Mem. Differential Equations Math. Phys. 6(1995), 127–129

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ON OSCILLATORY PROPERTIES OF THE n-TH ORDER SYSTEM OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

(Reported on November 27, 1995)

Consider the system

$$x_i'(t) = f_i\Big(t, x_1(\delta_{i1}(t)), \dots, x_n(\delta_{in}(t))\Big), \quad (i = 1, \dots, n), \tag{1}$$

where $n \geq 2$, the vector function $(f_i)_{i=1}^n : R_+ \times R^n \to R^n$ satisfies the local Caratheodory conditions, $\delta_{ij} : R_+ \to R$ are nondecreasing and

$$\lim_{t\to+\infty} \delta_{ij}(t) = +\infty \quad (i, j = 1, \dots, n), \quad \delta_{i,i+1} \in C'(R_+, R) \quad (i = 1, \dots, n-1).$$

Define $\sigma : R_+ \to R_+$ by

$$\sigma(t) = \inf\Big\{s : s \in R_+, s \geq t, \delta_{ij}(\xi) \geq t, \quad \text{for} \quad \xi \in [s, +\infty[\quad (i, j = 1, \dots, n)\Big\}.$$

Definition 1. A continuous vector function $X = (X_i)_{i=1}^n : [t_0, +\infty[\to R^n$ with $t_0 \in R_+$ is said to be a proper solution of the system (1) if it is locally absolutely continuous on $[\sigma(t_0), +\infty[$, almost everywhere on this interval the equality (1) is fulfilled, and

$$\sup\Big\{\|x(s)\| : s \in [t, \infty[\Big\} > 0, \quad \text{for} \quad t \in [t_0, +\infty[$$

Definition 2. A proper solution of the system (1) is said to be oscillatory if every component of this solution has a sequence of zeroes tending to $+\infty$. Otherwise the solution is said to be nonoscillatory.

Definition 3. We say that the system (1) has the property A provided its every proper solution is oscillatory if n is even, and either is oscillatory or satisfies

$$|x_i(t)| \downarrow 0, \quad \text{for} \quad t \uparrow +\infty, \quad (i = 1, \dots, n), \tag{2}$$

if n is odd.

Definition 4. We say that the system (1) has the property B provided its every proper solution either is oscillatory or satisfies either (2) or

$$|x_i(t)| \uparrow +\infty, \quad \text{for} \quad t \uparrow +\infty, \quad (i = 1, \dots, n) \tag{3}$$

if n is even, and either is oscillatory or satisfies (3) if n is odd.

1991 *Mathematics Subject Classification.* 34K15.

Key words and phrases. Oscillatory solution, nonoscillatory solution, property A, property B.

We will assume that there exist $\nu_i \in \{0;1\}$ such that

$$\begin{gathered}(-1)^{\nu_i} f_i(t,x_1,\dots,x_n)\operatorname{sign} x_{i+1} \ge p_i(t)|x_{i+1}|,\\ (-1)^{\nu_n} f_n(t,x_1,\dots,x_n)\operatorname{sign} x_1 \ge g(t,|x_1|), \quad \text{for} \quad t\in R_+; \quad x_1,\dots,x_n\in R,\end{gathered} \tag{4}$$

where the function $g \in K_{loc}(R_+ \times R_+; R_+)$ is nondecreasing in the second argument, $p_i \in L_{loc}(R_+, R_+)$ and

$$\int\limits_0^{+\infty} p_i(t)dt = +\infty \quad (i=1,\dots,n-1). \tag{5}$$

Besides, introduce the notation

$$\nu = \sum_{i=1}^{n} \nu_i.$$

$$\tau_i(t) = \delta_{i-1,i}(t) \quad (i=2,\dots,n), \quad \tau_1(t) = \tau_{n+1}(t) = \delta_{n1}(t).$$

$$\tau^*_{ji}(t) = \begin{cases} \tau_j\Big(\tau_{j-1}(\dots(\tau_{i+1}(t))\dots)\Big), & \text{if} \quad 1\le i<j\le n+1,\\ t, & \text{if} \quad i=j \quad (i=1,\dots,n),\end{cases}$$

$$\gamma^*_{ji}(t) = \inf\Big\{s : s\in R_+, \tau^*_{ki}(s)\ge t \ (k=i,\dots,j)\Big\} \quad (1\le i\le j\le n),$$

$$I^0 = 1, \quad I^j(s,t;p_{i+j-1},\dots,p_i) =$$

$$= \int\limits_t^s p_{i+j-1}(\tau_{i+j-1,i}(\xi))(\tau^{*'}_{i+j-1,i}(\xi) I^{j-1}(\xi,t;p_{i+j-2},\dots,p_i)d\xi,$$

$$J_0 = 1, \quad J^j(t,s;p_i,\dots,p_{i+j-1}) = \int\limits_s^t p_i(\xi) J^{j-1}(\tau_{i+1}(\xi),\tau_{i+1}(s);p_{i+1},\dots,p_{i+j-1}d\xi),$$

$$(i=1,\dots,n-1; \quad j=1,\dots,n-i).$$

Note that the functions $\gamma^*_{ji} : R\to R_+$ are increasing,

$$\begin{gathered}\gamma^*_{ki}(t) \ge \gamma^*_{ji}(t) \quad (1\le i\le j\le k\le n),\\ \gamma^*_{ji}(t)\ge t \quad (1\le i\le j\le n), \quad \text{for} \quad t\in R,\end{gathered}$$

and the expressions $I^j(s,t;p_{i+j-1},\dots,p_i)$ and $J^j(t,s;p_i,\dots,p_{i+j-1})$ have the meaning iff $t,s \ge \gamma^*_{i+j-1,i}(0)$ $(i=1,\dots,n-1; j=1,\dots,n-i)$.

Theorem -1. *Suppose that the conditions* (4) *and* (5) *are fulfilled, ν is odd and for every $l\in\{1,\dots,n-1\}$ such that $l+n$ is odd, the equation*

$$v'(t) = I^{n-l}\Big(\tau^*_{l1}(t), t_{*l}; p_{n-1},\dots,p_l\Big) g\Big(\tau^*_{n1}(t), z_l(\tau^*_{n+1,1}(t))\Big)\tau^{*'}_{n1}(t), \tag{6}$$

with $z_l(t) = \dfrac{J^l(t,\gamma^*_{l1}(0);p_1,\dots,p_l)}{J'(\tau^*_{l1}(t),0;p_l)}$, $t_{*l} = \gamma^*_{n-1,l}(0)$, *has no positive proper solution. In the case where n is odd, let, moreover,*

$$\int\limits_{\gamma^*_{n1}(0)}^{+\infty} I^{n-1}\Big(\xi,\gamma^*_{n-1,1}(0);p_{n-1,\dots,p_1}\Big) g\Big(\tau^*_{n1}(\xi),c\Big)\tau^{*'}_{n1}(\xi)d\xi = +\infty, \quad \textit{for} \quad c>0. \tag{7}$$

Then the system (1) *has the property A.*

Theorem 0. *Suppose that the conditions* (4) *and* (5) *are fulfilled,* ν *is even and for every* $l \in \{1, \dots, n-2\}$ *such that* $l+n$ *is even, the equation* (6) *has no positive proper solution. Let, moreover,*

$$\int_{\gamma(0)}^{+\infty} g\Big(t, cJ^{n-1}(\tau_1(t), \gamma^*_{n1}(0); p_1, \dots, p_{n-1})\Big)\,dt = +\infty. \tag{8}$$

for any $c > 0$, *and, in the case where* n *is even, the condition* (7) *be fulfilled. Then the system* (1) *has the property* B.

Consider now the case where the inequalities

$$(-1)^{\nu_i} f_i(t, x_1, \dots, x_n) \operatorname{sign} x_{i+1} \ge p_i(t)|x_{i+1}| \quad (i = 1, \dots, n; x_{n+1=x_1}) \tag{9}$$

are fulfilled, where $p_i \in L_{loc}(R_+, R_+)$ $(i = 1, \dots, n)$ and (5) holds.

Theorem 1. *Suppose that* (12) *is fulfilled,* ν *is odd and for every* $i \in \{1, \dots, n-1\}$ *such that* $i+n$ *is odd, the inequalites*

$$\begin{aligned}\lim_{t\to+\infty} \sup \frac{I^{n-i}(t, t_{*i}, p_{n-1}, \dots, p_i)}{I^{n-i-1}\Big(\tau_{i+1}(t), \tau_{i+1}(t_{*i}); p_{n-1}, \dots, p_{i+1}\Big)} \times \\ \times \int_t^{+\infty} I^{n-i-1}\Big(\tau_{i+1}(s), \tau_{i+1}(t_{*i}); p_{n-1}, \dots, p_{i+1}\Big) \times \\ \times \frac{J^i\Big(\tau^*_{n+1}(s), \gamma^*_{i1}(0); p_1, \dots, p_i\Big)}{J'\Big(\tau^*_{i1}(\tau^*_{n+1}(s)), 0; p_i\Big)} p_n(\tau^*_{ni}(s)){\tau^*_{ni}}'(s)ds > 1\end{aligned} \tag{10}$$

and

$$\tau^*_{i1}(\tau^*_{n+1,i}(t)) \ge t, \quad textfor \quad t \ge \gamma(0) \tag{11}$$

hold, where $t_{*i} = \gamma^*_{n-1,i}(0)$. *In the case where* n *is odd, let, moreover,*

$$\int_{\gamma^*_{n1}(0)}^{+\infty} I^{n-1}\Big(\xi, \gamma^*_{n-1,1}(0); p_{n-1}, \dots, p_1\Big) p_n\Big(\tau^*_{n1}(\xi)\Big){\tau^*_{n1}}'(\xi)d\xi = +\infty \tag{12}$$

Then the system (1) *has the property* A.

Theorem 2. *Suppose that* (12) *is fulfilled and for every* $i \in \{1, \dots, n-1\}$ *such that* $i+n$ *is even, the inequalities* (13) *and* (14) *are fulfilled. Let. moreover,*

$$\int_{\gamma(0)}^{+\infty} J^{n-1}\Big(\tau_1(t), \gamma^*_{n-1,1}(0); p_1, \dots, p_{n-1}\Big) p_n(t)dt = +\infty,$$

and, in the case where n *is odd,* (15) *hold. Then the system* (1) *has the property* B.

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