Memoirs on Differential Equations and Mathematical Physics
Volume 61, 2014, 21–36

Mouffak Benchohra and Sara Litimein

FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY IN FRÉCHET SPACES

Abstract. Sufficient conditions for the existence and uniqueness of a mild solution on a semi-infinite interval for functional integro-differential equations with state dependent delay are obtained.

2010 Mathematics Subject Classification. 34G20, 34K30.
Key words and phrases. Functional integro-differential equations, state-dependent delay, mild solution, fixed point, Fréchet space, contraction.

რეზიუმე. ფუნქციონალური ინტეგრო-დიფერენციალური განტოლებებისათვის მდგომარეობისაგან დამოკიდებული დაგვიანებით დადგენილია სუსტი ამონახსნის არსებობისა და ერთადერთობის საკმარისი პირობები ნახევრად უსასრულო შუალედში.

1. Introduction

The purpose of this paper is to prove the existence of mild solutions defined on the positive semi-infinite real interval $J := [0, +\infty)$, for functional integro-differential equations with state-dependent delay of the form

$$y'(t) = Ay(t) + f\left(t, y_{\rho(t,y_t)}, \int_0^t e(t,s,y_{\rho(s,y_s)})\,ds\right), \quad \text{a.e. } t \in J, \tag{1}$$

$$y_0 = \phi \in \mathcal{B}, \tag{2}$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(T(t)_{t\geq 0})$ on a Banach space $(E, |\cdot|)$ and $f : J \times \mathcal{B} \times E \to E$, $e : J \times J \times \mathcal{B} \to E$, $\rho : J \times \mathcal{B} \to \mathbb{R}$ and $\phi \in \mathcal{B}$ are the given function. For any continuous function y defined on $(-\infty, +\infty)$ and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in (-\infty, 0]$. Here $y_t(\,\cdot\,)$ represents the history of the state from each time $\theta \in (-\infty, 0]$ up to the present time t. We assume that the histories y_t belong to some abstract phase space $\mathcal{B}$ to be specified later.

Integro-differential equations have attracted great interest due to their applications in characterizing many problems in physics, fluid dynamics, biological models and chemical kinetics. Qualitative properties such as the existence, uniqueness and stability for various functional integro-differential equations have been extensively studied by many researchers (see, for instance, [3, 4, 7, 18, 21, 23, 25]. Likewise, the functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equation has received a significant amount of attention in the last years (we refer to [2,5,6,8,13–15] and the references therein).

In the literature, the problem (1)–(2) has been studied by several authors without delay or with delay depending only on time. A method to reduce integro-differential equations with unbounded memory to systems of functional differential equations with bounded memory without integrals and analysis of stability of partial functional integro-differential equations on this basis was presented in [1]. An important study of functional differential equations with state dependent delay was presented in [11]. Hernández [12] has discussed the existence of mild solutions of partial neutral integro-differential equations with an infinite delay. Ravichandran and Mallika [21] investigated the fractional problem. Gunasekar *et al.* [19] have discussed the existence of mild solutions for an impulsive semilinear neutral functional integro-differential equations with infinite delay in Banach spaces by using the Hausdorff measure of noncompactness. When A depends on time, Marcos *et al.* [22] have discussed the case of the existence of solutions for a class of impulsive differential equations by using the fixed point theory of compact and condensing operators. Yan [26] investigated the existence of solutions for semilinear evolution integro-differential equations with nonlocal conditions. Recently, Hong-Kun [17] studied the existence of strong solutions of a nonlinear neutral integro-differential problem on an unbounded interval.

The main purpose of the paper is to establish a global uniqueness of solutions for the problem (1)–(2). Our approach here is based on a recent Frigon–Granas nonlinear alternative of Leray–Schauder type in Fréchet spaces [9] combined with the semigroup theory.

2. Preliminaries

We introduce notations, definitions and theorems which are used throughout this paper.

Let $C([0,+\infty);E)$ be the space of continuous functions from $[0,+\infty)$ into E and $B(E)$ be the space of all bounded linear operators from E into E, with the usual supremum norm

$$N \in B(E), \quad \|N\|_{B(E)} = \sup\big\{|N(y)| : \ |y| = 1\big\}.$$

A measurable function $y : [0,+\infty) \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. For the Bochner integral properties, see the classical monograph of Yosida [24].

Let $L^1([0,+\infty),E)$ denote the Banach space of measurable functions $y : [0,+\infty) \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{+\infty} |y(t)|\, dt.$$

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [10] and follow the terminology used in [16]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty,0]$ into E, and satisfying the following axioms:

(A_1) If $y : (-\infty,b) \to E, b > 0$, is continuous on $[0,b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0,b)$ the following conditions hold:

(i) $y_t \in \mathcal{B}$;

(ii) there exists a positive constant H such that $|y(t)| \le H\|y_t\|_{\mathcal{B}}$;

(iii) there exist two functions $K(\,\cdot\,), M(\,\cdot\,) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that

$$\|y_t\|_{\mathcal{B}} \le K(t) \sup\big\{|y(s)| : \ 0 \le s \le t\big\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}$-valued continuous function on $[0,b]$.

(A_3) The space $\mathcal{B}$ is complete.

Denote $K_b = \sup\{K(t) : \ t \in [0,b]\}$ and $M_b = \sup\{M(t) : \ t \in [0,b]\}$.

Remark 2.1.

1. (ii) is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.
2. Since $\| \cdot \|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.
3. From the equivalence in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$: We necessarily have that $\phi(0) = \psi(0)$.

We now indicate some examples of phase spaces. For other details we refer, for instance, to the book due to Hino *et al.* [16].

Example 2.2. Let:

BC be the space of bounded continuous functions defined from $(-\infty, 0]$ to E;

BUC be the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E;

$$C^{\infty} := \Big\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } E\Big\};$$
$$C^{0} := \Big\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0\Big\},$$

be endowed with the uniform norm

$$\|\phi\| = \sup\big\{|\phi(\theta)| : \ \theta \leq 0\big\}.$$

We have that the spaces BUC, C^{∞} and C^{0} satisfy conditions (A_1)–(A_3). However, BC satisfies (A_1), (A_3) but does not satisfy (A_2).

Example 2.3. The spaces C_g, UC_g, C_g^{∞} and C_g^{0}.

Let g be a positive continuous function on $(-\infty, 0]$. We define:

$$C_g := \Big\{\phi \in C((-\infty, 0], E) : \ \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty, 0]\Big\};$$
$$C_g^{0} := \Big\{\phi \in C_g : \ \lim_{\theta \to -\infty} \frac{\phi(\theta)}{g(\theta)} = 0\Big\},$$

endowed with the uniform norm

$$\|\phi\| = \sup\Big\{\frac{|\phi(\theta)|}{g(\theta)} : \ \theta \leq 0\Big\}.$$

Then we have that the spaces C_g and C_g^{0} satisfy conditions (A_3). We consider the following condition on the function g.

(g_1) For all $a > 0$,

$$\sup_{0 \leq t \leq a} \sup\Big\{\frac{g(t+\theta)}{g(\theta)} : \ -\infty < \theta \leq -t\Big\} < \infty.$$

They satisfy conditions (A_1) and (A_2) if (g_1) holds.

Example 2.4. The space C_γ.

For any real positive constant γ, we define the functional space C_γ by

$$C_\gamma := \Big\{\phi \in C\big((-\infty,0],E\big) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E\Big\}$$

endowed with the norm

$$\|\phi\| = \sup\Big\{e^{\gamma\theta}|\phi(\theta)| : \ \theta \le 0\Big\}.$$

Then in the space C_γ the axioms $(A_1)-(A_3)$ are satisfied.

Definition 2.5. A function $f : J\times\mathcal{B}\times E \to E$ is said to be an L^1 -Carathéodory function if it satisfies:

(i) for each $t \in J$, the function $f(t,\cdot,\cdot) : \mathcal{B}\times E \to E$ is continuous;

(ii) for each $(y,z) \in \mathcal{B}\times E$, the function $f(\cdot,y,z) : J \to E$ is measurable;

(iii) for every positive integer k, there exists $h_k \in L^1(J;\mathbb{R}^+)$ such that

$$|f(t,y,z)| \le h_k(t)$$

for all $\|y\|_{\mathcal{B}} \le k$, $|z| \le k$ and almost every $t \in J$.

Let E be a Banach space and $B(E)$ be the Banach space of linear bounded operators.

Definition 2.6. A one parameter family $\{T(t) \mid t \ge 0\} \subset B(E)$ of bounded linear operators from $E \to E$ is a semigroup of bounded linear operator on E if satisfying the conditions:

(i) $T(t)T(s) = T(t+s)$, for $t,s \ge 0$;

(ii) $T(0) = I$.

Definition 2.7. Let $T(t)$ be a semigroup defined on E. A linear operator A defined by

$$D(A) = \Big\{x \in E| \lim_{h\to0^+} \frac{T(h)(x)-x}{h} \text{ exists in } E\Big\},$$

and

$$A(x) = \lim_{h\to0^+} \frac{T(h)x-x}{h} \quad \text{for } x \in D(A),$$

is the infinitesimal generator of the semigroup $T(t)$. $D(A)$ is called the domain of A.

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \cdots \quad \text{for every } x \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \le \overline{M}_n \quad \text{for all } y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by: $x \sim_n y$ if and only if $\|x-y\|_n = 0$ for $x, y \in X$. We denote

$$X^n = \big(X|_{\sim_n}, \|\cdot\|_n\big)$$

the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote by $[x]_n$ the equivalence class of x of the subset X^n and we define $Y^n = \{[x]_n : x \in Y\}$. We denote by $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n.

The following definition is the appropriate concept of contraction in X.

Definition 2.8 ([9]). A function $f : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in [0,1)$ such that

$$\|f(x) - f(y)\|_n \le k_n \|x-y\|_n \ \text{ for all } \ x, y \in X.$$

The corresponding nonlinear alternative result is the following

Theorem 2.9 (Nonlinear Alternative of Granas–Frigon, [9]). *Let X be a Fréchet space and $Y \subset X$ a closed subset and let $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds:*

($C1$) *N has a unique fixed point;*

($C2$) *there exists $\lambda \in [0,1)$, $n \in \mathbb{N}$ and $x \in \partial_n Y^n$ such that $\|x - \lambda N(x)\|_n = 0$.*

3. Existence results

3.1. Mild solutions.

Definition 3.1. We say that the function $y : (-\infty, +\infty) \to E$ is a mild solution of (1)–(2) if $y(t) = \phi(t)$ for all $t \le 0$ and y satisfies the following integral equation:

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s) f\left(s, y_{\rho(s,y_s)}, \int_0^s e(s,\tau,y_{\rho(\tau,y_\tau)})\, d\tau\right) ds \qquad (3)$$

$$\text{for each } \ t \ge 0.$$

Set

$$\mathcal{R}(\rho^-) = \Big\{\rho(s,\phi) : \ (s,\phi) \in J \times \mathcal{B}, \ \rho(s,\phi) \le 0\Big\}.$$

For each $b \in (0,\infty)$, we assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous. Additionally, we introduce the following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \le L^\phi(t) \|\phi\|_{\mathcal{B}} \ \text{ for every } \ t \in \mathcal{R}(\rho^-).$$

Remark 3.2. The condition (H_ϕ) is frequently verified by the functions continuous and bounded. For more details, see for instance, [16].

Lemma 3.3 ([15, Lemma 2.4]). *If $y : (-\infty, b] \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_b + L^{\phi})\|\phi\|_{\mathcal{B}} + K_b \sup\Big\{|y(\theta)| : \ \theta \in [0, \max\{0, s\}]\Big\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^{\phi} = \sup_{t \in \mathcal{R}(\rho^-)} L^{\phi}(t)$.

We introduce the following hypotheses:

$(H1)$ There exists a constant $\widehat{M} \geq 1$ such that

$$\|T(t)\|_{B(E)} \leq \widehat{M} \ \text{ for every } \ t \geq 0.$$

(H_f) (i) There exist a function $p \in L^1_{loc}(J; \mathbb{R}_+)$ and a continuous non-decreasing function $\psi : [0, \infty) \to (0, \infty)$ such that:

$$|f(t, \delta, w)| \leq p(t)\psi\big(\|\delta\|_{\mathcal{B}} + \|w\|\big) \ \text{ for every } \ (t, \delta, x) \in J \times \mathcal{B} \times E.$$

(ii) For all $R > 0$, there exists $l_R \in L^1_{loc}(J; \mathbb{R}_+)$ such that

$$\big|f(t, \delta_1, w_1) - f(t, \delta_2, w_2)\big| \leq l_R(t)\Big(\|\delta_1 - \delta_2\|_{\mathcal{B}} + \|w_1 - w_2\|\Big)$$

where $(t, \delta_i, w_i) \in J \times \mathcal{B} \times E$, $i = 1, 2$.

(H_e) (i) There exist a function $m \in L^1_{loc}(J; \mathbb{R}_+)$ and a continuous non-decreasing function $\Omega : \mathbb{R}_+ \to (0, \infty)$ such that:

$$|e(t, s, \delta)| \leq m(s)\Omega\big(\|\delta\|_{\mathcal{B}}\big) \ \text{ for all } \ (t, s, \delta) \in J \times J \times \mathcal{B}.$$

(ii) There exists a constant $C_1 > 0$ such that

$$\left| \int_0^t \big[e(t, s, x) - e(t, s, y)\big]\, ds \right| \leq C_1 \|x - y\|_{\mathcal{B}} \quad \text{for } \ (t, s) \in J, \ \ (x, y) \in \mathcal{B}.$$

Consider the space

$$B_{+\infty} = \Big\{y : \ \mathbb{R} \to E : \ y\big|_{[0,T]} \text{ continuous for } T > 0 \text{ and } y_0 \in \mathcal{B}\Big\},$$

where $y|_{[0,T]}$ is the restriction of y to the real compact interval $[0, T]$.

Let us fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms by

$$\|y\|_n := \sup\Big\{e^{-\tau \ L^*_n(t)}|y(t)| : \ t \in [0, n]\Big\},$$

where

$$L^*_n(t) = \int_0^t \bar{l}_n(s)\, ds, \ \ \bar{l}_n(t) = (1 + C_1)K_n\widehat{M}l_n(t)$$

and l_n is the function from $(H_f)(ii)$.

Then $B_{+\infty}$ is a Fréchet space with this family of semi-norms $\|\cdot\|_{n\in\mathbb{N}}$.

Theorem 3.4. *Assume that* $(H1), (H_f)$, (H_e) *and* (H_ϕ) *hold, and suppose that for* $n \in \mathbb{N}$,

$$\int\limits_{w(0)}^{+\infty} \frac{ds}{\psi(s) + \Omega(s)} > \int\limits_0^n \vartheta(s)\, ds. \tag{4}$$

Then the problem (1)–(2) *has a unique mild solution on* $(-\infty, +\infty)$.

Proof. We transform the problem (1)–(2) into a fixed-point problem. Consider the operator $N : B_{+\infty} \to B_{+\infty}$ defined by

$$N(y)(t) =$$

$$= \begin{cases} \phi(t), & \text{if } t \le 0, \\ T(t)\phi(0) + \displaystyle\int\limits_0^t T(t-s) f\left(s, y_{\rho(s,y_s)}, \int\limits_0^s e(s,\tau,y_{\rho(\tau,y_\tau)})\, d\tau\right) ds, & \\ & \text{if } t \in J. \end{cases} \tag{5}$$

Clearly, fixed points of the operator N are mild solutions of the problem (1)–(2).

For $\phi \in \mathcal{B}$, we define the function $x(\,\cdot\,) : (-\infty, +\infty) \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \le 0, \\ T(t)\phi(0), & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$ with $z_0 = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \le 0, \\ z(t), & \text{if } t \in J. \end{cases}$$

If $y(\,\cdot\,)$ satisfies (3), we can decompose it as $y(t) = \overline{z}(t) + x(t)$, $t \ge 0$, which implies that $y_t = \overline{z}_t + x_t$, for every $t \in J$ and the function $z(\,\cdot\,)$ satisfies

$$z(t) = \int\limits_0^t T(t-s) f\Bigg(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)},$$

$$\int\limits_0^s e\Big(s,\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\Big)\, d\tau\Bigg)\, ds \ \text{ for } \ t \in J.$$

Let

$$B^0_{+\infty} = \big\{z \in B_{+\infty} :\ z_0 = 0 \in \mathcal{B}\big\}.$$

For any $z \in B^0_{+\infty}$, we have

$$\|z\|_{+\infty} = \|z_0\|_{\mathcal{B}} + \sup\big\{|z(s)| : \ 0 \le s < +\infty\big\} = \\ = \sup\big\{|z(s)| : \ 0 \le s < +\infty\big\}.$$

Thus $(B^0_{+\infty}, \|\cdot\|_{+\infty})$ is a Banach space. We define the operator $F : B^0_{+\infty} \to B^0_{+\infty}$ by

$$F(z)(t) = \int_0^t T(t-s) f\Big(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}, \\ \int_0^s e\Big(s,\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\Big)\, d\tau\Big)\, ds \ \text{ for } \ t \in J.$$

Obviously, the operator N has a fixed point is equivalent to F has one, so it turns to prove that F has a fixed point. Let $z \in B^0_{+\infty}$ be such that $z = \lambda F(z)$ for some $\lambda \in [0,1)$. By the hypotheses $(H1)$, $(H_f(i))$ and $(H_e(i))$, for each $t \in [0,n]$, we have

$$|z(t)| \le \int_0^t \|T(t-s)\|_{B(E)} \Big| f\Big(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}, \\ \int_0^s e\Big(s,\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\Big)\, d\tau\Big)\Big|\, ds \le \\ \le \widehat{M} \int_0^t p(s)\psi\Big(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + \\ + \int_0^s m(\tau)\Omega\Big(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\Big)\, d\tau\Big)\, ds \le \\ \le \widehat{M} \int_0^t p(s)\psi\Big(K_n|z(s)| + \big(M_n + L^\phi + K_n MH\big)\|\phi\|_{\mathcal{B}} + \\ + \int_0^s m(\tau)\Omega\Big(K_n|z(s)| + \big(M_n + L^\phi + K_n MH\big)\|\phi\|_{\mathcal{B}}\Big)\, d\tau\Big)\, ds.$$

Set

$$c_n := \big(M_n + K_n + L^\phi + K_n MH\big)\|\phi\|_{\mathcal{B}}.$$

Then we have

$$|z(t)| \le M \int_0^t p(s)\psi\Big(K_n|z(s)| + c_n + \int_0^s m(\tau)\Omega(K_n|z(s)| + c_n)\, d\tau\Big)\, ds.$$

Thus

$$K_n|z(t)| + c_n \leq$$
$$\leq c_n + K_n\widehat{M}\int_0^t p(s)\psi\Big(K_n|z(s)| + c_n + \int_0^s m(\tau)\Omega(K_n|z(s)| + c_n)\,d\tau\Big)\,ds.$$

We consider the function μ defined by

$$\mu(t) := \sup\Big\{K_n|z(s)| + c_n : \ 0 \leq s \leq t\Big\}, \quad 0 \leq t < +\infty.$$

Let $t^\star \in [0,t]$ be such that $\mu(t) = K_n|z(t^\star)| + c_n\|\phi\|_{\mathcal{B}}$. By the previous inequality, we have

$$\mu(t) \leq c_n + K_n\widehat{M}\int_0^t p(s)\psi\Big(\mu(s) + \int_0^s m(\tau)\Omega(\mu(\tau))\,d\tau\Big)\,ds \qquad \text{for } \ t \in [0,n].$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have $\mu(t) \leq v(t)$ for all $t \in [0,n]$. This leads us to the following inequality:

$$v(t) \leq c_n + K_n\widehat{M}\int_0^t p(s)\psi\Big(v(s) + \int_0^s m(\tau)\Omega\big(v(\tau)\,d\tau\big)\,ds\Big) \quad \text{for } \ t \in [0,n],$$

whence

$$v'(t) \leq \widehat{M}\ K_n\ p(t)\psi\Big(v(t) + \int_0^t m(\tau)\Omega\big(v(\tau)\,d\tau\big)\Big).$$

Next, we consider the function

$$w(t) = v(t) + \int_0^t m(\tau)\Omega(v(\tau))\,d\tau,$$

thus we have that $v(0) = w(0)$ and $v(t) \leq w(t)$ for all $t \in [0,n]$.

Using the nondecreasing character of ψ, we get

$$w'(t) = v'(t) + p(t)\Omega(v(t)) \leq$$
$$\leq \widehat{M}K_n p(t)\psi(w(t)) + m(t)\Omega(w(t)) \ \text{ a.e. } \ t \in [0,n].$$

We define the function $\vartheta(t) = \max\big\{\widehat{M}K_n p(t), m(t)\big\}$, $t \in [0,n]$, which implies that

$$\frac{w'(t)}{\psi(w'(t)) + \Omega(w(t))} \leq \vartheta(t).$$

From condition (4), we have

$$\int\limits_{w(0)}^{w(t)} \frac{ds}{\psi(s)+\Omega(s)} \le \int\limits_0^t \vartheta(s)\,ds \le \int\limits_{w(0)}^{+\infty} \frac{ds}{\psi(s)+\Omega(s)}\,.$$

Thus, for every $t \in [0,n]$, there exists a constant Λ_n such that $w(t) \le \Lambda_n$ and hence, $\mu(t) \le \Lambda_n$. Since $\|z\|_n \le \mu(t)$, we have $\|z\|_n \le \Lambda_n$.

Set

$$Z = \Big\{ z \in B^0_{+\infty} : \sup_{0\le t\le n} |z(t)| \le \Lambda_n + 1,\ \forall\, n \in \mathbb{N} \Big\}.$$

Clearly, Z is a closed subset of $B^0_{+\infty}$.

We shall show that $F : Z \to B^0_{+\infty}$ is a contraction operator. Indeed, consider $z, \overline{z} \in Z$, thus using $(H1)$ and $(H3)$ for each $t \in [0,n]$ and $n \in \mathbb{N}$,

$$\big|F(z)(t) - F(\overline{z})(t)\big| \le \int\limits_0^t \big\|T(t-s)\big\|_{B(E)} \times$$

$$\times \bigg| f\bigg(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}, \int\limits_0^s e\big(s,\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\big)\,d\tau\bigg) -$$

$$-f\bigg(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)},$$

$$\int\limits_0^s e\big(s,\tau,\overline{z}_{\rho(\tau,\overline{z}_\tau+x_\tau)} + x_{\rho(\tau,\overline{z}_\tau+x_\tau)}\big)\,d\tau\bigg)\bigg|\,ds \le$$

$$\le \int\limits_0^t \widehat{M} l_n(s)\Big(\big\|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,\overline{z}_s+x_s)}\big\|_{\mathcal{B}} +$$

$$+C_1\big\|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,\overline{z}_s+x_s)}\big\|_{\mathcal{B}}\Big)\,ds.$$

Using (H_ϕ) and Lemma 3.3, we obtain

$$\big|F(z)(t) - F(\overline{z})(t)\big| \le$$

$$\le \int\limits_0^t \widehat{M} l_n(s)\Big(K_n|z(s)-\overline{z}(s)| + C_1\big(K_n|z(s)-\overline{z}(s)|\big)\Big)\,ds \le$$

$$\le \int\limits_0^t \widehat{M} l_n(s)[1+C_1]K_n|z(s)-\overline{z}(s)|\,ds \le$$

$$\le \int\limits_0^t \Big[\overline{l}_n(s)\ e^{\tau L^*_n(s)}\Big]\,\Big[e^{-\tau L^*_n(s)}\big|z(s)-\overline{z}(s)\big|\Big]\,ds \le$$

$$\leq \int_0^t \Big[\frac{e^{\tau L_n^*(s)}}{\tau}\Big]' ds \, \|z-\overline{z}\|_n \leq \frac{1}{\tau}\, e^{\tau L_n^*(t)} \|z-\overline{z}\|_n.$$

Therefore,

$$\|F(z)-F(\overline{z})\|_n \leq \frac{1}{\tau}\, \|z-\overline{z}\|_n.$$

So, the operator F is a contraction for all $n \in \mathbb{N}$. By the choice of Z, there is no $z \in \partial Z^n$ such that $z = \lambda F(z)$, $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 2.9 does not hold. The nonlinear alternative due to Frigon and Granas shows that $(C1)$ holds. Thus, we conclude that the operator F has a unique fixed-point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty, +\infty)$ is a fixed point of the operator N, which is the unique mild solution of the problem (1)–(2). □

4. An Example

To apply our results, we consider the following partial differential equation:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,\xi) = \dfrac{\partial^2 v}{\partial \xi^2}(t,\xi) + \\ \quad + m\Big(t, v\big(t-\sigma(v(t,0)),\xi\big), \displaystyle\int_0^t \eta\big(t,s,v\big(s-\sigma(v(s,0)),\xi\big)\big)\,ds\Big), \\ \qquad\qquad t \in [0,\infty),\ \ \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0,\ \ t \in [0,\infty), \\ v(\theta,\xi) = v_0(\theta,\xi),\ \ \theta \in (-\infty,0],\ \ \xi \in [0,\pi], \end{cases} \tag{6}$$

where v_0 and $\sigma \in C(\mathbb{R},[0,\infty))$ are continuous. Take $E = L^2[0,\pi]$ and define $A : D(A) \subset E \to E$ by $Aw = w''$ with the domain

$$D(A) = \Big\{ w \in E,\, w, w' \text{ are absolutely continuous}, \\ w'' \in E,\ w(0) = w(\pi) = 0 \Big\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} -n^2 (w,w_n) w_n, \ \ w \in D(A),$$

where $w_n(s) = \sqrt{\frac{2}{\pi}}\, \sin ns$, $n = 1,2,\ldots$, is the orthogonal set of eigenvalues of A. It is well known (see [20]) that A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$ in E and is given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t}(w,w_n)w_n, \ \ w \in E.$$

Since the analytic semigroup $T(t)$ is compact for $t > 0$, there exists a constant $M \geq 1$ such that $\|T(t)\| \leq M$.

Theorem 4.1. *Let $\mathcal{B} = BUC(\mathbb{R}_-, E)$ and $\phi \in \mathcal{B}$. Assume that condition (H_ϕ) holds. The function $m : J \times J \times [0,\pi] \to [0,\pi]$, $\sigma : \mathbb{R} \to \mathbb{R}^+$, $\eta : J \times J \times [0,\pi] \to [0,\pi]$ are continuous. Then there exists a unique mild solution of* (6)*.*

Proof. From the above assumptions, we have that the functions

$$\begin{aligned} f(t,\varphi,x)(\xi) &= m\Big(t, \varphi(0,\xi), \int_0^t \eta\big(t,s,\varphi(0,\xi)\big)\, ds\Big), \\ e(t,s,\varphi)(\xi) &= \eta\big(t,s,\varphi(0,\xi)\big), \\ \rho(t,\varphi) &= t - \sigma(\varphi(0,0)) \end{aligned}$$

are well defined, permitting to transform system (6) into the abstract system (1)–(2). Moreover, the function f is a bounded linear operator. Now the existence of mild solutions can be deduced from a direct application of Theorem 3.4. From Remark 3.2, we have the following result. ☐

Corollary 4.2. *Let $\varphi \in \mathcal{B}$ be continuous and bounded. Then there exists a unique mild solution of* (6) *on* $(-\infty, +\infty)$*.*

Acknowledgements

The authors are grateful to the referee for the helpful remarks and suggestions.

References

1. R. P. Agarwal, A. Domoshnitsky, and Ya. Goltser, Stability of partial functional integro-differential equations. *J. Dyn. Control Syst.* **12** (2006), No. 1, 1–31.
2. W. G. Aiello, H. I. Freedman, and J. Wu, Analysis of a model representing stage-structured population growth with state-dependent time delay. *SIAM J. Appl. Math.* **52** (1992), No. 3, 855–869.
3. K. Balachandran and N. Annapoorani, Existence results for impulsive neutral evolution integrodifferential equations with infinite delay. *Nonlinear Anal. Hybrid Syst.* **3** (2009), No. 4, 674–684.
4. K. Balachandran and R. R. Kumar, Existence of solutions of integrodifferential evolution equations with time varying delays. *Appl. Math. E-Notes* **7** (2007), 1–8 (electronic).
5. M. Bartha, Periodic solutions for differential equations with state-dependent delay and positive feedback. *Nonlinear Anal.* **53** (2003), No. 6, 839–857.
6. Yu. Cao, J. Fan, and T. C. Gard, The effects of state-dependent time delay on a stage-structured population growth model. *Nonlinear Anal.* **19** (1992), No. 2, 95–105.
7. Y.-K. Chang, M. M. Arjunan, and V. Kavitha, Existence results for neutral functional integrodifferential equations with infinite delay via fractional operators. *J. Appl. Math. Comput.* **36** (2011), No. 1-2, 201–218.
8. A. Domoshnitsky, M. Drakhlin, and E. Litsyn, On equations with delay depending on solution. *Nonlinear Anal.* **49** (2002), No. 5, Ser. A: Theory Methods, 689–701.
9. M. Frigon and A. Granas, Résultats de type Leray–Schauder pour des contractions sur des espaces de Fréchet. (French) Dedicated to the memory of Gilles Fournier (Sherbrooke, PQ, 1997). *Ann. Sci. Math. Québec* **22** (1998), No. 2, 161–168.

10. J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay. *Funkcial. Ekvac.* **21** (1978), No. 1, 11–41.
11. F. Hartung, T. Krisztin, H.-O. Walther, and J. Wu, Functional differential equations with state-dependent delays: theory and applications. *Handbook of differential equations: ordinary differential equations. Vol. III*, 435–545, Handb. Differ. Equ., *Elsevier/North-Holland, Amsterdam*, 2006.
12. E. Hernández, Existence results for partial neutral integro-differential equations with unbounded delay. *Cadernos de Matemática* **03** (2002), No. 1, 1-16.
13. E. Hernández, A. Prokopczyk, and L. Ladeira, A note on partial functional differential equations with state-dependent delay. *Nonlinear Anal. Real World Appl.* **7** (2006), No. 4, 510–519.
14. E. Hernández, M. Pierri, and G. Goncalves, Existence results for an impulsive abstract partial differential equation with state-dependent delay. *Comput. Math. Appl.* **52** (2006), No. 3-4, 411–420.
15. E. Hernández, R. Sakthivel, and S. Tanaka Aki, Existence results for impulsive evolution differential equations with state-dependent delay. *Electron. J. Differential Equations* **2008**, No. 28, 11 pp.
16. Y. Hino, S. Murakami, and T. Naito, Functional-differential equations with infinite delay. Lecture Notes in Mathematics, 1473. *Springer-Verlag, Berlin*, 1991.
17. G. Marino, P. Pietramala, and H.-K. Xu, Nonlinear neutral integrodifferential equations on unbounded intervals. *Int. Math. Forum* **1** (2006), No. 17-20, 933–946.
18. R. S. Jain and M. B. Dhakne, On global existence of solutions for abstract nonlinear functional integro-differential equations with nonlocal condition. *Contemporary Math. Stat.*, 2013, No. 1, 44–53; doi:10.7726/cms.2013.1004.
19. T. Gunasekar, F. P. Samuel, and M. M. Arjunan, Existence results for impulsive neutral functional integrodifferential equation with infinite delay. J. Nonlinear Sci. Appl. 6 (2013), no. 4, 234–243.
20. A. Pazy, Semigroups of operators in Banach spaces. *Equadiff 82 (Würzburg, 1982)*, 508–524, Lecture Notes in Math., 1017, *Springer, Berlin*, 1983.
21. Ch. Ravichandran and D. Baleanu, Existence results for fractional neutral functional integro-differential evolution equations with infinite delay in Banach spaces. *Adv. Difference Equations* **2013** (2013); doi:10.1186/1687-1847-2013-215.
22. M. N. Rabelo, M. Henrique, and G. Siracusa, Existence of integro-differential solutions for a class of abstract partial impulsive differential equations. *J. Inequal. Appl.* **2011**, 2011:135, 19 pp.
23. K. Sathiyanathan and T. Nandha, Global existence results on general integro-differential evolution equations in Banach space. *Appl. Math.* **4** (2013), 149–154.
24. K. Yosida, Functional analysis. Sixth edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 123. *Springer-Verlag, Berlin–New York*, 1980.
25. E. M. Roshdy, On asymptotic behaviour of solutions of integrodifferential equations. *Int. J. Res. Rev. Appl. Sci.* **7** (2011), No. 1, 103–105.
26. Z. Yan, On solutions of semilinear evolution integrodifferential equations with nonlocal conditions. *Tamkang J. Math.* **40** (2009), No. 3, 257–269.

(Received May 17, 2013; revised February 5, 2014)

Authors' addresses:

Mouffak Benchohra

1. Laboratory of Mathematics, University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria.

2. Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.
E-mail: benchohra@univ-sba.dz

Sara Litimein
Laboratory of Mathematics, University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria.
E-mail: sara_litimein@yahoo.fr