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INTEGRO-DIFFERENTIAL EQUATIONS OF PRANDTL TYPE IN THE BESSEL POTENTIAL SPACES

Dedicated to Professor Boris Khvedelidze, a mathematician, teacher and mentor, on the occasion of his 100th birthday anniversary

Abstract. The purpose of the present research is to investigate the Fredholm criteria for the Prandtl-type integro-differential equation with piecewise-continuous coefficients in the Bessel potential spaces $\mathbb{H}^s_p(\mathbb{R})$.

We reduce the integro-differential equations to an equivalent system of Mellin type convolution equation. Applying the recent results to Mellin convolution equations with meromorphic kernels in Bessel potential spaces obtained by V. Didenko & R. Duduchava [3] and R. Duduchava [9], the Fredholm criteria (and in some cases, the unique solvability criteria) of the above-mentioned integro-differential equations are obtained.

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რეზიუმე. წინამდებარე სტატიის მიზანია გამოიკვლიოს ფრედჰოლმურობის კრიტერიუმი პრანდტლის ტიპის ინტეგრო-დიფერენციალური განტოლებისათვის უბან-უბან უწყვეტი კოეფიციენტებით ბესელის პოტენციალთა სივრცეში $\mathbb{H}^s_p(\mathbb{R})$. გამოსაკვლევი ინტეგრო-დიფერენციალური განტოლებების სისტემა დაიყვანება ექვივალენტურ მელინის კონვოლუციის ტიპის სისტემაზე, რომლისთვისაც გამოყენება ვ. დიდენკოს, რ. დუდუჩავას [3] და რ. დუდუჩავას [9] მიერ ბოლო დროს მიღებული შედეგები მელინის კონვოლუციის ტიპის განტოლებებისათვის მერომორფული ბირთვებით ბესელის პოტენციათა სივრცეებში, სადაც დადგენილია ფრედჰოლმურობის (და, რიგ შემთხვევებში, ერთადერთი ამოხსნადობის) კრიტერიუმები ზემოთ ხსენებული ინტეგრო-დიფერენციალური განტოლებებისათვის.

Introduction and the Formulation of the Main Theorem

We study the following integro-differential equation in the Bessel potential space setting

$$\varphi(t) - \frac{a(t)}{\pi} \int_{\mathbb{R}} \frac{\varphi'(\tau)}{\tau - t}\, d\tau = f(t), \tag{1}$$

$$\varphi \in \mathbb{H}^s_p(\mathbb{R}),\ \ \varphi(0) = 0,\ \ f \in \mathbb{H}^{s-1}_p(\mathbb{R}),\ \ \frac{1}{p} < s < 1 + \frac{1}{p},\ \ 1 < p < \infty,$$

where $a(t)$ is a piecewise-constant coefficient: $a(t) = a_-$ for $t < 0$ and $a(t) = a_+$ for $t > 0$. Such boundary integral equations occur as an equivalent reformulation of many problems in the classical two-dimensional elasticity (stringers attached to plates, rigid inclusions in elastic plates, stamps applied to elastic plates etc., see [16]) in aerodynamics (airfoil equation) and in many other problems. In Section 1 we expose an example from Section 6, [18], where the model initial stringer problem was considered and solved in a spaceless setting by a somewhat different method, namely by the method of complex analysis. We endow the example with the non-classical setting when the displacement vector $u + iv$ is sought in the Bessel potential space $\mathbb{H}^{s+1/p}_p$ and the stresses σ_x, σ_y, τ_{xy} belong to the Bessel potential space $\mathbb{H}^{s+1/p-1}_p$.

Based on the investigations from [3,9], in Section 4 is defined the symbol $\mathcal{A}^s_p(\omega)$ of the equation (1), which is a continuous 2×2 matrix-function on the infinite rectangle $\mathfrak{R}$. For an elliptic symbol $\inf\limits_{\omega \in \mathfrak{R}} |\det \mathcal{A}^s_p(\omega)| \neq 0$, the increment of the argument $\frac{1}{2\pi} \arg \det \mathcal{A}^s_p(\omega)$ is an integer and called the index $\operatorname{ind} \det \mathcal{A}^s_p$. The following theorem is the main result for the equation (1) in the present paper.

Theorem 0.1. *Let,* $1 < p < \infty$, $-1 \leqslant s \leqslant 1$, $a_\pm \in \mathbb{C}$.

The equation (1) *is Fredholm if and only if the following two conditions hold:*

(i) *The coefficients* $a_\pm$ *are not negative reals:* $a_\pm \in \mathbb{C} \setminus \overline{\mathbb{R}^-}$, $\overline{\mathbb{R}^-} := (-\infty, 0]$;

(ii) *The parameters* p *and* s *are not the solutions to the following transcendental equation:*

$$\cos^2 \frac{\pi}{p} \sin^2 \pi \left(\frac{1}{p} + s\right) - \sin^2 \frac{\pi}{p} = 0. \tag{2}$$

If the conditions i and ii hold and $1 < p < 4$, *then the equation* (1) *has a unique solution for all* $1 < p < 4$ *and arbitrary* $-1 \leqslant s \leqslant 1$.

If the conditions i and ii hold and $4 \leqslant p < \infty$, *then the transcendental equation* (2) *has two pairs of solutions* $\{p, s_p\}$ *and* $\{p, s_p - 1\}$, *where* $s_p > 0$, $s_p - 1 < 0$. *Then the equation* (1) *has*

(i) *a unique solution for all* $s_p - 1 < s < s_p$;

(ii) *a unique solution for all right-hand sides which are orthogonal to the solution of the dual homogeneous equation for all* $s_p < s \leqslant 1$ (*the equation has index* -1)*;*

(iii) *a non-unique solution for all right-hand sides provided* $-1 \leqslant s < s_p - 1$*; the homogeneous equation has one linearly independent solution* (*the equation has index* 1)*.*

The same method which we use in the present paper, applies also to the equations with complex conjugated unknown functions

$$\begin{gathered}
a_1(x)\varphi(x) + a_2(x)\varphi'(x) + a_3(x)\overline{\varphi(x)} + a_4(x)\overline{\varphi'(x)} \\
+\frac{a_5(x)}{\pi i}\int\limits_{\Gamma}\frac{\varphi(t)}{t-x}\,dt + \frac{a_6(x)}{\pi}\int\limits_{\Gamma}\frac{\varphi'(t)}{t-x}\,dt + \frac{a_7(x)}{\pi i}\int\limits_{\Gamma}\frac{\overline{\varphi(t)}}{t-x}\,dt \\
+\frac{a_8(x)}{\pi}\int\limits_{\Gamma}\frac{\overline{\varphi'(t)}}{t-x}\,dt + \frac{a_9(x)}{\pi i}\overline{\int\limits_{\Gamma}\frac{\varphi(t)}{t-x}\,dt} \\
+\frac{a_{10}(x)}{\pi}\overline{\int\limits_{\Gamma}\frac{\varphi'(t)}{t-x}\,dt} = f(x), \quad x \in \Gamma, \qquad (3) \\
\varphi \in \mathbb{H}^1_p(\Gamma), \quad f \in \mathbb{L}_p(\Gamma), \quad a_j \in PC(\Gamma), \quad j = 1, \ldots, 10,
\end{gathered}$$

where Γ is a union of smooth curves, open or closed, including infinite beams (e.g. $\mathbb{R}$). Such equations occur in many problems of elasticity (see e.g. [6–8, 17]).

For the investigation of equation (1) on $\mathbb{R}$ we first convert it into a system of Mellin convolution equations with constant coefficients on the semi-axes $\mathbb{R}^+$. Then the results on Mellin convolution equations in the Bessel potential spaces (see [3,9]) are applied and provide the criteria for the initial equation to have the Fredholm property and write formula for the index.

For the investigation of equation (3) first a quasi-localization is applied, which assigns to it at each point $t \in \Gamma$ the same equation, but either on the axes $\mathbb{R}$ with piecewise constant coefficients, which have jumps only at 0, or on the beam $\mathbb{R}^+$ with constant coefficients ("freezing coefficients" at the localization points; see details in [1, 2, 4, 15]). The obtained equations are investigated just as in the case of equation (1). It is proved that equation (3) is Fredholm one, if and only if all local equations are Fredholm (the local and global Fredholmness for the localized equations coincide).

The details of this investigation will be available in a forthcoming publication.

The present paper is organized as follows: in Section 1 we describe the stringer problem which leads to the integro-differential equation (1) we are going to investigate. In Section 2 we observe Fourier convolution operators in the Bessel potential spaces. The key result on commutants of the Mellin convolution operators and Bessel potentials is represented in Section 3. In the Section 4 we investigate integro-differential equation (1) in the Bessel potential space $\mathbb{B}^s_p(\mathbb{R})$ and prove the key results, including Theorem 0.1.

1. The Integro-Differential Equation of the Stringer Problem

In the present section we expose some details how the Prandtl-type equation (1) is derived as an equivalent boundary integral equation for a model stringer problem. The procedure is very well described in the literature and we only expose some details to show in which space is it correct to look for a solution of a boundary integral equation. In foregoing papers the space where solution belongs was either ignored (see e.g. [16, 18]), or a solution was sought in the Lebesgue space $\mathbb{L}_p$ (see e.g. [6–8]). It should be noted here that the Fredholm property of equation (1) might be essentially different in Lebesgue and Bessel potential spaces (see [3, 9] and Section 3 below).

Suppose a piecewise homogenous thin elastic plate, consisting of two semi-infinite parts occupy the upper $\operatorname{Im} z > 0$ and the lower $\operatorname{Im} z < 0$ complex half-planes of the variable $z = x + iy$. It is reinforced along the junction line $y = 0$. A piecewise homogenous infinite elastic stringer consists of two semi-infinite bars $x > 0$ and $x < 0$, joined to one another and having elastic moduli E_- and E_+ and small cross sections S_- and S_+, respectively. The plates have thicknesses h_-, h_+, Poisson's ratios ν_-, ν_+ and share moduli μ_-, μ_+. Here and below the subscript $+$ corresponds to the plate occupying the upper half-plane $\operatorname{Im} z > 0$ and the subscript $-$ corresponds to the plate occupying the lower half-plane $\operatorname{Im} z < 0$. The plates are joined so that their middle surfaces are identical. The stringer is attached ideally rigidly to the plates and symmetrically both with respect to the junction line of the plates and with respect to their middle surfaces.

Problem S: *Find complex potentials that describe the stress state of the plates and the contact stresses under the stringer.*

To write the corresponding boundary integral equation we follow [18] and apply the complex potentials.

First, we write the equilibrium equations in the interval $[x, x + \Delta x]$:

$$N(x + \Delta x) - N(x) + \big[h_- \tau^+_{xy}(x) - h_- + \tau^-_{xy}(x)\big]\Delta x = 0, \qquad (4)$$
$$\big[h_- \sigma^+_{xy}(x) - h_- + \sigma^-_{xy}(x)\big]\Delta x = 0.$$

After dividing both sides by Δx and taking the limit as $\Delta x \to 0$, we obtain

$$N'(x) + h_- \tau^+_{xy}(x) - h_- + \tau^-_{xy}(x) = 0, \;\; h_- \sigma^+_y(x) - h_- + \sigma^-_y(x) = 0, \quad (5)$$

where N is the normal stress in the stringer calculated for the entire thickness of the stringer, τ^-_{xy} and σ^+_{xy} are the share and normal stresses in the plates calculated per unit thickness of the plates. $N(x) = E_- S_- \varepsilon(x)$ at $x > 0$ and $N(x) = E_+ S_+ \varepsilon(x)$ at $x < 0$. The stringer is rigidly attached to the plates. Within the model adopted, this is taken into account by equating the displacement vector $u + iv$ of points in the stringer and the displacement vectors $(u + iv)^+$ and $(u + iv)^-$ of the corresponding points in the upper and lower plates on the line $y = 0$. Thus, we obtain the following system of boundary conditions:

$$\begin{gathered}A(x)u''(x)+h_-\tau^+_{xy}(x)-h_-+\tau^-_{xy}(x)=0, \quad h_-\sigma^+_y(x)-h_-+\sigma^-_y(x)=0,\\ (u+iv)(x)=(u+iv)^-(x)=(u+iv)^+(x), \quad x\in\mathbb{R}\setminus\{0\},\end{gathered} \tag{6}$$

where $A(x) = E_-S_-$ for $x < 0$ and $A(x) = E_+S_+$ for $x > 0$. Conditions (6) must be supplemented with the equilibrium condition of the stringer

$$\int\limits_{-\infty}^{\infty}\big[h_-\tau^+_{xy}(x)-h_-+\tau^-_{xy}(x)\big]\,dx+P_\infty=0.$$

It is natural to look for a weak solution. Namely, in the classical setting the displacement vector $u + iv$ belongs to the Sobolev (energy) space $\mathbb{H}^1$ and the stresses σ_x, σ_y, τ_{xy}, which are compiled of the partial derivatives of the displacement vector $u + iv$ in the plates with respect to the variable x in the Hilbert space $\mathbb{L}_2$:

$$u+iv\in\mathbb{H}^1(\mathbb{C}^-\cup\mathbb{C}^+), \quad \sigma_x,\sigma_y,\tau_{x,y}\in\mathbb{L}_2(\mathbb{C}), \tag{7}$$

where $\mathbb{C}^-$ denotes the lower and $\mathbb{C}^+$ the upper complex half-planes.

The displacement vector $u+iv$ and the stresses σ_x, σ_y, τ_{xy} are found by means of the Kolosov–Muskhelishvili's formulae

$$\begin{gathered}\sigma_x(z)+\sigma_y(z)=4\operatorname{Re}\Phi_\pm(z),\\ \sigma_y(z)-i\tau_{xy}(z)=\Phi_\pm(z)-\Phi_\pm(\overline{z})+(z-\overline{z})\overline{\Phi_\pm(z)},\\ 2\mu_\pm\frac{d}{dx}\big[u(z)+iv(z)\big]=\Phi_\pm(z)-\Phi_\pm(\overline{z})+(z-\overline{z})\overline{\Phi_\pm(z)}, \quad \pm\operatorname{Im}z>0,\\ \Phi_\pm(z)=\begin{cases}\Phi^+_\pm(z), & \operatorname{Im}z>0,\\ \Phi^-_\pm(z), & \operatorname{Im}z<0,\end{cases} \qquad \mu_\pm=\frac{3-\nu_\pm}{1+\nu_\pm}\,,\end{gathered} \tag{8}$$

where $\Phi_\pm(z)$ are piecewise holomorphic functions (complex potentials) with a line of discontinuity along the real axis and they vanish at infinity. Based on the representation of the potentials as the Cauchy integrals,

$$\Phi^+_-(z)=\frac{1}{2\pi(1+\delta\kappa_-)}\int\limits_{-\infty}^{\infty}\frac{g(t)}{t-z}\,dt, \quad \Phi^-_-(z)=\frac{\kappa_+}{2\pi(\kappa_++\delta)}\int\limits_{-\infty}^{\infty}\frac{g(t)}{t-z}\,dt, \tag{9}$$

for the unknown density we derive the following equation from (8):

$$\begin{gathered}g(x)-\frac{a(x)}{\pi}\frac{d}{dx}\int\limits_{-\infty}^{\infty}\frac{g(t)}{t-x}\,dt=g(x)-\frac{a(x)}{\pi}\int\limits_{-\infty}^{\infty}\frac{g'(t)}{t-x}\,dt=0,\\ g\in\mathbb{H}^{-\frac{1}{2}}(\mathbb{R}), \quad x\in\mathbb{R}\end{gathered} \tag{10}$$

(see [18, Section 6] for details), where

$$\begin{aligned}
&\delta=\frac{h_+\mu_+}{h_-\mu_-},\quad g(x)=\frac{A(x)}{2h_-\mu_-}\frac{d}{dx}\,\mathrm{Re}\left[\kappa_-\Phi^+_-(x)+\Phi^-_-(x)\right],\\
&a(x)=a_\pm\ \text{ for }\ \pm x>0,\ \ a_\pm:=\frac{E_\pm S_\pm}{4h_+\mu_+}\,\frac{\kappa_+(\kappa_-+\delta)+\kappa_-(1+\kappa_+\delta)}{(\kappa_++\delta)(1+\kappa_+\delta)}>0.
\end{aligned}\tag{11}$$

Equation (10) coincides with (1) and in the classical setting (7) due to the Kolosov–Muskhelishvili's formulae (8), we have $\Phi_\pm \in \mathbb{L}_2(\mathbb{C}^\pm)$. Then, due to the representation formulae (9), the unknown function g in equation (11) has to be found in the trace space

$$g \in \mathbb{H}^{-1/2}(\mathbb{R}). \tag{12}$$

In the non-classical setting,

$$u+iv \in \mathbb{H}^s_p(\mathbb{C}^-\cup\mathbb{C}^+),\ \ \sigma_x,\sigma_y,\tau_{x,y}\in\mathbb{H}^{s-1}_p(\mathbb{C}),\ \ 1<p<\infty,\ \ s>\frac{1}{p} \tag{13}$$

(we should impose the constraint $s>1/p$ to ensure the existence of the trace $(u+iv)^+$ on the boundary), the integral equation (11) has to be solved in the trace space

$$g\in\mathbb{H}^{s-1/p-1}_p(\mathbb{R}). \tag{14}$$

2. Fourier Convolution Operators in the Bessel Potential Spaces $\mathbb{H}^s_p(\mathbb{R}^+)$

To formulate the next theorem we need to introduce the Fourier convolution and Bessel potential operators.

Let $a\in\mathbb{L}_{\infty,loc}(\mathbb{R})$ be a locally bounded $m\times m$ matrix function. The Fourier convolution operator (FCO) with the symbol a is defined by

$$W^0_a := \mathcal{F}^{-1}a\mathcal{F}. \tag{15}$$

Here

$$\mathcal{F}u(\xi):=\int_{\mathbb{R}^n} e^{i\xi x}u(x)\,dx,\ \ \xi\in\mathbb{R}^n. \tag{16}$$

is the Fourier transformation and

$$\mathcal{F}^{-1}v(\xi):=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\xi x}v(\xi)\,d\xi,\ \ x\in\mathbb{R}^n. \tag{17}$$

is its inverse transformation. If the operator

$$W^0_a:\mathbb{H}^s_p(\mathbb{R})\longrightarrow\mathbb{H}^{s-r}_p(\mathbb{R}) \tag{18}$$

is bounded, we say that a is an $\mathbb{L}_p$-multiplier of order r and use "$\mathbb{L}_p$-multiplier" if the order is 0. The set of all $\mathbb{L}_p$-multipliers of order r (of order 0) is denoted by $\mathfrak{M}^r_p(\mathbb{R})$ (by $\mathfrak{M}_p(\mathbb{R})$, respectively). Let

$$\widetilde{\mathfrak{M}}^r_p(\mathbb{R}):=\bigcap_{p-\varepsilon<q<p+\varepsilon}\mathfrak{M}^r_q(\mathbb{R}),\quad \widetilde{\mathfrak{M}}_p(\mathbb{R}):=\bigcap_{p-\varepsilon<q<p+\varepsilon}\mathfrak{M}_q(\mathbb{R}).$$

For an $\mathbb{L}_p$-multiplier of order r, $a \in \mathfrak{M}_p^r(\mathbb{R})$, the Fourier convolution operator (FCO) on the semi-axis $\mathbb{R}^+$ is defined by the equality

$$W_a = r_+ W_a^0 : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{s-r}(\mathbb{R}^+), \tag{19}$$

where $r_+ := r_{\mathbb{R}^+} : \mathbb{H}_p^s(\mathbb{R}) \longrightarrow \mathbb{H}_p^s(\mathbb{R}^+)$ is the restriction operator to the semi-axes $\mathbb{R}^+$.

We did not use in the definition of the class of multipliers $\mathfrak{M}_p^r(\mathbb{R})$ the parameter $s \in \mathbb{R}$. This is due to the fact that $\mathfrak{M}_p^r(\mathbb{R})$ is independent of s: if the operator W_a in (19) is bounded for some $s \in \mathbb{R}$, it is bounded for all other values of s.

Another definition of the multiplier class $\mathfrak{M}_p^r(\mathbb{R})$ is written as follows: $a \in \mathfrak{M}_p^r(\mathbb{R})$ if and only if $\lambda^{-r}a \in \mathfrak{M}_p(\overline{\mathbb{R}}) = \mathfrak{M}_p^0(\overline{\mathbb{R}})$, where $\lambda^r(\xi) := (1+|\xi|^2)^{r/2}$. This assertion is one of the consequences of Theorem 2.1 below.

The Bessel potential operators are defined as follows:

$$\begin{aligned} \mathbf{\Lambda}_\gamma^r &= W_{\lambda_\gamma^r}^0 : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}_p^{s-r}(\mathbb{R}^+), \\ \mathbf{\Lambda}_{-\gamma}^r &= r_+ W_{\lambda_{-\gamma}^r}^0 \ell : \mathbb{H}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{s-r}(\mathbb{R}^+), \\ \lambda_{\pm\gamma}^r(\xi) &:= (\xi \pm \gamma)^r, \quad \xi \in \mathbb{R}, \quad \operatorname{Im}\gamma > 0, \end{aligned} \tag{20}$$

and they arrange isomorphisms of the corresponding spaces (see [6,9]). Here, $\ell : \mathbb{H}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^s(\mathbb{R})$ is some extension operator and the final result is independent of the choice of an extension ℓ (we did not needed the extension operator in (19), since the space $\widetilde{\mathbb{H}}_p^s(\mathbb{R}^+)$ is automatically embedded in $\mathbb{H}_p^s(\mathbb{R})$ by extending the functions with 0).

Theorem 2.1. *Let* $1 < p < \infty$. *Then*

1. *For any* $r, s \in \mathbb{R}$, $\gamma \in \mathbb{C}$, $\operatorname{Im}\gamma > 0$ *the convolution operators*

$$\begin{aligned} \mathbf{\Lambda}_\gamma^r &= W_{\lambda_\gamma^r} : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}_p^{s-r}(\mathbb{R}^+), \\ \mathbf{\Lambda}_{-\gamma}^r &= r_+ W_{\lambda_{-\gamma}^r}^0 \ell : \mathbb{H}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{s-r}(\mathbb{R}^+), \\ \lambda_{\pm\gamma}^r(\xi) &:= (\xi \pm \gamma)^r, \quad \xi \in \mathbb{R}, \quad \operatorname{Im}\gamma > 0, \end{aligned} \tag{21}$$

arrange isomorphisms of the corresponding spaces (*see* [6,14]). *Here,* $\ell : \mathbb{H}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^s(\mathbb{R})$ *is some extension operator and the final result is independent of the choice of an extension* ℓ. r_+ *is the restriction from the axes* $\mathbb{R}$ *to the semi-axes* $\mathbb{R}^+$.

2. *For any operator* $\mathbf{A} : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{s-r}(\mathbb{R}^+)$ *of order* r, *the following diagram is commutative*

$$\begin{tikzcd} \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \arrow[r, "\mathbf{A}"] \arrow[d, "\mathbf{\Lambda}_\gamma^{-s}"'] & \mathbb{H}_p^{s-r}(\mathbb{R}^+) \arrow[d, "\mathbf{L}_{-\gamma}^{s-r}"] \\ \mathbb{L}_p(\mathbb{R}^+) \arrow[r, "\mathbf{\Lambda}_{-\gamma}^{s-r}\mathbf{A}\mathbf{\Lambda}_\gamma^{-s}"'] & \mathbb{L}_p(\mathbb{R}^+) \end{tikzcd} . \tag{22}$$

Diagram (22) *provides an equivalent lifting of the operator* $\mathbf{A}$ *of order* r *to the operator* $\mathbf{\Lambda}^{s-r}_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *of order* 0.

3. *For any bounded convolution operator* $W_a : \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+)$ *of order* r *and for any pair of complex numbers* γ_1, γ_2 *such that* $\operatorname{Im}\gamma_j > 0$, $j = 1, 2$, *the lifted operator*

$$\begin{gathered}\mathbf{\Lambda}^{\mu}_{-\gamma_1} W_a \mathbf{\Lambda}^{\nu}_{\gamma_2} = W_{a_{\mu,\nu}} : \mathbb{H}^{s+\nu}_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r-\mu}_p(\mathbb{R}^+),\\ a_{\mu,\nu}(\xi) := (\xi-\gamma_1)^{\mu} a(\xi)(\xi+\gamma_2)^{\nu}\end{gathered} \tag{23}$$

is again a Fourier convolution.

In particular, the lifted operator $W_{a_{s-r,-s}} = \mathbf{\Lambda}^{s-r}_{-\gamma} W_a \mathbf{\Lambda}^{-s}_{\gamma} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *has the symbol*

$$a_{s-r,-s}(\xi) = \lambda^{s-r}_{-\gamma}(\xi) a(\xi) \lambda^{-s}_{\gamma}(\xi) = \Big(\frac{\xi-\gamma}{\xi+\gamma}\Big)^{s-r} \frac{a(\xi)}{(\xi+i)^r}\,.$$

Remark 2.2. For any pair of multipliers $a \in \mathfrak{M}^r_p(\mathbb{R})$, $b \in \mathfrak{M}^s_p(\mathbb{R})$ the corresponding convolution operators on the axes W^0_a and W^0_b have the property $W^0_a W^0_b = W^0_b W^0_a = W^0_{ab}$.

For the corresponding Wiener–Hopf operators on the half-axes a similar equality

$$W_a W_b = W_{ab} \tag{24}$$

holds if and only if either the function $a(\xi)$ has an analytic extension in the lower half-plane, or the function $b(\xi)$ has an analytic extension in the upper half-plane (see [6]).

Note that actually (23) is a consequence of (24).

Let $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ denote the one point compactification of the real axes $\mathbb{R}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ denote the two point compactification of $\mathbb{R}$. By $C(\dot{\mathbb{R}})$ (by $C(\overline{\mathbb{R}})$, respectively) we denote the space of continuous functions $g(x)$ on $\mathbb{R}$ which have the equal limits at the infinity $g(-\infty) = g(+\infty)$ (limits at the infinity may be different $g(-\infty) \neq g(+\infty)$. By $PC(\dot{\mathbb{R}})$ is denoted the space of piecewise-continuous functions on $\dot{\mathbb{R}}$, having the limits $a\ (t \pm 0)$ at all points $t \in \dot{\mathbb{R}}$, including the infinity.

Proposition 2.3 ([6, Lemma 7.1] and [10, Proposition 1.2])**.** *Let* $1 < p < \infty$, $a \in C(\dot{\mathbb{R}}^+)$, $b \in C(\dot{\mathbb{R}}) \cap \widetilde{\mathfrak{M}}_p(\dot{\mathbb{R}})$ *and* $a(\infty) = b(\infty) = 0$. *Then the operators* $aW_b, W_b\, aI : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *are compact.*

Moreover, these operators are bounded in all Bessel potential space, and, due to Krasnoselskij interpolation theorem for compact operators, are compact in these spaces.

Proposition 2.4 ([6, Lemma 7.4] and [10, Lemma 1.2])**.** *Let* $1 < p < \infty$ *and let* a *and* b *satisfy at least one of the conditions*

(i) $a \in C(\overline{\mathbb{R}}^+)$, $b \in \widetilde{\mathfrak{M}}_p(\mathbb{R}) \cap PC(\overline{\mathbb{R}})$;

(ii) $a \in PC(\overline{\mathbb{R}}^+)$, $b \in C\widetilde{\mathfrak{M}}_p(\overline{\mathbb{R}})$.

Then the commutants $[aI, W_b]$ *and* $[aI, \mathfrak{M}^0_b]$, *where* $\mathfrak{M}^0_b$ *is a Mellin convolution operator* (*see the next Section* 3), *are compact operators in the space* $\mathbb{L}_p(\mathbb{R}^+)$.

Moreover, these operators are compact in all Bessel potential and Besov spaces, where they are bounded, due to Krasnoselskij interpolation theorem for compact operators.

The differentiation is a Fourier convolution operator with the symbol $-i\xi$:

$$r_+\partial_t\psi = r_+\partial_t\mathcal{F}^{-1}\mathcal{F}\psi = r_+\mathcal{F}^{-1}(-i\xi)\mathcal{F}\psi = W_{-i\xi}\psi, \quad \psi \in C^\infty_0(\mathbb{R}^+). \tag{25}$$

Using (25) and (20), we get

$$r_+(\partial_t\mathbf{\Lambda}^{-1}_{\pm\gamma} - I) = r_+(\mathbf{\Lambda}^{-1}_{\pm\gamma}\partial_t - I) = W_g, \quad g(\xi) := \frac{\xi}{\xi \pm \gamma} - 1, \quad \xi \in \mathbb{R}.$$

The symbol $g(\xi)$ is infinitely smooth and vanishes at infinity: $g(\pm\infty) = 0$. Then, due to Proposition 2.3, the operators

$$v_0\big[r_+(\partial_t\mathbf{\Lambda}^{-1}_{\pm\gamma} - I)\big], \quad \big[r_+(\partial_t\mathbf{\Lambda}^{-1}_{\pm\gamma} - I)\big]v_0I \tag{26}$$

are compact for all $v_0 \in C^\infty_0(\mathbb{R}^+)$ (and even for all sufficiently smooth $v_0 \in C^m(\mathbb{R}^+)$) which vanish at infinity $v_0(\infty) = 0$. The compactness of the operators in (26) imply the local invertibility of ∂_t (with the local inverse $\mathbf{\Lambda}^{-1}_{\pm\gamma}$) even at all finite points $t \in \mathbb{R}^+$.

3. Mellin Convolution Operators in the Bessel Potential Spaces $\mathbb{H}^s_p(\mathbb{R}^+)$

In the present section we expose auxiliary results from [9] (also see [3, 6, 10]), which are essential for the investigation of boundary integral equation (1).

Consider a Mellin convolution operator $\mathfrak{M}^0_a$ in the Bessel potential spaces

$$\mathfrak{M}^0_a := \mathcal{M}^{-1}_\beta a\mathcal{M}_\beta : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+), \tag{27}$$

where

$$\mathcal{M}_\beta v(\xi) := \int\limits_0^\infty \tau^{\beta - i\xi}v(\tau)\frac{d\tau}{\tau}\,, \quad \xi \in \mathbb{R},$$

$$\mathcal{M}^{-1}_\beta u(t) := \frac{1}{2\pi}\int\limits_{-\infty}^\infty t^{i\xi-\beta}u(\xi)\,d\xi, \quad t \in \mathbb{R}^+.$$

are the Mellin transformation and the inverse to it.

The symbol $a(\xi)$ of this operator is an $n\times n$ matrix function $a \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$ continuous on the real axis $\mathbb{R}$ with the only possible jump at infinity.

The most important example of a Mellin convolution operator is an integral operator of the form

$$\mathfrak{M}^0_a \mathbf{u}(t) := c_0 \mathbf{u}(t) + \frac{c_1}{\pi i} \int_0^\infty \frac{\mathbf{u}(\tau)}{\tau - t}\, dt + \int_0^\infty \mathcal{K}\Big(\frac{t}{\tau}\Big) \mathbf{u}(\tau)\, \frac{d\tau}{\tau} \tag{28}$$

with $n \times n$ matrix coefficients and $n \times n$ matrix kernel. $\mathfrak{M}^0_a$ is a bounded operator in the weighted Lebesgue space of vector-functions

$$\mathfrak{M}^0_a : \mathbb{L}_p(t^\gamma, \mathbb{R}^+) \longrightarrow \mathbb{L}_p(t^\gamma,\ \mathbb{R}^+), \ \ 1 < p < \infty, \ \ -1 < \gamma < p - 1, \tag{29}$$

endowed with the norm

$$\big\| u \mid \ \mathbb{L}_p(t^\gamma, \mathbb{R}^+) \big\| := \Bigg[\int_0^\infty t^\gamma |u(t)|^p \, dt \Bigg]^{1/p}$$

under the following constraint on the kernel (on each entry of the matrix kernel)

$$\int_0^\infty t^{\beta - 1} \mathcal{K}(t)\, dt < \infty, \ \ \beta := \frac{1 + \gamma}{p}\,, \ \ 0 < \beta < 1 \tag{30}$$

(cf. [6]). The symbol of the operator (28) is the Mellin transform of the kernel

$$\begin{aligned} a(\xi) &:= c_0 + c_1 \coth \pi(i\beta + \xi) + \mathcal{M}_\beta \mathcal{K}(\xi) \\ &:= c_0 + c_1 \coth \pi(i\beta + \xi) + \int_0^\infty t^{\beta - i\xi} \mathcal{K}(t) \frac{dt}{t}\,, \ \ \xi \in \mathbb{R}, \end{aligned} \tag{31}$$

and the symbol is responsible for the Fredholm properties of the operator.

Obviously,

$$\mathfrak{M}^0_a \mathfrak{M}^0_b \varphi = \mathfrak{M}^0_{ab} \varphi = \mathfrak{M}^0_b \mathfrak{M}^0_a \varphi, \ \ \varphi \in C^\infty_0(\mathbb{R}^+),$$

for arbitrary $a \in \mathfrak{M}^r_p(\mathbb{R})$ and $b \in \mathfrak{M}^s_p(\mathbb{R})$.

Theorem 3.1. *Let $1 < p < \infty$ and $-1 < \gamma < p-1$ (or $1 \leqslant p \leqslant \infty$ provided $c_1 = 0$ in (28)). The operator $\mathfrak{M}^0_a$ in (28)–(29) with a symbol $a \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$, is a Fredholm operator if and only if its symbol is invertible (is elliptic)*

$$\inf_{\xi \in \mathbb{R}} |\det a(\xi)| > 0. \tag{32}$$

If the symbol is elliptic, the operator is invertible and $\mathfrak{M}^0_{a^{-1}}$ is the inverse.

Things are different in the Bessel potential spaces. Let us recall some results from [9, Section 2].

Consider the kernels which are meromorphic functions on the complex plane $\mathbb{C}$, vanishing at infinity,

$$\mathcal{K}(t) := \sum_{j=0}^N \frac{d_j}{(t - c_j)^{m_j}}\,, \tag{33}$$

having poles at $c_0, c_1, \dots, c_N \in \mathbb{C} \setminus \{0\}$ and complex coefficients $d_j \in \mathbb{C}$.

Definition 3.2 (see [9]). We call a kernel $\mathcal{K}(t)$ in (33) admissible if for those poles $c_0, \dots, c_\ell$ which belong to the positive semi-axes $\arg c_0 = \cdots = \arg c_\ell = 0$, the corresponding multiplicities is one $m_0 = \cdots = m_\ell = 1$.

For example: The Mellin convolution operator

$$\mathbf{K}_c^m v(t) := \frac{1}{\pi} \int_0^\infty \frac{\tau^{m-1} v(\tau)}{(t - c\tau)^m}\, d\tau, \quad -\pi < \arg c < \pi, \quad v \in \mathbb{L}_p(\mathbb{R}^+) \tag{34}$$

has an admissible kernel for arbitrary $m = 1, 2, \dots$ if the following constraint holds: for a real $\arg c = 0$ and positive $c > 0$ necessarily $m = 1$.

Proposition 3.3 (see [9, Corollary 2.3, Theorem 2.4]). *Let $1 < p < \infty$ and $-1 < \gamma < p - 1$ (or $1 \leqslant p \leqslant \infty$ provided $c_1 = 0$ in (28)), $s \in \mathbb{R}$ and $\mathcal{K}(t)$ in (33) be an admissible kernel. Then the Mellin convolution operator*

$$\mathfrak{M}_a^0 \mathbf{u}(t) := c_0 \mathbf{u}(t) + \int_0^\infty \mathcal{K}\Big(\frac{t}{\tau}\Big) \mathbf{u}(\tau) \frac{d\tau}{\tau}$$

is bounded in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ and, also, in the Bessel potential space in the following setting:

$$\mathfrak{M}_a^0 : r\widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^s(\mathbb{R}^+). \tag{35}$$

Theorem 3.4 ([3, Theorem 5.1] and [9]). *Let $s \in \mathbb{R}$ and $1 < p < \infty$.*

If $-\pi \leqslant \arg c < \pi$, $\arg c \neq 0$, $0 < \arg \gamma < \pi$ and $0 < \arg(-c\,\gamma) < \pi$, the Mellin convolution operator between the Bessel potential spaces

$$\mathbf{K}_c^1 : \widetilde{\mathbb{H}}_p^r(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^r(\mathbb{R}^+) \tag{36}$$

is lifted to the equivalent operator

$$\mathbf{\Lambda}_{-\gamma}^s \mathbf{K}_c^1 \mathbf{\Lambda}_\gamma^{-s} = c^{-s} \mathbf{K}_c^1 W_{g^s_{-c\gamma,\gamma}} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+), \tag{37}$$

where $c^{-s} = |c|^{-s} e^{-is \arg c}$ and

$$g^s_{\delta,\mu}(\xi) := \Big(\frac{\xi + \delta}{\xi + \mu}\Big)^s. \tag{38}$$

If $-\pi \leqslant \arg c < \pi$, $\arg c \neq 0$, $0 < \arg \gamma < \pi$ and $-\pi < \arg(-c\,\gamma) < 0$, the Mellin convolution operator between the Bessel potential spaces (36) is lifted to the equivalent operator

$$\begin{aligned} \mathbf{\Lambda}_{-\gamma}^s \mathbf{K}_c^1 \mathbf{\Lambda}_\gamma^{-s} &= c^{-s} W_{g^s_{-\gamma,-\gamma_0}} \mathbf{K}_c^1 W_{g^s_{-c\gamma_0,\gamma}} \\ &= c^{-s} \mathbf{K}_c^1 W_{g^s_{-\gamma,-\gamma_0} g^s_{-c\gamma_0,\gamma}} + \mathbf{T} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+), \end{aligned} \tag{39}$$

where $\mathbf{T} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ is a compact operator.

Let us consider the Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$ generated by Mellin convolution and Fourier convolution operators, i.e. by the operators

$$\mathbf{A} := \sum_{j=1}^{m} W_{d_j} \mathfrak{M}^0_{a_j} W_{b_j} \tag{40}$$

and their compositions in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$. Here, $\mathfrak{M}^0_{a_j}$ are the Mellin convolution operators with continuous $N \times N$ matrix symbols $a_j \in C\mathfrak{M}_p(\overline{R})$, W_{b_j}, W_{d_j} are Fourier convolution operators with $N \times N$ matrix symbols $b_j, d_j \in C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\}) := C\mathfrak{M}_p(\overline{\mathbb{R}}^- \cup \overline{\mathbb{R}}^+)$. The algebra of $N \times N$ matrix $\mathbb{L}_p$-multipliers $C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\})$ consists of those piecewise-continuous $N \times N$ matrix multipliers $b \in \mathfrak{M}_p(\mathbb{R}) \cap PC(\overline{\mathbb{R}})$ which are continuous on the semi-axis $\mathbb{R}^-$ and $\mathbb{R}^+$, but might have finite jump discontinuities at 0 and at infinity.

To define the symbol of the operator $\mathbf{A}$ in (40) which governs the Fredholm property and the index of $\mathbf{A}$ (see Theorem 3.5, below) we consider the infinite clockwise oriented "rectangle" $\mathfrak{R} := \Gamma_1 \cup \Gamma_2^- \cup \Gamma_2^+ \cup \Gamma_3$, where (cf. Figure 1)

$$\Gamma_1 := \overline{\mathbb{R}} \times \{+\infty\}, \;\; \Gamma_2^{\pm} := \{\pm\infty\} \times \overline{\mathbb{R}}^+, \;\; \Gamma_3 := \overline{\mathbb{R}} \times \{0\}.$$

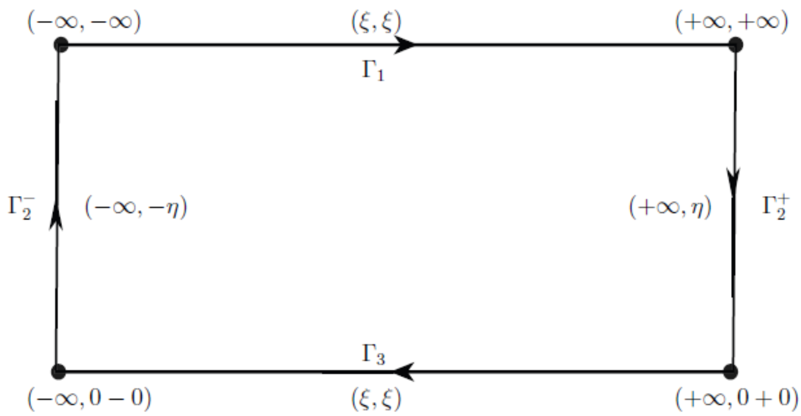

FIGURE 1. The domain $\mathfrak{R}$ of definition of the symbol $\mathcal{A}^s_p(\omega)$.

The symbol $\mathcal{A}_p(\omega)$ of the operator $\boldsymbol{A}$ in (40) is a function on the set $\mathfrak{R}$, viz.

$$\mathcal{A}_p(\omega) := \begin{cases} \sum_{j=1}^{m} (d_j)_p(\infty,\xi) a_j(\xi) (b_j)_p(\infty,\xi), & \omega = (\xi,\xi) \in \overline{\Gamma_1}, \\ \sum_{j=1}^{m} d_j(\eta) a_j(+\infty) b_j(\eta), & \omega = (+\infty,\eta) \in \Gamma_2^+, \\ \sum_{j=1}^{m} d_j(-\eta) a_j(-\infty) b_j(-\eta), & \omega = (-\infty,\eta) \in \Gamma_2^-, \\ \sum_{j=1}^{m} (d_j)_p(0,\xi) a_j(\xi) (b_j)_p(0,\xi), & \omega = (\xi,0) \in \overline{\Gamma_3}. \end{cases} \tag{41}$$

The connecting function $g_p(\infty,\xi)$ in (41) for a piecewise continuous function $g \in PC(\overline{\mathbb{R}})$ is defined as follows:

$$\begin{aligned} g_p(x,\xi) &:= \frac{1}{2}\big[g(x+0)+g(x-0)\big] - \frac{i}{2}\big[g(x+0)-g(x-0)\big]\cot\pi\Big(\frac{1}{p}-i\xi\Big) \\ &= e^{i\pi\,\frac{g_x^+ + g_x^-}{2}}\,\frac{\cos\pi\big(\frac{1}{p}+\frac{g_x^+ - g_x^-}{2}-i\xi\big)}{\sin\pi\big(\frac{1}{p}-i\xi\big)}\,,\quad \xi\in\mathbb{R}, \end{aligned} \tag{42}$$

$$g_x^{\pm} := \frac{1}{\pi i}\ln\, g(x\pm 0),\quad \operatorname{Re} g_x^{\pm} = \frac{1}{\pi}\arg\, g(x\pm 0),\quad x \in \dot{\mathbb{R}} := \mathbb{R}\cup\{\infty\}.$$

The function $g_p(\infty,\xi)$ fills up the discontinuity (the jump) of $g(\xi)$ at ∞ between $g(-\infty)$ and $g(+\infty)$ with an oriented arc of the circle (see [9, Section 4] for further details).

The symbol $\mathcal{A}_p(\omega)$ is continuous on the rectangle $\mathfrak{R}$ and if it is elliptic

$$\inf_{\omega\in\mathfrak{R}} \big|\det \mathcal{A}_p(\omega)\big| > 0, \tag{43}$$

the increment of the argument $(1/2\pi)\arg\mathcal{A}_p(\omega)$ when ω ranges through $\mathfrak{R}$ in the positive direction, is an integer. This integer is called the winding number or the index of the symbol and is denoted by $\operatorname{ind}\det\mathcal{A}_p$.

Theorem 3.5 ([9, Theorem 4.13])**.** *Let $1 < p < \infty$ and let $\mathbf{A}$ be defined by* (40)*. Then $\mathbf{A} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ is a Fredholm operator if and only if its symbol $\mathcal{A}_p(\omega)$ is elliptic. If $\mathbf{A}$ is Fredholm, the index of the operator is*

$$\operatorname{Ind}\mathbf{A} = -\operatorname{ind}\det\mathcal{A}_p. \tag{44}$$

4. Investigation of the Integro-Differential Equation (1)

For the investigation of equation (1) we apply the approach developed in [11] and the localization technique.

Proof of Theorem 0.1. Let us introduce the notation

$$\varphi_1(t) := \varphi(-t),\quad f_1(t) = f(-t),$$
$$\varphi_2(t) := \varphi(t),\quad f_2(t) = f(t)\ \text{ for }\ t > 0.$$

Then $\varphi_1'(t) := -\varphi'(-t)$, $\varphi_2'(t) := \varphi'(t)$ and the integral equation (1) is then written in the following form:

$$\begin{cases} \varphi_1(t) + \dfrac{a_-}{\pi}\displaystyle\int_0^\infty \dfrac{\varphi_1'(\tau)}{t-\tau}\,d\tau - \dfrac{a_-}{\pi}\displaystyle\int_0^\infty \dfrac{\varphi_2'(\tau)}{t+\tau}\,d\tau = f_1(t), \\ \varphi_2(t) - \dfrac{a_+}{\pi}\displaystyle\int_0^\infty \dfrac{\varphi_1'(\tau)}{t+\tau}\,d\tau + \dfrac{a_+}{\pi}\displaystyle\int_0^\infty \dfrac{\varphi_2'(\tau)}{t-\tau}\,d\tau = f_2(t), \end{cases} \quad t \in \mathbb{R}^+, \tag{45}$$
$$\varphi_1, \varphi_2 \in \mathbb{H}_p^s(\mathbb{R}^+), \quad f_1, f_2 \in \mathbb{H}_p^{s-1}(\mathbb{R}^+).$$

Moreover, by physical arguments (the system (45) is an equivalent reformulation of the Problem S (see (11), (13)) and we can assume that:

(i) a solution to the system (45) vanishes at 0

$$\varphi_1, \varphi_2 \in \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+). \tag{46}$$

(ii) the system (45) has a unique solution φ_1, φ_2 in the classical setting $s = 1/2$, $p = 2$:

$$\varphi_1, \varphi_2 \in \widetilde{\mathbb{H}}^{1/2}(\mathbb{R}^+), \quad f_1, f_2 \in \mathbb{H}^{-1/2}(\mathbb{R}^+). \tag{47}$$

The system of integral equations (45) is of Mellin type,

$$\begin{cases} \varphi_1(t) + a_-\Big[\boldsymbol{K}_1^1\,\varphi_1'(t) - \boldsymbol{K}_{-1}^1\,\varphi_2'(t)\Big] = f_1(t), \\ \varphi_2(t) - a_+\Big[\boldsymbol{K}_{-1}^1\,\varphi_1'(t) - \boldsymbol{K}_1^1\,\varphi_2'(t)\Big] = f_2(t), \end{cases} \tag{48}$$
$$\varphi_1, \varphi_2 \in \widetilde{\mathbb{H}}_p^s(\mathbb{R}+), \quad f_1, f_2 \in \mathbb{H}_p^{s-1}(\mathbb{R}^+),$$

where

$$\mathbf{K}_c^1\varphi(t) := \frac{1}{\pi}\int_0^\infty \frac{\varphi(\tau)}{t - c\,\tau}\,d\tau, \quad 0 < |\arg c| \leqslant \pi, \quad \varphi \in \mathbb{L}_p(\mathbb{R}^+),$$

is a Mellin convolutions operator with a meromorphic kernel (see Definition 3.2).

Since $\varphi_j \in \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+)$, $f_j \in \mathbb{H}_p^{s-1}(\mathbb{R}^+)$, $j = 1, 2$, we introduce new functions

$$\varphi_1 = \Lambda_\gamma^{-s}\psi_1, \quad \varphi_2 = \Lambda_\gamma^{-s}\psi_2, \quad f_1 = \Lambda_{-\gamma}^{-s+1}g_1 \quad f_2 = \Lambda_{-\gamma}^{-s+1}g_2,$$
$$\operatorname{Im}\gamma > 0, \quad \psi_1, \psi_2, g_1, g_2 \in \mathbb{L}_p(\mathbb{R}^+),$$

use the equality

$$\frac{d\varphi(t)}{dt} = \varphi'(t) = (\boldsymbol{W}_{-i\xi}\varphi)(t)$$

and get

$$\begin{cases} \Lambda_\gamma^{-s}\psi_1 + a_-\Big[\boldsymbol{K}_1^1\boldsymbol{W}_{-i\xi}\Lambda_\gamma^{-s}\psi_1 - \boldsymbol{K}_{-1}^1\boldsymbol{W}_{-i\xi}\Lambda_\gamma^{-s}\psi_2\Big] = \Lambda_{-\gamma}^{-s+1}g_1, \\ \Lambda_\gamma^{-s}\psi_2 - a_+\Big[\boldsymbol{K}_{-1}^1\boldsymbol{W}_{-i\xi}\Lambda_\gamma^{-s}\psi_1 - \boldsymbol{K}_1^1\boldsymbol{W}_{-i\xi}\Lambda_\gamma^{-s}\psi_2\Big] = \Lambda_{-\gamma}^{-s+1}g_2. \end{cases}$$

Here, the pair of functions ψ_1, ψ_2 is unknown while the pair g_1, g_2 is known (prescribed).

The system is already lifted to the $\mathbb{L}_p$-space setting, and we will write it in a convenient form by applying the Bessel potential operator Λ^{s-1}_{γ} to the both parts of the equations:

$$\begin{cases} \Lambda^{s-1}_{-\gamma}\Lambda^{-s}_{\gamma}\psi_1 + a_-\Big[\Lambda^{s-1}_{-\gamma}\boldsymbol{K}^1_1\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_1 - \Lambda^{s-1}_{-\gamma}\boldsymbol{K}^1_{-1}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_2\Big] \\ \qquad\qquad = \Lambda^{s-1}_{-\gamma}\Lambda^{-s+1}_{-\gamma}g_1 = g_1, \\ \Lambda^{s-1}_{-\gamma}\Lambda^{-s}_{\gamma}\psi_2 - a_+\Big[\Lambda^{s-1}_{-\gamma}\boldsymbol{K}^1_{-1}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_1 - \Lambda^{s-1}_{-\gamma}\boldsymbol{K}^1_1\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_2\Big] \\ \qquad\qquad = \Lambda^{s-1}_{-\gamma}\Lambda^{-s+1}_{-\gamma}g_2 = g_2, \end{cases}$$
$$\psi_1, \psi_2, g_1, g_2 \in \mathbb{L}_p(\mathbb{R}^+),$$

since $\Lambda^{s-1}_{-\gamma}\Lambda^{-s+1}_{-\gamma}u = u$ (see [6]). By using the equality

$$\boldsymbol{\Lambda}^r_{-\gamma}\mathbf{K}^1_c\boldsymbol{\Lambda}^{-r}_{\gamma} = c^{-r}\mathbf{K}^1_c\boldsymbol{\Lambda}^r_{-c\gamma}\boldsymbol{\Lambda}^{-r}_{\gamma}$$

proved in [3, Theorem 5.1] for arbitrary $c \in \mathbb{C}$ and again the equality $\Lambda^{s-1}_{-\gamma}\Lambda^{-s+1}_{\gamma} = I$, we rewrite the system in the following form:

$$\begin{cases} \Lambda^{s-1}_{-\gamma}\Lambda^{-s}_{\gamma}\psi_1 \\ \quad +a_-\Big[\boldsymbol{K}^1_1\Lambda^{s-1}_{-\gamma}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_1 - (-1)^{-s+1}\boldsymbol{K}^1_{-1}\Lambda^{s-1}_{\gamma}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_2\Big] = g_1, \\ \Lambda^{s-1}_{-\gamma}\Lambda^{-s}_{\gamma}\psi_2 \\ \quad -a_+\Big[(-1)^{-s+1}\boldsymbol{K}^1_{-1}\Lambda^{s-1}_{\gamma}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_1 - \boldsymbol{K}^1_1\Lambda^{s-1}_{-\gamma}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_2\Big] = g_2, \end{cases}$$
$$\psi_1, \psi_2, g_1, g_2 \in \mathbb{L}_p(\mathbb{R}^+).$$

Next, we apply the equalities

$$\Lambda^r_{\mu} = \boldsymbol{W}_{(\xi+\mu)^r}, \quad \boldsymbol{W}_a\boldsymbol{W}_b = \boldsymbol{W}_{ab},$$

where the second one holds if $a(\xi)$ has an analytic extension in the lower half-plane or $b(\xi)$ has an analytic extension in the upper half-plane (see [3,6] for details). The above equalities imply, in particular, that

$$\Lambda^{s-1}_{-\gamma}\Lambda^{-s}_{\gamma} = \boldsymbol{W}_{(\xi-\gamma)^{s-1}}\boldsymbol{W}_{(\xi+\gamma)^{-s}} = \boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s\,\frac{1}{\xi-\gamma}},$$
$$\Lambda^{s-1}_{\gamma}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma} = \boldsymbol{W}_{\frac{-i\xi}{\xi+\gamma}}.$$

Finally, we arrive to the following system of convolution equations, which is an equivalent reformulation of the system (48) in the $\mathbb{L}_p$-space setting:

$$\begin{cases} \boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s \frac{1}{\xi-\gamma}}\psi_1 + a_-\boldsymbol{K}^1_1\boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s \frac{-i\xi}{\xi-\gamma}}\psi_1 \\ \qquad\qquad +(-1)^s a_-\boldsymbol{K}^1_{-1}\boldsymbol{W}_{\frac{-i\xi}{\xi+\gamma}}\psi_2 = g_1, \\ \boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s \frac{1}{\xi-\gamma}}\psi_2 + (-1)^s a_+\boldsymbol{K}^1_{-1}\boldsymbol{W}_{\frac{-i\xi}{\xi+\gamma}}\psi_1 \\ \qquad\qquad +a_+\boldsymbol{K}^1_1\boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s \frac{-i\xi}{\xi-\gamma}}\psi_2 = g_2, \end{cases} \tag{49}$$
$$\psi_1, \psi_2, g_1, g_2 \in \mathbb{L}_p(\mathbb{R}^+).$$

Let us rewrite the system (49) as follows

$$\boldsymbol{A}\Psi = \mathbf{G}, \quad \Psi := \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \in \mathbb{L}_p(\mathbb{R}^+), \quad \mathbf{G} := \begin{pmatrix}g_1\\ g_2\end{pmatrix} \in \mathbb{L}_p(\mathbb{R}^+)$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s \frac{1}{\xi-\gamma}} + a_-\boldsymbol{K}^1_1\boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s \frac{-i\xi}{\xi-\gamma}} & e^{\pi si}a_-\boldsymbol{K}^1_{-1}\boldsymbol{W}_{\frac{-i\xi}{\xi+\gamma}} \\ e^{\pi si}a_+\boldsymbol{K}^1_{-1}\boldsymbol{W}_{\frac{-i\xi}{\xi+\gamma}} & \boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s \frac{1}{\xi-\gamma}} + a_+\boldsymbol{K}^1_1\boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s \frac{-i\xi}{\xi-\gamma}} \end{bmatrix}$$

According to [9, Formulae (41), (81)], the symbols of operators $\boldsymbol{K}^1_1 = \mathfrak{M}^0_{\mathcal{K}^1_1}$ and $\boldsymbol{K}^1_{-1} = \mathfrak{M}^0_{\mathcal{K}^1_{-1}}$ are, respectively,

$$\mathcal{K}^1_1(\xi) = -i\coth\pi(i\beta+\xi) = -\cot\pi(\beta-i\xi), \ \ \mathcal{K}^1_{-1}(\xi) = \frac{1}{\sin\pi(\beta-i\xi)}.$$

With the shorthand notation,

$$b_1(\xi) = \Big(\frac{\xi-\gamma}{\xi+\gamma}\Big)^s \frac{1}{\xi-\gamma}, \ \ b_2(\xi) = \Big(\frac{\xi-\gamma}{\xi+\gamma}\Big)^s \frac{-i\xi}{\xi-\gamma}, \ \ b_3(\xi) = \frac{-i\xi}{\xi+\gamma},$$

we rewrite the operator $\boldsymbol{A}$ as follows

$$\boldsymbol{A} = \begin{bmatrix}\boldsymbol{A}^-_1 & \boldsymbol{A}^-_2\\ \boldsymbol{A}^+_2 & \boldsymbol{A}^+_1\end{bmatrix} = \begin{bmatrix}\boldsymbol{W}_{b_1} + a_-\mathfrak{M}^0_{\mathcal{K}^1_1}\boldsymbol{W}_{b_2} & e^{\pi si}a_-\mathfrak{M}^0_{\mathcal{K}^1_{-1}}\boldsymbol{W}_{b_3}\\ e^{\pi si}a_+\mathfrak{M}^0_{\mathcal{K}^1_{-1}}\boldsymbol{W}_{b_3} & \boldsymbol{W}_{b_1} + a_+\mathfrak{M}^0_{\mathcal{K}^1_1}\boldsymbol{W}_{b_2}\end{bmatrix} \tag{50}$$

and investigate the operator $\boldsymbol{A} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$.

It is easy to see that the functions $b_1(\xi)$, $b_2(\xi)$ $b_3(\xi)$ and $\mathcal{K}^1_{\pm 1}$ have the following limits:

$$b_1(\pm\infty) = 0, \quad b_1(0) = -\frac{e^{\pi si}}{\gamma},$$
$$b_2(-\infty) = -i, \quad b_2(+\infty) = -ie^{2\pi si}, \quad b_2(0) = 0,$$
$$b_3(\pm\infty) = -i, \quad b_3(0) = 0,$$
$$\mathcal{K}^1_1(\pm\infty) = \pm i, \quad \mathcal{K}^1_{-1}(\pm\infty) = 0.$$

Then, according to [9, Formulae (85), (86)] (also see the earlier paper [10]), the symbol of the operators $\boldsymbol{A}_1^{\pm}$ and $\boldsymbol{A}_2^{\pm}$ in (50) are written as follows:

$$\mathcal{A}_p^s(\omega) = \begin{bmatrix} (\mathcal{A}_1^-)_p^s(\omega) & (\mathcal{A}_2^-)_p^s(\omega) \\ (\mathcal{A}_2^+)_p^s(\omega) & (\mathcal{A}_1^+)_p^s(\omega) \end{bmatrix},$$

where

$$\begin{aligned}
(\mathcal{A}_1^{\pm})_p^s(\omega) &= ia_{\pm}\cot\pi(\beta-i\xi)\Big(\frac{e^{2\pi si}+1}{2}+\frac{e^{2\pi si}-1}{2i}\,\cot\pi(\beta-i\xi)\Big)\\
&= ia_{\pm}e^{\pi si}\,\frac{\cot\pi(\beta-i\xi)\sin\pi(\beta-i\xi+s)}{\sin\pi(\beta-i\xi)}\\
&= ia_{\pm}e^{\pi si}\,\frac{\cos\pi(\beta-i\xi)\sin\pi(\beta-i\xi+s)}{\sin^2\pi(\beta-i\xi)}\,,\\
(\mathcal{A}_2^{\pm})_p^s(\omega) &= \frac{-ia_{\pm}e^{\pi si}}{\sin\pi(\beta-i\xi)} \quad \text{if } \omega=(\xi,\infty)\in\overline{\Gamma_1},\\
(\mathcal{A}_1^{\pm})_p^s(\omega) &= \Big(\frac{-\eta-\gamma}{-\eta+\gamma}\Big)^s\,\frac{1}{-\eta-\gamma} - ia_{\pm}\Big(\frac{-\eta-\gamma}{-\eta+\gamma}\Big)^s\,\frac{i\eta}{-\eta-\gamma}\\
&= -\Big(\frac{\eta-\gamma}{\eta+\gamma}\Big)^{-s}\,\frac{1+a_{\pm}\eta}{\eta+\gamma}\,,\\
(\mathcal{A}_2^{\pm})_p^s(\omega) &= 0 \ \text{ if } \ \omega=(+\infty,\eta)\in\overline{\Gamma_2^-}\,,\\
(\mathcal{A}_1^{\pm})_p^s(\omega) &= \Big(\frac{\eta-\gamma}{\eta+\gamma}\Big)^s\,\frac{1}{\eta-\gamma} + ia_{\pm}\Big(\frac{\eta-\gamma}{\eta+\gamma}\Big)^s\,\frac{-i\eta}{\eta-\gamma}\\
&= \Big(\frac{\eta-\gamma}{\eta+\gamma}\Big)^s\,\frac{1+a_{\pm}\eta}{\eta-\gamma},\\
(\mathcal{A}_2^{\pm})_p^s(\omega) &= 0 \ \text{ if } \ \omega=(-\infty,\eta)\in\overline{\Gamma_2^+}\,,\\
(\mathcal{A}_1^{\pm})_p^s(\omega) &= -\frac{e^{\pi si}}{\gamma}\\
(\mathcal{A}_2^{\pm})_p^s(\omega) &= 0 \ \text{ if } \ \omega=(\xi,\infty)\in\overline{\Gamma_3},
\end{aligned}$$

and $\beta=\frac{1}{p}$, $\xi\in\mathbb{R}$, $\eta\in\mathbb{R}^+$. Then

$$\det\mathcal{A}_p^s(\omega)$$
$$= \begin{cases} -a_-a_+e^{2\pi si}\,\dfrac{\cos^2\pi(\beta-i\xi)\sin^2\pi(\beta-i\xi+s)-\sin^2\pi(\beta-i\xi)}{\sin^4\pi(\beta-i\xi)}\,, \\ \qquad\qquad \omega=(\xi,\infty)\in\overline{\Gamma_1}, \\ \mp\Big(\dfrac{\eta-\gamma}{\eta+\gamma}\Big)^{\mp2s}\,\dfrac{(1+a_-\eta)(1+a_+\eta)}{\eta^2-\gamma^2}\,, \quad \omega=(\pm\infty,\eta)\in\Gamma_2^{\pm}, \\ \dfrac{e^{2\pi si}}{\gamma^2}\,, \quad \omega=(\xi,0)\in\overline{\Gamma_3}. \end{cases} \tag{51}$$

The symbol $\mathcal{A}_p^s(\omega)$ is non-elliptic (i.e., $\det\mathcal{A}_p^s(\omega)=0$) if and only if

(i) $(1+a_-\eta)(1+a_+\eta)\neq 0$ for all $0<\eta<\infty$. This condition holds if and only if coefficients $a_\pm$ are not negative reals: $a_\pm\in\mathbb{C}\setminus\overline{\mathbb{R}^-}$;

(ii) The parameters p and s are solutions to the equation
$$\cos^2\pi\Big(\frac{1}{p}-i\xi\Big)\sin^2\pi\Big(\frac{1}{p}+s-i\xi\Big)-\sin^2\pi\Big(\frac{1}{p}-i\xi\Big)=0. \tag{52}$$

By analyzing the transcendental equation (52), we come to the following conclusions.

(52) have a solution only for $\xi=0$ and (52) transforms into an equivalent transcendental equation (2).

For $1<p<4$, (2) has no solution for any $-1\leqslant s\leqslant 1$, because if we write it in an equivalent form
$$\sin^2\pi\Big(\frac{1}{p}+s\Big)=\tan^2\frac{\pi}{p} \tag{53}$$
the right-hand side is more than 1, while the left-hand side is less than or is equal to 1. On the other hand, in the classical setting $p=2$, $s=-\frac{1}{2}$ equation (1) has a unique solution (see (47)). Since this pair belongs to the quadrat $1<p<4$, $-1\leqslant s\leqslant 1$, where equation (1) is Fredholm, it has the same kernel and co-kernel in all these cases, i.e., is uniquely solvable for all $1<p<4$ and all $-1\leqslant s\leqslant 1$ (see [5] and [12] for the proof of the assertion).

For $4\leqslant p<\infty$, (53) has, due to the periodicity, two pairs of solutions $\{p,s_p\}$ and $\{p,s_p-1\}$, where $s_p>0$, $s_p-1<0$. It can be shown that for $s_p-1<s<s_p$, for $-1\leqslant s<s_p-1$, and for $s_p<s\leqslant 1$ the symbol $\mathcal{A}_p^s$ has index 0, $+1$ and -1, respectively. Manipulating with the properties of kernels and co-kernels in embedded spaces, we can prove easily that equation (1) has, respectively, no kernel and co-kernel (is uniquely solvable), has no kernel, but 1-dimensional co-kernel (has a unique solution for all right-hand sides which are orthogonal to the solution of the dual homogeneous equation) and has the 1-dimensional kernel, but no co-kernel (has a non-unique solution for all right-hand sides), respectively. □

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