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Short Communications

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ON THE SOLVABILITY OF THE ANTIPERIODIC PROBLEM FOR LINEAR SYSTEMS OF IMPULSIVE EQUATIONS

Abstract. The antiperiodic boundary value problem for systems of linear impulsive equations is considered. The Green type theorem on the unique solvability of the problem is established, and its solution is represented. The effective necessary and sufficient (among them spectral sufficient) conditions for the unique solvability of the problem are also given.

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In the present paper, we consider the system of linear impulsive equations on the real axis with a finite number of impulses points

$$\frac{dx}{dt} = P(t)x + p(t) \quad \text{for a.e. } t \in \mathbb{R}, \tag{1}$$

$$x(\tau_{kj}+) - x(\tau_{kj}-) = Q_{kj}x(\tau_{kj}-) + q_{kj} \quad (j = 1, \dots, m_0; \;\; k = 0, \pm 1, \pm 2, \dots) \tag{2}$$

under the ω-antiperiodic condition

$$x(t + \omega) = -x(t) \quad \text{for} \;\; t \in \mathbb{R}, \tag{3}$$

where $k\omega \leq \tau_{k1} < \cdots < \tau_{km_0} < (k+1)\omega$, $\tau_{k+1\,j} = \tau_{k\,j} + \omega$ ($j = 1, \dots, m_0$; $k = 0, \pm 1, \pm 2, \dots$), m_0 is a fixed natural number, ω is a fixed positive number, $P \in L_{loc}(\mathbb{R}; \mathbb{R}^{n\times n})$ is a ω-periodic matrix-function, $p \in L_{loc}(\mathbb{R}; \mathbb{R}^n)$ is a ω-antiperiodic vector-function, $Q_{kj} \in \mathbb{R}^{n\times n}$ ($j = 1, \dots, m$; $k = 0, \pm 1, \pm 2, \dots$) and $q_{kj} \in \mathbb{R}^n$ ($j = 1, \dots, m$; $k = 0, \pm 1, \pm 2, \dots$) are, respectively, constant $n \times n$-matrices and n-vectors.

Below we present the Green type theorem on the solvability of the problem (1), (2); (3) and give representation of its solution. In addition, we give effective necessary and sufficient (spectral type) conditions for the unique solvability of the problem. The general linear boundary value problem for the system (1), (2) and the nonlinear problems for impulsive systems are investigated sufficiently well in [1,5,6,8–11,16–18] (see also the references therein), where, in particular, the Green type theorems for the unique solvability have been obtained. Some questions of periodic problems for the system (1), (2) are investigated in [10, 11, 16–18]. Moreover, they are a particular case of the problems considered in [3, 4, 6, 19]. As to the antiperiodic problem, it is rather far from completeness. Thus the problem under consideration what follows, is actual.

In the paper we establish some spectral conditions for the unique solvability of the problem which follows from the analogous results for the generalized linear differential systems.

In the paper, the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$; $[a, b]$ and $]a, b[$ ($a, b \in \mathbb{R}$) are, respectively, closed and open intervals. $\mathbb{Z}$ is a set of all integers.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\|=\max\limits_{j=1,\dots,m}\sum\limits_{i=1}^{n}|x_{ij}|$.

$O_{n\times m}$ (or O) is the zero $n\times m$ matrix.

If $X=(x_{ij})_{i,j=1}^{n,m}\in\mathbb{R}^{n\times m}$, then $|X|=(|x_{ij}|)_{i,j=1}^{n,m}$.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n=\mathbb{R}_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times n}$, then:

X^{-1} is the matrix inverse to X;

$\det X$ is the determinant of X;

$r(X)$ is the spectral radius of X;

I_n is the identity $n\times n$-matrix.

The inequalities between the real matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

If $X:[a,b]\to\mathbb{R}^{n\times m}$ is a matrix-function, then $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t ($X(a-)=X(a)$, $X(b+)=X(b)$).

$L([a,b];\mathbb{R}^{n\times m})$ is the set of all measurable and Lebesgue integrable on $[a,b]$ matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$;

$L_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{R}\to\mathbb{R}^{n\times m}$ whose restrictions on every closed interval $[a,b]$ from $\mathbb{R}$ belong to $L([a,b],\mathbb{R}^{n\times n})$.

$C([a,b];\mathbb{R}^{n\times l})$ is the set of all continuous on $[a,b]$ matrix-functions $X:[a,b]\to\mathbb{R}^{n\times l}$;

$C_{loc}(\mathbb{R},\mathbb{R}^{n\times l})$ is the set of all matrix-functions $X:\mathbb{R}\to\mathbb{R}^{n\times l}$ whose restrictions on every closed interval $[a,b]$ from $\mathbb{R}$ belong to $C([a,b],\mathbb{R}^{n\times l})$.

$\widetilde{C}([a,b];\mathbb{R}^{n\times l})$ is the set of all absolutely continuous on $[a,b]$ matrix-functions $X:[a,b]\to\mathbb{R}^{n\times l}$;

$\widetilde{C}([a,b];\mathbb{R}^{n\times l};\tau_1,\dots,\tau_m)$, where $\tau_1,\dots,\tau_m\in[a,b]$, is the set of all matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$, having the one-sided limits $X(\tau_k-)$ $(k=1,\dots,m)$ and $X(\tau_k+)$ $(k=1,\dots,m)$, whose restriction on an arbitrary closed interval $[c,d]$ from $[a,b]\setminus\{\tau_k\}_{k=1}^m$ belong to $\widetilde{C}([c,d];\mathbb{R}^{n\times l})$.

For the pair $\{X;\{Y_l\}_{l=1}^m\}$, consisting of the matrix-function $X\in L([0,\omega],\mathbb{R}^{n\times n})$ and a sequence of constant $n\times n$ matrices $Y_1,\dots,Y_m$, we put

$$\begin{aligned}
&\big[(X;\{Y_l\}_{l=1}^m)(t)\big]_0=I_n \ \text{ for } \ 0\le t\le\omega,\\
&\big[(X;\{Y_l\}_{l=1}^m)(0)\big]_i=O_{n\times n} \ \ (i=1,2,\dots),\\
&\big[(X;\{Y_l\}_{l=1}^m)(t)\big]_{i+1}=\int\limits_0^t X(\tau)\cdot\big[(X;\{Y_l\}_{l=1}^m)(\tau)\big]_i\,d\tau\\
&\qquad+\sum_{a\le\tau_l<t}Y_l\cdot\big[(X;\{Y_l\}_{l=1}^m)(\tau_l)\big]_i \ \text{ for } \ 0<t\le\omega \ \ (i=1,2,\dots).
\end{aligned}\tag{4}$$

We say that the pair $\{X;\{Y_l\}_{l=1}^m\}$ satisfies the Lappo–Danilevskiĭ condition, if the matrices $Y_1,\dots,Y_m$ are pairwise permutable and there exists $t_0\in[a,b]$ such that

$$\int\limits_{t_0}^t X(\tau)\,dX(\tau)=\int\limits_{t_0}^t dX(\tau)\cdot X(\tau) \ \text{ for } \ t\in[0,\omega]$$

and

$$X(t)Y_l=Y_lX(t) \ \text{ for } \ t\in[0,\omega] \ \ (l=1,\dots,m).$$

Under a solution of the system (1), (2) we understand a continuous from the left vector-function $x:\mathbb{R}\to\mathbb{R}^n$ whose restrictions on $[k\omega,(k+1)\omega]$ belong to $\widetilde{C}([k\omega,(k+1)\omega];\mathbb{R}^n;\tau_{k1},\dots,\tau_{km_0})$ for every $k\in\mathbb{Z}$ and satisfying both the system (1) for a.e. $t\in\mathbb{R}$ and the equality (2) for every $j\in\{1,\dots,m_0\}$.

In the sequel, we assume everywhere that $P(t+\omega)=P(t)$ and $q(t+\omega)=-q(t)$ for $t\in\mathbb{R}$, $\tau_{0j}=\tau_j$, $q_{0j}=q_j$, $Q_{kj}=Q_j$ and $q_{k+1j}=-q_{kj}$ $(j=1,\dots,m_0;\ k=0,\pm1,\pm2,\dots)$. Moreover, we assume that

$$\det(I_n+Q_j)\neq 0 \ \ (j=1,\dots,m_0).\tag{5}$$

Note that the condition (5) guarantees the unique solvability of the system (1), (2) under the Cauchy condition $x(t_0)=c_0$.

Alongside with the system (1), (2), we consider the corresponding homogeneous system

$$\frac{dx}{dt} = P(t)x \ \text{ for a.e. } \ t \in \mathbb{R}, \tag{1_0}$$

$$x(\tau_{kj}+) - x(\tau_{kj}-) = Q_{kj}x(\tau_{kj}-) \ \ (j = 1, \dots, m_0; \ k = 0, \pm 1, \pm 2, \dots). \tag{2_0}$$

Moreover, along with (3) we consider the condition

$$x(0) = -x(\omega). \tag{6}$$

Proposition 1. *The following statements are valid:*

(a) *if x is a solution of the system* (1), (2)*, then the function $y(t) = -x(t+\omega)$ $(t \in \mathbb{R})$ is a solution of the system* (1), (2)*, as well;*

(b) *the problem* (1), (2); (3) *is solvable if and only if the system* (1), (2) *on the closed interval $[0, \omega]$ has a solution satisfying the boundary condition* (6)*. Moreover, the set of restrictions of solutions of the problem* (1), (2); (3) *on $[0, \omega]$ coincides with the set of solutions of the problem* (1), (2); (6)*.*

Based on this proposition we give the following definition.

Let

$$D = I_n + Y(\omega),$$

where Y is the fundamental matrix of the problem $(1_0), (2_0); (6)$ under the condition $Y(0) = I_n$.

Definition 1. Let $\det D \neq 0$. A matrix-function $\mathcal{G} : [0, \omega] \times [0, \omega] \to \mathbb{R}^{n\times n}$ is said to be the Green matrix of the problem $(1_0), (2_0); (6)$ if:

(a) for every $s \in]0, \omega[$, the matrix-function $\mathcal{G}(\,\cdot\,, s)$ satisfies the impulsive homogeneous matrix equation

$$\frac{dX}{dt} = P(t)X \ \text{ for a. e. } \ t \in \mathbb{R},$$
$$X(\tau_j+) - X(\tau_j-) = Q_j X(\tau_j-) \ \ (j = 1, \dots, m_0);$$

(b)

$$\mathcal{G}(t, t+) - \mathcal{G}(t, t-) = Y(t)D^{-1}Y(\omega)Y^{-1}(t) \ \text{ for } \ t \in]0, \omega[\setminus \{\tau_1, \dots, \tau_{m_0}\},$$
$$\mathcal{G}(\tau_j, \tau_j+) - \mathcal{G}(\tau_j, \tau_j-) = Y(\tau_j)D^{-1}Y(\omega)Y^{-1}(\tau_j)(I_n + Q_j)^{-1} \ \ (j = 1, \dots, m_0);$$

(c)

$$\mathcal{G}(t+, t) - \mathcal{G}(t-, t) = I_n \ \text{ for } \ t \in]0, \omega[\setminus \{\tau_1, \dots, \tau_{m_0}\},$$
$$\mathcal{G}(\tau_j+, \tau_j) - \mathcal{G}(\tau_j-, \tau_j) = I_n + Q_j Y(\tau_j)D^{-1}\big(I_n + Y^{-1}(\tau_j)\big) \ \ (j = 1, \dots, m_0);$$

(d)

$$\mathcal{G}(t, \cdot) \in \widetilde{C}\big([0, \omega]; \mathbb{R}^{n\times n}; \tau_1, \dots, \tau_{m_0}\big) \ \text{ for } \ t \in [0, \omega];$$

(e) the equality

$$\int\limits_0^\omega \big(\mathcal{G}(0, s) + \mathcal{G}(\omega, s)\big) \cdot p(s)\, ds + \sum_{j=1}^{m_0} \big(\mathcal{G}(0, \tau_j+) + \mathcal{G}(\omega, \tau_j+)\big) \cdot q_j = 0$$

holds for every $p \in L([0, \omega], \mathbb{R}^n)$ and $q_1, \dots, q_{m_0} \in \mathbb{R}^n$.

The Green matrix of the problem $(1_0), (2_0); (6)$ exists and is unique in the following sense. If $\mathcal{G}(t, s)$ and $\mathcal{G}_1(t, s)$ are two matrix-functions satisfying the conditions (a)–(e) of Definition 1, then

$$\mathcal{G}(t, s) - \mathcal{G}_1(t, s) \equiv Y(t)H_*(s),$$

where $H_* \in \widetilde{C}([0, \omega]; \mathbb{R}^{n\times n}; \tau_1, \dots, \tau_{m_0})$ is a matrix-function such that

$$H_*(s+) = H_*(s-) = C = const \ \text{ for } \ s \in [0, \omega],$$

and $C \in \mathbb{R}^{n\times n}$ is a constant matrix.

In particular, the matrix-function $\mathcal{G}$ defined by

$$\mathcal{G}(t,s)=\begin{cases} Y(t)D^{-1}(I_n+Y^{-1}(s)) & \text{for } 0\le s<t\le\omega,\\ Y(t)D^{-1}\big(I_n-Y(\omega)Y^{-1}(s)\big) & \text{for } 0\le t<s\le\omega,\\ \text{arbitrary} & \text{for } t=s \end{cases}$$

is the Green matrix of the problem $(1_0),(2_0);(6)$.

Theorem 1. *The problem* (1),(2) *has a unique ω-antiperiodic solution x if and only if the corresponding homogeneous system $(1_0),(2_0)$ has only the trivial solution satisfying the condition* (6), *i.e., when*

$$\det(I_n+Y(\omega))\neq 0. \tag{7}$$

If the last condition holds, then the solution x admits the notation

$$x(t)=\int\limits_0^\omega \mathcal{G}(t,s)\cdot p(s)\,ds+\sum_{j=1}^{m_0}\mathcal{G}(t,\tau_j+)\cdot q_j \ \ \textit{for}\ \ t\in[0,\omega], \tag{8}$$

where $\mathcal{G}:[0,\omega]\times[0,\omega]\to\mathbb{R}^{n\times n}$ is the Green matrix $\mathcal{G}$ of the problem $(1_0),(2_0);(6)$ on $[0,\omega]$.

Corollary 1. *Let the pair $\{P,\{Q_j\}_{j=1}^{m_0}\}$ satisfy the Lappo–Danilevskiĭ condition. Then the problem* (1),(2) *has a unique ω-antiperiodic solution if and only if*

$$\det\bigg(I_n+\exp\bigg(\int\limits_0^\omega P(s)\,ds\bigg)\prod_{j=1}^{m_0}(I_n+Q_j)\bigg)\neq 0.$$

Note that if the pair $\{P,\{Q_j\}_{j=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then

$$Y(t)\equiv\exp\bigg(\int\limits_0^\omega P(s)\,ds\bigg)\prod_{j=1}^{m_0}(I_n+Q_j)$$

and, therefore, the condition (7) is of the form given in the corollary.

Remark 1. If the system $(1_0),(2_0)$ has a nontrivial ω-antiperiodic solution, then there exist the vector-function $p\in L_{loc}(\mathbb{R},\mathbb{R}^n)$ and constant vectors q_{kj} ($j=1,\dots,m_0$; $k=0,\pm1,\pm2,\dots$) such that $q(t+\omega)=-q(t)$ for $t\in\mathbb{R}$, $q_{k+1j}=-q_{kj}$ ($j=1,\dots,m_0$; $k=0,\pm1,\pm2,\dots$), but the system (1), (2) has no ω-antiperiodic solution.

In general, it is quite difficult to verify the condition (7) directly even in the case where one is able to write out the fundamental matrix of the system $(1_0),(2_0)$ explicitly. Therefore it is important to find of effective conditions which would guarantee the absence of nontrivial ω-antiperiodic solutions of the homogeneous system $(1_0),(2_0)$. Below we give the results concerning the subset question. Analogous results have been obtained by T. Kiguradze for the ordinary differential equations (see [12, 13]).

Theorem 2. *The system* (1), (2) *has a unique ω-antiperiodic solution if and only if there exist natural numbers k and m such that the matrix*

$$M_k=-\sum_{i=0}^{k-1}\big[(P;\{Q_l\}_{l=1}^{m_0})(\omega)\big]_i$$

is nonsingular and

$$r(M_{k,m})<1, \tag{9}$$

where

$$M_{k,m}=\big[(P;\{Q_l\}_{l=1}^{m_0})(\omega)\big]_m+\sum_{i=0}^{m-1}\big[(P;\{Q_l\}_{l=1}^{m_0})(\omega)\big]_i\cdot|M_k^{-1}|\big[(P;\{Q_l\}_{l=1}^{m_0})(\omega)\big]_k,$$

and $[(P;\{Q_l\}_{l=1}^{m_0})(\omega)]_i$ $(i=0,\dots,m-1)$ are defined by (4).

Corollary 2. *Let there exist a natural j such that*

$$\big[(P;\{Q_l\}_{l=1}^{m_0})(\omega)\big]_j = 0 \ \ (i=1,\dots,j)$$

and

$$\det\Big(\big[(P;\{Q_l\}_{l=1}^{m_0})(\omega)\big]_{j+1}\Big) \neq 0,$$

where $[(P;\{Q_l\}_{l=1}^{m_0})(\omega)]_i$ $(i=0,\dots,m-1)$ *are defined by* (4)*. Then there exists* $\varepsilon_0>0$ *such that the system*

$$\frac{dx}{dt} = \varepsilon P(t)x + p(t) \ \text{ for a.e. } t\in\mathbb{R},$$
$$x(\tau_{kj}+) - x(\tau_{kj}-) = \varepsilon Q_j x(\tau_{kj}-) + q_{kj} \ \ (j=1,\dots,m_0;\ k=0,\pm1,\pm2,\dots)$$

have one and only one ω*-antiperiodic solution for every* $\varepsilon\in]0,\varepsilon_0[$.

Theorem 3. *Let the homogeneous system*

$$\frac{dx}{dt} = P_0(t)x \ \text{ for a. e. } t\in\mathbb{R},, \tag{10_0}$$
$$x(\tau_{kj}+) - x(\tau_{kj}-) = Q_{0kj}x(\tau_{kj}-) \ \ (j=1,\dots,m_0;\ \ k=0,\pm1,\pm2,\dots) \tag{11_0}$$

has only the trivial ω*-antiperiodic solution, where* $P_0\in L_{loc}(\mathbb{R};\mathbb{R}^{n\times n})$ *is* $\omega>0$*-periodic matrix-function,* $Q_{0kj}\in\mathbb{R}^{n\times n}$ $(j=1,\dots,m;\ k=0,\pm1,\pm2,\dots)$ *are constant* $n\times n$*-matrices such that* $Q_{0kj}=Q_{0j}$ $(j=1,\dots,m_0;\ k=0,\pm1,\pm2,\dots)$ *and*

$$\det(I_n+Q_{0j})\neq 0 \ \ (j=1,\dots,m_0).$$

Let, moreover, the matrix-function $P_0\in L_{loc}(\mathbb{R};\mathbb{R}^{n\times n})$ *and constant matrices* $Q_j (j=1,\dots,m_0)$ *admit the estimate*

$$\int_0^\omega |\mathcal{G}_0(t,\tau)|\,|P(\tau)-P_0(\tau)|\,d\tau + \sum_{j=1}^{m_0} |\mathcal{G}_0(t,\tau_j+)(Q_j-Q_{0j})| \leq M \ \text{ for } t\in[0,\omega],$$

where $\mathcal{G}_0(t,\tau)$ *is the Green matrix of the problem* $(10_0),(11_0);(6)$*, and* $M\in\mathbb{R}_+^{n\times n}$ *is a constant matrix such that*

$$r(M)<1.$$

Then the system (1), (2) *has one and only one* ω*-antiperiodic solution.*

The representation (8) can be replaced by a more simple and suitable form by introducing the concept of the Green matrix for the problem $(1_0),(2_0);(3)$.

Definition 2. The matrix-function $\mathcal{G}_\omega:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^{n\times n}$ is said to be the Green matrix of the problem $(1_0),(2_0);(3)$ if:

(a) $\mathcal{G}_\omega(t+\omega,\tau+\omega)=\mathcal{G}_\omega(t,\tau)$, $\mathcal{G}_\omega(t,t+\omega)+\mathcal{G}_\omega(t,\tau)=-I_n$ for $t,\tau\in\mathbb{R}$;

(b) the matrix-function $\mathcal{G}_\omega(\,\cdot\,,\tau):\mathbb{R}\to\mathbb{R}^{n\times n}$ is a fundamental matrix of the system $(1_0),(2_0)$ for every $\tau\in\mathbb{R}$.

Proposition 2. *Let the problem* $(1_0),(2_0)$ *have only a trivial solution. Then there exists the unique Green matrix of the problem, which has the form*

$$\mathcal{G}_\omega(t,\tau) = -Y(t)(I_n+Y^{-1}(\omega))^{-1}Y^{-1}(\tau) \ \text{ for } t,\tau\in\mathbb{R}.$$

Theorem 4. *Let the condition*

$$\det(I_n\pm Q_j)\neq 0 \ \ (j=1,\dots,m_0)$$

hold and the boundary value problem $(1_0),(2_0);(3)$ *have only a trivial solution. Then the* ω*-antiperiodic problem* (1), (2); (3) *has a unique solution* x *admitting the representation*

$$x(t) = \int_t^{t+\omega} \mathcal{G}_\omega(t,\tau)p(\tau)\,d\tau + \sum_{t\le\tau_{kj}<(k+1)\omega} \mathcal{G}_\omega(t,\tau_{kj})(I_n-Q_j^2)^{-1}q_{kj}$$
$$+ \sum_{(k+1)\omega\le\tau_{k+1\,j}<t+\omega} \mathcal{G}_\omega(t,\tau_{k+1\,j})(I_n-Q_j^2)^{-1}q_{k+1\,j} \ \text{ for } t\in(k\omega,(k+1)\omega] \ \ (k=0;\pm1;\pm;\dots), \tag{12}$$

where $\mathcal{G}_\omega$ *is the Green matrix of the problem* $(1_0), (2_0); (3)$.

Using the properties of the Green matrix $\mathcal{G}_\omega(t,\tau)$ (see Definition 2 (a)), the representation (12) can be rewriten in the form

$$x(t)=\int\limits_t^{t+\omega}\mathcal{G}_\omega(t,\tau)p(\tau)\,d\tau+(-1)^{k+1}\sum_{0\le\tau_j<t-k\omega}\mathcal{G}_\omega(t-\omega,\tau_j)(I_n-Q_j^2)^{-1}q_j$$

$$+(-1)^k\sum_{t-k\omega\le\tau_j<\omega}\mathcal{G}_\omega(t-k\omega,\tau_j)(I_n-Q_j^2)^{-1}q_j \ \text{ for } \ t\in(k\omega,(k+1)\omega] \ \ (k=0;\pm1;\pm;\dots).$$

Note that the results obtained in the paper, follow from the corresponding results given in [7] for the generalized differential system of the form

$$dx(t)=dA(t)\cdot x(t)+df(t)$$

since the impulsive system (1), (2) is the particular case of the last system under the assumptions that

$$A(0)=O_{n\times n}, \ \ A(t)=\int\limits_0^t P(\tau)\,d\tau+\sum_{0\le\tau_j<t}Q_j \ \text{ for } \ t\in(0,\omega],$$

$$f(0)=0, \ \ f(t)=\int\limits_0^t p(\tau)\,d\tau+\sum_{0\le\tau_j<t}q_j \ \text{ for } \ t\in(0,\omega],$$

and

$$A(t+\omega)=A(t) \ \text{ and } \ f(t+\omega)=-f(t) \ \text{ for } \ t\in\mathbb{R}\setminus[0,\omega].$$

It is not difficult to verify that

$$A(t)=\int\limits_{k\omega}^t P(\tau)\,d\tau+\sum_{k\omega\le\tau_{kj}<t}Q_j+kA(\omega) \ \text{ for } \ t\in(k\omega,(k+1)\omega]$$

and

$$f(t)=\int\limits_{k\omega}^t p(\tau)\,d\tau+\sum_{k\omega\le\tau_{kj}<t}q_j+\varphi(k)f(\omega) \ \text{ for } \ t\in(k\omega,(k+1)\omega] \quad (k=0;\pm1,\pm2,\dots),$$

where $\varphi(k)=0$ if k is an even integer, and $\varphi(k)=1$ if k is an odd one.

The theory of generalized ordinary differential equations has been introduced by J. Kurzweil [14,15] in connection with the investigation of the well-posed problem for the Cauchy problem for ordinary differential equations.

Finally, we note that, to a considerable extent, the interest to the theory of generalized ordinary differential equations has been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see [1, 2, 5, 7, 19] and the references therein).

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References

1. Sh. Akhalaia, M. Ashordia and N. Kekelia, On the necessary and sufficient conditions for the stability of linear generalized ordinary differential, linear impulsive and linear difference systems. *Georgian Math. J.* **16** (2009), no. 4, 597–616.
2. M. Ashordia, On the stability of solutions of the multipoint boundary value problem for the system of generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **6** (1995), 1–57, 134.
3. M. Ashordia, On the question of solvability of the periodic boundary value problem for a system of generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **11** (1997), 159–162.
4. M. Ashordia, On existence of solutions of the periodic boundary value problem for nonlinear system of generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **16** (1999), 150–153.

5. M. Ashordia, On the general and multipoint boundary value problems for linear systems of generalized ordinary differential equations, linear impulse and linear difference systems. *Mem. Differential Equations Math. Phys.* **36** (2005), 1–80.
6. M. Ashordia, On the two-point boundary value problems for linear impulsive systems with singularities. *Georgian Math. J.* **19** (2012), no. 1, 19–40.
7. M. Ashordia, Antiperiodic boundary value problem for systems of linear generalized differential equations. *Mem. Differ. Equ. Math. Phys.* **66** (2015), 141–152.
8. M. Ashordia and G. Ekhvaia, On the solvability of a multipoint boundary value problem for systems of nonlinear impulsive equations with finite and fixed points of impulses actions. *Mem. Differential Equations Math. Phys.* **43** (2008), 153–158.
9. M. Ashordia, G. Ekhvaia and N. Kekelia, On the solvability of general boundary value problems for systems of nonlinear impulsive equations with finite and fixed points of impulse actions. *Bound. Value Probl.* **2014**, 2014:157, 17 pp.
10. D. D. Baĭnov and P. S. Simeonov, Systems with impulse effect. Stability, theory and applications. *Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York*, 1989.
11. M. Benchohra, J. Henderson and S. Ntouyas, Impulsive differential equations and inclusions. *Contemporary Mathematics and Its Applications*, 2. *Hindawi Publishing Corporation, New York*, 2006.
12. I. Kiguradze, The initial value problem and boundary value problems for systems of ordinary differential equations. Vol. I. Linear theory. (Russian) *Metsniereba, Tbilisi*, 1997.
13. T. Kiguradze, Some boundary value problems for systems of linear partial differential equations of hyperbolic type. *Mem. Differential Equations Math. Phys.* **1** (1994), 144 pp.
14. J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter. (Russian) *Czechoslovak Math. J.* **7(82)** (1957), 418–449.
15. J. Kurzweil, Generalized ordinary differential equations. *Czechoslovak Math. J.* **8(83)** (1958), 360–388.
16. V. Lakshmikantham, D. D. Baĭnov and P. S. Simeonov, Theory of impulsive differential equations. *Series in Modern Applied Mathematics*, 6. *World Scientific Publishing Co., Inc., Teaneck, NJ*, 1989.
17. N. A. Perestyuk, V. A. Plotnikov, A. M. Samoilenko and N. V. Skripnik, Differential equations with impulse effects. Multivalued right-hand sides with discontinuities. *de Gruyter Studies in Mathematics*, 40. *Walter de Gruyter & Co., Berlin*, 2011.
18. A. M. Samoĭlenko and N. A. Perestyuk, Impulsive differential equations. *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*, 14. *World Scientific Publishing Co., Inc., River Edge, NJ*, 1995.
19. Š. Schwabik, M. Tvrdý and O. Vejvoda, Differential and integral equations. Boundary value problems and adjoints. *D. Reidel Publishing Co., Dordrecht–Boston, Mass.–London*, 1979.

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