (11.10) has a nontrivial solution

$$u(t) = \begin{cases} 1 - y_0 + \dfrac{y_0 - x_0(1-y_0)}{t_1 - a}\,(t-a) & \text{for } t \in [a, t_1[\,, \\ 1 - (1-y_0)\displaystyle\int_t^b p(s)\,ds & \text{for } t \in [t_1, t_2[\,, \\ 1 - y_0 + \displaystyle\int_t^b g(s)\,ds & \text{for } t \in [t_2, b]. \end{cases}$$

According to Remark 1.1 (see p. 11) there exists $q_0 \in L([a,b];R)$ such that the problem (11.11) has no solution. Now, if we define the operator G by (11.12), then the problem (11.3) has no solution.

Let $(x_0, y_0) \in D_6$, $a < t_1 < t_2 < t_3 < t_4 < b$, and choose $p, g \in L([a,b];R_+)$ such that

$$\int_a^{t_1} p(s)\,ds = x_0 - 2\sqrt{1-y_0}\,, \quad \int_{t_1}^{t_2} p(s)\,ds = \sqrt{1-y_0}\,, \quad \int_{t_2}^{t_3} p(s)\,ds = 0,$$

$$\int_{t_3}^{t_4} p(s)\,ds = \sqrt{1-y_0}\,, \quad \int_{t_4}^{b} p(s)\,ds = 0, \quad \int_a^{t_1} g(s)\,ds = 0,$$

$$\int_{t_1}^{t_2} g(s)\,ds = y_0, \quad \int_{t_2}^{b} g(s)\,ds = 0.$$

Put $\mu \equiv a$,

$$\tau(t) = \begin{cases} t_4 & \text{for } t \in [a, t_1[\,, \\ t_3 & \text{for } t \in [t_1, t_3[\,, \\ a & \text{for } t \in [t_3, b], \end{cases}$$

and

$$G(v)(t) = \begin{cases} q_0(t) & \text{for } t \in [a, t_1[\,, \\ 0 & \text{for } t \in [t_1, t_2[\cup [t_3, t_4[\,, \\ v(t)|v(t)| & \text{for } t \in [t_2, t_3[\cup [t_4, b], \end{cases}$$

where $q_0 \in L([a,b];R)$ is such that

$$\int_a^{t_1} q_0(s)\,ds \geq \frac{x_0 - \sqrt{1-y_0}}{b - t_4} + \frac{1}{t_3 - t_2}\,.$$

Then the problem (11.3) has no solution. Indeed, assume on the contrary that (11.3) has a solution u. Then from (11.3) we get

$$u(t_1) = u(a) + u(t_4)\big(x_0 - 2\sqrt{1-y_0}\,\big) + \int_a^{t_1} q_0(s)\,ds, \tag{11.18}$$

$$u(t_2) = u(t_1) + u(t_3)\sqrt{1-y_0} - u(a)y_0, \tag{11.19}$$

$$u(t_2) = \frac{u(t_3)}{1+|u(t_3)|(t_3-t_2)}\,, \tag{11.20}$$

$$u(t_4) = u(t_3) + u(a)\sqrt{1-y_0}\,, \tag{11.21}$$

$$u(t_4) = \frac{u(b)}{1+|u(b)|(b-t_4)}\,. \tag{11.22}$$

The equalities (11.18), (11.19), and (11.21) imply

$$\int_a^{t_1} q_0(s)\,ds = u(t_2) - u(t_4)\big(x_0 - \sqrt{1-y_0}\,\big).$$

Hence, by virtue of (11.20) and (11.22), we get

$$\begin{aligned}\int_a^{t_1} q_0(s)\,ds &= \frac{u(t_3)}{1+|u(t_3)|(t_3-t_2)} - \frac{(x_0-\sqrt{1-y_0}\,)u(b)}{1+|u(b)|(b-t_4)} \leq \\ &\leq \frac{|u(t_3)|}{1+|u(t_3)|(t_3-t_2)} + \frac{(x_0-\sqrt{1-y_0}\,)|u(b)|}{1+|u(b)|(b-t_4)} < \\ &< \frac{1}{t_3-t_2} + \frac{x_0-\sqrt{1-y_0}}{b-t_4}\,,\end{aligned}$$

a contradiction.

Consequently, we have shown that the strict inequalities (8.14) and (8.15) in Theorems 8.6 and 8.12 (the inequalities (10.11) and (10.12) in Corollaries 10.6 and 10.12) cannot be replaced by the nonstrict ones.

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