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SOLVABILITY OF PERIODIC BOUNDARY VALUE PROBLEMS OF FRACTIONAL DIFFERENTIAL SYSTEMS WITH IMPULSE EFFECTS

Abstract. Two new classes of periodic boundary value problems of coupled impulsive fractional differential equations are proposed. Sufficient conditions are given for the existence of solutions of these problems. The analysis relies on the well known Schauder's fixed point theorem. The obtained results show that the Riemann–Liouville fractional derivative and the Caputo's fractional derivative have similar properties. Examples are given to illustrate the main theorems.

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რეზიუმე. შეუღლებული იმპულსური წილადწარმოებულიანი დიფერენციალური განტოლებებისთვის შემოტანილია პერიოდული სასაზღვრო ამოცანების ორი ახალი კლასი. მოცემულია ამ ამოცანების ამონახსნის არსებობის საკმარისი პირობები. ანალიზი ეყრდნობა შაუდერის ცნობილ თეორემას უძრავი წერტილის შესახებ. მიღებული შედეგები აჩვენებს, რომ რიმან-ლიუვილის და კაპუტოს წილად წარმოებულებს აქვს მსგავსი თვისებები. მთავარი თეორემების საილუსტრაციოდ მოყვანილია მაგალითები.

1 Introduction

The fractional derivatives serve an excellent tool for the description of hereditary properties of various materials and processes. Fractional differential equations arise naturally in many engineering and scientific disciplines such as physics, chemistry, biology, electrochemistry, electromagnetic, control theory, economics, signal and image processing, aerodynamics, and porous media. The boundary value problems for nonlinear fractional differential equations have been addressed by several researchers during the last decades. There have been many results obtained on the existence of solutions of boundary value problems for nonlinear fractional differential equations (see [7, 8, 33, 35, 36, 47, 55, 58]).

Applications of fractional order differential systems are in many fields, as for example, rheology, mechanics, chemistry, physics, bioengineering, robotics and many others (see [11]). Diethehm [12] proposed the model of the type (which is called a multi-order fractional differential system):

$$\begin{cases} {}^cD_{0+}^{n_i} y_i(t) = f_i\big(t, y_1(t), \dots, y_n(t)\big), & i = 1, 2, \dots, n, \\ y_j(0) = y_{j,0}, & j = 1, 2, \dots, n. \end{cases} \tag{1.1}$$

Here ${}^cD_{0+}^{*}$ is the standard Caputo's fractional derivative. This system contains many models as special cases, see Chen's fractional order system [51, 52] with a double scroll attractor, Genesio-Tesi fractional-order system [19], Lu's fractional order system [13], Volta's fractional-order system [38, 39], Rossler's fractional-order system [27] and so on. Other applications of fractional differential systems may be seen in Chapter 10 in [40].

In [16, 37, 49], the fractional order nonlinear dynamical model of interpersonal relationships

$$\begin{cases} D^\alpha x_1(t) + \alpha_1 x_1(t) = A_1 + \beta_1 x_2(t)\big(1 - \varepsilon x_2^2(t)\big), \\ D^\alpha x_2(t) + \alpha_2 x_2(t) = A_2 + \beta_2 x_1(t)\big(1 - \varepsilon x_1^2(t)\big), \end{cases} \tag{1.2}$$

was proposed, where $0 < \alpha \le 1$, $\alpha_i > 0$, β_i, A_i $(i = 1, 2)$, ε are the real constants. These parameters are oblivion, reaction, and attraction constants. The variables x_1 and x_2 are the measures of love of individuals and for their respective partners, where positive and negative measures represent feelings. In the equations in (1.2), we assume that feelings decay exponentially fast in the absence of partners. The parameters specify the romantic style of individuals 1 and 2. For instance, α_i describes the extent to which individual i is encouraged by his/her own feeling. In other words, α_i indicates the degree to which an individual has internalized a sense of his/her self-worth. In addition, it can be used as the level of anxiety and dependency on other person's approval in romantic relationships. The parameter β_i represents the extent to which individual i is encouraged by his/her partner, and/or expects his/her partner to be supportive. It measures the tendency to seek or avoid closeness in a romantic relationship. Therefore, the term $-\alpha_i x_i$ says that the love measure of i, in the absence of the partner, decays exponentially and α_i is the time required for love to decay (see [37]).

From the viewpoint of the theoretics and practice, it is natural for mathematicians to investigate the impulsive fractional differential equations. In recent years, many authors [1, 9, 15, 17, 20, 22, 25, 26, 28, 29, 34, 41, 46, 47, 54] studied the existence or uniqueness of solutions of impulsive initial or boundary value problems for fractional differential equations. For examples, impulsive anti-periodic boundary value problems (see [2–4, 43]), impulsive periodic boundary value problems (see [44]), impulsive initial value problems (see [10, 14, 31, 50]), two-point, three-point or multi-point impulsive boundary value problems (see [5, 45, 57]), impulsive boundary value problems on infinite intervals (see [56]). However, there has been no papers concerned with the solvability of periodic boundary value problems of impulsive fractional differential systems.

In [9], the authors have studied the solvability of the following periodic boundary value problem:

$$\begin{cases} D_{t_i^+}^\alpha x(t) - \lambda x(t) = p(t) f(t, x(t)), \quad t \in (t_i, t_{i+1}), \;\; i = 0, 1, \dots, m, \\ x(1) - \lim\limits_{t \to 0} t^{1-\alpha} x(t) = 0, \\ \lim\limits_{t \to t_i^+} (t - t_i)^{1-\alpha} \big[x(t) - x(t_i)\big] = I_i(x(t_i)), \;\; i = 1, 2, \dots, m, \end{cases}$$

where $\alpha \in (0,1)$, $0 = t_0 < t_1 < \dots < t_{m+1} = 1$, $D^{\alpha}_{t_i^+}$ is the standard Riemann–Liouville fractional derivative, $I_i : R \to R$ is continuous, $\lambda \neq 0$, f is continuous at every points in $(t_i, t_{i+1}] \times R$ and for every function $v \in C^0(t_i, t_{i+1}]$ the limit $\lim_{t \to t_i^+} v(t)$ exists (finite), then $\lim_{t \to t_i^+} f(t, (t-t_i)^{\alpha-1} v(t))$ exists (finite).

In [24], Liu studied the existence of solutions of the following periodic type boundary value problem of nonlinear singular fractional differential equation

$$\begin{cases} D^{\beta}_{0^+}\left[\Phi(\rho(t) D^{\alpha}_{0^+} u(t))\right] = q(t) f\big(t, u(t), D^{\alpha}_{0^+} u(t)\big), \quad t \in (0,1), \\ \lim\limits_{t \to 1} t^{1-\alpha} u(t) - \lim\limits_{t \to 0} t^{1-\alpha} u(t) = \int\limits_0^1 G\big(s, u(s), D^{\alpha}_{0^+} u(s)\big)\, ds, \\ \lim\limits_{t \to 1} t^{1-\beta} \Phi\big(\rho(t) D^{\alpha}_{0^+} u(t)\big) - \lim\limits_{t \to 0} t^{1-\beta} \Phi\big(\rho(t) D^{\alpha}_{0^+} u(t)\big) = \int\limits_0^1 H(s, u(s), D^{\alpha}_{0^+} u(s))\, ds, \\ \lim\limits_{t \to t_1^+} u(t) = I\big(t_1, u(t_1), D^{\alpha}_{0^+} u(t_1)\big), \\ \lim\limits_{t \to t_1^+} \Phi\big(\rho(t) D^{\alpha}_{0^+} u(t)\big) = J\big(t_1, u(t_1), D^{\alpha}_{0^+} u(t_1)\big). \end{cases}$$

where $0 < \alpha, \beta \le 1$, $D^{\alpha}_{0^+}$ (or $D^{\beta}_{0^+}$) is the Riemann–Liouville fractional derivative of order α (or β), $\Phi : R \to R$ is a sup-multiplicative-like function with supporting function ω, its inverse function is denoted by $\Phi^{-1} : R \to R$ with supporting function ν, $0 < t_1 < 1$, $I, J : (0,1) \times R^2 \to R$ are continuous functions, $\phi, \psi : (0,1) \to R$ with $\phi|_{(0,t_1]}, \rho|_{(0,t_1]} \in L^1(0,t_1)$ and $\phi|_{(t_1,1]}, \rho|_{(t_1,1]} \in L^1(t_1,1)$, $\rho : (0,1) \to [0,+\infty)$ with $\rho|_{(0,t_1]} \in C^0(0,t_1]$ and $\rho|_{(t_1,1]} \in C^0(t_1,1)$ satisfies that there exist numbers $L > 0$ and $k > -\alpha$ such that $\rho(t) \ge \frac{t^{-k}\nu(t^{\beta-1})}{L}$ for all $t \in (0,1)$, $t \neq t_1$, $q : (0,1) \to R$ with $q|_{(0,t_1]} \in C^0(0,t_1]$ and $q|_{(t_1,1]} \in C^0(t_1,1)$ and there exist numbers $L_1 > 0$ and $k_1 > -\beta$ such that $|q(t)| \le L_1 t^{k_1}$ for all $t \in (0,1)$, f, G, H defined on $(0,t_1) \bigcup (t_1,1) \times R \times R$ are **impulsive Carathéodory functions** that may be singular at $t = 0$, t_1 and 1.

One knows that both of the fractional derivatives (the Riemann–Liouville fractional derivative and the Caputo's fractional derivative) are actually nonlocal operators because integrals are nonlocal operators. Moreover, calculating time fractional derivatives of a function at some time requires all the past history and hence fractional derivatives can be used for modeling systems with memory. In [9], the fractional derivative has a variable base points t_i $(i = 0,1,2,\dots,m)$. This action may short the memory time. However, in applications, fractional differential equation involves fractional derivative that has a constant base point.

In this paper, we discuss the following impulsive periodic boundary value problems of singular fractional differential systems with a constant base point $t = 0$:

$$\begin{cases} D^{\alpha_1}_{0^+} x(t) - \lambda_1 x(t) = p_1(t) f_1\big(t, x(t), y(t)\big), \quad t \in (t_i, t_{i+1}), \quad i \in N_0, \\ D^{\alpha_2}_{0^+} y(t) - \lambda_2 y(t) = p_2(t) f_2\big(t, x(t), y(t)\big), \quad t \in (t_i, t_{i+1}), \quad i \in N_0, \\ x(1) - \lim\limits_{t \to 0} t^{1-\alpha_1} x(t) = 0, \\ y(1) - \lim\limits_{t \to 0} t^{1-\alpha_2} y(t) = 0, \\ \lim\limits_{t \to t_i^+} (t-t_i)^{1-\alpha_1} x(t) = I\big(t_i, x(t_i), y(t_i)\big), \quad i \in N, \\ \lim\limits_{t \to t_i^+} (t-t_i)^{1-\alpha_2} y(t) = J\big(t_i, x(t_i), y(t_i)\big), \quad i \in N, \end{cases} \tag{1.3}$$

and

$$\begin{cases} {}^cD_{0^+}^{\alpha_1}x(t)-\lambda_1x(t)=p_3(t)f_3\big(t,x(t),y(t)\big), & t\in(t_i,t_{i+1}),\ \ i\in N_0,\\ {}^cD_{0^+}^{\alpha_2}y(t)-\lambda_2y(t)=p_4(t)f_4\big(t,x(t),y(t)\big), & t\in(t_i,t_{i+1})),\ \ i\in N_0,\\ x(1)-\lim\limits_{t\to0}x(t)=0,\\ y(1)-\lim\limits_{t\to0}y(t)=0,\\ \lim\limits_{t\to t_i^+}x(t)=I\big(t_i,x(t_i),y(t_i)\big), & i\in N,\\ \lim\limits_{t\to t_i^+}y(t)=J\big(t_i,x(t_i),y(t_i)\big), & i\in N, \end{cases} \tag{1.4}$$

where

(a) $0<\alpha_1,\ \alpha_2<1$, $\lambda_1,\lambda_2\in R$, D^* is the standard Riemann–Liouville fractional derivative of order $*>0$, ${}^cD^*$ is the standard Caputo's fractional derivative of order $*>0$;

(b) m is a positive integer, $0=t_0<t_1<t_2<\cdots<t_{m-1}<t_m<t_{m+1}=1$, $N_0=\{0,1,2,\dots,m\}$ and $N=\{1,2,\dots,m\}$;

(c) $p_1,\ p_2$ are continuous on $(0,1)$ and $p_1,p_2\in L^1(0,1)$ and there exist constants $k_j>-1$, $l_j\in(-\alpha_j,0]$ with $1+k_j+l_j>0$ $(j=1,2)$ such that $|p_j(t)|\le t^{k_j}(1-t)^{l_j}$ for all $t\in(0,1)$, $j=1,2$;

(c1) p_3,p_4 are continuous on $(0,1)$ and $p_1,p_2\in L^1(0,1)$ and there exist constants $k_j>-1$, $l_j\in(-\alpha_j,0]$ with $\alpha_j+k_j+l_j>0$ $(j=1,2)$ such that $|p_j(t)|\le t^{k_j}(1-t)^{l_j}$ for all $t\in(0,1)$, $j=3,4$;

(d) f_1,f_2 defined on $\bigcup\limits_{i=0}^{m}(t_i,t_{i+1})\times R^2$ are **I-Carathéodory functions** (see the definition in Section 2), $I,J:\{t_i:\ i\in N\}\times R^2\to R$ are **discrete I-Carathéodory functions**;

(d1) $f_3,\ f_4$ defined on $\bigcup\limits_{i=0}^{m}(t_i,t_{i+1})\times R^2$ are **II-Carathéodory functions** (see the definition in Section 2), $I,J:\{t_i:\ i\in N\}\times R^2\to R$ are **discrete II-Carathéodory functions**.

A pair of functions $x,y:(0,1]\to R$ is called a solution of BVP (1.3) if

$$x\big|_{(t_i,t_{i+1}]}\in C^0(t_i,t_{i+1}],\quad y\big|_{(t_i,t_{i+1}]}\in C^0(t_i,t_{i+1}],\ \ i\in N_0, \tag{1.5}$$

and the limits

$$\lim_{t\to t_i^+}(t-t_i)^{1-\alpha_1}x(t),\quad \lim_{t\to t_i^+}(t-t_i)^{1-\alpha_2}y(t),\ \ i\in N_0,$$

exist and x, y satisfy all equations in (1.3).

A pair of functions $x,y:(0,1]\to R$ is called a solution of BVP (1.4) if the limits

$$\lim_{t\to t_i^+}x(t),\quad \lim_{t\to t_i^+}y(t),\ \ i\in N_0,$$

exist and x, y satisfy all equations in (1.4).

To the best of the authors knowledge, no one has studied the existence of solutions for BVPs (1.3) and (1.4). We obtain results on the existence of at least one solution for BVPs (1.3) and (1.4), respectively. Two examples are given to illustrate the efficiency of the main theorems.

The remainder of this paper is organized as follows: in Section 2, we present preliminary results. In Sections 3 and 4, the existence theorems and their proofs on BVPs (1.3) and (1.4) are given, respectively. Finally, we present examples to show the applications of the main theorems.

2 Preliminaries

For the convenience of the readers, we firstly present the necessary definitions from the fractional calculus theory. These definitions and results can be found in [23, 40].

Let the Gamma function, Beta function and two classical Mittag–Leffler special functions be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}\,dx, \quad \mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx,$$

$$E_{\delta,\delta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\delta k+\delta)}, \quad E_{\delta,1}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\delta k+1)},$$

respectively, for $\alpha > 0$, $p > 0$, $q > 0$, $\delta > 0$. We note that $E_{\delta,\delta}(x) > 0$ for all $x \in R$ and $E_{\delta,\delta}(x)$ is strictly increasing in x. Then for $x > 0$, we have

$$E_{\delta,\delta}(-x) < E_{\delta,\delta}(0) = \frac{1}{\Gamma(\delta)} < E_{\delta,\delta}(x).$$

Definition 2.1 ([23]). Let $c \in R$. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $g : (c,\infty) \to R$ is given by

$$I_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)}\int_c^t (t-s)^{\alpha-1}g(s)\,ds,$$

provided that the right-hand side exists.

Definition 2.2 ([23]). Let $c \in R$. The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (c,\infty) \to R$ is given by

$$D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_c^t \frac{g(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

where $\alpha < n \le \alpha+1$, i.e., $n = \lceil\alpha\rceil$, provided that the right-hand side exists.

Lemma 2.1 ([23]). *Let $\alpha < n \le \alpha + 1$, $u \in C^0(c,\infty) \cap L^1(c,\infty)$. Then*

$$I_{c^+}^{\alpha} D_{c^+}^{\alpha} u(t) = u(t) + C_1(t-c)^{\alpha-1} + C_2(t-c)^{\alpha-2} + \cdots + C_n(t-c)^{\alpha-n},$$

where $C_i \in R$, $i = 1, 2, \ldots, n$.

We use the function space

$$X = \Big\{x : \ (0,1] \to R : \ x\big|_{(t_i,t_{i+1}]} \in C^0(t_i,t_{i+1}], \ i \in N_0, \\ \text{there exist the limits } \lim_{t\to t_i^+}(t-t_i)^{1-\alpha_1}x(t), \ i \in N_0\Big\}.$$

Define

$$\|x\| = \|x\|_X = \max\Big\{\sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{1-\alpha_1}|x(t)| : \ i \in N_0\Big\}.$$

Lemma 2.2. *X is a Banach space with the norm $\|\cdot\|$ defined.*

Proof. In fact, it is easy to see that X is a normed linear space with the norm $\|\cdot\|$. Let $\{x_u\}$ be a Cauchy sequence in X. Then $\|x_u - x_v\| \to 0$, $u, v \to +\infty$. It follows that

$$\sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{1-\alpha_1}\big|x_u(t) - x_v(t)\big| \longrightarrow 0, \quad v, u \to +\infty, \quad i \in N_0.$$

Define $x_i = x\big|_{(t_i,t_{i+1}]}$ and

$$(t-t_i)^{1-\alpha_1}\overline{x}_i(t) = \begin{cases} \lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha_1}x(t), & t=t_i, \\ (t-t_i)^{1-\alpha_1}x(t), & t\in(t_i,t_{i+1}]. \end{cases}$$

We know that $t \to (t-t_i)^{1-\alpha_1}\overline{x}(t)$ is continuous on $[t_i,t_{i+1}]$. Thus $t\to(t-t_i)^{1-\alpha_1}\overline{x}_{u,i}(t)$ is a Cauchy sequence in $C[t_i,t_{i+1}]$. Then $(t-t_i)^{1-\alpha_1}\overline{x}_{u,i}(t)$ uniformly converges to some $x_{0,i}$ in $C[t_i,t_{i+1}]$ as $u\to+\infty$. It follows that

$$\sup_{t\in[t_i,t_{i+1}]}\left|(t-t_i)^{1-\alpha_1}\overline{x}_{u,i}(t)-x_{0,i}\right| \longrightarrow 0,\ \ u\to+\infty,\ \ i\in N_0.$$

That is,

$$\sup_{t\in[t_i,t_{i+1}]}(t-t_i)^{1-\alpha_1}\left|\overline{x}_{u,i}(t)-(t-t_i)^{\alpha_1-1}x_{0,i}\right| \longrightarrow 0,\ \ u\to+\infty,\ \ i\in N_0.$$

Let $x_0(t)=(t-t_i)^{\alpha_1-1}x_{0,i}(t)$ for $t\in(t_i,t_{i+1}]$, $i\in N_0$. It is easy to see that $x_0\in X$ and $x_u\to x_0$ as $u\to+\infty$ in X. It follows that X is a Banach space. The proof is complete. □

Define

$$Y=\Big\{y:\ (0,1]\to R:\ y\big|_{(t_i,t_{i+1}]}\in C^0(t_i,t_{i+1}],\ i\in N_0,\ \text{there exist the limits } \lim_{t\to t_i^+}(t-t_i)^{1-\alpha_2}y(t),\ i\in N_0,\ \Big\}$$

with the norm

$$\|y\|=\|y\|_Y=\max\Big\{\sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{1-\alpha_2}|y(t)|:\ i\in N_0\Big\}.$$

Then Y is a Banach space. Choose $E=X\times Y$ with the norm $\|(x,y)\|=\max\{\|x\|_X,\|y\|_Y\}$. Then E is a Banach space. We will seek for solutions of BVP (1.3) in E.

Definition 2.3. We call $F:\bigcup\limits_{i=0}^{m}(t_i,t_{i+1})\times R^2\to R$ an **I-Carathéodory function** if it satisfies the following conditions:

(i) $t\to F(t,(t-t_i)^{\alpha_1-1}u,(t-t_i)^{\alpha_2-1}v)$ are measurable on (t_i,t_{i+1}), $i\in N_0$ for any $(u,v)\in R^2$;

(ii) $(u,v)\to F\left(t,(t-t_i)^{\alpha_1-1}u,(t-t_i)^{\alpha_2-1}v\right)$ are continuous on R^2 for all $t\in(t_i,t_{i+1})$, $i\in N_0$;

(iii) for each $r>0$, there exists $M_r\ge 0$ such that $|F(t,(t-t_i)^{\alpha_1-1}u,(t-t_i)^{\alpha_2-1}v)|\le M_r$ for all $t\in(t_i,t_{i+1})$, $i\in N_0$ and $|u|,|v|\le r$.

We call $G:\{t_i:\ \ i\in N\}\times R^2\to R$ a **discrete I-Carathéodory function** if it satisfies the following conditions:

(i) $(u,v)\to G(t_i,(t_i-t_{i-1})^{\alpha_1-1}u,(t_i-t_{i-1})^{\alpha_2-1}v)$, $i\in N$ are continuous on R^2;

(ii) for each $r>0$, there exists $M_r\ge0$ such that $|G(t_i,(t_i-t_{i-1})^{\alpha_1-1}u,(t_i-t_{i-1})^{\alpha_2-1}v)|\le M_r$ for almost all $i\in N$ and $|u|,|v|\le r$.

Definition 2.4. We call $F:\bigcup\limits_{i=0}^{m}(t_i,t_{i+1})\times R^2\to R$ a **II-Carathéodory function** if it satisfies the following conditions:

(i) $t\to F(t,u,v)$ are measurable on (t_i,t_{i+1}), $i\in N_0$ for any $(u,v)\in R^2$;

(ii) $(u,v)\to F(t,u,v)$ are continuous on R^2 for all $t\in(t_i,t_{i+1})$, $i\in N_0$;

(iii) for each $r>0$, there exists $M_r\ge0$ such that $|F(t,u,v)|\le M_r$ for all $t\in(t_i,t_{i+1})$, $i\in N_0$ and $|u|,|v|\le r$.

We call $G : \{t_i : \ i \in N\} \times R^2 \to R$ a **discrete II-Carathéodory function** if it satisfies the following conditions:

(i) $(u,v) \to G(t_i,u,v)$, $i \in N$ are continuous on R^2;

(ii) for each $r > 0$, there exists $M_r \geq 0$ such that $|G(t_i,u,v)| \leq M_r$ for almost all $i \in N$ and $|u|,|v| \leq r$.

We also use the function space

$$X_1 = \Big\{x : \ (0,1] \to R : \ x\big|_{(t_i,t_{i+1}]} \in C^0(t_i,t_{i+1}], \ i \in N_0, \text{ there exist the limits } \lim_{t \to t_i^+} x(t), \ i \in N_0\Big\}.$$

Define

$$\|x\| = \|x\|_{X_1} = \max\Big\{\sup_{t \in (t_i,t_{i+1}]} |x(t)| : \ i \in N_0\Big\}.$$

Then X_1 is the Banach space with the norm $\|\cdot\|_{X_1}$ defined. Choose $E_1 = X_1 \times X_1$ with the norm $\|(x,y)\| = \max\{\|x\|_{X_1}, \|y\|_{X_1}\}$. Then E_1 is a Banach space. We will seek for solutions of BVP (1.4) in E_1.

To ease expression, we denote $\delta_{\alpha,\lambda}(t,t_i) = (t-t_i)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-t_i)^\alpha)$ for $t \in (t_i,t_{i+1}]$ and $\alpha \in (0,1]$ and $\lambda \in R$.

3 Solvability of BVP (1.3)

In this section, we study the solvability of BVP (1.3) by seeking solutions in the Banach space E.

Lemma 3.1. *Suppose that $\sigma \in L^1(0,1)$ and there exist numbers $k_1 > -1$ and $\max\{-\alpha_1, -k_1-1\} < l_1 \leq 0$ such that $|\sigma(t)| \leq t^{k_1}(1-t)^{l_1}$ for all $t \in (0,1)$. Then $x \in X$ is a solution of*

$$D_{0^+}^{\alpha_1} x(t) - \lambda_1 x(t) = \sigma(t), \ \ t \in (t_i,t_{i+1}), \ \ i \in N_0, \tag{3.1}$$

if and only if there exist constants A_i $(i \in N_0)$ such that

$$x(t) = \Gamma(\alpha_1)\sum_{j=0}^{i} A_j \delta_{\alpha_1,\lambda_1}(t,t_j) + \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds, \ \ t \in (t_i,t_{i+1}], \ \ i \in N_0. \tag{3.2}$$

Proof. We do two steps:

Step 1. Suppose that $x \in X$ is a solution of (3.1). By (3.26) in [8], we know that there exist numbers A_0 such that

$$x(t) = \Gamma(\alpha_1)A_0\delta_{\alpha_1,\lambda_1}(t,0) + \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds, \ \ t \in (0,t_1]. \tag{3.3}$$

We know that (3.2) holds when $i = 0$. Now suppose that (3.2) holds for $i = 0,1,2,\dots,n$ $(n \leq m-1)$. We will prove that (3.2) holds for $i = n+1$. Suppose that

$$x(t) = \Phi(t) + \Gamma(\alpha_1)\sum_{j=0}^{n} A_j \delta_{\alpha,\lambda_1}(t,t_j) + \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds, \ \ t \in (t_{n+1},t_{n+2}]. \tag{3.4}$$

It is easy to check that for $t \in (t_{n+1},t_{n+2}]$

$$\int_0^t \frac{x(s)}{(t-s)^{\alpha_1}}\,ds = \sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} \frac{x(s)}{(t-s)^{\alpha_1}}\,ds + \int_{t_{n+1}}^{t} \frac{x(s)}{(t-s)^{\alpha_1}}\,ds$$

$$= \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{\sum_{u=0}^{j} A_u \Gamma(\alpha_1)(s-t_u)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-t_u)^{\alpha_1}) + \int_0^s (s-v)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\,dv}{(t-s)^{\alpha_1}}\,ds$$

$$+ \int_{t_{n+1}}^{t} \frac{\Phi(s) + \sum_{j=0}^{n} A_j \Gamma(\alpha_1)(s-t_j)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-t_j)^{\alpha_1}) + \int_0^s (s-v)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\,dv}{(t-s)^{\alpha_1}}\,ds$$

$$= \int_{t_{n+1}}^{t} \frac{\Phi(s)}{(t-s)^{\alpha_1}}\,ds + \sum_{j=0}^{n}\sum_{u=0}^{j} A_u \Gamma(\alpha_1) \int_{t_j}^{t_{j+1}} \frac{(s-t_u)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-t_u)^{\alpha_1})}{(t-s)^{\alpha_1}}\,ds$$

$$+ \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \int_0^s \frac{(s-v)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)dv}{(t-s)^{\alpha_1}}\,ds$$

$$+ \sum_{j=0}^{n} A_j \Gamma(\alpha_1) \int_{t_{n+1}}^{t} \frac{(s-t_j)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-t_j)^{\alpha_1})}{(t-s)^{\alpha_1}}\,ds$$

$$+ \int_{t_{n+1}}^{t} \int_0^s \frac{(s-v)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\,dv}{(t-s)^{\alpha_1}}\,ds$$

$$= \int_{t_{n+1}}^{t} \frac{\Phi(s)}{(t-s)^{\alpha}}\,ds + \sum_{j=0}^{n}\sum_{u=0}^{j} A_u \Gamma(\alpha_1) \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1}(s-t_u)^{\alpha_1-1} \sum_{w=0}^{+\infty} \frac{\lambda_1^w (s-t_u)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\,ds$$

$$+ \sum_{j=0}^{n} A_j \Gamma(\alpha_1) \int_{t_{n+1}}^{t} (t-s)^{-\alpha_1}(s-t_j)^{\alpha_1-1} \sum_{w=0}^{+\infty} \frac{\lambda_1^w (s-t_j)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\,ds$$

$$+ \sum_{j=0}^{n} \int_0^{t_j} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1}(s-v)^{\alpha_1-1} \sum_{w=0}^{+\infty} \frac{\lambda_1^w (s-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\,ds\,\sigma(v)\,dv$$

$$+ \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \int_{v}^{t_{j+1}} (t-s)^{-\alpha_1}(s-v)^{\alpha_1-1} \sum_{w=0}^{+\infty} \frac{\lambda_1^w (s-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\,ds\sigma(v)\,dv$$

$$+ \int_0^{t_{n+1}} \int_{t_{n+1}}^{t} (t-s)^{-\alpha_1}(s-v)^{\alpha_1-1} \sum_{w=0}^{+\infty} \frac{\lambda_1^w (s-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\,ds\,\sigma(v)\,dv$$

$$+ \int_{t_{n+1}}^{t} \int_{v}^{t} (t-s)^{-\alpha_1}(s-v)^{\alpha_1-1} \sum_{w=0}^{+\infty} \frac{\lambda_1^w (s-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\,ds\,\sigma(v)\,dv$$

$$= \int_{t_{n+1}}^{t} \frac{\Phi(s)}{(t-s)^{\alpha}}\,ds + \sum_{j=0}^{n}\sum_{u=0}^{j} A_u \Gamma(\alpha_1) \sum_{w=0}^{+\infty} \frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1}(s-t_u)^{\alpha_1 w+\alpha_1-1}\,ds$$

$$+ \sum_{j=0}^{n} A_j \Gamma(\alpha_1) \sum_{w=0}^{+\infty} \frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))} \int_{t_{n+1}}^{t} (t-s)^{-\alpha_1}(s-t_j)^{\alpha_1 w+\alpha_1-1}\,ds$$

$$+ \sum_{j=0}^{n} \int_0^{t_j} \sum_{w=0}^{+\infty} \frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1}(s-v)^{\alpha_1 w+\alpha_1-1}\,ds\,\sigma(v)\,dv$$

$$+\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\sum_{w=0}^{+\infty}\frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))}\int_{v}^{t_{j+1}}(t-s)^{-\alpha_1}(s-v)^{\alpha_1 w+\alpha_1-1}\,ds\,\sigma(v)\,dv$$

$$+\int_{0}^{t_{n+1}}\sum_{w=0}^{+\infty}\frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))}\int_{t_{n+1}}^{t}(t-s)^{-\alpha_1}(s-v)^{\alpha_1 w+\alpha_1-1}\,ds\,\sigma(v)\,dv$$

$$+\int_{t_{n+1}}^{t}\sum_{w=0}^{+\infty}\frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))}\int_{v}^{t}(t-s)^{-\alpha_1}(s-v)^{\alpha_1 w+\alpha_1-1}\,ds\,\sigma(v)\,dv$$

$$=\int_{t_{n+1}}^{t}\frac{\Phi(s)}{(t-s)^{\alpha}}\,ds+\sum_{j=0}^{n}\sum_{u=0}^{j}A_u\Gamma(\alpha_1)\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-t_u)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_j-t_u}{t-t_u}}^{\frac{t_{j+1}-t_u}{t-t_u}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega$$

$$+\sum_{j=0}^{n}A_j\Gamma(\alpha_1)\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-t_j)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_{n+1}-t_j}{t-t_j}}^{1}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega$$

$$+\sum_{j=0}^{n}\int_{0}^{t_j}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_j-v}{t-v}}^{\frac{t_{j+1}-v}{t-v}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{0}^{\frac{t_{j+1}-v}{t-v}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\int_{0}^{t_{n+1}}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_{n+1}-v}{t-v}}^{1}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\int_{t_{n+1}}^{t}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{0}^{1}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$=\int_{t_{n+1}}^{t}\frac{\Phi(s)}{(t-s)^{\alpha}}\,ds+\sum_{u=0}^{n}A_u\Gamma(\alpha_1)\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-t_u)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{0}^{\frac{t_{n+1}-t_u}{t-t_u}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega$$

$$+\sum_{j=0}^{n}A_j\Gamma(\alpha_1)\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-t_j)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_{n+1}-t_j}{t-t_j}}^{1}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega$$

$$+\sum_{j=1}^{n}\int_{0}^{t_j}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_j-v}{t-v}}^{\frac{t_{j+1}-v}{t-v}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{0}^{\frac{t_{j+1}-v}{t-v}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\int_0^{t_{n+1}}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_{n+1}-v}{t-v}}^{1}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\int_{t_{n+1}}^{t}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_0^1(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$=\int_{t_{n+1}}^{t}\frac{\Phi(s)}{(t-s)^{\alpha}}\,ds+\sum_{u=0}^{n}A_u\Gamma(\alpha_1)\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-t_u)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_0^1(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega$$

$$+\sum_{j=1}^{n}\int_0^{t_j}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_j-v}{t-v}}^{\frac{t_{j+1}-v}{t-v}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_0^{\frac{t_{j+1}-v}{t-v}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\int_0^{t_{n+1}}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_{n+1}-v}{t-v}}^{1}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\int_{t_{n+1}}^{t}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_0^1(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$=\int_{t_{n+1}}^{t}\frac{\Phi(s)}{(t-s)^{\alpha}}\,ds+\sum_{u=0}^{n}A_u\Gamma(\alpha_1)\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-t_u)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_0^1(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega$$

$$+\int_0^{t}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_0^1(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv.$$

One can see that for $t\in(t_{n+1},t_{n+2}]$

$$\frac{1}{\Gamma(1-\alpha_1)}\left(\int_0^t\frac{x(s)}{(t-s)^{\alpha}}\,ds\right)'=\frac{1}{\Gamma(1-\alpha_1)}\left(\int_{t_{n+1}}^{t}\frac{\Phi(s)}{(t-s)^{\alpha}}\,ds\right)'$$

$$+\frac{1}{\Gamma(1-\alpha_1)}\sum_{u=0}^{n}A_u\Gamma(\alpha_1)\sum_{w=1}^{+\infty}\frac{\lambda_1^w\alpha_1 w(t-t_u)^{\alpha_1 w-1}}{\Gamma(\alpha_1(w+1))}\int_0^1(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega$$

$$+\frac{1}{\Gamma(1-\alpha_1)}\int_0^t\sum_{w=1}^{+\infty}\frac{\lambda_1^w\alpha_1 w(t-v)^{\alpha_1 w-1}}{\Gamma(\alpha_1(w+1))}\int_0^1(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\frac{1}{\Gamma(1-\alpha_1)}\frac{1}{\Gamma(\alpha_1)}\int_0^1(1-\omega)^{-\alpha_1}\omega^{\alpha_1-1}\,d\omega\,\sigma(t)$$

$$=\sigma(t)+D_{t_{n+1}^+}^{\alpha_1}\Phi(t)+\frac{1}{\Gamma(1-\alpha_1)}\sum_{u=0}^{n}A_u\Gamma(\alpha_1)\sum_{w=1}^{+\infty}\frac{\lambda_1^w\alpha_1 w(t-t_u)^{\alpha_1 w-1}}{\Gamma(\alpha_1(w+1))}\,\mathbf{B}(1-\alpha_1,\alpha_1 w+\alpha_1)$$

$$+\frac{1}{\Gamma(1-\alpha_1)}\int\limits_0^t\sum_{w=1}^{+\infty}\frac{\lambda_1^w\alpha_1 w(t-v)^{\alpha_1 w-1}}{\Gamma(\alpha_1(w+1))}\mathbf{B}(1-\alpha_1,\alpha_1 w+\alpha_1)\sigma(v)\,dv$$

$$=\sigma(t)+D_{t_{n+1}^+}^{\alpha_1}\Phi(t)+\sum_{u=0}^{n}A_u\Gamma(\alpha_1)\sum_{w=1}^{+\infty}\frac{\lambda_1^w\alpha_1 w(t-t_u)^{\alpha_1 w-1}}{\Gamma(\alpha_1 w+1))}+\int\limits_0^t\sum_{w=1}^{+\infty}\frac{\lambda_1^w\alpha_1 w(t-v)^{\alpha_1 w-1}}{\Gamma(\alpha_1 w+1))}\sigma(v)\,dv$$

$$=\sigma(t)+D_{t_{n+1}^+}^{\alpha_1}\Phi(t)+\lambda_1\sum_{j=0}^{n}A_j\Gamma(\alpha_1)(t-t_j)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-t_j)^{\alpha_1})$$

$$+\lambda_1\int\limits_0^t(t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\,ds.$$

So,

$$\sigma(t)=D_{0^+}^{\alpha_1}x(t)-\lambda_1 x(t)=\frac{1}{\Gamma(1-\alpha_1)}\left(\int\limits_0^t\frac{x(s)}{(t-s)^{\alpha}}\,ds\right)'$$

$$-\lambda_1\Bigg(\Phi(t)+\sum_{j=0}^{n}A_j\Gamma(\alpha_1)(t-t_j)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-t_j)^{\alpha_1})$$

$$+\int\limits_0^t(t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\,ds\Bigg)$$

$$=\sigma(t)+D_{t_{n+1}^+}^{\alpha_1}\Phi(t)-\lambda_1\Phi(t).$$

It follows that $D_{t_{n+1}^+}^{\alpha_1}\Phi(t)-\lambda_1\Phi(t)=0$ for $t\in(t_{n+1},t_{n+2}]$. Hence there exists a constant $A_{n+1}\in R$ such that $\Phi(t)=A_{n+1}\Gamma(\alpha_1)(t-t_{n+1})^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-t_{n+1})^{\alpha_1})$. Substituting Φ into (9), we know that (3.2) holds for $i=n+1$. Thus, due to induction, (3.2) holds for all $i\in N_0$.

Step 2. Now suppose that x satisfies (3.2). We prove that $x\in X$ and x satisfies (3.1).

Firstly, we prove that $x\in X$. In fact, we have for $0<t_1<t_2\le 1$ that

$$\left|\int\limits_0^{t_2}(t_2-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t_2-s)^{\alpha_1})\sigma(s)\,ds-\int\limits_0^{t_1}(t_1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t_1-s)^{\alpha_1})\sigma(s)\,ds\right|$$

$$\le\int\limits_{t_1}^{t_2}(t_2-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t_2-s)^{\alpha_1})s^{k_1}(1-s)^{l_1}\,ds$$

$$+\int\limits_0^{t_1}\left|(t_2-s)^{\alpha_1-1}-(t_1-s)^{\alpha_1-1}\right|E_{\alpha_1,\alpha_1}(\lambda_1(t_2-s)^{\alpha_1})s^{k_1}(1-s)^{l_1}\,ds$$

$$+\int\limits_0^{t_1}(t_1-s)^{\alpha_1-1}\left|E_{\alpha_1,\alpha_1}(\lambda_1(t_2-s)^{\alpha_1})-E_{\alpha_1,\alpha_1}(\lambda_1(t_1-s)^{\alpha_1})\right|s^{k_1}(1-s)^{l_1}\,ds$$

$$\le\sum_{w=0}^{+\infty}\frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))}\int\limits_{t_1}^{t_2}(t_2-s)^{\alpha w+\alpha_1+l_1-1}s^{k_1}\,ds$$

$$+\sum_{w=0}^{+\infty}\frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))}\int\limits_0^{t_1}\left[(t_1-s)^{\alpha_1-1}-(t_2-s)^{\alpha_1-1}\right](t_2-s)^{\alpha_1 w}s^{k_1}(1-s)^{l_1}\,ds$$

$$+\int_0^{t_1}(t_1-s)^{\alpha_1-1}\big|E_{\alpha_1,\alpha_1}(\lambda_1(t_2-s)^{\alpha_1})-E_{\alpha_1,\alpha_1}(\lambda_1(t_1-s)^{\alpha_1})\big|s^{k_1}(1-s)^{l_1}\,ds$$

$$=: M_1+M_2+M_3.$$

We see that the first term M_1 satisfies

$$M_1=\sum_{w=0}^{+\infty}\frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))}\,t_2^{\alpha_1 w+\alpha_1+k_1+l_1}\int_{\frac{t_1}{t_2}}^{1}(1-\omega)^{\alpha w+\alpha_1+l_1-1}\omega^{k_1}\,d\omega$$

$$\le E_{\alpha_1,\alpha_1}(\lambda_1 t_2^{\alpha_1})\int_{\frac{t_1}{t_2}}^{1}(1-\omega)^{\alpha_1+l_1-1}\omega^{k_1}\,d\omega\longrightarrow 0\,'\ \text{as}\ \ t_1\to t_2.$$

The second term M_2 satisfies

$$M_2\le\sum_{w=0}^{+\infty}\frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))}\int_0^{t_1}\big[(t_1-s)^{\alpha_1-1}-(t_2-s)^{\alpha_1-1}\big](1-s)^{l_1}s^{k_1}\,ds$$

$$\le\sum_{w=0}^{+\infty}\frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))}\int_0^{t_1}(t_1-s)^{\alpha_1-1}(t_2-s)^{l_1}s^{k_1}\,ds-\sum_{w=0}^{+\infty}\frac{\lambda^w}{\Gamma(\alpha_1(w+1))}\int_0^{t_1}(t_2-s)^{\alpha_1+l_1-1}s^{k_1}\,ds$$

$$\le\sum_{w=0}^{+\infty}\frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))}\int_0^{t_1}(t_1-s)^{\alpha_1+l_1-1}s^{k_1}\,ds-\sum_{w=0}^{+\infty}\frac{\lambda^w}{\Gamma(\alpha_1(w+1))}\int_0^{t_1}(t_2-s)^{\alpha_1+l_1-1}s^{k_1}\,ds$$

$$=\sum_{w=0}^{+\infty}\frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))}\,t_1^{\alpha_1+k_1+l_1}\int_0^{1}(1-\omega)^{\alpha_1+l_1-1}\omega^{k_1}\,d\omega$$

$$-\sum_{w=0}^{+\infty}\frac{\lambda^w}{\Gamma(\alpha_1(w+1))}\,t_2^{\alpha_1+k_1+l_1}\int_0^{\frac{t_1}{t_2}}(1-\omega)^{\alpha_1+l_1-1}\omega^{k_1}\,d\omega$$

$$=E_{\alpha_1,\alpha_1}(\lambda_1)\big[t_1^{\alpha_1+k_1+l_1}-t_2^{\alpha_1+k_1+l_1}\big]\mathbf{B}(\alpha_1+l_1,k_1+1)$$

$$-E_{\alpha_1,\alpha_1}(\lambda_1)t_2^{\alpha_1+k_1+l_1}\int_{\frac{t_1}{t_2}}^{1}(1-\omega)^{\alpha_1+l_1-1}\omega^{k_1}\,d\omega\longrightarrow 0\ \ \text{as}\ \ t_1\to t_2.$$

Now for the third term M_3, we know that $E_{\alpha_1,\alpha_1}(\lambda_1 x^\alpha)$ is uniformly continuous on $[0,t_2]$, thus for each $\varepsilon>0$ there exists $\delta>0$ such that $|E_{\alpha_1,\alpha_1}(\lambda_1 x_1^\alpha)-E_{\alpha_1,\alpha_1}(\lambda_1 x_2^\alpha)|<\varepsilon$ for $|x_1-x_2|<\delta$. Then for $s\in[0,t_1]$ and $|t_1-t_2|<\delta$, we have $t_1-s,t_2-s\in[0,t_2]$ and $|(t_1-s)-(t_2-s)|<\delta$. So $|E_{\alpha_1,\alpha_1}(\lambda_1(t_2-s)^{\alpha_1})-E_{\alpha_1,\alpha_1}(\lambda_1(t_1-s)^{\alpha_1})|<\varepsilon$. Hence,

$$M_3\le\int_0^{t_1}(t_1-s)^{\alpha_1+l_1-1}\big|E_{\alpha_1,\alpha_1}(\lambda_1(t_2-s)^{\alpha_1})-E_{\alpha_1,\alpha_1}(\lambda_1(t_1-s)^{\alpha_1})\big|s^{k_1}\,ds$$

$$\le\varepsilon\int_0^{t_1}(t_1-s)^{\alpha_1+l_1-1}s^{k}\,ds\le\varepsilon\mathbf{B}(\alpha_1+l_1,k_1+1).$$

Thus $M_3\to 0$ as $t_1\to t_2$.

So, $t\longrightarrow\int_0^t(t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\,ds$ is continuous on $(0,1]$. Then $x\in C^0(t_i,t_{i+1}]$, $i\in N_0$. For $t\in(0,t_1]$, we get

$$\begin{aligned}
&t^{1-\alpha_1}\left|\int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\,ds\right| \le t^{1-\alpha_1}\int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})s^{k_1}(1-s)^{l_1}\,ds\\
&= t^{1-\alpha_1}\int_0^t (t-s)^{\alpha_1-1}\sum_{i=0}^{+\infty}\frac{\lambda_1^i(t-s)^{\alpha_1 i}}{\Gamma(\alpha_1 i+\alpha_1)}\,s^{k_1}(1-s)^{l_1}\,ds \le t^{1-\alpha_1}\int_0^t (t-s)^{\alpha_1+l_1-1}\sum_{i=0}^{+\infty}\frac{\lambda_1^i(t-s)^{\alpha_1 i}}{\Gamma(\alpha_1 i+\alpha_1)}\,s^{k_1}\,ds\\
&= t^{1-\alpha_1}\sum_{i=0}^{+\infty}\frac{\lambda_1^i}{\Gamma(\alpha_1 i+\alpha_1)}\int_0^t (t-s)^{\alpha_1+\alpha_1 i+l_1-1}s^{k_1}\,ds\\
&= t^{1-\alpha_1}\sum_{i=0}^{+\infty}\frac{\lambda_1^i}{\Gamma(\alpha_1 i+\alpha_1)}\,t^{\alpha_1+\alpha_1 i+l_1+k_1}\int_0^1 (1-w)^{\alpha_1+\alpha_1 i+l_1-1}w^{k_1}\,dw\\
&\le t^{1-\alpha_1}\sum_{i=0}^{+\infty}\frac{\lambda_1^i}{\Gamma(\alpha_1 i+\alpha_1)}\,t^{\alpha_1+\alpha_1 i+l_1+k_1}\int_0^1 (1-w)^{\alpha_1+l_1-1}w^{k_1}\,dw\\
&= t^{1+l_1+k_1}\mathbf{B}(\alpha_1+l_1,k_1+1)\sum_{i=0}^{+\infty}\frac{\lambda_1^i t^{\alpha_1 i}}{\Gamma(\alpha_1 i+\alpha_1)} = t^{1+l_1+k_1}\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,\alpha_1}(\lambda_1 t^{\alpha_1}).
\end{aligned}$$

From $k_1+l_1+1>0$, we get

$$\lim_{t\to 0} t^{1-\alpha_1}\left|\int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\,ds\right| = 0.$$

Then $\lim_{t\to 0^+} t^{1-\alpha_1}x(t)$ exists. It is similar to prove that $\lim_{t\to t_i^+}(t-t_i)^{1-\alpha_1}x(t)$, $i\in N$, exist. Hence $x\in X$.

By direct computation, similarly to Step 1, we can show for $t\in(t_i,t_{i+1}]$, $i\in N_0$, that

$$D_{0^+}^{\alpha_1}x(t)-\lambda_1 x(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_0^t (t-s)^{-\alpha_1}x(s)\,ds\right)' - \lambda_1 x(t) = \sigma(t).$$

So, x is a solution of (3.1) in X. □

Lemma 3.2. *Suppose that* $\Delta = 1-\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(\lambda_1)\neq 0$, $\sigma\in L^1(0,1)$ *and there exist numbers* $k_1>-1$ *and* $\max\{-\alpha_1,-k_1-1\}<l_1\le 0$ *such that* $|\sigma(t)|\le t^{k_1}(1-t)^{l_1}$ *for all* $t\in(0,1)$. *Then* $x\in X$ *is a solution of*

$$\begin{cases} D_{0^+}^{\alpha_1}x(t)-\lambda_1 x(t)=\sigma(t), & t\in(t_i,t_{i+1}),\ \ i\in N_0,\\ x(1)-\lim_{t\to 0} t^{1-\alpha_1}x(t)=0, & \\ \lim_{t\to t_i^+}(t-t_i)^{1-\alpha_1}x(t)=I_i, & i\in N, \end{cases} \tag{3.5}$$

if and only if $x\in X$ *and*

$$\begin{aligned}
x(t) &= \frac{\Gamma(\alpha_1)^2\delta_{\alpha_1,\lambda_1}(t,0)}{\Delta}\sum_{j=1}^m I_j\delta_{\alpha_1,\lambda_1}(1,t_j) + \frac{\Gamma(\alpha_1)\delta_{\alpha_1,\lambda_1}(t,0)}{\Delta}\int_0^1 \delta_{\alpha_1,\lambda_1}(1,s)\sigma(s)\,ds\\
&\quad + \Gamma(\alpha_1)\sum_{j=1}^i I_j\delta_{\alpha_1,\lambda_1}(t,t_j) + \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds,\ \ t\in(t_i,t_{i+1}],\ \ i\in N_0.
\end{aligned} \tag{3.6}$$

Proof. Let x be a solution of (3.5). Similarly to Step 2 in Lemma 3.1, since $l \le 0$ for $t \in (0, t_1]$, one can see that

$$t^{1-\alpha_1}\left|\int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds\right| = t^{1-\alpha_1}\left|\int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\,ds\right|$$
$$\le t^{1+l_1+k_1}\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,\alpha_1}(\lambda_1 t^{\alpha_1}).$$

From $k_1 + l_1 + 1 > 0$, we get

$$\lim_{t\to 0} t^{1-\alpha_1}\left|\int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds\right| = 0. \tag{3.7}$$

From Lemma 3.1, there exist constants $A_i \in R$ $(i \in N_0)$ such that

$$x(t) = \Gamma(\alpha_1)\sum_{j=0}^{i} A_j\delta_{\alpha_1,\lambda_1}(t,t_j) + \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds, \quad t \in (t_i,t_{i+1}], \quad i \in N_0. \tag{3.8}$$

Note that $E_{\alpha_1,\alpha_1}(0) = \frac{1}{\Gamma(\alpha_1)}$. It follows from the boundary conditions and the impulse assumption in (3.5) that

$$\Gamma(\alpha_1)\sum_{j=0}^{m} A_j\delta_{\alpha_1,\lambda_1}(1,t_j) + \int_0^1 \delta_{\alpha_1,\lambda_1}(1,s)\sigma(s)\,ds - A_0 = 0, \quad A_i = I_i, \quad i \in N.$$

It follows that

$$A_0 = \frac{1}{\Delta}\left(\Gamma(\alpha_1)\sum_{j=1}^{m} I_j\delta_{\alpha_1,\lambda_1}(1,t_j) + \int_0^1 \delta_{\alpha_1,\lambda_1}(1,s)\sigma(s)\,ds\right).$$

Substituting A_i $(i \in N_0)$ into (3.8), we obviously get (3.6) by changing the term order.

On the other hand, if x satisfies (3.6), then it is easy to see that $x\big|_{(t_i,t_{i+1}]}$ is continuous and the limit $\lim_{t\to t_{i+}}(t-t_i)^{1-\alpha_1}x(t)$ exists for $i \in N_0$. Using Lemma 3.1, we can prove that x satisfies (3.5). □

Lemma 3.3. *Suppose that* $\nabla = 1 - \Gamma(\alpha_2)E_{\alpha_2,\alpha_2}(\lambda_2) \neq 0$, $\sigma \in L^1(0,1)$ *and there exist numbers* $k_2 > -1$ *and* $\max\{-\alpha_2, -k_2 - 1\} < l_2 \le 0$ *such that* $|\sigma(t)| \le t^{k_2}(1-t)^{l_2}$ *for all* $t \in (0,1)$. *Then* $x \in Y$ *is a solution of*

$$\begin{cases} D_{0^+}^{\alpha_2}x(t) - \lambda_2 x(t) = \sigma(t), & t \in (t_i,t_{i+1}), \quad i \in N_0,\\ x(1) - \lim_{t\to 0} t^{1-\alpha_2}x(t)\,dt = 0, & \\ \lim_{t\to t_i^+}(t-t_i)^{1-\alpha_2}x(t) = J_i, & i \in N \end{cases} \tag{3.9}$$

if and only if $x \in Y$ *and*

$$x(t) = \frac{\Gamma(\alpha_2)^2\delta_{\alpha_2,\lambda_2}(t,0)}{\nabla}\sum_{j=1}^{m} J(t_j,x(t_j),y(t_j))\delta_{\alpha_2,\lambda_2}(1,t_j) + \frac{\Gamma(\alpha_2)\delta_{\alpha_2,\lambda_2}(t,0)}{\nabla}\int_0^1 \delta_{\alpha_2,\lambda_2}(1,s)\sigma(s)\,ds$$
$$+\,\Gamma(\alpha_2)\sum_{j=1}^{i} J(t_j,x(t_j),y(t_j))\delta_{\alpha_2,\lambda_2}(t,t_j) + \int_0^t \delta_{\alpha_2,\lambda_2}(t,s)\sigma(s)\,ds, \quad t \in (t_i,t_{i+1}], \quad i \in N_0. \tag{3.10}$$

Proof. The proof is similar to that of Lemma 3.2, and hence we omit it. □

Let Δ, ∇ be defined in Lemmas 3.2 and 3.3. Define the nonlinear operator T on E by $T(x,y)(t) = ((T_1(x,y))(t), (T_2(x,y))(t))$ with

$$\begin{aligned}(T_1(x,y))(t) &= \frac{\Gamma(\alpha_1)^2\delta_{\alpha_1,\lambda_1}(t,0)}{\Delta}\sum_{j=1}^{m}\delta_{\alpha_1,\lambda_1}(1,t_j)I(t_j,x(t_j),y(t_j))\\ &+ \frac{\Gamma(\alpha_1)\delta_{\alpha_1,\lambda_1}(t,0)}{\Delta}\int_0^1 \delta_{\alpha_1,\lambda_1}(1,s)p_1(s)f_1(s,x(s),y(s))\,ds\\ &+ \Gamma(\alpha_1)\sum_{j=1}^{i} I(t_j,x(t_j),y(t_j))\delta_{\alpha_1,\lambda_1}(t,t_j)\\ &+ \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)p_1(s)f_1(s,x(s),y(s))\,ds, \quad t\in(t_i,t_{i+1}], \quad i\in N_0,\end{aligned}$$

and

$$\begin{aligned}(T_2(x,y))(t) &= \frac{\Gamma(\alpha_2)^2\delta_{\alpha_2,\lambda_2}(t,0)}{\nabla}\sum_{j=1}^{m}\delta_{\alpha_2,\lambda_2}(1,t_j)J(t_j,x(t_j),y(t_j))\\ &+ \frac{\Gamma(\alpha_2)\delta_{\alpha_2,\lambda_2}(t,0)}{\nabla}\int_0^1 \delta_{\alpha_2,\lambda_2}(1,s)p_2(s)f_2(s,x(s),y(s))\,ds\\ &+ \Gamma(\alpha_2)\sum_{j=1}^{i} J(t_j,x(t_j),y(t_j))\delta_{\alpha_2,\lambda_2}(t,t_j)\\ &+ \int_0^t \delta_{\alpha_2,\lambda_2}(t,s)p_2(s)f_2(s,x(s),y(s))\,ds, \quad t\in(t_i,t_{i+1}], \quad i\in N_0,\end{aligned}$$

for $(x,y)\in E$.

Lemma 3.4. *Suppose that f_1, f_2 are* **I-Carathéodory functions***, I, J are discrete I-Carathéodory functions. Then $(x,y)\in E$ is a solution of BVP* (1.3) *if and only if (x,y) is a fixed point of T in E and $T: E\to E$ is well defined and is completely continuous.*

Proof. It is easy to see from Lemmas 3.2 and 3.3 that $(x,y)\in E$ is a solution of BVP (1.3) if and only if (x,y) is a fixed point of T in E. We divide the remainder of the proof into the following steps:

Step (i). We prove that $T: E\to E$ is well defined. Similarly to the proofs of Lemma 3.1, we can prove that $T_1(x,y)\in X$ and $T_2(x,y)\in Y$ for all $(x,y)\in E$. Thus $T: E\to E$ is well defined.

Step (ii). We prove that T is continuous.

Let $(x_n,y_n)\in E$ with $(x_n,y_n)\to(x_0,y_0)$ as $n\to\infty$. We can show that $T(x_n,y_n)\to T(x_0,y_0)$ as $n\to\infty$ by using the dominant convergence theorem. We refer the readers to the papers [42, 48, 53].

Step (iii). Prove that T is compact, i.e., $T(\overline{\Omega})$ is relatively compact for every bounded subset $\Omega\subset E$.

Let Ω be a bounded open nonempty subset of E. Then there exists $r>0$ such that

$$\|(x,y)\| = \max\Big\{\sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{1-\alpha_1}|x(t)|, \sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{1-\alpha_2}|y(t)|:\ i\in N_0\Big\}\le r<+\infty \quad (3.11)$$

holds for all $(x,y)\in\overline{\Omega}$. Since f_1, f_2 are **I-Carathéodory functions** and I, J are **discrete I-Cara-**

théodory functions, there exist constants $M_I, M_J, M_{f_1}, M_{f_2} \geq 0$ such that

$$\begin{aligned}
\big|f_1(t,x(t),y(t))\big| &= \Big|f\Big(t,\frac{(t-t_i)^{1-\alpha_1}x(t)}{(t-t_i)^{1-\alpha_1}},\frac{(t-t_i)^{1-\alpha_2}y(t)}{(t-t_i)^{1-\alpha_2}}\Big)\Big| \leq M_{f_1},\ \ t\in(t_i,t_{i+1}),\ \ i\in N_0,\\
\big|f_2(t,x(t),y(t))\big| &\leq M_{f_2},\ \ t\in(t_i,t_{i+1}),\ \ i\in N_0,\\
\big|I(t_i,x(t_i),y(t_i))\big| &= \Big|I\Big(t_i,\frac{(t_i-t_{i-1})^{1-\alpha_1}x(t_i)}{(t_i-t_{i-1})^{1-\alpha_1}},\frac{(t_i-t_{i-1})^{1-\alpha_2}y(t_i)}{(t_i-t_{i-1})^{1-\alpha_2}}\Big)\Big| \leq M_I,\ \ i\in N,\\
\big|J(t_i,x(t_i),y(t_i))\big| &\leq M_J,\ \ i\in N.
\end{aligned} \tag{3.12}$$

This step is done by three sub-steps.

Sub-step (iii1). Prove that $T(\overline{\Omega})$ is uniformly bounded.

Using the definition of T_1, for $t\in(0,t_1]$ we have

$$\begin{aligned}
t^{1-\alpha_1}|(T_1(x,y))(t)| &\leq \frac{\Gamma(\alpha_1)^2E_{\alpha_1,\alpha_1}(|\lambda_1|t_1^{\alpha_1})}{|\Delta|}\sum_{j=1}^{m}(1-t_j)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1})M_I\\
&\quad+\frac{\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(|\lambda_1|t_1^{\alpha_1})}{|\Delta|}\int_0^1(1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(1-s)^{\alpha_1})s^{k_1}(1-s)^{l_1}\,ds\,M_{f_1}\\
&\quad+t^{1-\alpha_1}\int_0^t(t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})s^{k_1}(1-s)^{l_1}\,ds\,M_{f_1}\\
&\leq \frac{\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}(1-t_m)^{\alpha_1-1}m\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(|\lambda_1|)M_I\\
&\quad+\Big(\frac{\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}+1\Big)\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,\alpha_1}(|\lambda_1|)M_{f_1}.
\end{aligned}$$

Similarly, we can prove for $t\in(t_i,t_{i+1}]$, $i\in N$, that

$$\begin{aligned}
&(t-t_i)^{1-\alpha_1}|(T_1(x,y))(t)|\\
&\quad\leq (t_{i+1}-t_i)^{1-\alpha_1}\,\frac{\Gamma(\alpha_1)^2t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|t_{i+1}^{\alpha_1})}{|\Delta|}\sum_{j=1}^{m}(1-t_j)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1})M_I\\
&\qquad+(t_{i+1}-t_i)^{1-\alpha_1}\,\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|t_{i+1}^{\alpha_1})}{|\Delta|}\,\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,\alpha_1}(\lambda_1)M_{f_1}\\
&\qquad+\Gamma(\alpha_1)\sum_{j=1}^{i}E_{\alpha_1,\alpha_1}(|\lambda_1|(t_{i+1}-t_j)^{\alpha_1})M_I\\
&\qquad+\max\big\{1,t_1^{\alpha_1+k_1+l_1}\big\}\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,\alpha_1}(|\lambda_1|t_{i+1}^{\alpha_1})M_{f_1}\\
&\quad\leq\Big(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}(1-t_m)^{\alpha_1-1}+1\Big)m\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(|\lambda_1|)M_I\\
&\qquad+\Big(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}+\max\big\{1,t_1^{\alpha_1+k_1+l_1}\big\}\Big)\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,\alpha_1}(|\lambda_1|)M_{f_1}.
\end{aligned}$$

Then

$$\begin{aligned}
\|T_1(x,y)\|_X &\leq \Big(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}(1-t_m)^{\alpha_1-1}+1\Big)m\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(|\lambda_1|)M_I\\
&\quad+\Big(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}+\max\big\{1,t_1^{\alpha_1+k_1+l_1}\big\}\Big)\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,\alpha_1}(|\lambda_1|)M_{f_1}.
\end{aligned}$$

We can also find that

$$\|T_2(x,y)\| \le \Big(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|}(1-t_m)^{\alpha_2-1}+1\Big)m\Gamma(\alpha_2)E_{\alpha_2,\alpha_2}(|\lambda_2|)M_J + \Big(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|}+\max\big\{1,t_1^{\alpha_2+k_2+l_2}\big\}\Big)\mathbf{B}(\alpha_2+l_2,k_2+1)E_{\alpha_2,\alpha_2}(|\lambda_2|)M_{f_2}.$$

From the above discussion, $T(\overline{\Omega})$ is uniformly bounded.

Sub-step (iii2). Prove that both $\{(t-t_i)^{1-\alpha_1}(T_1(x,y))(t):\ (x,y)\in\overline{\Omega}\}$ and $\{(t-t_i)^{1-\alpha_2}(T_2(x,y))(t):\ (x,y)\in\overline{\Omega}\}$ are equi-continuous on each subinterval $[a,b]\subseteq(t_i,t_{i+1}]$, $i\in N_0$, respectively.

Let $s_2\le s_1$ and $s_1,s_2\in[a,b]\subseteq(0,t_1]$. We can prove

$$\begin{aligned}
&\big|s_1^{1-\alpha_1}(T_1(x,y))(s_1)-s_2^{1-\alpha_1}(T_1(x,y))(s_2)\big|\\
&\le \Gamma(\alpha_1)^2\Big|\frac{E_{\alpha_1,\alpha_1}(\lambda_1 s_1^{\alpha_1})-E_{\alpha_1,\alpha_1}(\lambda_1 s_2^{\alpha_1})}{\Delta}\Big|\sum_{j=1}^{m}(1-t_j)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1})M_I\\
&+\Gamma(\alpha_1)\Big|\frac{E_{\alpha_1,\alpha_1}(\lambda_1 s_1^{\alpha_1})-E_{\alpha_1,\alpha_1}(\lambda_1 s_2^{\alpha_1})}{\Delta}\Big|\int_0^1(1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(1-s)^{\alpha_1})s^{k_1}(1-s)^{l_1}\,ds\\
&+\Big|s_1^{1-\alpha_1}\int_0^{s_1}(s_1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(s_1-s)^{\alpha_1})p_1(s)f_1(s,x(s),y(s))\,ds\\
&\quad - s_2^{1-\alpha_1}\int_0^{s_2}(s_2-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(s_2-s)^{\alpha_1})p_1(s)f_1(s,x(s),y(s))\,ds\Big|.
\end{aligned}$$

(i) It is easy to show that $E_{\alpha_1,\alpha_1}(\lambda_1 x^{\alpha_1})$ is uniformly continuous on $[0,b]$. Then for any $\varepsilon>0$, there exists $\delta>0$ such that $|E_{\alpha_1,\alpha_1}(\lambda_1 s_1^{\alpha_1})-E_{\alpha_1,\alpha_1}(\lambda_1 s_2^{\alpha_1})|<\varepsilon$ for all $s_1,s_2\in[0,b]$ with $|s_1-s_2|<\delta$.

(ii) Both $\sum\limits_{j=1}^{m}(1-t_j)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1})M_I$ and $\int\limits_0^1(1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(1-s)^{\alpha_1})s^{k_1}(1-s)^{l_1}\,ds$ are constants.

(iii) It is easy to prove that

$$\begin{aligned}
&x^\mu-y^\mu\le(x-y)^\mu \ \text{ for all } \ \mu\in(0,1],\ \ x\ge y>0,\\
&x^\mu-y^\mu\le\mu x^{\mu-1}(x-y) \ \text{ for all } \ \mu>1,\ \ x\ge y>0,\\
&h_1(x)=xc^{x-1} \ \text{ is increasing on } \Big(0,-\frac{1}{\ln c}\Big) \text{ and decreasing on } \Big(-\frac{1}{\ln c},+\infty\Big),\ \ c\in(0,1),\\
&h_2(x)=-x\ln x\ge\min\big\{-(b-a)\ln(b-a),-b\ln b\big\} \ \text{ for all } \ x\in[b-a,b].
\end{aligned}$$

It follows that

$$h_1(x)\le\max\Big\{1,\frac{1}{e}\,\frac{1}{-c\ln c}\Big\}\le\max\Big\{1,\frac{1}{e}\,\frac{1}{\min\{-(b-a)\ln(b-a),-b\ln b\}}\Big\},\ \ x>1,\ \ c\in[b-a,b].$$

Then we have for $s_2,s_1\in[a,b]$ with $s_1>s_2$ that

$$\begin{aligned}
&\Big|s_1^{1-\alpha_1}\int_0^{s_1}(s_1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(s_1-s)^{\alpha_1})p_1(s)f_1(s,x(s),y(s))\,ds\\
&\quad - s_2^{1-\alpha_1}\int_0^{s_2}(s_2-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(s_2-s)^{\alpha_1})p_1(s)f_1(s,x(s),y(s))\,ds\Big|\\
&=\Big|s_1^{1-\alpha_1}\int_0^{s_1}(s_1-s)^{\alpha_1-1}\sum_{k=0}^{+\infty}\frac{\lambda_1^k(s_1-s)^{\alpha_1 k}}{\Gamma(\alpha_1(k+1))}\,p_1(s)f_1(s,x(s),y(s))\,ds
\end{aligned}$$

$$- s_2^{1-\alpha_1}\int\limits_0^{s_2}(s_2-s)^{\alpha_1-1}\sum_{k=0}^{+\infty}\frac{\lambda_1^k(s_2-s)^{\alpha_1 k}}{\Gamma(\alpha(k+1))}\,p_1(s)f_1(s,x(s),y(s))\,ds\Big|$$

$$= \Big|\sum_{k=0}^{+\infty}\frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))}\,s_1^{1-\alpha_1}\int\limits_0^{s_1}(s_1-s)^{\alpha_1(k+1)-1}p_1(s)f_1(s,x(s),y(s))\,ds$$

$$-\sum_{k=0}^{+\infty}\frac{|\lambda_1|^k}{\Gamma(\alpha(k+1))}\,s_2^{1-\alpha_1}\int\limits_0^{s_2}(s_2-s)^{\alpha_1(k+1)-1}p_1(s)f_1(s,x(s),y(s))\,ds\Big|$$

$$\le \sum_{k=0}^{+\infty}\frac{|\lambda_1|^k}{\Gamma(\alpha(k+1))}\Big|s_1^{1-\alpha_1}\int\limits_0^{s_1}(s_1-s)^{\alpha_1(k+1)-1}p_1(s)f_1(s,x(s),y(s))\,ds$$

$$- s_2^{1-\alpha_1}\int\limits_0^{s_2}(s_2-s)^{\alpha_1(k+1)-1}p_1(s)f_1(s,x(s),y(s))\,ds\Big|$$

$$\le \sum_{k=0}^{+\infty}\frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))}\Big[\big|s_1^{1-\alpha_1}-s_2^{1-\alpha_1}\big|\int\limits_0^{s_1}(s_1-s)^{\alpha_1(k+1)-1}|p_1(s)|\,|f_1(s,x(s),y(s))|\,ds$$

$$+ s_2^{1-\alpha_1}\int\limits_{s_2}^{s_1}(s_1-s)^{\alpha_1(k+1)-1}|p_1(s)|\,|f_1(s,x(s),y(s))|\,ds$$

$$+ s_2^{1-\alpha_1}\int\limits_0^{s_2}\big|(s_1-s)^{\alpha_1(k+1)-1}-(s_2-s)^{\alpha_1(k+1)-1}\big||p_1(s)|\,|f_1(s,x(s),y(s))|\,ds\Big]$$

$$\le M_{f_1}\sum_{k=0}^{+\infty}\frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))}\Big[\big|s_1^{1-\alpha_1}-s_2^{1-\alpha_1}\big|\int\limits_0^{s_1}(s_1-s)^{\alpha_1(k+1)-1}s^{k_1}(1-s)^{l_1}\,ds$$

$$+ s_2^{1-\alpha_1}\int\limits_{s_2}^{s_1}(s_1-s)^{\alpha_1(k+1)-1}s^{k_1}(1-s)^{l_1}\,ds$$

$$+ s_2^{1-\alpha_1}\int\limits_0^{s_2}\big|(s_1-s)^{\alpha_1(k+1)-1}-(s_2-s)^{\alpha_1(k+1)-1}\big|s^{k_1}(1-s)^{l_1}\,ds\Big]$$

$$\le M_{f_1}\sum_{k=0}^{+\infty}\frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))}\Big[\big|s_1^{1-\alpha_1}-s_2^{1-\alpha_1}\big|\int\limits_0^{s_1}(s_1-s)^{\alpha_1(k+1)+l_1-1}s^{k_1}\,ds$$

$$+ s_2^{1-\alpha_1}\int\limits_{s_2}^{s_1}(s_1-s)^{\alpha_1(k+1)+l_1-1}s^{k_1}\,ds$$

$$+ s_2^{1-\alpha_1}\int\limits_0^{s_2}\big|(s_1-s)^{\alpha_1(k+1)-1}-(s_2-s)^{\alpha_1(k+1)-1}\big|s^{k_1}(1-s)^{l_1}\,ds\Big]$$

$$= M_{f_1}\sum_{k=0}^{+\infty}\frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))}\Big[\big|s_1^{1-\alpha_1}-s_2^{1-\alpha_1}\big|s_1^{\alpha_1(k+1)+k_1+l_1}\int\limits_0^1(1-w)^{\alpha_1(k+1)+l_1-1}w^{k_1}\,dw$$

$$+ s_2^{1-\alpha_1}s_1^{\alpha_1(k+1)+k_1+l_1}\int\limits_{\frac{s_2}{s_1}}^1(1-w)^{\alpha_1(k+1)+l_1-1}w^{k_1}\,dw$$

$$+ s_2^{1-\alpha_1}\int_0^{s_2}\left|(s_1-s)^{\alpha_1(k+1)-1}-(s_2-s)^{\alpha_1(k+1)-1}\right|s^{k_1}(1-s)^{l_1}\,ds\Big]$$

$$\le M_{f_1}\sum_{k=0}^{+\infty}\frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))}\Big[\left|s_1^{1-\alpha_1}-s_2^{1-\alpha_1}\right|\max\left\{a^{\alpha_1+k_1+l_1},b^{\alpha_1+k_1+l_1}\right\}\int_0^1(1-w)^{\alpha_1+l_1-1}w^{k_1}\,dw$$

$$+\max\left\{a^{\alpha_1+k_1+l_1},b^{\alpha_1+k_1+l_1}\right\}\int_{\frac{s_2}{s_1}}^1(1-w)^{\alpha_1+l_1-1}w^{k_1}\,dw$$

$$+ s_2^{1-\alpha_1}\int_0^{s_2}\left|(s_1-s)^{\alpha_1(k+1)-1}-(s_2-s)^{\alpha_1(k+1)-1}\right|s^{k_1}(1-s)^{l_1}\,ds\Big].$$

One can see

$$\left|s_1^{1-\alpha_1}-s_2^{1-\alpha_1}\right|\longrightarrow 0,\quad \int_{\frac{s_2}{s_1}}^1(1-w)^{\alpha_1+l_1-1}w^{k_1}\,dw\longrightarrow 0 \ \text{ as } \ s_1\to s_2.$$

On the other hand, for $\alpha_1(k+1)-1\in(0,1]$ we have

$$s_2^{1-\alpha_1}\int_0^{s_2}\left|(s_1-s)^{\alpha_1(k+1)-1}-(s_2-s)^{\alpha_1(k+1)-1}\right|s^{k_1}(1-s)^{l_1}\,ds$$

$$\le (s_1-s_2)^{\alpha_1(k+1)-1}\int_0^{s_2}s^{k_1}(s_2-s)^{l_1}\,ds=(s_1-s_2)^{\alpha_1(k+1)-1}s_2^{k_1+l_1+1}\int_0^1(1-w)^{l_1}w^{k_1}\,ds$$

$$\le (s_1-s_2)^{\alpha_1(k+1)-1}\int_0^1(1-w)^{l_1}w^{k_1}\,dw\longrightarrow 0$$

$$\text{as } s_1\to s_2 \text{ uniformly for } \ k=\left[\frac{1}{\alpha_1}\right]-1,\left[\frac{1}{\alpha_1}\right],\dots,\left[\frac{2}{\alpha_1}\right]-1.$$

For $\alpha_1(k+1)-1>1$,

$$s_2^{1-\alpha_1}\int_0^{s_2}\left|(s_1-s)^{\alpha_1(k+1)-1}-(s_2-s)^{\alpha_1(k+1)-1}\right|s^{k_1}(1-s)^{l_1}\,ds$$

$$\le (s_1-s_2)\int_0^{s_2}(\alpha_1(k+1)-1)(s_1-s)^{\alpha_1(k+1)-2}s^{k_1}(s_2-s)^{l_1}\,ds$$

$$\le (s_1-s_2)\max\left\{1,\frac{1}{e}\,\frac{1}{\min\{-(b-a)\ln(b-a),-b\ln b\}}\right\}\int_0^{s_2}s^{k_1}(s_2-s)^{l_1}\,ds$$

$$\le (s_1-s_2)\max\left\{1,\frac{1}{e}\,\frac{1}{\min\{-(b-a)\ln(b-a),-b\ln b\}}\right\}s_2^{k_1+l_1+1}\int_0^1 w^{k_1}(1-w)^{l_1}\,dw$$

$$\le (s_1-s_2)\max\left\{1,\frac{1}{e}\,\frac{1}{\min\{-(b-a)\ln(b-a),-b\ln b\}}\right\}\int_0^1 w^{k_1}(1-w)^{l_1}\,dw\longrightarrow 0 \ \text{ as } \ s_1\to s_2.$$

For $\alpha_1(k+1)-1\le 0$,

$$
\begin{aligned}
s_2^{1-\alpha_1}\int_0^{s_2}&\big|(s_1-s)^{\alpha_1(k+1)-1}-(s_2-s)^{\alpha_1(k+1)-1}\big|s^{k_1}(1-s)^{l_1}\,ds\\
&\le \int_0^{s_2}\big[(s_2-s)^{\alpha_1(k+1)-1}-(s_1-s)^{\alpha_1(k+1)-1}\big]s^{k_1}(s_2-s)^{l_1}\,ds\\
&\le \int_0^{s_2}(s_2-s)^{\alpha_1(k+1)+l_1-1}s^{k_1}ds-\int_0^{s_2}(s_1-s)^{\alpha_1(k+1)-1}s^{k_1}(s_2-s)^{l_1}\,ds\\
&\le \int_0^{s_2}(s_2-s)^{\alpha_1(k+1)+l_1-1}s^{k_1}ds-\int_0^{s_2}(s_1-s)^{\alpha_1(k+1)+l_1-1}s^{k_1}\,ds\\
&=s_2^{\alpha_1(k+1)+k_1+l_1}\int_0^1(1-w)^{\alpha_1(k+1)+l_1-1}w^{k_1}\,dw-s_1^{\alpha_1(k+1)+k_1+l_1}\int_0^{\frac{s_2}{s_1}}(1-w)^{\alpha_1(k+1)+l_1-1}w^{k_1}\,dw\\
&\le \big|s_2^{\alpha_1(k+1)+k_1+l_1}-s_1^{\alpha_1(k+1)+k_1+l_1}\big|\int_0^1(1-w)^{\alpha_1(k+1)+l_1-1}w^{k_1}\,dw\\
&\qquad+s_1^{\alpha_1(k+1)+k_1+l_1}\int_{\frac{s_2}{s_1}}^1(1-w)^{\alpha_1(k+1)+l_1-1}w^{k_1}\,dw\\
&\le \big|s_2^{\alpha_1(k+1)+k_1+l_1}-s_1^{\alpha_1(k+1)+k_1+l_1}\big|\int_0^1(1-w)^{\alpha_1+l_1-1}w^{k_1}\,dw\\
&\qquad+\max\big\{a^{\alpha_1+k_1+l_1},b^{\alpha_1+k_1+l_1}\big\}\int_{\frac{s_2}{s_1}}^1(1-w)^{\alpha_1+l_1-1}w^{k_1}\,dw\longrightarrow 0\\
&\qquad\text{as } s_1\to s_2 \text{ uniformly for } k=0,1,2,\dots,\Big[\frac{1}{\alpha_1}\Big]-1.
\end{aligned}
$$

From the above discussion, there exists $\delta>0$ such that

$$
\big|s_1^{1-\alpha_1}-s_2^{1-\alpha_1}\big|<\varepsilon,\quad \int_{\frac{s_2}{s_1}}^1(1-w)^{\alpha_1+l_1-1}w^{k_1}\,dw<\varepsilon,
$$

$$
s_2^{1-\alpha_1}\int_0^{s_2}\big|(s_1-s)^{\alpha_1(k+1)-1}-(s_2-s)^{\alpha_1(k+1)-1}\big|s^{k_1}(1-s)^{l_1}\,ds<\varepsilon
$$

hold for all $s_1,s_2\in[a,b]$, $0<s_1-s_2<\delta$, $k=0,1,2,\dots$. Then

$$
\begin{aligned}
\bigg|s_1^{1-\alpha_1}&\int_0^{s_1}(s_1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(s_1-s)^{\alpha_1})p_1(s)f_1(s,x(s),y(s))\,ds\\
&-s_2^{1-\alpha_1}\int_0^{s_2}(s_2-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(s_2-s)^{\alpha_1})p_1(s)f_1(s,x(s),y(s))\,ds\bigg|
\end{aligned}
$$

$$< \varepsilon M_{f_1} \sum_{k=0}^{+\infty} \frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))} \left[\max\left\{a^{\alpha_1+k_1+l_1}, b^{\alpha_1+k_1+l_1}\right\} \int_0^1 (1-w)^{\alpha_1+l_1-1} w^{k_1}\, dw \right.$$
$$\left. + \max\left\{a^{\alpha_1+k_1+l_1}, b^{\alpha_1+k_1+l_1}\right\} + 1 \right]$$
$$= \varepsilon M_{f_1} E_{\alpha_1,\alpha_1}(|\lambda_1|) \left[\max\left\{a^{\alpha_1+k_1+l_1}, b^{\alpha_1+k_1+l_1}\right\} \int_0^1 (1-w)^{\alpha_1+l_1-1} w^{k_1}\, dw \right.$$
$$\left. + \max\left\{a^{\alpha_1+k_1+l_1}, b^{\alpha_1+k_1+l_1}\right\} + 1 \right], \quad s_1, s_2 \in [a,b], \ \ 0 < s_1 - s_2 < \delta, \ \ k = 0,1,2,\dots .$$

From (i), (ii) and (iii), we have

$$\left| s_1^{1-\alpha_1}(T_1(x,y))(s_1) - s_2^{1-\alpha_1}(T_1(x,y))(s_2) \right| \longrightarrow 0 \ \text{ uniformly in } \ [a,b] \subset (0,t_1] \ \text{ as } \ s_1 \to s_2.$$

Similarly, we can show that for $s_1, s_2 \in [a,b] \subseteq (t_i, t_{i+1}]$, $i \in N$,

$$\left| (s_1 - t_i)^{1-\alpha_1}(T_1(x,y))(s_1) - (s_2 - t_i)^{1-\alpha_1}(T_1(x,y))(s_2) \right| \longrightarrow 0 \ \text{ uniformly as } \ s_1 \to s_2,$$

and for $s_1, s_2 \in [a,b] \subseteq (t_i, t_{i+1}]$, $i \in N_0$,

$$\left| (s_1 - t_i)^{1-\alpha_2}(T_2(x,y))(s_1) - (s_2 - t_i)^{1-\alpha_2}(T_2(x,y))(s_2) \right| \longrightarrow 0 \ \text{ uniformly as } \ s_1 \to s_2.$$

All of these expressions complete this step.

Sub-step (iii3). Prove that both $\{(t-t_i)^{1-\alpha_1}(T_1(x,y))(t) : \ (x,y) \in \overline{\Omega}\}$ and $\{(t-t_i)^{1-\alpha_2}(T_2(x,y))(t) : (x,y) \in \overline{\Omega}\}$ are equi-convergent at $t = t_i$, $i \in N_0$, respectively.

For $i \in N_0$, since $\lim\limits_{t \to t_i^+} (t-t_i)^{1-\alpha_1}(T_1(x,y))(t)$ exists, we can easily see that $\{(t-t_i)^{1-\alpha_1}(T_1(x,y))(t) : (x,y) \in \overline{\Omega})\}$ is equi-convergent at $t = t_i$, $i \in N_0$. Similarly, we can show that $(t-t_i)^{1-\alpha_2}(T_2(x,y))(t) : (x,y) \in (\overline{\Omega})\}$ is equi-convergent at $t = t_i$, $i \in N_0$. So $T(\overline{\Omega})$ is relatively compact. Then T is completely continuous. □

Now, we prove the first theorem of this paper by using the Schauder's fixed point theorem [30]. We need the following assumptions:

(D1) f_1, f_2 are **I-Carathéodory functions**, I, J are discrete I-Carathéodory functions and there exist non-decreasing functions $\phi_i, \psi_i : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, measurable functions $\phi_i, \overline{\phi}_i, \overline{\psi}_i : (0,1) \to R$ $(i = 1,2)$ and constants I_j, J_j such that

$$\left| f_1\Big(t, \frac{x}{(t-t_i)^{1-\alpha_1}}, \frac{y}{(t-t_i)^{1-\alpha_2}}\Big) - \phi_1(t) \right| \le \overline{\phi}_1(|x|,|y|), \ \ t \in (t_i, t_{i+1}), \ \ x,y \in R, \ \ i \in N_0,$$
$$\left| f_2\Big(t, \frac{x}{(t-t_i)^{1-\alpha_1}}, \frac{y}{(t-t_i)^{1-\alpha_2}}\Big) - \phi_2(t) \right| \le \overline{\phi}_2(|x|,|y|), t \in (t_i, t_{i+1}), x,y \in R, i \in N_0,$$
$$\left| I\Big(t_j, \frac{x}{(t_j-t_{j-1})^{1-\alpha_1}}, \frac{y}{(t_j-t_{j-1})^{1-\alpha_2}}\Big) - I_j \right| \le \overline{\psi}_1(|x|,|y|), \ \ j \in N, \ \ x,y \in R,$$
$$\left| J\Big(t_j, \frac{x}{(t_j-t_{j-1})^{1-\alpha_1}}, \frac{y}{(t_j-t_{j-1})^{1-\alpha_2}}\Big) - J_j \right| \le \overline{\psi}_2(|x|,|y|), \ \ j \in N, \ \ x,y \in R.$$

(D2) f_1, f_2 are **I-Carathéodory functions**, I, J are discrete I-Carathéodory functions and there exist nonnegative constants I_i, J_i, b_i, a_i, B_i, A_i, τ_j, σ_j $(j = 1,2,\dots,n)$ and measurable functions

$\phi_i:(0,1)\to R$ $(i=1,2)$ such that

$$\Big|f_1\Big(t,\frac{x}{(t-t_i)^{1-\alpha_1}},\frac{y}{(t-t_i)^{1-\alpha_2}}\Big)-\phi_1(t)\Big|\le\sum_{j=1}^n a_j|x|^{\tau_j}|y|^{\sigma_j},\ \ t\in(t_i,t_{i+1}),\ \ x,y\in R,\ \ i\in N_0,$$

$$\Big|f_2\Big(t,\frac{x}{(t-t_i)^{1-\alpha_1}},\frac{y}{(t-t_i)^{1-\alpha_2}}\Big)-\phi_2(t)\Big|\le\sum_{j=1}^n b_j|x|^{\tau_j}|y|^{\sigma_j},\ \ t\in(t_i,t_{i+1}),\ \ x,y\in R,\ \ i\in N_0,$$

$$\Big|I\Big(t_j,\frac{x}{(t_j-t_{j-1})^{1-\alpha_1}},\frac{y}{(t_j-t_{j-1})^{1-\alpha_2}}\Big)-I_j\Big|\le\sum_{j=1}^n A_j|x|^{\tau_j}|y|^{\sigma_j},\ \ j\in N,\ \ x,y\in R,$$

$$\Big|J\Big(t_j,\frac{x}{(t_j-t_{j-1})^{1-\alpha_1}},\frac{y}{(t_j-t_{j-1})^{1-\alpha_2}}\Big)-J_j\Big|\le\sum_{j=1}^n B_j|x|^{\tau_j}|y|^{\sigma_j},\ \ j\in N,\ \ x,y\in R.$$

Define

$$\Phi_1(t)=\frac{\Gamma(\alpha_1)^2\delta_{\alpha_1,\lambda_1}(t,0)}{\Delta}\sum_{j=1}^m\delta_{\alpha_1,\lambda_1}(1,t_j)I_j+\frac{\Gamma(\alpha_1)\delta_{\alpha_1,\lambda_1}(t,0)}{\Delta}\int_0^1\delta_{\alpha_1,\lambda_1}(1,s)p_1(s)\phi_1(s)\,ds$$
$$+\Gamma(\alpha_1)\sum_{j=1}^i\delta_{\alpha_1,\lambda_1}(t,t_j)I_j+\int_0^t\delta_{\alpha_1,\lambda_1}(t,s)p_1(s)\phi_1(s)\,ds,\ \ t\in(t_i,t_{i+1}],\ \ i\in N_0,$$

and

$$\Phi_2(t)=\frac{\Gamma(\alpha_2)^2\delta_{\alpha_2,\lambda_2}(t,0)}{\nabla}\sum_{j=1}^m\delta_{\alpha_2,\lambda_2}(1,t_j)J_j+\frac{\Gamma(\alpha_2)\delta_{\alpha_2,\lambda_2}(t,0)}{\nabla}\int_0^1\delta_{\alpha_2,\lambda_2}(1,s)p_2(s)\phi_2(s)\,ds$$
$$+\Gamma(\alpha_2)\sum_{j=1}^i\delta_{\alpha_2,\lambda_2}(t,t_j)J_j+\int_0^t\delta_{\alpha_2,\lambda_2}(t,s)p_2(s)\phi_2(s)\,ds,\ \ t\in(t_i,t_{i+1}],\ \ i\in N_0.$$

Denote

$$P_1=\Big(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}+\max\big\{1,t_1^{\alpha_1+k_1+l_1}\big\}\Big)\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,\alpha_1}(|\lambda_1|),$$
$$Q_1=\Big(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}(1-t_m)^{\alpha_1-1}+1\Big)m\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(|\lambda_1|),$$
$$P_2=\Big(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|}+\max\big\{1,t_1^{\alpha_2+k_2+l_2}\big\}\Big)\mathbf{B}(\alpha_2+l_2,k_2+1)E_{\alpha_2,\alpha_2}(|\lambda_2|),$$
$$Q_2=\Big(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|}(1-t_m)^{\alpha_2-1}+1\Big)m\Gamma(\alpha_2)E_{\alpha_2,\alpha_2}(|\lambda_2|).$$

Theorem 3.1. *Suppose that* (D1) *holds. Then BVP* (1.3) *has at least one solution if*

$$\begin{aligned}P_1\overline{\phi}_1\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)+Q_1\overline{\psi}_1\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)\le r_1,\\ P_2\overline{\phi}_2\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)+Q_2\overline{\psi}_2\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)\le r_2\end{aligned}\tag{3.13}$$

has a couple of positive solutions (r_1,r_2).

Proof. To apply the Schauder's fixed point theorem, we have to define a closed convex bounded subset Ω of E such that $T(\Omega)\subseteq\Omega$.

Let $r_1>0$, $r_2>0$, denote $\Omega=\{(x,y)\in E:\ \|x-\Phi_1\|\le r_1,\ \|y-\Phi_2\|\le r_2\}$. For $(x,y)\in\Omega$, we get

$$\|x\|\le\|x-\Phi_1\|+\|\Phi_1\|\le r_1+\|\Phi_1\|,\quad \|y\|\le\|y-\Phi_2\|+\|\Phi_2\|\le r_2+\|\Phi_2\|.$$

Then (D1) implies that

$$\begin{aligned}\big|f_1(t,x(t),y(t))-\phi_1(t)\big| &= \Big|f\Big(t,\frac{(t-t_i)^{1-\alpha_1}x(t)}{(t-t_i)^{1-\alpha_1}},\frac{(t-t_i)^{1-\alpha_2}y(t)}{(t-t_i)^{1-\alpha_1}}\Big)-\phi_1(t)\Big|\\ &\le \overline{\phi}_1\big(|(t-t_i)^{1-\alpha_1}x(t)|,|(t-t_i)^{1-\alpha_2}y(t)|\big)\le \phi_1\big(\|x\|,\|y\|\big)\\ &\le \overline{\phi}_1\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big),\ \ t\in(t_i,t_{i+1}),\ \ i\in N_0,\\ \big|f_2(t,x(t),y(t))-\phi_2(t)\big| &\le \overline{\phi}_2\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big),\ \ t\in(t_i,t_{i+1}),\ \ i\in N_0,\\ \big|I(t_j,x(t_j),y(t_j))-I_j\big| &\le \overline{\psi}_1\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big),\ \ j\in N,\\ \big|J(t_j,x(t_j),y(t_j))-J_j\big| &\le \overline{\psi}_2\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big),\ \ j\in N.\end{aligned}$$

By the definition of T, using the above inequalities, similarly to Step (iii1) in the proof of Lemma 3.4, we get

$$\begin{aligned}\|T_1(x,y)-\Phi_1\| &\le \Big(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}(1-t_m)^{\alpha_1-1}+1\Big)\\ &\qquad\times m\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(|\lambda_1|)\overline{\psi}_1\big(r_1+\|\Phi_1\|,\ r_2+\|\Phi_2\|\big)\\ &\quad+\Big(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}+\max\big\{1,t_1^{\alpha_1+k_1+l_1}\big\}\Big)\\ &\qquad\times \mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,\alpha_1}(|\lambda_1|)\overline{\phi}_1\big(r_1+\|\Phi_1\|,\ r_2+\|\Phi_2\|\big)\\ &= P_1\overline{\phi}_1\big(r_1+\|\Phi_1\|,\ r_2+\|\Phi_2\|\big)+Q_1\overline{\psi}_1\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|T_2(x,y)-\Phi_2\| &\le \Big(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|}(1-t_m)^{\alpha_2-1}+1\Big)\\ &\qquad\times m\Gamma(\alpha_2)E_{\alpha_2,\alpha_2}(|\lambda_2|)\overline{\psi}_2\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)\\ &\quad+\Big(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|}+\max\big\{1,t_1^{\alpha_2+k_2+l_2}\big\}\Big)\\ &\qquad\times \mathbf{B}(\alpha_2+l_2,k_2+1)E_{\alpha_2,\alpha_2}(|\lambda_2|)\overline{\phi}_2\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)\\ &= P_2\overline{\phi}_2\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)+Q_2\overline{\psi}_2\big(r_1+\|\Phi_1\|,\ r_2+\|\Phi_2\|\big).\end{aligned}$$

From the assumption, the inequality of system (3.13) has positive solution (r_1,r_2). We choose $\Omega=\{(x,y)\in E:\ \|x-\Phi_1\|\le r_1,\ \|y-\Phi_2\|\le r_2\}$. Then we get $T(\Omega)\subset\Omega$. Hence the Schauder's fixed point theorem implies that T has a fixed point $(x,y)\in\Omega$. So (x,y) is a solution of BVP (1.3). □

Denote

$$\begin{aligned}P_j &= \Big(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}\,(1-t_m)^{\alpha_1-1}+1\Big)m\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(|\lambda_1|)A_j\\ &\quad+\Big(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}+\max\big\{1,t_1^{\alpha_1+k_1+l_1}\big\}\Big)\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,\alpha_1}(|\lambda_1|)a_j,\ j=1,2,\dots,n,\\ Q_j &= \Big(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|}\,(1-t_m)^{\alpha_2-1}+1\Big)m\Gamma(\alpha_2)E_{\alpha_2,\alpha_2}(|\lambda_2|)B_j\\ &\quad+\Big(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|}+\max\big\{1,t_1^{\alpha_2+k_2+l_2}\big\}\Big)\mathbf{B}(\alpha_2+l_2,k_2+1)E_{\alpha_2,\alpha_2}(|\lambda_2|)b_j,\ j=1,2,\dots,n.\end{aligned}$$

Theorem 3.2. *Suppose that* (D2) *holds. Then BVP* (1.3) *has at least one solution if*

$$\sum_{j=1}^{n}P_j\big[r_1+\|\Phi_1\|\big]^{\tau_j}\big[r_2+\|\Phi_2\|\big]^{\sigma_j}\le r_1,\quad \sum_{j=1}^{n}Q_j\big[r_1+\|\Phi_1\|\big]^{\tau_j}\big[r_2+\|\Phi_2\|\big]^{\sigma_j}\le r_2 \tag{3.14}$$

has a couple of positive solutions (r_1,r_2).

Proof. To apply the Schauder's fixed point theorem, we have to define a closed convex bounded subset Ω of E such that $T(\Omega) \subseteq \Omega$.

Let $r_1 > 0, r_2 > 0$, denote $\Omega = \{(x,y) \in E : \ \|x - \Phi_1\| \le r_1, \ \|y - \Phi_2\| \le r_2\}$. For $(x,y) \in \Omega$, we get

$$\|x\| \le \|x - \Phi_1\| + \|\Phi_1\| \le r_1 + \|\Phi_1\|, \quad \|y\| \le \|y - \Phi_2\| + \|\Phi_2\| \le r_2 + \|\Phi_2\|.$$

Then (D2) implies that

$$\begin{aligned}
\big|f_1(t,x(t),y(t)) - \phi_1(t)\big| &\le \sum_{j=1}^{n} a_j \big[r_1 + \|\Phi_1\|\big]^{\tau_j} \big[r_2 + \|\Phi_2\|\big]^{\sigma_j}, \ \ t \in (t_i, t_{i+1}), \ \ i \in N_0, \\
\big|f_2(t,x(t),y(t)) - \phi_2(t)\big| &\le \sum_{j=1}^{n} b_j \big[r_1 + \|\Phi_1\|\big]^{\tau_j} \big[r_2 + \|\Phi_2\|\big]^{\sigma_j}, \ \ t \in (t_i, t_{i+1}), \ \ i \in N_0, \\
\big|I(t_j,x(t_j),y(t_j)) - I_j\big| &\le \sum_{j=1}^{n} A_j \big[r_1 + \|\Phi_1\|\big]^{\tau_j} \big[r_2 + \|\Phi_2\|\big]^{\sigma_j}, \ \ j \in N, \\
\big|J(t_j,x(t_j),y(t_j)) - J_j\big| &\le \sum_{j=1}^{n} B_j \big[r_1 + \|\Phi_1\|\big]^{\tau_j} \big[r_2 + \|\Phi_2\|\big]^{\sigma_j}, \ \ j \in N.
\end{aligned}$$

By the definition of T, using the above inequalities, similarly to Step (iii1) in the proof of Lemma 3.4, we get

$$\begin{aligned}
\|T_1(x,y) - \Phi_1\| \le& \Big(\frac{\Gamma(\alpha_1) t_1^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|} (1 - t_m)^{\alpha_1 - 1} + 1\Big) \\
& \times m\Gamma(\alpha_1) E_{\alpha_1,\alpha_1}(|\lambda_1|) \sum_{j=1}^{n} A_j \big[r_1 + \|\Phi_1\|\big]^{\tau_j} \big[r_2 + \|\Phi_2\|\big]^{\sigma_j} \\
& + \Big(\frac{\Gamma(\alpha_1) t_1^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|} + \max\big\{1, t_1^{\alpha_1 + k_1 + l_1}\big\}\Big) \\
& \times \mathbf{B}(\alpha_1 + l_1, k_1 + 1) E_{\alpha_1,\alpha_1}(|\lambda_1|) \sum_{j=1}^{n} a_j \big[r_1 + \|\Phi_1\|\big]^{\tau_j} \big[r_2 + \|\Phi_2\|\big]^{\sigma_j} \\
=& \sum_{j=1}^{n} P_j \big[r_1 + \|\Phi_1\|\big]^{\tau_j} \big[r_2 + \|\Phi_2\|\big]^{\sigma_j}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|T_2(x,y) - \Phi_2\| \le& \Big(\frac{\Gamma(\alpha_2) t_1^{\alpha_2 - 1} E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|} (1 - t_m)^{\alpha_2 - 1} + 1\Big) \\
& \times m\Gamma(\alpha_2) E_{\alpha_2,\alpha_2}(|\lambda_2|) \sum_{j=1}^{n} B_j \big[r_1 + \|\Phi_1\|\big]^{\tau_j} \big[r_2 + \|\Phi_2\|\big]^{\sigma_j} \\
& + \Big(\frac{\Gamma(\alpha_2) t_1^{\alpha_2 - 1} E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|} + \max\big\{1, t_1^{\alpha_2 + k_2 + l_2}\big\}\Big) \\
& \times \mathbf{B}(\alpha_2 + l_2, k_2 + 1) E_{\alpha_2,\alpha_2}(|\lambda_2|) \sum_{j=1}^{n} b_j \big[r_1 + \|\Phi_1\|\big]^{\tau_j} \big[r_2 + \|\Phi_2\|\big]^{\sigma_j} \\
=& \sum_{j=1}^{n} Q_j \big[r_1 + \|\Phi_1\|\big]^{\tau_j} \big[r_2 + \|\Phi_2\|\big]^{\sigma_j}.
\end{aligned}$$

From the assumption, the inequality of system (3.14) has positive solution (r_1, r_2). We choose $\Omega = \{(x,y) \in E : \ \|x - \Phi_1\| \le r_1, \ \|y - \Phi_2\| \le r_2\}$. Then we get $T(\Omega) \subset \Omega$. Hence the Schauder's fixed point theorem implies that T has a fixed point $(x,y) \in \Omega$. So (x,y) is a solution of BVP (1.3). $\square$

Remark 3.1. Suppose that (D2) holds. Fix $r_1 > 0$ and $r_2 > 0$. Since

$$\lim_{A_j,a_j\to 0}\sum_{j=1}^{n} Q_j\big[r_1+\|\Phi_1\|\big]^{\tau_j}\big[r_2+\|\Phi_2\|\big]^{\sigma_j} = \lim_{B_j,b_j\to 0}\sum_{j=1}^{n} P_j\big[r_1+\|\Phi_1\|\big]^{\tau_j}\big[r_2+\|\Phi_2\|\big]^{\sigma_j} = 0,$$

(3.14) holds for sufficiently small nonnegative constants b_i, a_i $(i = 1, 2, \dots, n)$, B_i, A_i $(i = 1, 2, \dots, n)$. So it is easy to see that BVP (1.3) has at least one solution if the nonnegative constants b_i, a_i $(i = 1, 2, \dots, n)$, B_i, A_i $(i = 1, 2, \dots, n)$ are sufficiently small.

4 Solvability of BVP (1.4)

In this section, we study the solvability of BVP (1.4). We will seek for solutions of BVP (1.4) in E_1.

Lemma 4.1. *Suppose that $\sigma \in L^1(0,1)$ and there exist numbers $k > -1$ and $\max\{-\alpha_1, -k-1\} < l \le 0$ such that $|\sigma(t)| \le t^k(1-t)^l$ for all $t \in (0,1)$. Then $x \in X_1$ is a solution of*

$$ {}^cD_{0^+}^{\alpha_1} x(t) - \lambda_1 x(t) = \sigma(t), \quad t \in (t_i, t_{i+1}), \quad i \in N_0, \tag{4.1}$$

if and only if there exist constants c_i $(i \in N_0)$ such that

$$x(t) = \Gamma(\alpha_1)\sum_{j=0}^{i} c_j E_{\alpha_1,1}(\lambda_1(t-t_j)^{\alpha_1}) + \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds, \quad t \in (t_i, t_{i+1}], \quad i \in N_0. \tag{4.2}$$

Proof. Suppose that x satisfies (4.2). Firstly, we prove that $\int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds$ is convergent. In fact, we have for $t \in (t_i, t_{i+1}]$ that

$$\begin{aligned}
\left|\int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds\right| &= \left|\int_0^t (t-s)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\,ds\right| \\
&\le \int_0^t (t-s)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})|\sigma(s)|\,ds \\
&\le \int_0^t (t-s)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})s^k(1-s)^l\,ds \\
&= \sum_{j=0}^{+\infty}\frac{\lambda_1^j}{((j+1)\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}(t-s)^{\alpha_1 j}s^k(1-s)^l\,ds \\
&\le \sum_{j=0}^{+\infty}\frac{\lambda_1^j}{((j+1)\alpha_1)}\int_0^t (t-s)^{\alpha_1+l-1}(t-s)^{\alpha_1 j}s^k\,ds \\
&= \sum_{j=0}^{+\infty}\frac{\lambda_1^j}{((j+1)\alpha_1)}\,t^{\alpha_1+\alpha_1 j+k+l}\int_0^1 (1-w)^{\alpha_1+\alpha_1 j+l-1}w^k\,dw \\
&\le \sum_{j=0}^{+\infty}\frac{\lambda_1^j t^{\alpha_1 j}}{((j+1)\alpha_1)}\,t^{\alpha_1+k+l}\int_0^1 (1-w)^{\alpha_1+l-1}w^k\,dw \\
&= t^{\alpha_1+k+l}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1 t^{\alpha_1})\mathbf{B}(\alpha_1+l,k+1).
\end{aligned}$$

One knows that $x\big|_{(t_i,t_{i+1}]}$ is continuous. Now, we prove that x satisfies (4.1). In fact, for $t\in(t_0,t_1]$, we have

$$
\begin{aligned}
{}^cD_{0^+}^{\alpha_1}x(t) &= \frac{1}{(1-\alpha_1)}\int_0^t (t-s)^{-\alpha_1}\Big(c_0\mathbf{E}_{\alpha_1,1}(\lambda_1(s)^{\alpha_1}) \\
&\quad + \int_0^s (s-v)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\,dv\Big)'\,ds \\
&= \frac{1}{(1-\alpha_1)}\int_0^t (t-s)^{-\alpha_1}\Big(c_0\sum_{j=0}^{+\infty}\frac{\lambda_1^j s^{\alpha_1 j}}{(\alpha_1 j+1)}+\sum_{j=0}^{+\infty}\frac{\lambda_1^j}{((j+1)\alpha_1)}\int_0^s (s-v)^{\alpha_1-1}(s-v)^{\alpha_1 j}\sigma(v)\,dv\Big)'\,ds \\
&= \frac{c_0}{(1-\alpha_1)}\sum_{j=1}^{+\infty}\int_0^t (t-s)^{-\alpha_1}\frac{\alpha_1 j\lambda_1^j s^{\alpha_1 j-1}}{\Gamma(\alpha_1 j+1)}\,ds \\
&\quad + \frac{1}{(1-\alpha_1)}\sum_{j=0}^{+\infty}\frac{\lambda_1^j}{((j+1)\alpha_1)}\int_0^t (t-s)^{-\alpha_1}\Big(\int_0^s (s-v)^{\alpha_1+\alpha_1 j-1}\sigma(v)\,dv\Big)'\,ds \\
&= \frac{c_0}{(1-\alpha_1)}\sum_{j=1}^{+\infty}\frac{\alpha_1 j\lambda_1^j}{(\alpha_1 j+1)}t^{-\alpha_1+\alpha_1 j}\int_0^1 (1-w)^{-\alpha_1}w^{\alpha_1 j-1}\,dw+\sum_{j=0}^{+\infty}\lambda_1^j\,{}^cD_{0^+}^{\alpha_1}I_{0^+}^{\alpha_1+\alpha_1 j}\sigma(t) \\
&= \frac{c_0}{(1-\alpha_1)}\sum_{j=1}^{+\infty}\frac{\alpha_1 j\lambda_1^j}{(\alpha_1 j+1)}t^{-\alpha_1+\alpha_1 j}\,\frac{(1-\alpha_1)\,(\alpha_1 j)}{(1+\alpha_1 j-\alpha_1)}+\sigma(t)+\sum_{j=1}^{+\infty}\lambda_1^j\int_0^t\frac{(t-s)^{\alpha_1 j-1}}{(\alpha_1 j)}\sigma(s)\,ds \\
&= \lambda_1 c_0\sum_{j=1}^{+\infty}\frac{\alpha_1 j\lambda_1^{j-1}}{(\alpha_1 j+1)}t^{\alpha_1(j-1)}\,\frac{(\alpha_1 j)}{(1+\alpha_1 j-\alpha_1)}+\sigma(t)+\lambda_1\int_0^t (t-s)^{\alpha_1-1}\sum_{j=1}^{+\infty}\lambda_1^{j-1}\frac{(t-s)^{\alpha_1(j-1)}}{(\alpha_1 j)}\sigma(s)\,ds \\
&= \lambda_1 c_0\sum_{j=1}^{+\infty}\frac{\lambda_1^{j-1}t^{\alpha_1(j-1)}}{(\alpha_1 j+1)}\,\frac{(\alpha_1 j+1)}{(\alpha_1(j-1)+1)}+\sigma(t)+\lambda_1\int_0^t (t-s)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\,ds \\
&= \lambda_1 x(t)+\sigma(t).
\end{aligned}
$$

Now, suppose that $t\in(t_i,t_{i+1}]$ $(i\geq 1)$. Similarly we have

$$
\begin{aligned}
{}^cD_{0^+}^{\alpha_1}x(t) &= \frac{1}{(1-\alpha_1)}\int_0^t (t-s)^{-\alpha_1}x'(s)\,ds \\
&= \frac{1}{(1-\alpha_1)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha_1}x'(s)\,ds+\int_{t_i}^t (t-s)^{-\alpha_1}x'(s)\,ds\Big] \\
&= \frac{1}{(1-\alpha_1)}\Big[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha_1} \\
&\quad\times\Big(\sum_{\kappa=0}^{j}c_\kappa\mathbf{E}_{\alpha_1,1}(\lambda_1(s-t_\kappa)^{\alpha})+\int_0^s (s-v)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\,dv\Big)'\,ds \\
&\quad+\int_{t_i}^t (t-s)^{-\alpha_1}\Big(\sum_{\kappa=0}^{i}c_\kappa\mathbf{E}_{\alpha_1,1}(\lambda_1(s-t_\kappa)^{\alpha_1})+\int_0^s (s-v)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\,dv\Big)'\,ds\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(1-\alpha_1)} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1} \Big[\sum_{\kappa=0}^{j} c_\kappa \mathbf{E}_{\alpha_1,1}(\lambda_1 (s-t_\kappa)^{\alpha_1}) \Big]' \, ds \\
&\quad + \frac{1}{(1-\alpha_1)} \int_{t_i}^{t} (t-s)^{-\alpha_1} \Big[\sum_{\kappa=0}^{i} c_\kappa \mathbf{E}_{\alpha_1,1}(\lambda_1 (s-t_\kappa)^{\alpha_1}) \Big]' \, ds \\
&\quad + \frac{1}{(1-\alpha_1)} \int_{t_0}^{t} (t-s)^{-\alpha_1} \Big[\int_0^s (s-v)^{\alpha_1-1} \mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1 (s-v)^{\alpha_1}) \sigma(v) \, dv \Big]' \, ds \\
&= \frac{1}{(1-\alpha_1)} \sum_{j=0}^{i-1} \sum_{\kappa=0}^{j} c_\kappa \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1} \Big[\sum_{m=0}^{+\infty} \frac{\lambda_1^m (s-t_\kappa)^{m\alpha_1}}{(m\alpha_1+1)} \Big]' \, ds \\
&\quad + \frac{1}{(1-\alpha_1)} \sum_{\kappa=0}^{i} c_\kappa \int_{t_i}^{t} (t-s)^{-\alpha_1} \Big[\sum_{m=0}^{+\infty} \frac{\lambda_1^m (s-t_\kappa)^{m\alpha_1}}{(m\alpha_1+1)} \Big]' \, ds \\
&\quad + \frac{1}{(1-\alpha_1)} \sum_{m=0}^{+\infty} \frac{\lambda_1^m}{(\alpha_1(m+1))} \int_0^t (t-s)^{-\alpha_1} \Big[\int_0^s (s-v)^{\alpha_1+m\alpha_1-1} \sigma(v) \, dv \Big]' \, ds \\
&= \frac{1}{(1-\alpha_1)} \sum_{j=0}^{i-1} \sum_{\kappa=0}^{j} c_\kappa \sum_{m=1}^{+\infty} \frac{\lambda_1^m m\alpha_1}{(m\alpha_1+1)} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1} (s-t_\kappa)^{m\alpha_1-1} \, ds \\
&\quad + \frac{1}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{m\alpha_1 \lambda_1^m}{(m\alpha_1+1)} \sum_{\kappa=0}^{i} c_\kappa \int_{t_i}^{t} (t-s)^{-\alpha_1} (s-t_\kappa)^{m\alpha_1-1} \, ds + \sum_{m=0}^{+\infty} \lambda_1^{mc} D_{0^+}^{\alpha_1} I_{0^+}^{\alpha_1(m+1)} \sigma(t) \\
&= \frac{1}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{\lambda_1^m m\alpha_1}{(m\alpha_1+1)} \sum_{j=0}^{i-1} \sum_{\kappa=0}^{j} c_\kappa (t-t_\kappa)^{m\alpha_1-\alpha_1} \int_{\frac{t_j-t_\kappa}{t-t_\kappa}}^{\frac{t_{j+1}-t_\kappa}{t-t_\kappa}} (1-w)^{-\alpha_1} w^{m\alpha_1-1} \, dw \\
&\quad + \frac{1}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{m\alpha_1 \lambda_1^m}{(m\alpha_1+1)} \sum_{\kappa=0}^{i} c_\kappa (t-t_\kappa)^{\alpha_1 m-\alpha_1} \int_{\frac{t_i-y_\kappa}{t-t_\kappa}}^{1} (1-w)^{-\alpha_1} w^{m\alpha_1-1} \, dw + \sum_{m=0}^{+\infty} \lambda_1^m I_{0^+}^{\alpha_1 m} \sigma(t) \\
&= \frac{1}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{\lambda_1^m m\alpha_1}{(m\alpha_1+1)} \sum_{\kappa=0}^{i-1} c_\kappa (t-t_\kappa)^{m\alpha_1-\alpha_1} \sum_{j=\kappa}^{i-1} \int_{\frac{t_j-t_\kappa}{t-t_\kappa}}^{\frac{t_{j+1}-t_\kappa}{t-t_\kappa}} (1-w)^{-\alpha_1} w^{m\alpha_1-1} \, dw \\
&\quad + \frac{1}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{m\alpha_1 \lambda_1^m}{(m\alpha_1+1)} \sum_{\kappa=0}^{i-1} c_\kappa (t-t_\kappa)^{\alpha_1 m-\alpha_1} \int_{\frac{t_i-y_\kappa}{t-t_\kappa}}^{1} (1-w)^{-\alpha_1} w^{m\alpha_1-1} \, dw \\
&\quad + \frac{c_i}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{m\alpha_1 \lambda_1^m}{(m\alpha_1+1)} \int_0^1 (1-w)^{-\alpha_1} w^{m\alpha_1-1} \, dw + \sigma(t) \\
&\quad + \lambda_1 \int_0^t \sum_{m=1}^{+\infty} (t-s)^{\alpha_1-1} \frac{\lambda_1^{m-1} (t-s)^{\alpha_1(m-1)}}{(\alpha_1 m)} \sigma(s) \, ds \\
&= \frac{1}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{\lambda_1^m m\alpha_1}{(m\alpha_1+1)} \sum_{\kappa=0}^{i-1} c_\kappa (t-t_\kappa)^{m\alpha_1-\alpha_1} \int_0^1 (1-w)^{-\alpha_1} w^{m\alpha_1-1} \, dw
\end{aligned}
$$

$$+\frac{c_i}{(1-\alpha_1)}\sum_{m=1}^{+\infty}\frac{m\alpha_1\lambda_1^m}{(m\alpha_1+1)}\int_0^1(1-w)^{-\alpha_1}w^{m\alpha_1-1}\,dw+\sigma(t)$$

$$+\lambda_1\int_0^t\sum_{m=1}^{+\infty}(t-s)^{\alpha_1-1}\frac{\lambda_1^{m-1}(t-s)^{\alpha_1(m-1)}}{(\alpha_1 m)}\sigma(s)\,ds=\lambda_1x(t)+\sigma(t).$$

Thus, due to induction, we conclude that x satisfies (4.1) if x satisfies (4.2).

We now suppose that x is a solution of (4.1). We will prove that x satisfies (4.2). Since x is continuous on $(t_0,t_1]$ and the limit $\lim_{t\to0^+}x(t)$ exists, by Theorem 6.11 in [21], there exists $c_0\in R$ such that

$$x(t)=c_0\mathbf{E}_{\alpha_1,1}(\lambda_1t^{\alpha_1})+\int_0^t\delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds,\quad t\in(t_0,t_1].$$

This means that (4.2) holds for $i=0$. Suppose

$$x(t)=\Phi(t)+c_0\mathbf{E}_{\alpha_1,1}(\lambda_1t^{\alpha_1})+\int_0^t\delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds,\quad t\in(t_1,t_2]. \tag{4.3}$$

By ${}^cD_{0^+}^{\alpha_1}x(t)-\lambda_1x(t)=\sigma(t)$, $t\in(t_1,t_2]$, we get

$$\sigma(t)+\lambda_1x(t)={}^cD_{0^+}^{\alpha_1}x(t)=\frac{1}{(1-\alpha_1)}\int_0^t(t-s)^{-\alpha_1}x'(s)\,ds$$

$$=\frac{1}{(1-\alpha_1)}\left(\int_0^{t_1}(t-s)^{-\alpha_1}x'(s)\,ds+\int_{t_1}^t(t-s)^{-\alpha_1}x'(s)\,ds\right)$$

$$=\frac{1}{(1-\alpha_1)}\left[\int_0^{t_1}(t-s)^{-\alpha_1}\left(c_0\mathbf{E}_{\alpha_1,1}(\lambda_1s^{\alpha_1})+\int_0^s(s-v)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\,dv\right)\right]'ds$$

$$+\int_{t_1}^t(t-s)^{-\alpha_1}\left(\Phi(s)+c_0\mathbf{E}_{\alpha_1,1}(\lambda_1s^{\alpha_1})+\int_0^s(s-v)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\,dv\right)'ds$$

$$=\frac{1}{(1-\alpha_1)}\int_0^{t_1}(t-s)^{-\alpha_1}\big(c_0\mathbf{E}_{\alpha_1,1}(\lambda_1s^{\alpha_1})\big)'ds+\frac{1}{(1-\alpha_1)}\int_{t_1}^t(t-s)^{\alpha_1-1}\big(c_0\mathbf{E}_{\alpha_1,1}(\lambda_1s^{\alpha_1})\big)'ds$$

$$+\frac{1}{(1-\alpha_1)}\int_0^t(t-s)^{-\alpha_1}\left(\int_0^s(s-v)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\,dv\right)'ds$$

$$+\frac{1}{(1-\alpha_1)}\int_{t_1}^t(t-s)^{\alpha_1-1}\Phi'(s)\,ds$$

$$=\frac{c_0}{(1-\alpha_1)}\sum_{m=1}^{+\infty}\frac{m\alpha_1\lambda_1^m}{(m\alpha_1+1)}\int_0^{t_1}(t-s)^{-\alpha_1}s^{m\alpha_1-1}\,ds$$

$$+\frac{c_0}{(1-\alpha_1)}\sum_{m=1}^{+\infty}\frac{m\alpha_1\lambda_1^m}{(m\alpha_1+1)}\int_{t_1}^t(t-s)^{-\alpha_1}s^{m\alpha_1-1}\,ds$$

$$+\frac{1}{(1-\alpha_1)}\int_0^t(t-s)^{-\alpha_1}\left(\int_0^s(s-v)^{\alpha_1-1}\sum_{m=0}^{+\infty}\frac{\lambda_1^m(s-v)^{\alpha_1m}}{(\alpha_1(m+1))}\sigma(v)\,dv\right)'ds$$

$$+\frac{1}{(1-\alpha_1)}\int_{t_1}^{t}(t-s)^{\alpha_1-1}\Phi'(s)\,ds$$

$$=\sigma(t)+\lambda_1\int_0^t(t-s)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\,ds+\lambda_1c_0\mathbf{E}_{\alpha_1,1}(\lambda_1t^{\alpha_1})+{}^cD_{t_1^+}^{\alpha_1}\Phi(t)$$

$$=\sigma(t)+\lambda_1x(t)+{}^cD_{t_1^+}^{\alpha_1}\Phi(t)-\lambda_1\Phi(t).$$

It follows that ${}^cD_{t_1^+}^{\alpha_1}\Phi(t)-\lambda_1\Phi(t)=0$ for all $t\in(t_1,t_2]$. By Theorem 6.11 in [21], there exists $c_1\in R$ such that $\Phi(t)=c_1\mathbf{E}_{\alpha,1}(\lambda(t-t_1)^{\alpha})$ for $t\in(t_1,t_2]$. Substituting Φ into (22), we find that (4.2) holds for $i=2$. Now suppose that (4.2) holds for all $i=0,1,2,\dots,n\le m-1$. By a similar method used above, we can prove that (4.2) holds for $i=n+1$. Due to induction, x satisfies (4.2), $x\big|_{(t_i,t_{i+1}]}$ is continuous and $\lim_{t\to t_i^+}x(t)$ exists. □

Lemma 4.2. *Suppose that $\Delta=1-\Gamma(\alpha_1)E_{\alpha_1,1}(\lambda_1)\neq 0$, $\sigma\in L^1(0,1)$ and that there exist numbers $k>-1$ and $\alpha_1+k+l>0$ such that $|\sigma(t)|\le t^k(1-t)^l$ for all $t\in(0,1)$. Then $x\in X$ is a solution of*

$$\begin{cases}{}^cD_{0^+}^{\alpha_1}x(t)-\lambda_1x(t)=\sigma(t), & t\in(t_i,t_{i+1}),\ \ i\in N_0,\\ x(1)-\lim\limits_{t\to 0}x(t)=0, & \\ \lim\limits_{t\to t_i^+}x(t)=I_i, \quad i\in N, & \end{cases}\tag{4.4}$$

if and only if $x\in X$ and

$$x(t)=\frac{\Gamma(\alpha_1)^2E_{\alpha_1,1}(\lambda_1t^{\alpha_1})}{\Delta}\sum_{j=1}^{m}I_jE_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1})+\frac{\Gamma(\alpha_1)E_{\alpha_1,1}(\lambda_1t^{\alpha_1})}{\Delta}\int_0^1\delta_{\alpha_1,\lambda_1}(1,s)\sigma(s)\,ds$$

$$+\Gamma(\alpha_1)\sum_{j=1}^{i}I_jE_{\alpha_1,1}(\lambda_1(t-t_j)^{\alpha_1})+\int_0^t\delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds,\ \ t\in(t_i,t_{i+1}],\ \ i\in N_0.\tag{4.5}$$

Proof. Let x be a solution of (4.4). Similarly to Step 2 in Lemma 4.1 since $l\le 0$ for $t\in(0,t_1]$, one can see that

$$\left|\int_0^t\delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds\right|\le t^{\alpha_1+l+k}\mathbf{B}(\alpha_1+l,k+1)E_{\alpha_1,\alpha_1}(\lambda_1t^{\alpha_1}).$$

From $\alpha_1+k+l>0$, we get

$$\lim_{t\to 0}\left|\int_0^t(t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\,ds\right|=0.\tag{4.6}$$

Due to Lemma 4.1, there exist constants $A_i\in R(i\in N_0)$ such that

$$x(t)=\Gamma(\alpha_1)\sum_{j=0}^{i}A_jE_{\alpha_1,1}(\lambda_1(t-t_j)^{\alpha_1})+\int_0^t\delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds,\ \ t\in(t_i,t_{i+1}],\ \ i\in N_0.\tag{4.7}$$

Boundary conditions and the impulse assumption in (4.4) yield that

$$\Gamma(\alpha_1)\sum_{j=0}^{m}A_jE_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1})+\int_0^1\delta_{\alpha_1,\lambda_1}(1,s)\sigma(s)\,ds-A_0=0,\quad A_i=I_i,\ \ i\in N.$$

It follows that

$$A_0 = \frac{1}{\Delta}\bigg(\Gamma(\alpha_1)\sum_{j=1}^{m} I_j E_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1}) + \int_0^1 \delta_{\alpha_1,\lambda_1}(1,s)\sigma(s)\,ds\bigg).$$

Substituting A_i ($i \in N_0$) into (4.7), we obviously get (4.5) by changing the term order.

On the other hand, if x satisfies (4.5), then it is easy to see that $x\big|_{(t_i,t_{i+1}]}$ is continuous and the limit $\lim\limits_{t\to t_{i+}} x(t)$ exists for $i \in N_0$. Using Lemma 4.1, we can prove that x satisfies (4.4). □

Lemma 4.3. *Suppose that* $\nabla = 1-\Gamma(\alpha_2)E_{\alpha_2,1}(\lambda_2) \neq 0$, $\sigma \in L^1(0,1)$ *and there exist numbers* $k > -1$ *and* $\max\{-\alpha_2, -k-1\} < l \leq 0$ *such that* $|\sigma(t)| \leq t^k(1-t)^l$ *for all* $t \in (0,1)$. *Then* $x \in Y$ *is a solution of*

$$\begin{cases} {}^cD_{0^+}^{\alpha_2}x(t) - \lambda_2 x(t) = \sigma(t), \quad t \in (t_i, t_{i+1}), \ \ i \in N_0, \\ x(1) - \lim\limits_{t\to 0} x(t) = 0, \\ \lim\limits_{t\to t_i^+} x(t) = J_i, \quad i \in N, \end{cases} \tag{4.8}$$

if and only if $x \in Y$ *and*

$$x(t) = \frac{\Gamma(\alpha_2)^2 E_{\alpha_2,1}(\lambda_2 t^{\alpha_2})}{\nabla}\sum_{j=1}^{m} J_j E_{\alpha_2,1}(\lambda_2(1-t_j)^{\alpha_2}) + \frac{\Gamma(\alpha_2)E_{\alpha_2,1}(\lambda_2 t^{\alpha_2})}{\nabla}\int_0^1 \delta_{\alpha_2,\lambda_2}(1,s)\sigma(s)\,ds$$

$$+\,\Gamma(\alpha_2)\sum_{j=1}^{i} J_j E_{\alpha_2,1}(\lambda_2(t-t_j)^{\alpha_2}) + \int_0^t \delta_{\alpha_2,\lambda_2}(t,s)\sigma(s)\,ds, \ \ t \in (t_i,t_{i+1}], \ \ i \in N_0. \tag{4.9}$$

Proof. The proof is similar to that of Lemma 4.2 and we omit it here. □

Let Δ, ∇ be defined as in Lemmas 4.2 and 4.3. Define the nonlinear operator T on E by $T(x,y)(t) = ((T_1(x,y))(t), (T_2(x,y))(t))$ for $(x,y) \in E$ with

$$\begin{aligned}(T_1(x,y))(t) = {} & \frac{\Gamma(\alpha_1)^2 E_{\alpha_1,1}(\lambda_1 t^{\alpha_1})}{\Delta}\sum_{j=1}^{m} E_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1}) I(t_j, x(t_j), y(t_j)) \\ & + \frac{\Gamma(\alpha_1)E_{\alpha_1,1}(\lambda_1 t^{\alpha_1})}{\Delta}\int_0^1 \delta_{\alpha_1,\lambda_1}(1,s)p_3(s)f_3(s,x(s),y(s))\,ds \\ & + \Gamma(\alpha_1)\sum_{j=1}^{i} E_{\alpha_1,1}(\lambda_1(t-t_j)^{\alpha_1}) I(t_j, x(t_j), y(t_j)) \\ & + \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)p_3(s)f_3(s,x(s),y(s))\,ds, \ \ t \in (t_i,t_{i+1}], \ \ i \in N_0, \\ (T_2(x,y))(t) = {} & \frac{\Gamma(\alpha_2)^2 E_{\alpha_2,1}(\lambda_2 t^{\alpha_2})}{\nabla}\sum_{j=1}^{m} E_{\alpha_2,1}(\lambda_2(1-t_j)^{\alpha_2}) J(t_j, x(t_j), y(t_j)) \\ & + \frac{\Gamma(\alpha_2)E_{\alpha_2,1}(\lambda_2 t^{\alpha_2})}{\nabla}\int_0^1 \delta_{\alpha_2,\lambda_2}(1,s)p_4(s)f_4(s,x(s),y(s))\,ds \\ & + \Gamma(\alpha_2)\sum_{j=1}^{i} E_{\alpha_2,1}(\lambda_2(t-t_j)^{\alpha_2}) J(t_j, x(t_j), y(t_j)) \\ & + \int_0^t \delta_{\alpha_2,\lambda_2}(t,s)p_4(s)f_4(s,x(s),y(s))\,ds, \ \ t \in (t_i,t_{i+1}], \ \ i \in N_0.\end{aligned}$$

Lemma 4.4. *Suppose that f_3, f_4 are* **II-Carathéodory functions**, *I, J* **discrete II-Carathéodory functions**, *Then $(x,y)\in E$ is a solution of BVP (1.4) if and only if (x,y) is a fixed point of T in E_1 and $T: E_1\to E_1$ is well defined and completely continuous.*

Proof. The proof is similar to that of of Lemma 3.4 and is omitted. □

Now, we prove the main theorem in this section by using the Schauder's fixed point theorem [30]. We need the following assumptions:

(D3) f_3, f_4 are **II-Carathéodory functions**, I, J are discrete II-Carathéodory functions and there exist non-decreasing functions $\phi_i,\psi_i : [0,+\infty)\times[0,+\infty)\to[0,+\infty)$, measurable functions $\phi_i:(0,1)\to R$ $(i=1,2)$ and constants I_j, J_j such that

$$\begin{aligned}
\big|f_3(t,x,y)-\phi_3(t)\big| &\le \overline{\phi}_3(x,y),\ \ t\in(t_i,t_{i+1}),\ \ x,y\in R,\ \ i\in N_0,\\
\big|f_4(t,x,y)-\phi_4(t)\big| &\le \overline{\phi}_4(x,y),\ \ t\in(t_i,t_{i+1}),\ \ x,y\in R,\ \ i\in N_0,\\
\big|I(t_j,x,y)-I_j\big| &\le \overline{\psi}_3(x,y),\ \ j\in N,\ \ x,y\in R,\\
\big|J(t_j,x,y)-J_j\big| &\le \overline{\psi}_4(x,y),\ \ j\in N,\ \ x,y\in R.
\end{aligned}$$

(D4) f_3, f_4 are **II-Carathéodory functions**, I, J are discrete II-Carathéodory functions and there exist nonnegative constants I_i, J_i, b_i, a_i, B_i, A_i, τ_j, σ_j $(j=1,2,\ldots,n)$ and measurable functions $\phi_i:(0,1)\to R$ $(i=1,2)$ such that

$$\begin{aligned}
\big|f_3(t,x,y)-\phi_1(t)\big| &\le \sum_{j=1}^{n} a_j|x|^{\tau_j}|y|^{\sigma_j},\ \ t\in(t_i,t_{i+1}),\ \ x,y\in R,\ \ i\in N_0,\\
\big|f_4(t,x,y)-\phi_2(t)\big| &\le \sum_{j=1}^{n} b_j|x|^{\tau_j}|y|^{\sigma_j},\ \ t\in(t_i,t_{i+1}),\ \ x,y\in R,\ \ i\in N_0,\\
\big|I(t_j,x,y)-I_j\big| &\le \sum_{j=1}^{n} A_j|x|^{\tau_j}|y|^{\sigma_j},\ \ j\in N,\ \ x,y\in R,\\
\big|J(t_j,x,y)-J_j\big| &\le \sum_{j=1}^{n} A_j|x|^{\tau_j}|y|^{\sigma_j},\ \ j\in N,\ \ x,y\in R.
\end{aligned}$$

Define

$$\begin{aligned}
\Phi_1(t) &= \frac{\Gamma(\alpha_1)^2E_{\alpha_1,1}(\lambda_1t^{\alpha_1})}{\Delta}\sum_{j=1}^{m}E_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1})I_j\\
&\quad+\frac{\Gamma(\alpha_1)E_{\alpha_1,1}(\lambda_1t^{\alpha_1})}{\Delta}\int_0^1\delta\alpha_1,\lambda_1(1,s)p_1(s)\phi_1(s)\,ds+\Gamma(\alpha_1)\sum_{j=1}^{i}E_{\alpha_1,1}(\lambda_1(t-t_j)^{\alpha_1})I_j\\
&\quad+\int_0^t\delta_{\alpha_1,\lambda_1}(t,s)p_1(s)\phi_1(s)\,ds,\ \ t\in(t_i,t_{i+1}],\ \ i\in N_0,\\
\Phi_2(t) &= \frac{\Gamma(\alpha_2)^2E_{\alpha_2,1}(\lambda_2t^{\alpha_2})}{\nabla}\sum_{j=1}^{m}E_{\alpha_2,1}(\lambda_2(1-t_j)^{\alpha_2})J_j\\
&\quad+\frac{\Gamma(\alpha_2)E_{\alpha_2,1}(\lambda_2t^{\alpha_2})}{\nabla}\int_0^1\delta_{\alpha_2,\lambda_2}(1,s)p_2(s)\phi_2(s)\,ds+\Gamma(\alpha_2)\sum_{j=1}^{i}E_{\alpha_2,1}(\lambda_2(t-t_j)^{\alpha_2})J_j\\
&\quad+\int_0^t\delta_{\alpha_2,\lambda_2}(t,s)p_2(s)\phi_2(s)\,ds,\ \ t\in(t_i,t_{i+1}],\ \ i\in N_0.
\end{aligned}$$

Denote

$$\begin{aligned}
P_1 &= \frac{\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)}{|\Delta|}\,\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,1}(|\lambda_1|)+\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,1}(|\lambda_1|),\\
Q_1 &= \frac{\Gamma(\alpha_1)^2E_{\alpha_1,1}(|\lambda_1|}{|\Delta|}\,mE_{\alpha_1,1}(|\lambda_1|)+m\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(|\lambda_1|),\\
P_2 &= \frac{\Gamma(\alpha_2)E_{\alpha_2,1}(|\lambda_2|)}{|\nabla|}\,\mathbf{B}(\alpha_2+l_2,k_2+1)E_{\alpha_2,1}(|\lambda_2|)+\mathbf{B}(\alpha_2+l_2,k_2+1)E_{\alpha_2,1}(|\lambda_2|),\\
Q_2 &= \frac{\Gamma(\alpha_2)^2E_{\alpha_2,1}(|\lambda_2|}{|\nabla|}\,mE_{\alpha_2,1}(|\lambda_2|)+m\Gamma(\alpha_2)E_{\alpha_2,1}(|\lambda_2|).
\end{aligned}$$

Theorem 4.1. *Suppose that* (D3) *holds. Then BVP* (1.4) *has at least one solution if*

$$\begin{aligned}
&P_1\overline{\phi}_3\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)+Q_1\overline{\psi}_3\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)\le r_1,\\
&P_2\overline{\phi}_4\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)+Q_2\overline{\psi}_4\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)\le r_2
\end{aligned}\tag{4.10}$$

has a couple of positive solutions (r_1,r_2).

Proof. To apply the Schauder's fixed point theorem, we have to define a closed convex bounded subset Ω of E such that $T(\Omega)\subseteq\Omega$.

Let $r_1>0$, $r_2>0$, denote $\Omega=\{(x,y)\in E:\ \|x-\Phi_1\|\le r_1,\ \|y-\Phi_2\|\le r_2\}$. For $(x,y)\in\Omega$, we get

$$\|x\|\le\|x-\Phi_1\|+\|\Phi_1\|\le r_1+\|\Phi_1\|,\quad \|y\|\le\|y-\Phi_2\|+\|\Phi_2\|\le r_2+\|\Phi_2\|.$$

Then (D3) implies that

$$\begin{aligned}
\big|f_3(t,x(t),y(t))-\phi_3(t)\big| &\le \overline{\phi}_3\big(|x(t)|,|y(t)|\big)\le\phi_1\big(\|x\|,\|y\|\big)\\
&\le\overline{\phi}_3\big(r_1+\|\Phi_2\|,r_2+\|\Phi_2\|\big),\ \ t\in(t_i,t_{i+1}),\ \ i\in N_0,\\
\big|f_4(t,x(t),y(t))-\phi_4(t)\big| &\le\overline{\phi}_4\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big),\ \ t\in(t_i,t_{i+1}),\ \ i\in N_0,\\
\big|I(t_j,x(t_j),y(t_j))-I_j\big| &\le\overline{\psi}_3\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big),\ \ j\in N,\\
\big|J(t_j,x(t_j),y(t_j))-J_j\big| &\le\overline{\psi}_4\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big),\ \ j\in N.
\end{aligned}$$

By the definition of T, using the above inequalities and the first inequality in the proof of Lemma 4.2, we get

$$\begin{aligned}
\|T_1(x,y)-\Phi_1\| &\le \frac{\Gamma(\alpha_1)^2E_{\alpha_1,1}(\lambda_1t^{\alpha_1})}{|\Delta|}\sum_{j=1}^{m}E_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1})\big|I(t_j,x(t_j),y(t_j))-I_j\big|\\
&\quad+\frac{\Gamma(\alpha_1)E_{\alpha_1,1}(\lambda_1t^{\alpha_1})}{|\Delta|}\int_0^1\delta_{\alpha_1,\lambda_1}(1,s)|p_3(s)|\,\big|f_3(s,x(s),y(s))-\phi_3(s)\big|\,ds\\
&\quad+\Gamma(\alpha_1)\sum_{j=1}^{i}E_{\alpha_1,1}(\lambda_1(t-t_j)^{\alpha_1})\big|I(t_j,x(t_j),y(t_j))-I_j\big|\\
&\quad+\int_0^t\delta_{\alpha_1,\lambda_1}(t,s)|p_3(s)|\,\big|f_3(s,x(s),y(s))-\phi_3(s)\big|\,ds\\
&\le\frac{\Gamma(\alpha_1)^2E_{\alpha_1,1}(|\lambda_1|}{|\Delta|}\,mE_{\alpha_1,1}(|\lambda_1|)\big(\overline{\psi}_3(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|)\big)\\
&\quad+\frac{\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)}{|\Delta|}\int_0^1\delta_{\alpha_1,\lambda_1}(1,s)|p_3(s)|\,ds(v)\\
&\quad+m\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)\big(\overline{\psi}_3(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|)\big)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \delta_{\alpha_1,\lambda_1}(t,s)|p_3(s)|\,ds\,\big(\overline{\phi}_3(r_1+\|\Phi_2\|,r_2+\|\Phi_2\|)\big)\\
&\le \frac{\Gamma(\alpha_1)^2E_{\alpha_1,1}(|\lambda_1|)}{|\Delta|}\,mE_{\alpha_1,1}(|\lambda_1|)\big(\overline{\psi}_3(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|)\big)\\
&\quad+\frac{\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)}{|\Delta|}\,\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,1}(|\lambda_1|)\big(\overline{\phi}_3(r_1+\|\Phi_2\|,r_2+\|\Phi_2\|)\big)\\
&\quad+m\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)\big(\overline{\psi}_3(r_1+\|\Phi_1\|,\ r_2+\|\Phi_2\|)\big)\\
&\quad+t^{\alpha_1+l_1+k_1}\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,1}(\lambda_1t^{\alpha_1})\big(\overline{\phi}_3(r_1+\|\Phi_2\|,r_2+\|\Phi_2\|)\big)\\
&\le P_1\overline{\phi}_3\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)+Q_1\overline{\psi}_3\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big).
\end{aligned}
$$

Similarly, we have

$$
\|T_2(x,y)-\Phi_2\|\le P_2\overline{\phi}_4\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)+Q_2\overline{\psi}_4\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big).
$$

From the assumption, the inequality of system (4.10) has a positive solution (r_1,r_2). We choose $\Omega=\{(x,y)\in E:\ \|x-\Phi_1\|\le r_1,\ \|y-\Phi_2\|\le r_2\}$. Then we get $T(\Omega)\subset\Omega$. Hence the Schauder's fixed point theorem implies that T has a fixed point $(x,y)\in\Omega$. So (x,y) is a solution of BVP (1.4). □

Denote

$$
\begin{aligned}
P_j&=\Big(\frac{\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)}{|\Delta|}+1\Big)m\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)A_j\\
&\quad+\Big(\frac{\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)}{|\Delta|}+1\Big)\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,1}(|\lambda_1|)a_j,\quad j=1,2,\dots,n,\\
Q_j&=\Big(\frac{\Gamma(\alpha_2)E_{\alpha_2,1}(|\lambda_2|)}{|\nabla|}+1\Big)m\Gamma(\alpha_2)E_{\alpha_2,1}(|\lambda_2|)B_j\\
&\quad+\Big(\frac{\Gamma(\alpha_2)E_{\alpha_2,1}(|\lambda_2|)}{|\nabla|}+1\Big)\mathbf{B}(\alpha_2+l_2,k_2+1)E_{\alpha_2,1}(|\lambda_2|)b_j,\quad j=1,2,\dots,n.
\end{aligned}
$$

Theorem 4.2. *Suppose that* (D4) *holds. Then BVP* (1.4) *has at least one solution if*

$$
\sum_{j=1}^n P_j\big[r_1+\|\Phi_1\|\big]^{\tau_j}\big[r_2+\|\Phi_2\|\big]^{\sigma_j}\le r_1,\quad \sum_{j=1}^n Q_j\big[r_1+\|\Phi_1\|\big]^{\tau_j}\big[r_2+\|\Phi_2\|\big]^{\sigma_j}\le r_2 \tag{4.11}
$$

has a couple of positive solutions (r_1,r_2).

Proof. The proof is similar to that of Lemma 3.2 and is omitted. □

Remark 4.1. Suppose that (D4) holds. Fix $r_1>0$ and $r_2>0$. Since

$$
\lim_{A_j,a_j\to0}\sum_{j=1}^n Q_j\big[r_1+\|\Phi_1\|\big]^{\tau_j}\big[r_2+\|\Phi_2\|\big]^{\sigma_j}=\lim_{B_j,b_j\to0}\sum_{j=1}^n P_j\big[r_1+\|\Phi_1\|\big]^{\tau_j}\big[r_2+\|\Phi_2\|\big]^{\sigma_j}=0,
$$

(4.11) holds for sufficiently small nonnegative constants b_i, a_i $(i=1,2,\dots,n)$, B_i, A_i $(i=1,2,\dots,n)$, we know that BVP (1.4) has at least one solution if the nonnegative constants b_i, a_i $(i=1,2,\dots,n)$, B_i, A_i $(i=1,2,\dots,n)$ are sufficiently small.

5 Examples

To illustrate the usefulness of our main results, we present two examples to see that Theorems 3.1 and 4.1 can be readily applied.

Example 5.1. Consider the following impulsive boundary value problem

$$\begin{cases} D_{0^+}^{\frac{2}{5}}u(t)-u(t)=q_1(t)\big[c_1+a_1\big[(t-t_i)^{\frac{3}{5}}u(t)\big]^{\sigma}+a_2\big[(t-t_i)^{\frac{2}{5}}v(t)\big]^{\sigma}\big], & i=0,1,2,3,\\ D_{0^+}^{\frac{3}{5}}v(t)-v(t)=q_2(t)\big[c_2+b_1\big[(t-t_i)^{\frac{3}{5}}u(t)\big]^{\sigma}+b_2\big[(t-t_i)^{\frac{2}{5}}v(t)\big]^{\sigma}\big], & i=0,1,2,3,\\ u(1)-\lim\limits_{t\to 0}t^{\frac{3}{5}}u(t)=0, \quad v(1)-\lim\limits_{t\to 0}t^{\frac{2}{5}}v(t)=0, & \\ \lim\limits_{t\to t_i^+}(t-t_i)^{\frac{3}{5}}u(t)=\lim\limits_{t\to t_i^+}(t-t_i)^{\frac{2}{5}}v(t)=0, & i=1,2,3, \end{cases} \tag{5.1}$$

where $c_1,c_2\in R$, $b_1,a_1,b_2,a_2\geq 0$ are constants, $0=t_0<t_1=\frac{1}{4}<t_2=\frac{1}{3}<t_3=\frac{1}{2}<t_4=1$ with $m=3$, $q_1(t)=q_2(t)=t^{-\frac{1}{15}}(1-t)^{-\frac{1}{15}}$, $t\in(0,1)$, $\sigma>0$.

We apply Theorem 3.2 to get solutions of BVP (5.1). Corresponding to BVP (1.3), we have $\alpha_1=\frac{2}{5}$, $\alpha_2=\frac{3}{5}$, $\lambda_1=\lambda_2=1$, q_1, q_2 satisfy the condition $|q_i(t)|\leq t^{k_i}(1-t)^{l_i}$ with $k_i=l_i=-\frac{1}{15}$ $(i=1,2)$, f_1, f_2, I, J satisfy the following items:

$$\begin{gathered} f_1\big(t,(t-t_i)^{-\frac{3}{5}}x,(t-t_i)^{-\frac{2}{5}}y\big)=c_1+a_1x^{\sigma}+a_2y^{\sigma},\ \ t\in(t_i,t_{i+1}],\ \ i=0,1,2,3,\\ f_2\big(t,(t-t_i)^{-\frac{3}{5}}x,(t-t_i)^{-\frac{2}{5}}y\big)=c_2+b_1x^{\sigma}+b_2y^{\sigma},\ \ t\in(t_i,t_{i+1}],\ \ i=0,1,2,3,\\ I(t_i,x,y)=J(t_i,x,y)=0,\ \ i=1,2,3. \end{gathered}$$

It is easy to show that f_1, f_2 are I-Carathéodory functions, I, J are discrete I-Carathéodory functions. Furthermore, choose $\phi_1(t)=c_1$ and $\phi_2(t)=c_2$, $I_i=J_i=0$. It is easy to see

$$\begin{gathered} \Big|f_1\big(t,(t-t_i)^{-\frac{3}{5}}x,(t-t_i)^{-\frac{2}{5}}y\big)-\phi_1(t)\Big|\leq a_1|x|^{\sigma}+a_2|y|^{\sigma},\ \ t\in(t_i,t_{i+1}],\ \ i=0,1,2,3,\\ \Big|f_2\big(t,(t-t_i)^{-\frac{3}{5}}x,(t-t_i)^{-\frac{2}{5}}y\big)-\phi_1(t)\Big|\leq b_1|x|^{\sigma}+b_2|y|^{\sigma},\ \ t\in(t_i,t_{i+1}],\ \ i=0,1,2,3,\\ \big|I(t_i,x,y)-I_i\big|=|J(t_i,x,y)-J_i|=0,\ \ i=1,2,3. \end{gathered}$$

By the Mathlab tool we find that

$$\begin{aligned} 2<E_{\frac{2}{5},\frac{2}{5}}(1)&=\sum_{j=0}^{+\infty}\frac{1}{\Gamma(\frac{2}{5}\,(j+1))}=\frac{1}{\Gamma(\frac{2}{5})}+\frac{1}{\Gamma(\frac{4}{5})}+\frac{1}{\Gamma(\frac{6}{5})}+\cdots+\frac{1}{\Gamma(\frac{30}{5})}+\cdots\\ &\leq 7+\frac{1}{\Gamma(\frac{32}{5})}+\frac{1}{\Gamma(\frac{34}{5})}+\frac{1}{\Gamma(\frac{36}{5})}+\frac{1}{\Gamma(\frac{38}{5})}+\cdots\\ &\leq 7+\frac{1}{\Gamma(\frac{32}{5})}+\frac{1}{\Gamma(\frac{34}{5})}+\frac{1}{\Gamma(\frac{36}{5})}+\frac{5}{33\Gamma(\frac{32}{5})}+\frac{5}{35\Gamma(\frac{34}{5})}+\frac{5}{37\Gamma(\frac{36}{5})}+\cdots\\ &\leq 7+\frac{1}{\Gamma(\frac{32}{5})}\frac{1}{1-\frac{5}{33}}+\frac{1}{\Gamma(\frac{34}{5})}\frac{1}{1-\frac{5}{35}}+\frac{1}{\Gamma(\frac{36}{5})}\frac{1}{1-\frac{5}{37}}\leq 8,\\ 2<E_{\frac{3}{5},\frac{3}{5}}(1)&=\frac{1}{\Gamma(\frac{3}{5})}+\frac{1}{\Gamma(\frac{6}{5})}+\frac{1}{\Gamma(\frac{9}{5})}+\frac{1}{\Gamma(\frac{12}{5})}+\frac{1}{\Gamma(\frac{15}{5})}+\frac{1}{\Gamma(\frac{21}{5})}+\frac{1}{\Gamma(\frac{24}{5})}+\cdots\\ &\leq 4+\frac{5}{7}\frac{1}{\Gamma(\frac{7}{5})}+\frac{5}{10}\frac{5}{5}\frac{1}{\Gamma(\frac{5}{5})}+\frac{5}{13}\frac{5}{8}\frac{1}{\Gamma(\frac{8}{5})}+\frac{5}{16}\frac{5}{11}\frac{5}{6}\frac{1}{\Gamma(\frac{6}{5})}\\ &\quad+\frac{5}{19}\frac{5}{14}\frac{5}{9}\frac{1}{\Gamma(\frac{9}{5})}+\frac{5}{22}\frac{5}{17}\frac{5}{12}\frac{5}{7}\frac{1}{\Gamma(\frac{7}{5})}+\cdots\\ &\leq 4+\frac{5}{7}\frac{1}{\Gamma(\frac{7}{5})}+\frac{5}{10}\frac{1}{\Gamma(\frac{5}{5})}+\frac{5}{7}\frac{5}{7}\frac{1}{\Gamma(\frac{7}{5})}+\frac{5}{10}\frac{5}{10}\frac{1}{\Gamma(\frac{5}{5})}\\ &\quad+\frac{5}{7}\frac{5}{7}\frac{5}{7}\frac{1}{\Gamma(\frac{7}{5})}+\frac{5}{10}\frac{5}{10}\frac{5}{10}\frac{1}{\Gamma(\frac{5}{5})}+\cdots\\ &=4+\frac{1}{\Gamma(\frac{7}{5})}\frac{5}{7-5}+\frac{1}{\Gamma(\frac{5}{5})}\frac{5}{10-5}\leq 8. \end{aligned}$$

By direct computation, using $\alpha_j + k_j + l_j > 0$, we get

$$P_j = \left(\frac{\Gamma(\frac{2}{5})\sqrt[5]{64}\,E_{\frac{2}{5},\frac{2}{5}}(1)}{|1-\Gamma(\frac{2}{5})E_{\frac{2}{5},\frac{2}{5}}(1)|} + 1\right)\mathbf{B}\Big(\frac{1}{3},\frac{14}{15}\Big)E_{\frac{2}{5},\frac{2}{5}}(1)a_j < 1035a_j,\quad j=1,2,$$
$$Q_j = \left(\frac{\Gamma(\frac{3}{5})\sqrt[5]{16}\,E_{\frac{3}{5},\frac{3}{5}}(1)}{|1-\Gamma(\frac{3}{5})E_{\frac{3}{5},\frac{3}{5}}(1)|} + 1\right)\mathbf{B}\Big(\frac{8}{15},\frac{14}{15}\Big)E_{\frac{3}{5},\frac{3}{5}}(1)b_j < 16b_j,\quad j=1,2.$$

One finds that

$$\Phi_1(t) = c_1\,\frac{\Gamma(\alpha_1)\delta_{\alpha_1,\lambda_1}(t,0)}{\Delta}\int_0^1 \delta_{\alpha_1,\lambda_1}(1,s)p_1(s)\,ds + c_1\int_0^t \delta_{\alpha_1,\lambda_1}(t,s)p_1(s)\,ds$$

and

$$\Phi_2(t) = c_2\,\frac{\Gamma(\alpha_2)\delta_{\alpha_2,\lambda_2}(t,0)}{\nabla}\int_0^1 \delta_{\alpha_2,\lambda_2}(1,s)p_2(s)\,ds + c_2\int_0^t \delta_{\alpha_2,\lambda_2}(t,s)p_2(s)\,ds.$$

Then

$$\|\Phi_1\| \le |c_1|\left(\frac{\Gamma(\frac{2}{5})\sqrt[5]{64}\,E_{\frac{2}{5},\frac{2}{5}}(1)}{|1-\Gamma(\frac{2}{5})E_{\frac{2}{5},\frac{2}{5}}(1)|} + 1\right)\mathbf{B}\Big(\frac{1}{3},\frac{14}{15}\Big)E_{\frac{2}{5},\frac{2}{5}}(1) \le 1035|c_1|.$$

We can also get

$$\|\Phi_2\| \le |c_2|\left(\frac{\Gamma(\frac{3}{5})\sqrt[5]{16}\,E_{\frac{3}{5},\frac{3}{5}}(1)}{|1-\Gamma(\frac{3}{5})E_{\frac{3}{5},\frac{3}{5}}(1)|} + 1\right)\mathbf{B}\Big(\frac{8}{15},\frac{14}{15}\Big)E_{\frac{3}{5},\frac{3}{5}}(1) \le 16|c_2|.$$

Then Theorem 3.2 implies that BVP (5.1) has at least one solution if

$$\begin{aligned} 1035a_1\big[r_1 + 1035|c_1|\big]^\sigma + 1035a_2\big[r_2+16|c_2|\big]^\sigma &\le r_1,\\ 16b_1\big[r_1 + 1035|c_1|\big]^\sigma + 16b_2\big[r_2+16|c_2|\big]^\sigma &\le r_2 \end{aligned} \tag{5.2}$$

has a couple of positive solutions (r_1, r_2). So BVP (5.1) has at least one solution for every $c_1, c_2 \in R$ and sufficiently small a_i, b_i.

Example 5.2. Consider the following impulsive boundary value problem

$$\begin{cases} {}^cD_{0^+}^{\frac{2}{5}}u(t) - u(t) = q_1(t)\big[c_1 + a_1[u(t)]^\sigma + a_2[v(t)]^\sigma\big], & t\in(t_i,t_{i+1}),\ i=0,1,2,3,\\ {}^cD_{0^+}^{\frac{3}{5}}v(t) - v(t) = q_2(t)\big[c_2 + b_1[u(t)]^\sigma + a_2[v(t)]^\sigma\big], & t\in(t_i,t_{i+1}),\ i=0,1,2,3,\\ u(1) - \lim\limits_{t\to 0}u(t) = 0,\quad v(1)-\lim\limits_{t\to 0}v(t) = 0, & \\ \lim\limits_{t\to t_i^+}u(t) = \lim\limits_{t\to t_i^+}v(t) = t_i, & i\in N, \end{cases} \tag{5.3}$$

where $c_1, c_2 \in R$, $b_1, a_1, b_2, a_2 \ge 0$ are constants, $0 = t_0 < t_1 = \frac{1}{4} < t_2 = \frac{1}{3} < t_3 = \frac{1}{2} < t_4 = 1$ with $m = 3$, $q_1(t) = q_2(t) = t^{-\frac{1}{15}}(1-t)^{-\frac{1}{15}}$, $t \in (0,1)$.

We apply Theorem 4.2 to get solutions of BVP (5.3). Corresponding to BVP (1.4), we have $\alpha_i = \frac{2}{5}$, $\beta_i = \frac{3}{5}$, $\lambda_1 = \lambda_2 = 1$, q_1, q_2 satisfy the condition $|q_i(t)| \le t^{k_i}(1-t)^{l_1}$ with $k_i = l_i = -\frac{1}{15}$, f_3, f_4, I, J are defined by $f_3(t,x,y) = c_1 + a_1x^\sigma + a_2y^\sigma$, $f_4(t,x,y) = c_2 + b_1x^\sigma + b_2y^\sigma$ and $I(t_i,x,y) = J(t_i,x,y) = t_i$, $i = 1,2,3$.

It is easy to show that f_3, f_4 are II-Carathéodory functions, I, J are discrete II-Carathéodory functions. Choose $\phi_1(t) = c_1$ and $\phi_2(t) = c_2$, $I_i = J_i = t_i$. It is easy to see

$$\begin{aligned} \big|f_3(t,x,y) - \phi_1(t)\big| &\le a_1|x|^\sigma + a_2|y|^\sigma,\quad t\in(t_i,t_{i+1}),\quad x,y\in R,\\ \big|f_4(t,x,y) - \phi_2(t)\big| &\le b_1|x|^\sigma + b_2|y|^\sigma,\quad t\in(t_i,t_{i+1}),\quad x,y\in R, \end{aligned}$$

$$\big|I(t_i,x,y)-I_i\big| = \big|J(t_i,x,y)-J_i\big| = 0, \ \ i\in N, \ \ x,y\in R.$$

By the direct computation, we get

$$P_j = \left(\frac{\Gamma(\frac{2}{5})E_{\frac{2}{5},1}(1)}{|1-\Gamma(\frac{2}{5})E_{\frac{2}{5},1}(1)|}+1\right)\mathbf{B}\Big(\frac{1}{3},\frac{14}{15}\Big)E_{\frac{2}{5},1}(1)a_j < 465a_j, \ \ j=1,2,$$
$$Q_j = \left(\frac{\Gamma(\frac{3}{5})E_{\frac{3}{5},1}(1)}{|1-\Gamma(\frac{3}{5})E_{\frac{3}{5},1}(1)|}+1\right)\mathbf{B}\Big(\frac{8}{15},\frac{14}{15}\Big)E_{\frac{3}{5},1}(1)b_j < 203b_j, \ \ j=1,2.$$

Then Theorem 4.2 implies that BVP (5.3) has at least one solution if (5.2) holds. So BVP (5.3) has at least one solution for every $c_1, c_2 \in R$ and sufficiently small a_i, b_i.

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