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Chengjun Guo, Donal O'Regan,
Chengjiang Wang, and Ravi P. Agarwal

PERIODIC SOLUTIONS OF SUPERQUADRATIC NONAUTONOMOUS DIFFERENTIAL SYSTEMS WITH A DELAY

Abstract. The nonautonomous delay differential system

$$x'(t) = f(t, x(t - \tau)),$$

is considered, where $\tau > 0$, $f : R \times R^n \to R^n$ is a continuous vector function such that

$$f(t + 4\tau, x) = f(t, x), \quad f(t, x) = \nabla_x F(t, x).$$

Using the critical point theory, the conditions ensuring the existence of a nontrivial 4τ-periodic solution of that system are established in the case, where $F(t, x)$ is superquadratic in x.

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რეზიუმე. განხილულია არაავტონომიური დაგვიანებული დიფერენციალური სისტემა

$$x'(t) = f(t, x(t - \tau)),$$

სადაც $\tau > 0$, ხოლო $f : R \times R^n \to R^n$ უწყვეტი ვექტორული ფუნქციაა ისეთი, რომ

$$f(t + 4\tau, x) = f(t, x), \quad f(t, x) = \nabla_x F(t, x).$$

კრიტიკული წერტილის თეორიის გამოყენებით დადგენილია პირობები, რომლებიც უზრუნველყოფენ აღნიშნული სისტემის 4τ-პერიოდული ამონახსნის არსებობას იმ შემთხვევაში, როცა $F(t, x)$ არის x-ის მიმართ სუპერკვადრატული.

1. Introduction

This paper studies the existence of periodic solutions for the first-order delay differential equations (with superquadratic growth conditions)

$$x'(t) = f(t, x(t-\tau)), \tag{1.1}$$

where $f \in C(R \times R^n, R^n)$ and $\tau > 0$ is a given constant.

The results on the existence of periodic solutions for a functional differential equation were obtained by several authors, but there are only a few results on periodic solutions to delay differential equations using critical point theory. We refer the reader to [3–7, 9–13] and the references therein.

In this paper, we study periodic solutions of (1.1) under some superquadratic condition. We apply critical point theory directly in the study of periodic orbits of the system (1.1); we do not reduce the original existence problem (1.1) to an existence problem for an associated Hamiltonian system.

Throughout this paper, we always assume that:

(F_1) f is periodic with respect to the first variable with the period 4τ and is odd with respect to the phase variables, i.e.,

$$f(t+4\tau, x) = f(t,x), \quad f(t,-x) = -f(t,x)$$

for every $t \in R$ and $x \in R^n$;

(F_2) there exists a continuously differentiable τ-periodic function $F(t,x) \in C^1(R \times R^n, R^+)$ with respect to t, such that $\nabla_x F = f$.

For our first result we assume the following:

(H_1) there is a constant $\nu > 2$ such that

$$0 < \nu F(t,x) \le (x, f(t,x)) \quad \text{whenever} \quad x \ne 0.$$

Here and in the sequel, $(\cdot\,,\cdot) : R^n \times R^n \to R$ denotes the standard inner product in R^n and $|\cdot|$ the induced norm.

(H_2) there is a constant $a_1 > 0$ such that

$$|f(t,x)| \le a_1(x, f(t,x)), \ \ \forall\,|x| \ge 1.$$

Remark 1. Set $a_2 = \min\limits_{|x|=1,\ t\in[0,\tau]} F(t,x)$, $a_3 = \max\limits_{|x|\le 1,\ t\in[0,\tau]} F(t,x)$. We have from ($F_2$) and ($H_1$) that

$$F(t,x) \ge a_2|x|^\nu, \ \ \forall\,|x| \ge 1$$

and

$$F(t,x) \ge a_2|x|^\nu - a_3, \ \ \forall\,x \in R^n.$$

Remark 2. Choose $q > 2$. By (F_2) and (H_1), for any $\varepsilon > 0$, there exists $a_4 > 0$ such that

$$F(t,x) \le \varepsilon|x|^2 + a_4|x|^q, \ \ \forall\,(t,x) \in [0,\tau] \times R^n.$$

Theorem 1.1. *Assume* (F_1)–(F_2) *and* (H_1)–(H_2). *Then the system* (1.1) *possesses a nontrivial* 4τ*-periodic solution.*

It is easy to see that (H_1) does not include nonlinearities like

$$F(t,x) = |x|^2\big(\ln(1+|x|^p)\big)^q, \quad p,q>1. \tag{1.2}$$

In the theorem below we study periodic solutions of (1.1) under some superquadratic condition which covers a case like (1.2). We assume F satisfies the following conditions:

(V_1) $F(t,x) \geq 0$, for all $(t,x) \in [0,4\tau] \times R^n$;

(V_2) $F(t,x) = o(|x|^2)$ as $|x| \to 0$ uniformly in t;

(V_3) $\frac{F(t,x)}{|x|^2} \to +\infty$ as $|x| \to +\infty$ uniformly in t;

(V_4) there exist positive constants $\beta > 1$, $1 < \lambda < 1 + \frac{\beta-1}{\beta}$, c_1, c_2, c_3 and c_4 such that

$$(x, f(t,x)) - 2F(t,x) \geq c_1|x|^\beta - c_2, \quad (t,x) \in [0,4\tau] \times R^n, \tag{1.3}$$

$$|f(t,x)| \leq c_3|x|^\lambda + c_4, \quad (t,x) \in [0,4\tau] \times R^n. \tag{1.4}$$

Theorem 1.2. *Assume* (F_1)–(F_2) *and* (V_1)–(V_4). *Then* (1.1) *possesses a nontrivial* 4τ*-periodic solution.*

This paper is motivated by [6] where the existence and multiplicity of periodic solutions for the delay differential equations

$$x'(t) = -f(x(t-\tau))$$

have been discussed.

The paper is organized as follows. In Section 2, we establish a variational structure for (1.1) with a periodic boundary value condition, and we show that the existence of 4τ-periodic solutions is equivalent to the existence of critical points of some variational functional defined on a suitable Hilbert space. Our main results will be proved in Section 3.

2. Variational Structure

By means of the transformation

$$t = \frac{2\tau}{\pi}\, s, \quad x(t) = y(s) \tag{2.1}$$

the system (1.1) receives the form

$$y'(s) = g\Big(s, y\Big(s - \frac{\pi}{2}\Big)\Big),$$

where

$$g(s,y) = \frac{2\tau}{\pi}\, f\Big(\frac{2\tau}{\pi}\, s, y\Big),$$

and g is 2π-periodic with respect to the first variable. Therefore, without loss of generality, one can assume that $\tau = \frac{\pi}{2}$ and f is 2π-periodic with respect to the first variable. Thus (1.1) transforms to

$$x'(t) = f\Big(t, x\Big(t - \frac{\pi}{2}\Big)\Big), \tag{2.2}$$

and we seek for 2π-periodic solutions of (2.2) which, of course, correspond to 4τ-periodic solutions of (1.1).

Let $C^\infty(S^1,R^n)$ denote the space of 2π-periodic C^∞ functions on R with values in R^n. Any $x\in C^\infty(S^1,R^n)$ has the following Fourier expansion in the sense that it is convergent in the space $L^2(S^1,R^n)$,

$$x(t)=\frac{a_0}{\sqrt{2\pi}}+\frac{1}{\sqrt{\pi}}\sum_{k=1}^{+\infty}(a_k\cos kt+b_k\sin kt), \tag{2.3}$$

where $a_0,a_k,b_k\in R^n$ $(k=1,2,\dots)$.

Let $x\in L^2(S^1,R^n)$. If for every $z\in C^\infty(S^1,R^n)$,

$$\int_0^{2\pi}\big(x(t),z'(t)\big)\,dt=-\int_0^{2\pi}\big(y(t),z(t)\big)\,dt,$$

then y is called a weak derivative of x denoted by $y=\dot{x}(t)$. Here and in the sequel, $(\,\cdot\,,\,\cdot\,):R^n\times R^n\to R$ denotes the standard inner product in R^n and $|\cdot|$ the induced norm.

Let $H^{\frac12}(S^1,R^n)$ be the closure of $C^\infty(S^1,R^n)$ with respect to the Hilbert norm

$$\|x\|_{H^{\frac12}(S^1,R^n)}=\Big[|a_0|^2+\sum_{k=1}^{+\infty}(1+k)\big(|a_k|^2+|b_k|^2\big)\Big]^{\frac12}. \tag{2.4}$$

Now $H^{\frac12}(S^1,R^n)$ can also be obtained by interpolation from the Sobolev spaces $H^1(S^1,R^n)$ and $L^2(S^1,R^n)$. More specifically, for any $x\in L^2(S^1,R^n)$, if x has a Fourier expansion with the convergence in the space $L^2(S^1,R^n)$, then x has a representation as in (2.3). Thus, $x\in H^{\frac12}(S^1,R^n)$, if and only if $x\in L^2(S^1,R^n)$, and

$$|a_0|^2+\sum_{k=1}^{+\infty}(1+k)\big(|a_k|^2+|b_k|^2\big)<+\infty.$$

For any $x,y\in H^{\frac12}(S^1,R^n)$, $\langle\,\cdot\,,\,\cdot\,\rangle$ can be explicitly expressed by

$$\langle x,y\rangle_{H^{\frac12}(S^1,R^n)}=(a_0,\overline{a}_0)+\sum_{k=1}^{+\infty}(1+k)\big((a_k,\overline{a}_k)+(b_k,\overline{b}_k)\big), \tag{2.5}$$

where

$$y(t)=\frac{\overline{a}_0}{\sqrt{2\pi}}+\frac{1}{\sqrt{\pi}}\sum_{k=1}^{+\infty}(\overline{a}_k\cos kt+\overline{b}_k\sin kt).$$

From the definition of $H^{\frac12}(S^1,R^n)$, we have

$$|a_0|^2+\sum_{k=1}^{+\infty}(1+k)\big(|a_k|^2+|b_k|^2\big)<+\infty. \tag{2.6}$$

Furthermore, let $L^{\infty}_{2\pi}(R,R^n)$ denote the space of 2π-periodic essentially bounded (measurable) functions from R into R^n equipped with the norm

$$\|x\|_{L^{\infty}_{2\pi}} := \operatorname{ess\,sup}\big\{|z(t)| : \ t \in [0,2\pi]\big\}.$$

Set

$$E = \left\{ x \in H^{\frac{1}{2}}(S^1,R^n) : \ x\Big(t+\frac{\pi}{2}\Big) = -x\Big(t-\frac{\pi}{2}\Big) \right\}.$$

Lemma 2.1. *Let* $E = \big\{x \in H^{\frac{1}{2}}(S^1,R^n) : \ x(t+\frac{\pi}{2}) = -x(t-\frac{\pi}{2})\big\}$. *Then*

$$E = \left\{ x(t) = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{+\infty} \Big(a_{2k-1}\cos(2k-1)t + b_{2k-1}\sin(2k-1)t \Big) \right\}, \quad (2.7)$$

where $a_{2k-1}, b_{2k-1} \in R^n$.

Proof. For

$$x(t) = \frac{a_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{+\infty} (a_k \cos kt + b_k \sin kt) \in E,$$

we have $x(t) = -x(t+\pi)$, and this implies

$$a_0 = -a_0, \quad a_k = (-1)^{k+1} a_k, \quad b_k = (-1)^{k+1} b_k,$$

so (2.7) holds.

We define

$$\langle Ax, y\rangle = \frac{1}{2} \int_0^{2\pi} \Big(\dot{x}\Big(t+\frac{\pi}{2}\Big), y \Big)\, dt, \quad \forall\, x, y \in E, \quad (2.8)$$

$$\Phi(x) = \int_0^{2\pi} F(t, x(t))\, dt \quad (2.9)$$

and

$$I(x) = \int_0^{2\pi} \left[\frac{1}{2} \Big(\dot{x}\Big(t+\frac{\pi}{2}\Big), x(t) \Big) - F(t,x(t)) \right] dt = \frac{1}{2}\langle Ax, x\rangle - \Phi(x), \quad (2.10)$$

where $\dot{x}(t)$ denotes the weak derivative of $x(t)$. Then A has a sequence of eigenvalues

$$\cdots \xi^{(-m)} \le \cdots \le \xi^{(-2)} \le \xi^{(-1)} < 0 < \xi^{(1)} \le \xi^{(2)} \le \cdots \le \xi^{(m)} \cdots$$

with $\xi^{(m)} \to \infty$ and $\xi^{(-m)} \to -\infty$ as $m \to \infty$. Let φ^j be the eigenvector of A corresponding to $\xi^{(j)}$, $j = \pm 1, \pm 2, \dots, \pm m, \dots$. Set

$$E^0 = \ker(A),$$

$$E^- = \text{the negative eigenspace of } A,$$

$$E^+_k = \text{the positive eigenspace of } A.$$

Then $E = E^- \oplus E^0 \oplus E^+$. ◻

From the argument in [1, 2], we have

Lemma 2.2. *Assume* (F_1)–(F_2) *and* (H_1)–(H_2) (*or* (F_1)–(F_2) *and* (V_1)–(V_4)) *hold. Then the functional* I *is continuously differentiable on* $H^{\frac{1}{2}}(S^1, R^n)$ *and* $I'(x)$ *is defined by*

$$\langle I'(x), y\rangle_{H^{\frac{1}{2}}(S^1,R^n)} = \int_0^{2\pi} \Big(\dot{x}\Big(t+\frac{\pi}{2}\Big) - f(t,x), y\Big)\, dt, \quad y \in H^{\frac{1}{2}}(S^1, R^n). \tag{2.11}$$

In addition, we need the following observations, which are necessary in the proof of Theorem 1.1 and Theorem 1.2.

Lemma 2.3. *A is self-adjoint on E and $\Phi'(x) \in E$ for $\forall\, x \in E$.*

Proof. For any $x, y \in E$, by the Riesz representation theorem, Ax can be viewed as a function belonging to $E \subseteq H^{\frac{1}{2}}(S^1, R^n)$ such that $\langle Ax, y\rangle = (Ax)(y)$.

Combining (2.8) and $y(t) = -y(t-\pi)$, we have

$$\langle Ax, y\rangle_E = \int_0^{2\pi} \Big(\dot{x}\Big(t+\frac{\pi}{2}\Big), y(t)\Big)\, dt = -\int_0^{2\pi} \Big(x\Big(t+\frac{\pi}{2}\Big), \dot{y}(t)\Big)\, dt =$$

$$= -\int_0^{2\pi} \Big(x(t), \dot{y}\Big(t-\frac{\pi}{2}\Big)\Big)\, dt = \int_0^{2\pi} \Big(x(t), \dot{y}\Big(t+\frac{\pi}{2}\Big)\Big)\, dt = \langle x, Ay\rangle_E.$$

Thus A is self-adjoint on E.

Now $\forall\, x \in E$ and $y \in H^{\frac{1}{2}}(S^1, R^n)$, we have from (F_1), (F_2) and (2.9) that

$$\big\langle \Phi'(x(t+\pi)), y\big\rangle_E = \int_0^{2\pi} f\big((t, x(t+\pi)), y(t)\big)\, dt =$$

$$= \int_0^{2\pi} f\big((t, -x(t)), y(t)\big)\, dt = -\int_0^{2\pi} f\big((t, x(t)), y(t)\big)\, dt = -\big\langle \Phi'(x(t)), y\big\rangle_E.$$

Thus $\Phi'(x) \in E$ for $\forall\, x \in E$. □

Lemma 2.4. *The existence of 2π-periodic solutions $x(t)$ for* (2.2) *is equivalent to the existence of critical points of the functional I.*

Lemma 2.5 ([8])**.** *Let E be a real Hilbert space with $E = E_1 \oplus E_2$ and $E_1 = (E_2)^{\perp}$. Suppose $I \in C^1(E, R)$ satisfy the* **(PS)** *condition, and*

(C_1) *$I(u) = \frac{1}{2}(Lu, u) + b(u)$, where $Lu = L_1P_1u + L_2P_2u$, $L_i : E_i \longmapsto E_i$ is bounded and self-adjoint, P_i is the projector of E onto $E^{(i)}$, $i = 1, 2$;*

(C_2) *b' is compact;*

(C_3) *there exist a subspace* $\widetilde{E} \subset E$ *and sets* $S \subset E$, $Q \subset \widetilde{E}$ *and constants* $\widetilde{\alpha} > \omega$ *such that*

(i) $S \subset E_1$ *and* $I|_S \geq \widetilde{\alpha}$;

(ii) Q *is bounded and* $I|_{\partial Q} \leq \omega$;

(iii) S *and* ∂Q *link.*

Then I *possesses a critical value* $c \geq \alpha$ *given by*

$$c = \inf_{g \in \Gamma} \sup_{u \in Q} I(g(1, u)),$$

where

$$\Gamma \equiv \{g \in C([0,1] \times E, E) : \ g \text{ satisfies } (\Gamma_1)\text{–}(\Gamma_3)\},$$

(Γ_1) $g(0, u) = u$;

(Γ_2) $g(t, u) = u$ *for* $u \in \partial Q$;

(Γ_3) $g(t, u) = e^{\theta(t,u)L}u + \chi(t, u)$, *where* $\theta(t, u) \in C([0, 1] \times E, R)$ *and* χ *is compact.*

3. Proof of the Main Results

In order to prove Theorem 1.1 and Theorem 1.2, the following result in [8, p. 36, Proposition 6.6] will be used.

Proposition 3.1. *There is a positive constant* c_θ *such that for* $x \in E$ *the inequality*

$$\|x\|_{L^\theta_{2\pi}} \leq c_\theta \|x\|_{H^{\frac{1}{2}}(S^1, R^n)} \tag{3.1}$$

holds, where $\theta \in [1, +\infty)$.

Lemma 3.1. *Under the conditions of Theorem* 1.1, *I satisfies the* **(PS)** *condition.*

Proof. Assume that $\{x_n\}_{n\in\mathbf{N}}$ in $H^{\frac{1}{2}}(S^1, R^n)$ is a sequence such that $\{I(x_n)\}_{n\in\mathbf{N}}$ is bounded and $I'(x_n) \to 0$, as $n \to +\infty$. Then there exists a constant $d_1 > 0$ such that

$$|I(x_n)| \leq d_1, \quad \|I'(x_n)\|_{(H^{\frac{1}{2}}(S^1, R^n))^*} \to 0 \ \text{ as } \ n \to \infty, \tag{3.2}$$

where $(H^{\frac{1}{2}}(S^1, R^n))^*$ denotes the dual space of $H^{\frac{1}{2}}(S^1, R^n)$.

We first prove that $\{x_n\}_{n\in\mathbf{N}}$ is bounded. Since $x_n \in H^{\frac{1}{2}}(S^1, R^n)$, we have $x_n = x_n^0 + x_n^+ + x_n^- \in E^0 \oplus E^+ \oplus E^-$.

From (F_2), (H_1) and (2.8)–(2.10), noting Remark 1, there exist two positive constants d_2 and d_3 such that

$$2d_1 \geq 2I(x_n) - \langle I'(x_n), x_n\rangle = \int_0^{2\pi} \big[\big(x_n, f(t,x_n)\big) - 2F(t,x_n)\big]\, dt =$$

$$= \int_0^{2\pi} \Big[\big(x_n, f(t,x_n)\big) - \nu F(t,x_n) + (\nu-2)F(t,x_n)\Big]\, dt \geq$$

$$\geq \int_0^{2\pi} \big[d_2(\nu-2)|x_n(t)|^\nu - d_3\big]\, dt. \tag{3.3}$$

This implies

$$\int_0^{2\pi} |x_n(t)|^\nu\, dt \leq \frac{2d_1 + 2\pi d_3}{d_2(\nu-2)} = \widetilde{M}_0^*. \tag{3.4}$$

Consider $\{\|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)}\}_{n\in\mathbf{N}}$. Arguing indirectly, we suppose $\big\{\|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)}\big\}_{n\in\mathbf{N}}$ is unbounded. Then we have $\|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \to \infty$. Note $\dim(E^0) < +\infty$, and this implies that there are constants b_1 and b_2 such that

$$b_1\|x_n^0\|_{L^\nu_{2\pi}} \leq \|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \leq b_2\|x_n^0\|_{L^\nu_{2\pi}}. \tag{3.5}$$

From (3.5), we have

$$\|x_n\|_{L^\nu_{2\pi}} \geq \|x_n^0\|_{L^\nu_{2\pi}} \to +\infty \ \text{ as } \ \|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \to +\infty. \tag{3.6}$$

We have from (3.4) and (3.6) that

$$\widetilde{M}_0^* \geq \int_0^{2\pi} |x_n(t)|^\nu\, dt \geq \int_0^{2\pi} |x_n^0(t)|^\nu\, dt \longrightarrow +\infty, \ \text{ as } \ \|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \to +\infty. \tag{3.7}$$

This is a contradiction. Hence $\{\|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)}\}_{n\in\mathbf{N}}$ is bounded. Therefore there exists a constant $\widetilde{M}_1^* > 0$ such that

$$\|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \leq \widetilde{M}_1^*. \tag{3.8}$$

We have from (H_1) and (3.3) that

$$2d_1 \geq 2I(x_n) - \langle I'(x_n), x_n\rangle =$$

$$= \int_0^{2\pi} \Big[\big(x_n, f(t,x_n)\big) - 2F(t,x_n)\Big]\, dt \geq \int_0^{2\pi} \Big(1 - \frac{2}{\nu}\Big)\big(x_n, f(t,x_n)\big)\, dt. \tag{3.9}$$

This implies from (H_2) and (3.9) that

$$\widetilde{M_2^*} = \frac{2\nu d_1}{(\nu-2)} \geq \int_0^{2\pi} \big(x_n, f(t,x_n)\big)\, dt \geq \frac{1}{a_1} \int_{|x_n|\geq 1} |f(t,x_n)|\, dt. \qquad (3.10)$$

We now show that

$$\|x_n\|_{L^\infty_{2\pi}} \leq \widetilde{M_3^*}. \qquad (3.11)$$

If not, by passing to a subsequence, without the loss of generality, assume that there exist t_n and $\widetilde{t}_n$ such that

$$|x_n(t_n)| = M_n^*, \quad \lim_{n\to\infty} M_n^* = \infty, \quad |x_n(\widetilde{t}_n)| = \frac{\widetilde{M_0^*}\widetilde{M_4^*}}{2\pi},$$

where $\widetilde{M_4^*} \geq 2$ is a constant such that $\frac{\widetilde{M_0^*}\widetilde{M_4^*}}{2\pi} \geq 1$, and $\frac{\widetilde{M_0^*}\widetilde{M_4^*}}{2\pi} \leq |x_n(t)| \leq M_n^*$ for $t \in (\widetilde{t}_n, t_n) \subseteq [0,2\pi]$. (In fact, suppose we cannot find a $\widetilde{t}_n$ such that $|x_n(\widetilde{t}_n)| \leq \frac{\widetilde{M_0^*}\widetilde{M_4^*}}{2\pi}$. Then from (3.4) we have $\widetilde{M_0^*} \geq \int_0^{2\pi} |x_n(t)|^\nu\, dt \geq \int_0^{2\pi} |x_n(t)|\, dt \geq \widetilde{M_0^*}\widetilde{M_4^*}$, a contradiction.)

From (F_2) and (H_1), noting Remark 2, for any $\widetilde{\varepsilon} > 0$, there exists a constant $\widetilde{d}_4 > 0$ such that

$$|f(t,x)| \leq \widetilde{\varepsilon}|x| + \widetilde{d}_4, \ \ \forall\, |x| < 1, \ \text{ uniformly in } t. \qquad (3.12)$$

Set

$$\Lambda_n = \int_0^{2\pi} \Big| \dot{x_n}\Big(s+\frac{\pi}{2}\Big) - f(s, x_n(s)) \Big|\, ds.$$

We have from (2.11) and (3.2) that $\lim\limits_{n\to\infty} \Lambda_n = 0$.

Hence, by the periodicity of $x_n(t)$ and $f(t,x_n(t))$ with respect to t, (3.10) and (3.12), there exists a constant $d_4 > 0$ such that

$$M_n^* - \frac{\widetilde{M_0^*}\widetilde{M_4^*}}{2\pi} = |x_n(t_n)| - \big|x_n(\widetilde{t}_n)\big| = \int_{\widetilde{t}_n}^{t_n} \frac{d}{ds}\, |x_n(s)|\, ds \leq$$

$$\leq \int_{\widetilde{t}_n}^{t_n} |\dot{x_n}(s)|\, ds \leq \int_0^{2\pi} |\dot{x_n}(s)|\, ds = \int_0^{2\pi} \Big|\dot{x_n}\Big(s+\frac{\pi}{2}\Big)\Big|\, ds =$$

$$= \int_0^{2\pi} \Big| \dot{x_n}\Big(s+\frac{\pi}{2}\Big) - f(s,x_n(s)) + f(s,x_n(s)) \Big|\, ds \leq$$

$$\leq \int_0^{2\pi} \Big| \dot{x_n}\Big(s+\frac{\pi}{2}\Big) - f(s,x_n(s)) \Big|\, ds + \int_0^{2\pi} \big|f(s,x_n(s))\big|\, ds =$$

$$= \Bigg[\int\limits_{|x_n|\geq 1} \big|f(s,x_n(s))\big|\, ds + \int\limits_{|x_n|<1} \big|f(s,x_n(s))\big| ds \Bigg] + \Lambda_n \leq$$

$$\leq (a_1 \widetilde{M}_2^* + d_4) + \Lambda_n, \tag{3.13}$$

where a_1, d_4 and $\widetilde{M}_2^*$ are constants independent on n. However, we have $\Lambda_n \to 0$ and $M_n^* \to \infty$, as $n \to \infty$, which leads to a contradictions. Hence there exist two positive constants ℓ, $\widetilde{M}_3^*$ such that

$$\|x_n\|_{L^\infty_{2\pi}} \leq (a_1 \widetilde{M}_2^* + d_4) + \ell + \frac{\widetilde{M}_0^* \widetilde{M}_4^*}{2\pi} = \widetilde{M}_3^*. \tag{3.14}$$

This shows that (3.11) holds.

Using (H_1), (H_2), (2.9) and (3.11), there exists a constant $\widetilde{C}_3 > 0$ such that

$$\|x_n^+\|_{H^{\frac{1}{2}}(S^1,R^n)} \geq \langle I'(x_n), x_n^+\rangle = \langle Ax_n^+, x_n^+\rangle - \int\limits_0^{2\pi} \big[\big(x_n^+, f(t,x_n)\big)\big]\, dt \geq$$

$$\geq \langle Ax_n^+, x_n^+\rangle - \Bigg(\int\limits_{|x_n|\geq 1} + \int\limits_{|x_n|<1} \Bigg) |x_n^+|\, |f(t,x_n)|\, dt \geq$$

$$\geq \langle Ax_n^+, x_n^+\rangle - \int\limits_{|x_n|\geq 1} |x_n^+|\, |f(t,x_n)|\, dt - \widetilde{C}_3, \tag{3.15}$$

$$\|x_n^-\|_{H^{\frac{1}{2}}(S^1,R^n)} \geq -\langle I'(x_n), x_n^-\rangle = -\langle Ax_n^-, x_n^-\rangle + \int\limits_0^{2\pi} \big[\big(x_n^-, f(t,x_n)\big)\big]\, dt \geq$$

$$\geq -\langle Ax_n^-, x_n^-\rangle - \Bigg(\int\limits_{|x_n|\geq 1} + \int\limits_{|x_n|<1} \Bigg) |x_n^+|\, |f(t,x_n)|\, dt \geq$$

$$\geq -\langle Ax_n^-, x_n^-\rangle - \int\limits_{|x_n|\geq 1} |x_n^-|\, |f(t,x_n)|\, dt - \widetilde{C}_3. \tag{3.16}$$

From (3.11), (3.12) and (3.15), (3.16), we have

$$\|x_n^+\|_{H^{\frac{1}{2}}(S^1,R^n)} + \|x_n^-\|_{H^{\frac{1}{2}}(S^1,R^n)} \geq$$

$$\geq \langle Ax_n^+, x_n^+\rangle - \langle Ax_n^-, x_n^-\rangle - 2\|x_n\|_{L^\infty_{2\pi}} \int\limits_{|x_n|\geq 1} |f(t,x_n)|\, dt - 2\widetilde{C}_3 \geq$$

$$\geq \xi_1 \|x_n^+\|^2_{H^{\frac{1}{2}}(S^1,R^n)} - \xi_{-1} \|x_n^-\|^2_{H^{\frac{1}{2}}(S^1,R^n)} - 2a_1 \widetilde{M}_2^* \widetilde{M}_3^* - 2\widetilde{C}_3, \tag{3.17}$$

where ξ_1 is the smallest positive eigenvalue and ξ_{-1} is the largest negative eigenvalue of the operator A, respectively.

From (3.8) and (3.17), there exists a positive constant $\widetilde{D}_2 > 0$ such that

$$\begin{aligned}
&\widetilde{D}_2\Big(\|x_n^+\|_{H^{\frac{1}{2}}(S^1,R^n)} + \|x_n^-\|_{H^{\frac{1}{2}}(S^1,R^n)} + \|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)}\Big) \geq \\
&\quad \geq \|x_n^+\|_{H^{\frac{1}{2}}(S^1,R^n)} + \|x_n^-\|_{H^{\frac{1}{2}}(S^1,R^n)} + \xi\widetilde{M}_1^*\|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \geq \\
&\quad \geq \|x_n^+\|_{H^{\frac{1}{2}}(S^1,R^n)} + \|x_n^-\|_{H^{\frac{1}{2}}(S^1,R^n)} + \xi\|x_n^0\|^2_{H^{\frac{1}{2}}(S^1,R^n)} \geq \\
&\quad \geq \xi_1\|x_n^+\|^2_{H^{\frac{1}{2}}(S^1,R^n)} - \xi_{-1}\|x_n^-\|^2_{H^{\frac{1}{2}}(S^1,R^n)} + \\
&\qquad\quad + \xi\|x_n^0\|^2_{H^{\frac{1}{2}}(S^1,R^n)} - 2a_1\widetilde{M}_2^*\widetilde{M}_3^* - 2\widetilde{C}_3 \geq \\
&\quad \geq \xi\|x_n\|^2_{H^{\frac{1}{2}}(S^1,R^n)} - 2a_1\widetilde{M}_2^*\widetilde{M}_3^* - 2\widetilde{C}_3, \qquad (3.18)
\end{aligned}$$

here $\xi = \min\{\xi_1, -\xi_{-1}\}$. We have from (3.18) that

$$\xi\|x_n\|^2_{H^{\frac{1}{2}}(S^1,R^n)} - \widetilde{D}_2\|x_n\|_{H^{\frac{1}{2}}(S^1,R^n)} - 2a_1\widetilde{M}_2^*\widetilde{M}_3^* - 2\widetilde{C}_3 < 0.$$

This implies that $\{\|x_n\|_{H^{\frac{1}{2}}(S^1,R^n)}\}_{n\in\mathbf{N}}$ is bounded. Going, if necessary, to a subsequence, we can assume that there exists $x \in E_k$ such that $x_{k_n} \rightharpoonup x$ as $n \to +\infty$ in $H^{\frac{1}{2}}(S^1,R^n)$, which implies $x_n \to x$ uniformly on $[0,2\pi]$. Hence $(I'(x_n) - I'(x))(x_n - x) \to 0$ and $\|x_n - x\|_{L^2_{2\pi}} \to 0$. Set

$$\Phi = \int_0^{2\pi} \big(f(t,x_n(t)) - f(t,x(t)), x_n(t) - x(t)\big)\,dt.$$

It is easy to check that $\Phi \to 0$, as $n \to +\infty$. Moreover, an easy computation shows that

$$\big(I'(x_n) - I'(x)\big)(x_n - x) = \big\langle A(x_n - x), (x_n - x)\big\rangle - \Phi.$$

By (2.5), (2.8) and (2.10), this implies $\|x_n - x\|_{H^{\frac{1}{2}}(S^1,R^n)} \to 0$. □

Proof of Theorem 1.1. The proof will be divided into two steps.

Step 1. Choose $q > 2$. By (H_1), for any $\widehat{\varepsilon} > 0$, there exists $\widehat{M} > 0$ such that

$$F(t,x) \leq \widehat{\varepsilon}|x|^2 + \widehat{M}|x|^q, \quad \forall\,(t,x) \in [0,\frac{\pi}{2}] \times R^n. \qquad (3.19)$$

From (3.1) and (3.19), for $x \in E_1 = E^+$, there exists a positive constant c_q such that

$$\begin{aligned}
I(x) &= \frac{1}{2}\langle Ax,x\rangle - \int_0^{2\pi} F(t,x)\,dt \geq \frac{1}{2}\langle Ax,x\rangle - \big(\widehat{\varepsilon}\|x\|^2_{L^2_{2\pi}} + \widehat{M}\|x\|^q_{L^q_{2\pi}}\big) \geq \\
&\geq \frac{\xi_1}{2}\|x\|^2_{H^{\frac{1}{2}}(S^1,R^n)} - c_q\Big(\widehat{\varepsilon}\|x\|^2_{H^{\frac{1}{2}}(S^1,R^n)} + \widehat{M}\|x\|^q_{H^{\frac{1}{2}}(S^1,R^n)}\Big). \qquad (3.20)
\end{aligned}$$

Choose $\widehat{\varepsilon} = \frac{\xi_1}{8c_q}$, $\rho = (\frac{\xi_1}{8c_p\widehat{M}})^{\frac{1}{q-2}}$ and denote by B_ρ the closed ball in $H^{\frac{1}{2}}(S^1, R^n)$ of radius ρ centered at the origin. Let $S = \partial B_\rho \cap E_1$, then $I(x) \geq \widetilde{\alpha} = \frac{\xi_1 \rho^2}{4}$ for all $x \in S$, and (C_3)(i) of Lemma 2.5 holds.

Step 2. Let $e \in E^+$ with $\|e\|_{H^{\frac{1}{2}}(S^1,R^n)} = 1$ and $E_2 = E^- \oplus E^0$, $Q = E^- \oplus E^0 \oplus span\{e\}$.

For $x = x^0 + x^- \in E_2$, then

$$I(x+\gamma e) = \frac{1}{2}\langle A(x+\gamma e),(x+\gamma e)\rangle - \int_0^{2\pi} F(t, x+\gamma e)\,dt =$$

$$= \frac{\gamma^2}{2}\langle Ae, e\rangle + \frac{1}{2}\langle Ax^-, x^-\rangle - \int_0^{2\pi} F(t, x+\gamma e)\,dt. \tag{3.21}$$

By (H_1), it is clear that $I(x) \leq 0$ on $x \in E_2$. Since E^0 is finite dimensional, there exists $\widehat{b}_1 > 0$ such that

$$\|A\|^{\frac{1}{2}}\|e\|_{H^{\frac{1}{2}}(S^1,R^n)} \leq \widehat{b}_1\|e\|_{L^2}, \quad \|A\|^{\frac{1}{2}}\|x^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \leq \widehat{b}_1\|x^0\|_{L^2} \tag{3.22}$$

for all $x^0 \in E^0$. Moreover, by (H_1),

$$F(t,x) \geq \widehat{b}_1^2|x|^2 - \widehat{b}_2, \ \ \forall\,(t,x) \in \left[0, \frac{\pi}{2}\right] \times R^n. \tag{3.23}$$

We have from (3.23) that

$$\int_0^{2\pi} F(t, \gamma e + x)\,dt \geq \widehat{b}_1^2\|\gamma e + x\|_{L^2}^2 - \widehat{b}_2 2\pi \geq$$

$$\geq \widehat{b}_1^2\big(\|x^0\|_{L^2}^2 + \|x^-\|_{L^2}^2 + \gamma^2\|e\|_{L^2}^2\big) - \widehat{b}_2 2\pi. \tag{3.24}$$

By (2.10) and (3.24), for all $\gamma > 0$ and $x \in E_2$ we get

$$I(x+\gamma e) \leq \frac{1}{2}\big\langle A(x+\gamma e),(x+\gamma e)\big\rangle - \int_0^{2\pi} F(t, x+\gamma e)\,dt \leq$$

$$\leq \frac{\gamma^2}{2}\langle Ae, e\rangle + \frac{1}{2}\langle Ax^-, x^-\rangle - \|A\|\big(\|x^0\|_{H^{\frac{1}{2}}(S^1,R^n)}^2 + \gamma^2\big) + \widehat{b}_2 2\pi \leq$$

$$\leq \frac{\|A\|\gamma^2}{2} + \frac{\xi_{-1}}{2}\|x^-\|_{H^{\frac{1}{2}}(S^1,R^n)}^2 - \|A\|\big(\|x^0\|_{H^{\frac{1}{2}}(S^1,R^n)}^2 + \gamma^2\big) + \widehat{b}_2 2\pi \leq$$

$$\leq -\frac{\|A\|\gamma^2}{2} + \frac{\xi_{-1}}{2}\|x^-\|_{H^{\frac{1}{2}}(S^1,R^n)}^2 + \widehat{b}_2 2\pi. \tag{3.25}$$

Let

$$\gamma_1 = 2\sqrt{\frac{\widehat{b}_2\pi}{\|A\|}} \quad \text{and} \quad \gamma_2 = 2\sqrt{\frac{\widehat{b}_2\pi}{-\xi_{-1}}}\,.$$

Then $I(x+\gamma e) \leq 0$ if either $\gamma \geq \gamma_1$, or $\|x\|_{H^{\frac{1}{2}}(S^1,R^n)} \geq \gamma_2$. Consequently, $I|_{\partial Q} \leq 0$, where $Q = \{\gamma e; \gamma \in [0,\gamma_1]\} \oplus (B_{\gamma_2} \cap E_2)$. By Lemma 2.5, S and ∂Q link and (C_3)(ii) and (C_3)(iii) of Lemma 2.5 hold.

From (H_1), (C_1) and (C_2) of Lemma 2.5 are true, so by Lemma 2.5, I has a nonconstant critical point x^* such that $I(x^*) \geq \widetilde{\alpha} > 0$. Now x^* is a 2π-solution of (2.2), hence x^* is a 4τ-solution of (1.1). ☐

Lemma 3.2. *Under the conditions of Theorem* 1.2*, I satisfies the* **(PS)** *condition.*

Proof. We have from (F_2), (2.8)–(2.10) and (1.3) of (V_4) that

$$2d_1 \geq 2I(x_n) - \langle I'(x_n), x_n\rangle = \int_0^{2\pi} \left[(x_n, f(t,x_n)) - 2F(t,x_n)\right] dt \geq$$

$$\geq \int_0^{2\pi} \left[c_1|x_n(t)|^\beta - c_2\right] dt. \tag{3.26}$$

This implies

$$\int_0^{2\pi} |x_n(t)|^\beta \, dt \leq \frac{2d_1 + 2\pi c_2}{c_1} = \widetilde{M}_0. \tag{3.27}$$

Consider $\{\|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)}\}_{n\in\mathbf{N}}$. Arguing indirectly, we suppose $\{\|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)}\}_{n\in\mathbf{N}}$ is unbounded. Then we have $\|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \to \infty$. Note that $\dim(E^0) < +\infty$, and this implies that there are constants $\widetilde{b}_1$ and $\widetilde{b}_2$ such that

$$\widetilde{b}_1 \|x_n^0\|_{L^\beta_{2\pi}} \leq \|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \leq \widetilde{b}_2 \|x_n^0\|_{L^\beta_{2\pi}}. \tag{3.28}$$

From (3.28), we have

$$\|x_n\|_{L^\beta_{2\pi}} \geq \|x_n^0\|_{L^\beta_{2\pi}} \to +\infty \text{ as } \|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \to +\infty. \tag{3.29}$$

We have from (3.27) and (3.29) that

$$\widetilde{M}_0 \geq \int_0^{2\pi} |x_n(t)|^\beta \, dt \geq \int_0^{2\pi} |x_n^0(t)|^\beta \, dt \longrightarrow +\infty \text{ as } \|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \to +\infty. \tag{3.30}$$

This is a contradiction. Hence $\{\|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)}\}_{n\in\mathbf{N}}$ is bounded. Therefore there exists a constant $\widetilde{M}_1 > 0$ such that

$$\|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \leq \widetilde{M}_1. \tag{3.31}$$

Let $\alpha = \frac{\beta-1}{\beta(\lambda-1)}$, then

$$\begin{cases} 1 < \lambda < 1 + \frac{\beta-1}{\beta}, & 0 < \frac{(\lambda\alpha-1)}{\alpha} < 1, \\ \lambda\alpha - 1 = \alpha - \frac{1}{\beta}, & \alpha > 1. \end{cases} \tag{3.32}$$

Using (3.1) and (3.32), we have (here $\frac{1}{\alpha}+\frac{1}{\sigma}=1$)

$$\|x_n^+\|_{H^{\frac{1}{2}}(S^1,R^n)} \geq \big\langle I'(x_n), x_n^+\big\rangle \geq \langle Ax_n^+, x_n^+\rangle - \int\limits_0^{2\pi} |x_n^+|\,|f(t,x_n)|\,dt \geq$$

$$\geq \langle Ax_n^+, x_n^+\rangle - \Bigg(\int\limits_0^{2\pi} |f(t,x_n)|^\alpha\,dt\Bigg)^{\frac{1}{\alpha}} \Bigg(\int\limits_0^{2\pi} |x_n^+|^\sigma\,dt\Bigg)^{\frac{1}{\sigma}} \geq$$

$$\geq \langle Ax_n^+, x_n^+\rangle - \Bigg(\int\limits_0^{2\pi} |f(t,x_n)|^\alpha\,dt\Bigg)^{\frac{1}{\alpha}} c_\sigma \|x_n^+\|_{H^{\frac{1}{2}}(S^1,R^n)}, \quad (3.33)$$

$$\|x_n^-\|_{H^{\frac{1}{2}}(S^1,R^n)} \geq -\big\langle I'(x_n), x_n^-\big\rangle \geq -\langle Ax_n^-, x_n^-\rangle - \int\limits_0^{2\pi} |x_n^-|\,|f(t,x_n)|\,dt \geq$$

$$\geq -\langle Ax_n^-, x_n^-\rangle - \Bigg(\int\limits_0^{2\pi} |f(t,x_n)|^\alpha\,dt\Bigg)^{\frac{1}{\alpha}} \Bigg(\int\limits_0^{2\pi} |x_n^-|^\sigma\,dt\Bigg)^{\frac{1}{\sigma}} \geq$$

$$\geq -\langle Ax_n^-, x_n^-\rangle - \Bigg(\int\limits_0^{2\pi} |f(t,x_n)|^\alpha\,dt\Bigg)^{\frac{1}{\alpha}} c_\sigma \|x_n^-\|_{H^{\frac{1}{2}}(S^1,R^n)}. \quad (3.34)$$

By (1.4) of (V_4) and (3.1), there exist two constants $\widetilde{C}_1 > 0$ and $\widetilde{C}_2 > 0$ such that

$$\int\limits_0^{2\pi} |f(t,x_n)|^\alpha\,dt \leq \int\limits_0^{2\pi} \big[|c_3|x_n|^\lambda + c_4\big]^\alpha\,dt \leq \int\limits_0^{2\pi} c_3^\alpha |x_n|^{\lambda\alpha}\,dt + \widetilde{C}_1 \leq$$

$$\leq c_3^\alpha \Bigg(\int\limits_0^{2\pi} |x_n|^\beta\,dt\Bigg)^{\frac{1}{\beta}} \Bigg(\int\limits_0^{2\pi} |x_n|^{(\lambda\alpha-1)\,\frac{\beta}{\beta-1}}\,dt\Bigg)^{1-\frac{1}{\beta}} + \widetilde{C}_1 =$$

$$= c_3^\alpha \Bigg(\int\limits_{|x_n|\geq 1} |x_n|^\beta\,dt\Bigg)^{\frac{1}{\beta}} \Bigg(\int\limits_{|x_n|\geq 1} |x_n|^{(\lambda\alpha-1)\,\frac{\beta}{\beta-1}}\,dt\Bigg)^{1-\frac{1}{\beta}} + \widetilde{C}_1 +$$

$$+ c_3^\alpha \Bigg(\int\limits_{|x_n|<1} |x_n|^\beta\,dt\Bigg)^{\frac{1}{\beta}} \Bigg(\int\limits_{|x_n|<1} |x_n|^{(\lambda\alpha-1)\,\frac{\beta}{\beta-1}}\,dt\Bigg)^{1-\frac{1}{\beta}} \leq$$

$$\leq c_3^\alpha \big(c_{\frac{\beta(\lambda\alpha-1)}{\beta-1}}\big)^{\lambda\alpha-1} \Bigg(\int\limits_0^{2\pi} |x_n|^\beta\,dt\Bigg)^{\frac{1}{\beta}} \|x_n\|^{\lambda\alpha-1}_{H^{\frac{1}{2}}(S^1,R^n)} + \widetilde{C}_1 + \widetilde{C}_2. \quad (3.35)$$

From (3.27) and (3.33)–(3.35), we have

$$\|x_n^+\|_{H^{\frac{1}{2}}(S^1,R^n)} + \|x_n^-\|_{H^{\frac{1}{2}}(S^1,R^n)} \geq \langle Ax_n^+, x_n^+\rangle - \langle Ax_n^-, x_n^-\rangle -$$
$$- \Big(\int_0^{2\pi} |f(t,x_n)|^\alpha\, dt\Big)^{\frac{1}{\alpha}} c_\sigma \Big(\|x_n^+\|_{H^{\frac{1}{2}}(S^1,R^n)} + \|x_n^-\|_{H^{\frac{1}{2}}(S^1,R^n)}\Big) \geq$$
$$\geq \xi_1 \|x_n^+\|^2_{H^{\frac{1}{2}}(S^1,R^n)} - \xi_{-1}\|x_n^-\|^2_{H^{\frac{1}{2}}(S^1,R^n)} -$$
$$- 2c_\sigma \Big[\widetilde{D}_0 \|x_n\|^{\lambda\alpha-1}_{H^{\frac{1}{2}}(S^1,R^n)} + \widetilde{C}_1 + \widetilde{C}_2\Big]^{\frac{1}{\alpha}} \|x_n\|_{H^{\frac{1}{2}}(S^1,R^n)}, \tag{3.36}$$

where

$$\widetilde{D}_0 = c_3^\alpha \big(c_{\frac{\beta(\lambda\alpha-1)}{\beta-1}}\big)^{\lambda\alpha-1} (\widetilde{M}_0)^{\frac{1}{\beta}}.$$

From (3.31) and (3.36), there exists a positive constant $\widetilde{D}_1 > 0$ such that

$$\widetilde{D}_1 \Big(\|x_n^+\|_{H^{\frac{1}{2}}(S^1,R^n)} + \|x_n^-\|_{H^{\frac{1}{2}}(S^1,R^n)} + \|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)}\Big) \geq$$
$$\geq \|x_n^+\|_{H^{\frac{1}{2}}(S^1,R^n)} + \|x_n^-\|_{H^{\frac{1}{2}}(S^1,R^n)} + \xi \widetilde{M}_1 \|x_n^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \geq$$
$$\geq \|x_n^+\|_{H^{\frac{1}{2}}(S^1,R^n)} + \|x_n^-\|_{H^{\frac{1}{2}}(S^1,R^n)} + \xi \|x_n^0\|^2_{H^{\frac{1}{2}}(S^1,R^n)} \geq$$
$$\geq \xi_1 \|x_n^+\|^2_{H^{\frac{1}{2}}(S^1,R^n)} - \xi_{-1}\|x_n^-\|^2_{H^{\frac{1}{2}}(S^1,R^n)} + \xi \|x_n^0\|^2_{H^{\frac{1}{2}}(S^1,R^n)} -$$
$$- 2c_\sigma \Big[\widetilde{D}_0 \|x_n\|^{\lambda\alpha-1}_{H^{\frac{1}{2}}(S^1,R^n)} + \widetilde{C}_1 + \widetilde{C}_2\Big]^{\frac{1}{\alpha}} \|x_n\|_{H^{\frac{1}{2}}(S^1,R^n)} \geq$$
$$\geq \xi \Big(\|x_n^+\|^2_{H^{\frac{1}{2}}(S^1,R^n)} + \|x_n^-\|^2_{H^{\frac{1}{2}}(S^1,R^n)} + \|x_n^0\|^2_{H^{\frac{1}{2}}(S^1,R^n)}\Big) -$$
$$- 2c_\sigma \Big[\widetilde{D}_0 \|x_n\|^{\lambda\alpha-1}_{H^{\frac{1}{2}}(S^1,R^n)} + \widetilde{C}_1 + \widetilde{C}_2\Big]^{\frac{1}{\alpha}} \|x_n\|_{H^{\frac{1}{2}}(S^1,R^n)}. \tag{3.37}$$

From (3.37), we have

$$\widetilde{D}_1 \geq \xi \|x_n\|_{H^{\frac{1}{2}}(S^1,R^n)} - 2c_\sigma \Big[\widetilde{D}_0 \|x_n\|^{\lambda\alpha-1}_{H^{\frac{1}{2}}(S^1,R^n)} + \widetilde{C}_1 + \widetilde{C}_2\Big]^{\frac{1}{\alpha}}.$$

Since $0 < \frac{(\lambda\alpha-1)}{\alpha} < 1$, this implies that $\{\|x_n\|_{H^{\frac{1}{2}}(S^1,R^n)}\}_{n\in\mathbf{N}}$ is bounded. Using an argument similar to that in the proof of Lemma 3.1, we have $\|x_n - x\|_{H^{\frac{1}{2}}(S^1,R^n)} \to 0$. □

Proof of Theorem 1.1. The proof will be divided into two steps.

Step 1. By (V_2), (V_3) and (1.4) of (V_4), for any $\varepsilon > 0$, there exists $M = M(\varepsilon) > 0$ such that

$$F(t,x) \leq \varepsilon |x|^2 + M|x|^{\lambda+1}, \ \ \forall\, (t,x) \in \Big[0, \frac{\pi}{2}\Big] \times R^n. \tag{3.38}$$

From (3.1) and (3.38), for $x \in E_1 = E^+$, we have

$$I(x) = \frac{1}{2}\langle Ax, x\rangle - \int_0^{2\pi} F(t,x)\,dt \geq$$
$$\geq \frac{\xi_1}{2}\|x\|^2_{H^{\frac{1}{2}}(S^1,R^n)} - \Big(\varepsilon\|x\|^2_{H^{\frac{1}{2}}(S^1,R^n)} + c_{\lambda+1}M\|x\|^{\lambda+1}_{H^{\frac{1}{2}}(S^1,R^n)}\Big). \quad (3.39)$$

Choose $\varepsilon = \frac{\xi_1}{8}$, $\rho = (\frac{\xi_1}{8Mc_{\lambda+1}})^{\frac{1}{\lambda-1}}$ and denote by B_ρ the closed ball in $H^{\frac{1}{2}}(S^1, R^n)$ of radius ρ centered at the origin. Let $S = \partial B_\rho \cap E_1$, then $I(x) \geq \widetilde{\alpha} = \frac{\xi_1\rho^2}{4}$ for all $x \in S$, and (C_3)(i) of Lemma 2.5 holds.

Step 2. Let $e \in E^+$ with $\|e\|_{H^{\frac{1}{2}}(S^1,R^n)} = 1$ and $E_2 = E^- \oplus E^0$.

For $x = x^0 + x^+ \in E_2$, then

$$I(x+\gamma e) = \frac{1}{2}\big\langle A(x+\gamma e),(x+\gamma e)\big\rangle - \int_0^{2\pi} F(t, x+\gamma e)\,dt =$$
$$= \frac{\gamma^2}{2}\langle Ae, e\rangle + \frac{1}{2}\langle Ax^-, x^-\rangle - \int_0^{2\pi} F(t, x+\gamma e)\,dt. \quad (3.40)$$

By (V_1), it is obvious that $I(x) \leq 0$ on $x \in E_2$. Since E^0 is finite dimensional, there exists $\widehat{a}_1 > 0$ such that

$$\begin{aligned}&\|A\|^{\frac{1}{2}}\|e\|_{H^{\frac{1}{2}}(S^1,R^n)} \leq \widehat{a}_1\|e\|_{L^2},\\ &\|A\|^{\frac{1}{2}}\|x^0\|_{H^{\frac{1}{2}}(S^1,R^n)} \leq \widehat{a}_1\|x^0\|_{L^2}\end{aligned} \quad (3.41)$$

for all $x^0 \in E^0$. Moreover, by (V_2) and (V_3), there exists a positive constant $\widehat{a}_2$ such that

$$F(t,x) \geq \widehat{a}_1^2|x|^2 - \widehat{a}_2, \ \ \forall\,(t,x) \in [0,\pi]\times R^n. \quad (3.42)$$

It follows from (3.42) that

$$\int_0^{2\pi} F(t, \gamma e + x)\,dt \geq \widehat{a}_1^2\|\gamma e + x\|^2_{L^2} - \widehat{a}_2 2\pi \geq$$
$$\geq \widehat{a}_1^2\big(\|x^0\|^2_{L^2} + \|x^-\|^2_{L^2} + \gamma^2\|e\|^2_{L^2}\big) - \widehat{a}_2 2\pi. \quad (3.43)$$

By (3.43), for all $\gamma > 0$ and $x \in E_2$ we get

$$I(x+\gamma e) \leq \frac{1}{2}\big\langle A(x+\gamma e),(x+\gamma e)\big\rangle - \int_0^{2\pi} F(t, x+\gamma e)\,dt \leq$$
$$\leq \frac{\gamma^2}{2}\langle Ae, e\rangle + \frac{1}{2}\langle Ax^-, x^-\rangle - \|A\|\big(\|x^0\|^2_{H^{\frac{1}{2}}(S^1,R^n)} + \gamma^2\big) + \widehat{a}_2 2\pi \leq$$

$$\le \frac{\|A\|\gamma^2}{2} + \frac{\xi_-}{2}\|x^-\|^2_{H^{\frac{1}{2}}(S^1,R^n)} - \|A\|\big(\|x^0\|^2_{H^{\frac{1}{2}}(S^1,R^n)} + \gamma^2\big) + \widehat{a}_2 2\pi \le$$
$$\le -\frac{\|A\|\gamma^2}{2} + \frac{\xi_-}{2}\|x^-\|^2_{H^{\frac{1}{2}}(S^1,R^n)} + \widehat{a}_2 2\pi. \quad (3.44)$$

Let

$$\gamma_1 = 2\sqrt{\frac{\widehat{a}_2\pi}{\|A\|}} \quad \text{and} \quad \gamma_2 = \sqrt{\frac{2\widehat{a}_2\pi}{-\xi_-}}.$$

Then $I(x+\gamma e) \le 0$, if either $\gamma \ge \gamma_1$, or $\|x\|^2_{H^{\frac{1}{2}}(S^1,R^n)} \ge \gamma_2$. Consequently, $I|_{\partial Q} \le 0$, where $Q = \{\gamma e; \gamma \in [0,\gamma_1]\} \oplus (B_{\gamma_2} \cap E_2)$. By the definition of linking, S and ∂Q link and (C_3)(ii) and (C_3)(iii) of Lemma 2.5 hold.

From (V_2)–(V_3), (C_1) and (C_2) of Lemma 2.5 are true, thus by Lemma 2.5, I has a nonconstant critical point x^* such that $I(x^*) \ge \widetilde{\alpha} > 0$. Now x^* is a 2π-solution of (2.2), hence x^* is a 4τ-solution of (1.1). □

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References

1. V. Benci, On critical point theory for indefinite functionals in the presence of symmetries. *Trans. Amer. Math. Soc.* **274** (1982), No. 2, 533–572.
2. V. Benci and P. H. Rabinowitz, Critical point theorems for indefinite functionals. *Invent. Math.* **52** (1979), No. 3, 241–273.
3. Ch. Guo and Zh. Guo, Existence of multiple periodic solutions for a class of second-order delay differential equations. *Nonlinear Anal. Real World Appl.* **10** (2009), No. 5, 3285–3297.
4. Ch. Guo, D. O'Regan, and R. P. Agarwal, Existence of multiple periodic solutions for a class of first-order neutral differential equations. *Appl. Anal. Discrete Math.* **5** (2011), No. 1, 147–158.
5. Ch. J. Guo, Y. T. Xu, and Zh. M. Guo, Existence of multiple periodic solutions for a class of third-order neutral differential equations. (Chinese) *Acta Math. Sinica (Chin. Ser.)* **52** (2009), No. 4, 737–750.
6. Zh. Guo and J. Yu, Multiplicity results for periodic solutions to delay differential equations via critical point theory. *J. Differential Equations* **218** (2005), No. 1, 15–35.
7. J. Li and X. He, Proof and generalization of Kaplan-Yorke's conjecture under the condition $f'(0) > 0$ on periodic solution of differential delay equations. *Sci. China Ser. A* **42** (1999), No. 9, 957–964.
8. P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations. CBMS Regional Conference Series in Mathematics, 65. *Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI*, 1986.
9. X.-B. Shu, Y.-T. Xu, and L. Huang, Infinite periodic solutions to a class of second-order Sturm-Liouville neutral differential equations. *Nonlinear Anal.* **68** (2008), No. 4, 905–911.
10. K. Wu, X. Wu, and F. Zhou, Periodic solutions for a class of second order delay systems. *J. Dyn. Control Syst.* **19** (2013), No. 3, 421–437.

11. Y.-T. Xu and Zh.-M. Guo, Applications of a Z_p index theory to periodic solutions for a class of functional differential equations. *J. Math. Anal. Appl.* **257** (2001), No. 1, 189–205.
12. X. Zhang and D. Wang, Infinitely many periodic solutions to delay differential equations via critical point theory. *Abstr. Appl. Anal.* **2013**, Art. ID 526350, 7 pp.
13. Sh.-K. Zhang and L. Chen, Existence of periodic solutions for first-order delay differential equations via critical point theory. *Bound. Value Probl.* **2013**, Article ID 254, 13 p., electronic only (2013).

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Authors' addresses:

Chengjun Guo, Chengjiang Wang

School of Applied Mathematics, Guangdong University of Technology, Guangzhou, 510006, China.

Donal O'Regan

1. School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland.

2. Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia.

E-mail: donal.oregan@nuigalway.ie

Ravi P. Agarwal

1. Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia.

2. Department of Mathematics, Texas A and M University-Kingsville, Texas, 78363, USA.

E-mail: Ravi.Agarwal@tamuk.edu