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NINO PARTSVANIA

ON SOME NONLINEAR BOUNDARY VALUE PROBLEMS ON A FINITE AND AN INFINITE INTERVALS FOR SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. For nonlinear advanced functional differential systems, sufficient conditions for the solvability of nonlinear nonlocal boundary value problems on a finite and an infinite intervals are found.

რეზიუმე. არაწრფივი წინწასწრებული ფუნქციონალურ დიფერენციალური სისტემებისათვის სასრულ და უსასრულო შუალედებში დადგენილია არაწრფივ, არალოკალურ ამოცანათა ამოხსნადობის საკმარისი პირობები.

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Let $a > 0$, $\mathbb{R}_+ = [0, +\infty[$, and $\mathbb{R}_- =]-\infty, 0]$. In a finite interval $[0, a]$ and an infinite interval $\mathbb{R}_+$, we consider the boundary value problems

$$\frac{du_i(t)}{dt} = g_i\Big(t, u_1(\delta_{i1}(t)), \dots, u_n(\delta_{in}(t))\Big) \;\; (i = 1, \dots, n), \tag{1}$$

$$\varphi(u_1, \dots, u_n) = 0, \quad u_k(a) = \varphi_k(u_n(a)) \;\; (k = 1, \dots, n-1) \tag{2}$$

and

$$\frac{du_i(t)}{dt} = f_i\Big(t, u_1(\tau_{i1}(t)), \dots, u_n(\tau_{in}(t))\Big) \;\; (i = 1, \dots, n), \tag{3}$$

$$\varphi(u_1, \dots, u_n) = 0, \tag{4}$$

respectively.

Throughout the paper it is assumed that $g_i : [0, a] \times \mathbb{R}_+^n \to \mathbb{R}_-$ $(i = 1, \dots, n)$, $f_i : \mathbb{R}_+ \times \mathbb{R}_+^n \to \mathbb{R}_-$ $(i = 1, \dots, n)$, $\varphi_k : \mathbb{R}_+ \to \mathbb{R}_+$ $(k = 1, \dots, n-1)$, $\delta_{ik} : [0, a] \to [0, a]$ $(i, k = 1, \dots, n)$, and $\tau_{ik} : \mathbb{R}_+ \to \mathbb{R}_+$ $(i, k = 1, \dots, n)$ are continuous functions such that

$$\begin{gathered} g_i(t, 0, \dots, 0) = 0 \;\text{ for }\; 0 \le t \le a, \;\; \varphi_k(0) = 0 \;\; (i = 1, \dots, n; \; k = 1, \dots, n-1), \\ \delta_{ik}(t) > t \;\text{ for }\; a \le t < b \;\; (i, k = 1, \dots, n), \\ f_i(t, 0, \dots, 0) = 0 \;\text{ for }\; t \in \mathbb{R}_+ \;\; (i = 1, \dots, n), \end{gathered} \tag{5}$$

and

$$\tau_{ik}(t) > t \;\text{ for }\; t \in \mathbb{R}_+ \;\; (i, k = 1, \dots, n). \tag{6}$$

As for $\varphi : C([0, a]; \mathbb{R}_+^n) \to \mathbb{R}$, it is a continuous functional, bounded on every bounded subset of the set $C([0, a]; \mathbb{R}_+^n)$ and satisfies the inequality

$$\varphi(0, \dots, 0) < 0.$$

A continuously differentiable vector function $(u_1, \dots, u_n) : [0, a] \to \mathbb{R}_+^n$ $((u_1, \dots, u_n) : \mathbb{R}_+ \to \mathbb{R}_+^n)$ is said to be a solution of the problem $(1), (2)$ (of the problem $(3), (4)$) if it:

(i) satisfies the system (1) (the system (3)) in the interval $[0, a]$ (in the interval $\mathbb{R}_+$);

(ii) satisfies the equalities (2) (the equality (4)).

It is natural to call the problems (1), (2) and (3), (4) the Kneser type nonlinear problems on a finite and an infinite intervals. In the case, where $\delta_{ik}(t) \equiv t$ and $\tau_{ik}(t) \equiv t$ $(i,k = 1,\dots,n)$, such problems and their different particular cases are investigated in detail (see, e.g., [1–7, 9–16] and the references therein). For two-dimensional nonlinear advanced functional differential systems, the nonlinear Kneser problem is studied in [8]. In the case, where $n > 2$ and the functions δ_{ik} and τ_{ik} $(i,k = 1,\dots,n)$ satisfy the inequalities (5) and (6), the problems (1), (2) and (3), (4) remain practically unstudied.

In the paper, conditions guaranteeing, respectively, the solvability and unique solvability of the problem (1), (2) are established. On the basis of these results sufficient conditions for the solvability of the problem (3), (4) are found.

For any $\rho \geq 0$ and $m \in \{1,\dots,n\}$ we put

$$\mathbb{R}^m_{\rho+} = \left\{(x_1,\dots,x_m) \in \mathbb{R}^m : x_1 > \rho, \dots, x_m > \rho\right\}.$$

The following theorem is valid.

Theorem 1. *Let there exist numbers $m \in \{1,\dots,n\}$ and $\rho > 0$ such that*

$$\varphi(u_1,\dots,u_n) > 0 \ \textit{ for } \ (u_1,\dots,u_m) \in C([0,a];\mathbb{R}^m_{\rho+}), \ \ (u_1,\dots,u_n) \in C([0,a];\mathbb{R}^n_{+}) \tag{7}$$

and

$$\liminf_{x\to+\infty} \varphi_i(x) > \rho \ \ (i = 1,\dots,m_0), \ \ \textit{where } \ m_0 = \min\{m, n-1\}. \tag{8}$$

Then the problem (1), (2) *has at least one solution.*

The particular case of (2) are the boundary conditions

$$\sum_{i=1}^{n} \int_0^a \varphi_{0i}(u_i(s))\, d\sigma_i(s) = c, \ \ u_k(a) = \varphi_k(u_n(a)) \ \ (k = 1,\dots,n-1), \tag{9}$$

where $c \in \mathbb{R}$, $\varphi_{0i} : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1,\dots,n)$ are continuous functions, while $\sigma_i : [0,a] \to \mathbb{R}$ $(i = 1,\dots,n)$ is a nondecreasing functions such that

$$\varphi_{0i}(0) = 0, \ \ \sigma_i(b) - \sigma_i(a) > 0 \ \ (i = 1,\dots,n).$$

Theorem 1 yields the following corollary.

Corollary 1. *If either*

$$\lim_{x\to+\infty} \varphi_{01}(x) = +\infty, \quad \lim_{x\to+\infty} \varphi_1(x) = +\infty, \tag{10}$$

or

$$\lim_{x\to+\infty} \varphi_{0n}(x) = +\infty, \tag{11}$$

then the problem (1), (9) *has at least one solution.*

Theorem 2. *Let g_i $(i = 1,\dots,n)$ be nonincreasing in the phase variables functions satisfying the local Lipschitz conditions in those arguments, and φ_k $(k = 1,\dots,n-1)$ be increasing functions. Let, moreover,*

$$\begin{gathered} \varphi(u_1+v_1,\dots,u_n+v_n) > \varphi(u_1,\dots,u_n) \\ \textit{for } \ (u_1,\dots,u_n) \in C([0,a];\mathbb{R}^n_{+}), \ \ (v_1,\dots,v_n) \in C([0,a];\mathbb{R}^n_{0+}), \end{gathered} \tag{12}$$

and for some $m \in \{1,\dots,n\}$ and $\rho > 0$ the conditions (7), (8) *be fulfilled. Then the problem* (1), (2) *has one and only one solution.*

Corollary 2. *Let g_i $(i = 1,\dots,n)$ be nonincreasing in the phase variables functions satisfying the local Lipschitz conditions in those arguments, φ_{0i} $(i = 1,\dots,n)$ be nondecreasing functions, and φ_k $(k = 1,\dots,n-1)$ be increasing functions. Let, moreover, either φ_{01} be an increasing function and the condition* (10) *be satisfied, or φ_{0n} be an increasing function and the condition* (11) *be satisfied. Then the problem* (1), (9) *has one and only one solution.*

If for some $a_0 \in]0, a[$ the equalities

$$\delta_{ik}(t) = b \ \text{ for } \ a_0 \le t \le a \ \ (i, k = 1, \dots, n) \tag{13}$$

are fulfilled, then the unique solvability of the problem (1), (2) can be proved also in the case where the functions g_i $(i = 1, \dots, n)$ are not locally Lipschitz in the phase variables. More precisely, the following theorem is valid.

Theorem 3. *Let g_i $(i = 1, \dots, n)$ be nonincreasing in the phase variables functions, and φ_k $(k = 1, \dots, n-1)$ be increasing functions. Let, moreover, the conditions* (7), (8), (12) *and* (13) *hold. Then the problem* (1), (2) *has one and only one solution.*

As an example, we consider the differential system

$$\frac{du_i(t)}{dt} = \sum_{k=1}^{n} p_{ik}(t) u_k^{\lambda_{ik}}(\delta_{ik}(t)) \ \ (i = 1, \dots, n) \tag{14}$$

with the boundary conditions

$$\sum_{i=1}^{n} \ell_{0i} u_i^{\mu_{0i}}(0) = c_0, \quad u_k(a) = \ell_k u^{\mu_k}(a) \ \ (k = 1, \dots, n-1), \tag{15}$$

where $\ell_{0i} \ge 0$, $\mu_{0i} > 0$, $\ell_k > 0$, $\mu_k > 0$ $(i = 1, \dots, n;\ k = 1, \dots, n-1)$, $c_0 > 0$, and

$$\sum_{i=1}^{n} \ell_{0i} > 0.$$

Corollary 2 and Theorem 3 imply the following corollary.

Corollary 3. *Let either*

$$\lambda_{ik} \ge 1 \ \ (i, k = 1, \dots, n),$$

or for some $a_0 \in]0, a[$ the equalities (13) *hold. Then the problem* (14), (15) *has one and only one solution.*

Finally we give the theorems on the solvability of the problem (3), (4), i.e. of the nonlinear Kneser problem on the infinite interval.

Theorem 4. *Let there exist numbers $\rho > 0$, $b_0 > a$ and nondecreasing in the second argument continuous functions $f_{0k} : [a, b] \times \mathbb{R}_+ \to \mathbb{R}_+$ $(k = 1, \dots, n-1)$ such that*

$$\varphi(u_1, \dots, u_n) > 0 \ \text{ for } \ u_1 \in C([0, a]; \mathbb{R}_{\rho+}), \ \ (u_1, \dots, u_n) \in C([0, a]; \mathbb{R}_+^n),$$
$$f_k(t, x_1, \dots, x_n) \le -f_{0k}(t, x_{k+1}) \ \text{ for } \ a \le t \le b, \ \ (x_1, \dots, x_n) \in \mathbb{R}_+^n \ \ (k = 1, \dots, n-1),$$

and

$$\lim_{x \to +\infty} \int_{t_1}^{t_2} f_{0k}(s, x)\, ds = +\infty \ \text{ for } \ a \le t_1 < t_2 \le b \ \ (k = 1, \dots, n-1).$$

Then the problem (3), (4) *has at least one solution.*

Theorem 5. *Let there exist a number $\rho > 0$ such that*

$$\varphi(u_1, \dots, u_n) > 0 \ \text{ for } \ u_1 + \dots + u_n \in C([0, a]; \mathbb{R}_{\rho+}), \ \ (u_1, \dots, u_n) \in C([0, a]; \mathbb{R}_+^n).$$

Then the problem (3), (4) *has at least one solution.*

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Author's addresses:

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia;

2. International Black Sea University, 2 David Agmashenebeli Alley 13km, Tbilisi 0131, Georgia.

E-mail: ninopa@rmi.ge