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Sergey Shchogolev

ON THE BLOCK SEPARATION OF A LINEAR HOMOGENEOUS DIFFERENTIAL SYSTEM WITH OSCILLATING COEFFICIENTS IN A SPECIAL CASE

Abstract. For the linear homogeneous system of differential equations, coefficients of which are represented by an absolutely and uniformly convergent Fourier series with slowly varying coefficients and frequency, the conditions of existence of the linear transformation with coefficients of similar structure leading this system to a block-diagonal form in a special case are obtained.

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რეზიუმე. დიფერენციალურ განტოლებათა წრფივი ერთგვაროვანი სისტემისთვის, რომლის კოეფიციენტები წარმოიდგინება აბსოლუტურად და თანაბრად კრებადი ფურიეს მწკრივებით ნელა ცვლადი კოეფიციენტებით და სიხშირით, დადგენილია ანალოგიური სტრუქტურის კოეფიციენტების მქონე ისეთი წრფივი გარდაქმნის არსებობის პირობები, რომელსაც ერთ სპეციალურ შემთხვევაში ეს სისტემა დაჰყავს უჯრულ-დიაგონალურ ფორმამდე.

1 Introduction

This article continues the research started by the author in [1] on the problem of the block separation of the linear homogeneous system of differential equations, whose coefficients are represented by an absolutely and uniformly convergent Fourier series with slowly varying in some sense coefficients and frequency. Now we study a special case which by the conditions of the theorem proved in [1] is not covered.

2 Basic notations and definitions

Let $G = \{t, \varepsilon :\ t \in \mathbf{R},\ \varepsilon \in [0, \varepsilon_0],\ \varepsilon_0 \in \mathbf{R}^+\}$.

Definition 2.1. We say that a function $p(t, \varepsilon)$, generally complex-valued, belongs to the class $S(m; \varepsilon_0)$, $m \in \mathbf{N} \cup \{0\}$, if $t, \varepsilon \in G$ and

1) $p(t, \varepsilon) \in C^m(G)$ with respect to t;

2) $\dfrac{d^k p(t, \varepsilon)}{dt^k} = \varepsilon^k p_k^*(t, \varepsilon), \quad \sup\limits_G |p_k^*(t, \varepsilon)| < +\infty\ (0 \le k \le m)$.

Slowly variability of a function is understood in the sense of its belonging to the class $S(m; \varepsilon_0)$. As examples of functions of this class may serve, in general, complex-valued, bounded together with their derivatives up to and including the order m functions that depend on the “slow time” $\tau = \varepsilon t$: $\sin \tau$, $\operatorname{arctg} \tau$ etc.

Definition 2.2. We say that a function $f(t, \varepsilon, \theta(t, \varepsilon))$ belongs to the class $F(m; \varepsilon_0; \theta)$, $m \in \mathbf{N} \cup \{0\}$, if it can be represented as

$$f(t, \varepsilon, \theta(t, \varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t, \varepsilon) \exp(in\theta(t, \varepsilon)),$$

and

1) $f_n(t, \varepsilon) \in S(m; \varepsilon_0)$, $\dfrac{d^k f_n(t, \varepsilon)}{dt^k} = \varepsilon^k f_{nk}(t, \varepsilon)\ (n \in \mathbf{Z},\ 0 \le k \le m)$;

2) $\|f\|_{F(m;\varepsilon_0;\theta)} \overset{def}{=} \sum\limits_{k=0}^{m} \sum\limits_{n=-\infty}^{\infty} \sup\limits_G |f_{nk}(t, \varepsilon)| < +\infty$,

3) $\theta(t, \varepsilon) = \displaystyle\int_0^t \varphi(\tau, \varepsilon)\, d\tau$, $\varphi(t, \varepsilon) \in \mathbf{R}^+$, $\varphi(t, \varepsilon) \in S(m; \varepsilon_0)$, $\inf\limits_G \varphi(t, \varepsilon) > 0$.

Some properties of functions from the class $F(m; \varepsilon_0; \theta)$ are described in [1].

For any function $f(t, \varepsilon, \theta) \in F(m; \varepsilon_0; \theta)$ denote

$$\Gamma_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t, \varepsilon, u) \exp(-inu)\, du, \quad I(f) = f - \Gamma_0(f).$$

We say that the function $f(t, \varepsilon, \theta) \in F(m; \varepsilon_0; \theta)$ satisfies *condition* (A), if $\Gamma_0(f) \equiv 0$.

Let $A(t, \varepsilon, \theta) = (a_{js}(t, \varepsilon, \theta))_{j=\overline{1,M};\, s=\overline{1,K}}$, $a_{js} \in F(m; \varepsilon_0; \theta)$ $(j = \overline{1, M};\ s = \overline{1, K})$. Denote

$$\|A\|^*_{F(m;\varepsilon_0;\theta)} = \max_{1 \le j \le M} \sum_{l=1}^{K} \|a_{jl}(t, \varepsilon, \theta)\|_{F(m;\varepsilon_0;\theta)}.$$

3 Statement of the problem

We consider the system of differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= H_1(\varphi)x_1 + \mu(B_{11}(t,\varepsilon,\theta)x_1 + B_{12}(t,\varepsilon,\theta)x_2),\\ \frac{dx_2}{dt} &= H_2(\varphi)x_2 + \mu(B_{21}(t,\varepsilon,\theta)x_1 + B_{22}(t,\varepsilon,\theta)x_2),\end{aligned} \tag{3.1}$$

where $x_1 = \mathrm{colon}(x_{11},\dots,x_{1N_1})$, $x_2 = \mathrm{colon}(x_{21},\dots,x_{2N_2})$,

$$H_1(\varphi) = \begin{pmatrix} ip\varphi & 0 & \cdots & 0 & 0\\ 1 & ip\varphi & \cdots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \cdots & ip\varphi & 0\\ 0 & 0 & \cdots & 1 & ip\varphi\end{pmatrix}, \quad H_2(\varphi) = \begin{pmatrix} ir\varphi & 0 & \cdots & 0 & 0\\ 1 & ir\varphi & \cdots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \cdots & ir\varphi & 0\\ 0 & 0 & \cdots & 1 & ir\varphi\end{pmatrix}$$

are the Jordan blocks of dimensions N_1 and N_2, respectively ($N_1 + N_2 = N$); $p, r \in \mathbf{Z}$; $B_{jk}(t,\varepsilon,\theta)$ are the $(N_j \times N_k)$-matrices with elements from the class $F(m;\varepsilon;\theta)$; $\varphi(t,\varepsilon)$ is the function appearing in the definition of the class $F(m;\varepsilon;\theta)$; $\mu \in (0,1)$. In this sense, we are dealing with the resonance case.

Just as in [1], we study the question of the existence as well as the properties of the transformation of the form

$$x_j = L_{j1}(t,\varepsilon,\theta,\mu)\widetilde{x}_1 + L_{j2}(t,\varepsilon,\theta,\mu)\widetilde{x}_2, \quad j = 1,2, \tag{3.2}$$

where the elements L_{jk} $(j,k = 1,2)$ of $(N_j \times N_k)$-matrices belong to the class $F(m-1;\varepsilon_1;\theta)$ $(0 < \varepsilon_1 \le \varepsilon_0)$, reducing the system (3.1) to the form

$$\frac{d\widetilde{x}_1}{dt} = D_{N_1}(t,\varepsilon,\theta,\mu)\widetilde{x}_1, \quad \frac{d\widetilde{x}_2}{dt} = D_{N_2}(t,\varepsilon,\theta,\mu)\widetilde{x}_2, \tag{3.3}$$

where the elements D_{N_j} $(j = 1,2)$ of $(N_j \times N_j)$-matrices also belong to the class $F(m-1;\varepsilon^*;\theta)$.

Performing in the system (3.1) the transformation

$$x_1 = e^{ip\theta}y_1, \quad x_2 = e^{ir\theta}y_2,$$

where $y_1 = \mathrm{colon}(y_{11},\dots,y_{1N_1})$, $y_2 = \mathrm{colon}(y_{21},\dots,y_{2N_2})$, we obtain

$$\begin{aligned}\frac{dy_1}{dt} &= J_{N_1}y_1 + \mu(\widetilde{B}_{11}(t,\varepsilon,\theta)y_1 + \widetilde{B}_{12}(t,\varepsilon,\theta)y_2),\\ \frac{dy_2}{dt} &= J_{N_2}y_2 + \mu(\widetilde{B}_{21}(t,\varepsilon,\theta)y_1 + \widetilde{B}_{22}(t,\varepsilon,\theta)y_2),\end{aligned} \tag{3.4}$$

where

$$J_{N_1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix}, \quad J_{N_2} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix}$$

are the Jordan blocks of dimensions N_1 and N_2, respectively, whose diagonal elements are equal to zero, and all elements of matrices $\widetilde{B}_{jk}(t,\varepsilon,\theta)$ belong to the class $F(m;\varepsilon_0;\theta)$.

Thus, the problem of the existence of transformation (3.2) reduces to the problem of the existence of the transformation

$$y_1 = z_1 + \mu Q_{12}(t,\varepsilon,\theta,\mu)z_2, \quad y_2 = \mu Q_{21}(t,\varepsilon,\theta,\mu)z_1 + z_2, \tag{3.5}$$

leading the system (3.4) to the form

$$\frac{dz_1}{dt} = D_{N_1}(t,\varepsilon,\theta,\mu)z_1, \quad \frac{dz_2}{dt} = D_{N_2}(t,\varepsilon,\theta,\mu)z_2,$$

where D_{N_1}, D_{N_2} are matrices of dimensions $(N_1 \times N_1)$ and $(N_2 \times N_2)$, respectively.

The matrices Q_{12}, Q_{21} must satisfy the system of matrix-equations

$$\begin{aligned}\frac{dQ_{jk}}{dt} &= J_{N_j}Q_{jk} - Q_{jk}J_{N_k} + \widetilde{B}_{jk}(t,\varepsilon,\theta) \\ &+ \mu\big(\widetilde{B}_{jj}(t,\varepsilon,\theta)Q_{jk} - Q_{jk}\widetilde{B}_{kk}(t,\varepsilon,\theta)\big) - \mu^2 Q_{jk}\widetilde{B}_{kj}Q_{jk}, \ \ j,k = 1,2 \ \ (j \neq k).\end{aligned} \tag{3.6}$$

Then

$$\begin{aligned}D_{N_1} &= J_{N_1} + \mu\widetilde{B}_{11}(t,\varepsilon,\theta) + \mu^2\widetilde{B}_{12}(t,\varepsilon,\theta)Q_{21}(t,\varepsilon,\theta,\mu), \\ D_{N_2} &= J_{N_1} + \mu\widetilde{B}_{22}(t,\varepsilon,\theta) + \mu^2\widetilde{B}_{21}(t,\varepsilon,\theta)Q_{12}(t,\varepsilon,\theta,\mu).\end{aligned} \tag{3.7}$$

It is easy to see that the system (3.6) is divided into two independent matrix-equations, each of which has the form

$$\frac{dX}{dt} = J_M X - XJ_K + F(t,\varepsilon,\theta) + \mu\big(A(t,\varepsilon,\theta)X - XB(t,\varepsilon,\theta)\big) - \mu^2 XR(t,\varepsilon,\theta)X, \tag{3.8}$$

where $X = (x_{js})_{j=\overline{1,M};\, s=\overline{1,K}}$,

$$J_M = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \hdotsfor{5} \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad J_K = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \hdotsfor{5} \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

are the Jordan blocks of dimensions M and K, respectively, whose diagonal elements are equal to zero, $F = (f_{js})_{j=\overline{1,M};\, s=\overline{1,K}}$, $A = (a_{js})_{j,s=\overline{1,M}}$, $B = (b_{js})_{j,s=\overline{1,K}}$, $R = (r_{js})_{j=\overline{1,K};\, s=\overline{1,M}}$. All elements of matrices F, A, B, R belong to the class $F(m;\varepsilon_0;\theta)$.

Therefore the problem of the existence of transformation (3.5), where all elements of matrices Q_{12}, Q_{21} belong to the class $F(m-1;\varepsilon^*;\theta)$ $(0 < \varepsilon^* < \varepsilon_0)$, reduces to the problem of the existence of a particular solution X of the equation (3.8) such that $x_{js} \in F(m-1;\varepsilon^*;\theta)$ $(j = \overline{1,M};\ s = \overline{1,K})$.

In [1], the conditions of the existence of such a solution are obtained when one of the sets of assumptions I, II, III is fulfilled.

I. (1) $M < K$;

(2) $V_1(F) \equiv 0$, where $V_1 = \operatorname{colon}(v_{11}(t,\varepsilon),\dots,v_{1M}(t,\varepsilon))$,
$$v_{1j}(t,\varepsilon) = \sum_{s=1}^{j}\Gamma_0(f_{s,K-j+s}(t,\varepsilon,\theta)) \ (j = \overline{1,M});$$

(3) $\inf\limits_G \big|\Gamma_0(b_{1K}(t,\varepsilon,\theta))\big| > 0$.

II. (1) $M = K$;

(2) $V_2(F) \equiv 0$, where $V_2 = \operatorname{colon}(v_{21}(t,\varepsilon),\dots,v_{2M}(t,\varepsilon))$,
$$v_{2j}(t,\varepsilon) = \sum_{s=1}^{j}\Gamma_0(f_{s,K-j+s}(t,\varepsilon,\theta)) \ (j = \overline{1,M});$$

(3) $\inf\limits_G \big|\Gamma_0(a_{1M}(t,\varepsilon,\theta) - b_{1M}(t,\varepsilon,\theta))\big| > 0$.

III. (1) $M > K$;

(2) $V_3(F) \equiv 0$, where $V_3 = \operatorname{colon}(v_{31}(t,\varepsilon),\dots,v_{3K}(t,\varepsilon))$,
$$v_{3j}(t,\varepsilon) = \sum_{s=1}^{j}\Gamma_0(f_{s,K-j+s}(t,\varepsilon,\theta)) \ (j = \overline{1,K});$$

(3) $\inf\limits_G |\Gamma_0(a_{1M}(t,\varepsilon,\theta))| > 0$.

In this paper it is assumed that the condition (2) in each of sets I, II, III is satisfied. But instead of the condition (3) it is accordingly supposed that

$$\begin{aligned} \Gamma_0(b_{1K}(t,\varepsilon,\theta)) &\equiv 0 \ \ (M<K); \\ \Gamma_0(a_{1M}(t,\varepsilon,\theta)-b_{1M}(t,\varepsilon,\theta)) &\equiv 0 \ \ (M=K); \\ \Gamma_0(a_{1M}(t,\varepsilon,\theta)) &\equiv 0 \ \ (M>K). \end{aligned}$$

4 Auxiliary results

As in [1], along with the equation (3.8) we consider an auxiliary matrix-equation

$$\varphi(t,\varepsilon)\,\frac{d\Xi}{d\theta} = J_M\Xi - \Xi J_K + F(t,\varepsilon,\theta) + \mu\big(A(t,\varepsilon,\theta)\Xi - \Xi B(t,\varepsilon,\theta)\big) - \mu^2\Xi R(t,\varepsilon,\theta)\Xi, \tag{4.1}$$

where t, φ are considered as constants, $\Xi = (\xi_{js})_{j=\overline{1,M};\, s=\overline{1,K}}$, F, A, B, R are the same as in the equation (3.8).

In accordance with the Poincaré method of small parameter [2], we construct an approximate 2π-periodic with respect to θ solution of the equation (4.1) in the form of the sum

$$\Xi = \sum_{\nu=0}^{2q-1} \Xi_\nu(t,\varepsilon,\theta)\mu^\nu, \tag{4.2}$$

where $\Xi_\nu = (\xi_{\nu,js})_{j=\overline{1,M};\, s=\overline{1,K}}$. The coefficients Ξ_ν are determined from the following chain of linear nonhomogeneous matrix differential equations:

$$\varphi(t,\varepsilon)\,\frac{d\Xi_0}{d\theta} = J_M\Xi_0 - \Xi_0 J_K + F(t,\varepsilon,\theta), \tag{4.3}$$

$$\varphi(t,\varepsilon)\,\frac{d\Xi_1}{d\theta} = J_M\Xi_1 - \Xi_1 J_K + A(t,\varepsilon,\theta)\Xi_0 - \Xi_0 B(t,\varepsilon,\theta), \tag{4.4}$$

$$\varphi(t,\varepsilon)\,\frac{d\Xi_2}{d\theta} = J_M\Xi_2 - \Xi_2 J_K + A(t,\varepsilon,\theta)\Xi_1 - \Xi_1 B(t,\varepsilon,\theta) - \Xi_0 R(t,\varepsilon,\theta)\Xi_0, \tag{4.5}$$

$$\begin{aligned} \varphi(t,\varepsilon)\,\frac{d\Xi_\nu}{d\theta} = J_M\Xi_\nu - \Xi_\nu J_K &+ A(t,\varepsilon,\theta)\Xi_{\nu-1} - \Xi_{\nu-1}B(t,\varepsilon,\theta) \\ &- \sum_{l=0}^{\nu-2} \Xi_l R(t,\varepsilon,\theta)\Xi_{\nu-2-l}, \quad \nu = \overline{3, 2q-1}. \end{aligned}$$

First, we consider the case $M<K$.

In scalar form, the equation (4.3) can be written as a following system of differential equations:

$$\begin{aligned} \varphi(t,\varepsilon)\,\frac{d\xi_{0,1K}}{d\theta} &= f_{1K}(t,\varepsilon,\theta), \\ \varphi(t,\varepsilon)\,\frac{d\xi_{0,jK}}{d\theta} &= \xi_{0,j-1,K} + f_{jK}(t,\varepsilon,\theta) \ \ (j=\overline{2,M}), \\ \varphi(t,\varepsilon)\,\frac{d\xi_{0,1s}}{d\theta} &= -\xi_{0,1,s+1} + f_{1s}(t,\varepsilon,\theta) \ \ (s=\overline{1,K-1}), \\ \varphi(t,\varepsilon)\,\frac{d\xi_{0,js}}{d\theta} &= -\xi_{0,j-1,s} - \xi_{0,j,s+1} + f_{js}(t,\varepsilon,\theta) \ \ (j=\overline{2,M};\ \ s=\overline{1,K-1}). \end{aligned} \tag{4.6}$$

The condition I(2) ensures the existence of a 2π-periodic with respect to θ solution of the equation (4.3) of the form

$$\Xi_0(t,\varepsilon,\theta) = C_0^{(1)}(t,\varepsilon) + L_1(F(t,\varepsilon,\theta)), \tag{4.7}$$

where the $(M\times K)$-matrix $C_0^{(1)}(t,\varepsilon)$ has the form

$$C_0^{(1)}(t,\varepsilon) = \begin{pmatrix} c_{01}^{(1)}(t,\varepsilon) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ c_{02}^{(1)}(t,\varepsilon) & c_{01}^{(1)}(t,\varepsilon) & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{0M}^{(1)}(t,\varepsilon) & c_{0,M-1}^{(1)}(t,\varepsilon) & \cdots & c_{01}^{(1)}(t,\varepsilon) & 0 & \cdots & 0 \end{pmatrix} \tag{4.8}$$

with $c_{01}^{(1)}(t,\varepsilon),\dots,c_{0M}^{(1)}(t,\varepsilon)$ as yet unknown scalar functions of the class $S(m;\varepsilon_0)$, $L_1(F(t,\varepsilon,\theta)) = (\widetilde{\xi}_{0,js}(t,\varepsilon,\theta))_{j=\overline{1,M};\, s=\overline{1,K}}$, and $\widetilde{\xi}_{0,js}$ are defined from the following equalities:

$$
\begin{aligned}
&\widetilde{\xi}_{0,1K}(t,\varepsilon,\theta) = I(f_{1K}(t,\varepsilon,\theta)) + p_{1K}(t,\varepsilon),\\
&\widetilde{\xi}_{0,jK}(t,\varepsilon,\theta) = I\big(\widetilde{\xi}_{0,j-1,K}(t,\varepsilon,\theta) + f_{jK}(t,\varepsilon,\theta)\big) + p_{jK}(t,\varepsilon) \ \ (j=\overline{2,M}),\\
&\widetilde{\xi}_{0,11}(t,\varepsilon,\theta) = I\big(f_{11}(t,\varepsilon,\theta) - \widetilde{\xi}_{0,12}(t,\varepsilon,\theta)\big) + p_{11}(t,\varepsilon),\\
&\widetilde{\xi}_{0,1s}(t,\varepsilon,\theta) = I\big(f_{1s}(t,\varepsilon,\theta) - \widetilde{\xi}_{0,1,s+1}(t,\varepsilon,\theta)\big) + p_{1s}(t,\varepsilon) \ \ (s=\overline{1,K-1}),\\
&\widetilde{\xi}_{0,js}(t,\varepsilon,\theta) = I\big(\widetilde{\xi}_{0,j-1,s}(t,\varepsilon,\theta) - \widetilde{\xi}_{0,j,s+1}(t,\varepsilon,\theta) + f_{js}(t,\varepsilon,\theta)\big) + p_{js}(t,\varepsilon) \ \ (j=\overline{2,M};\ \ s=\overline{1,K-1}),
\end{aligned}
$$

where $p_{js}(t,\varepsilon)$ are the functions from the class $S(m;\varepsilon_0)$ determined from the condition: all right-hand sides of the equations in (4.6) must satisfy condition (A). It is easy to verify that $p_{js}(t,\varepsilon)$ can be represented as some linear combinations of functions $\Gamma_0(f_{\alpha\beta}(t,\varepsilon,\theta))$ ($\alpha=\overline{1,M}$; $\beta=\overline{1,K}$).

We now define the matrix $C_0^{(1)}(t,\varepsilon)$ from the condition

$$V_1\big(A(t,\varepsilon,\theta)\Xi_0 - \Xi_0 B(t,\varepsilon,\theta)\big) = 0.$$

By virtue of (4.7), this condition can be rewritten as

$$V_1\big(A(t,\varepsilon,\theta)C_0^{(1)} - C_0^{(1)}B(t,\varepsilon,\theta)\big) = V_1\Big(L_1(F(t,\varepsilon,\theta))B(t,\varepsilon,\theta) - A(t,\varepsilon,\theta)L_1(F(t,\varepsilon,\theta))\Big). \tag{4.9}$$

In scalar form, the condition (4.9) can be written as a triangular with respect to $c_{01}^{(1)},\dots,c_{0M}^{(1)}$ system of linear algebraic equations:

$$\sum_{l=1}^{j} g_{jl}^{(1)}(t,\varepsilon)c_{0l}^{(1)} = h_j^{(1)}(t,\varepsilon), \ \ j=\overline{1,M},$$

where $g_{jl}^{(1)}(t,\varepsilon), h_j^{(1)}(t,\varepsilon) \in S(m;\varepsilon_0)$ and $g_{jj}^{(1)}(t,\varepsilon) = \Gamma_0(b_{1K}(t,\varepsilon,\theta))$ ($j=\overline{1,M}$) are the know functions.

Suppose that

$$g_{jl}^{(1)}(t,\varepsilon) \equiv 0 \ \ (j,l=\overline{1,M}, \ \ l\le j), \tag{4.10}$$

$$h_j^{(1)}(t,\varepsilon) \equiv 0 \ \ (j=\overline{1,M}). \tag{4.11}$$

Then

$$V_1\big(A(t,\varepsilon,\theta)C_0 - C_0 B(t,\varepsilon,\theta)\big) = 0 \tag{4.12}$$

for any matrix C_0 of the form (4.8). Besides,

$$V_1\Big(A(t,\varepsilon,\theta)L_1(F(t,\varepsilon,\theta)) - L_1(F(t,\varepsilon,\theta))B(t,\varepsilon,\theta)\Big) = 0. \tag{4.13}$$

Therefore the equation (4.9) is satisfied for any matrix $C_0^{(1)}$ of the form (4.8).

The equalities (4.12), (4.13) ensure the existence of a 2π-periodic with respect to θ solution of the equation (4.4) having the form

$$\Xi_1(t,\varepsilon,\theta) = C_1^{(1)}(t,\varepsilon) + L_1\big(A(t,\varepsilon,\theta)\Xi_0 - \Xi_0 B(t,\varepsilon,\theta)\big), \tag{4.14}$$

where

$$C_1^{(1)}(t,\varepsilon) = \begin{pmatrix} c_{11}^{(1)}(t,\varepsilon) & 0 & \cdots & 0 & 0 & \cdots & 0\\ c_{12}^{(1)}(t,\varepsilon) & c_{11}^{(1)}(t,\varepsilon) & \cdots & 0 & 0 & \cdots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ c_{1M}^{(1)}(t,\varepsilon) & c_{1,M-1}^{(1)}(t,\varepsilon) & \cdots & c_{11}^{(1)}(t,\varepsilon) & 0 & \cdots & 0 \end{pmatrix}.$$

The solution (4.14) can be written as

$$\Xi_1(t,\varepsilon,\theta) = C_1^{(1)}(t,\varepsilon) + L_1\big(A(t,\varepsilon,\theta)C_0^{(1)} - C_0^{(1)}B(t,\varepsilon,\theta)\big) + F_1(t,\varepsilon,\theta), \tag{4.15}$$

where $F_1(t,\varepsilon,\theta) = L_1(AL_1(F) - L_1(F)B)$ does not depend on $C_0^{(1)}$.

We write down the conditions of the existence of a 2π-periodic with respect to θ solution of the equation (4.5):

$$V_1\Big(A(t,\varepsilon,\theta)\Xi_1 - \Xi_1 B(t,\varepsilon,\theta) - \Xi_0 R(t,\varepsilon,\theta)\Xi_0\Big) = 0.$$

Taking into account the equalities (4.7) and (4.15), this condition can be rewritten (for brevity, we omit the arguments t, ε, θ) as

$$\begin{aligned} V_1(AC_1^{(1)} - C_1^{(1)}B) + V_1\Big(AL_1(AC_0^{(1)} - C_0^{(1)}B) - L_1(AC_0^{(1)} - C_0^{(1)}B)B\Big) + V_1(AF_1 - F_1B) \\ - V_1(C_0^{(1)}RC_0^{(1)}) - V_1\big(L_1(F)RC_0^{(1)} + C_0^{(1)}RL_1(F)\big) - V_1(L_1(F)RL_1(F)) = 0. \end{aligned} \tag{4.16}$$

Due to (4.12), the condition (4.16) can be rewritten as

$$\begin{aligned} V_1\Big(AL_1(AC_0^{(1)} - C_0^{(1)}B) - L_1(AC_0^{(1)} - C_0^{(1)}B)B\Big) \\ - V_1\big(L_1(F)RC_0^{(1)} + C_0^{(1)}RL_1(F)\big) - V_1(C_0^{(1)}RC_0^{(1)}) + U^{(1)} = 0, \end{aligned} \tag{4.17}$$

where $U^{(1)} = U^{(1)}(t,\varepsilon)$ is the known M-vector that does not depend on $C_0^{(1)}$.

In scalar form, the equation (4.17) can be written as a nonlinear with respect to $c_{01}^{(1)},\dots,c_{0M}^{(1)}$ system of algebraic equations

$$\Phi_j^{(1)}(t,\varepsilon,c_{01}^{(1)},\dots,c_{0M}^{(1)}) = 0, \quad j = \overline{1,M}, \tag{4.18}$$

with quadratic nonlinearities.

Suppose that the system (4.18) has a solution $c_{01}^{(1)},\dots,c_{0M}^{(1)}$ such that

$$\inf_G \left| \det \frac{\partial(\Phi_1^{(1)},\dots,\Phi_M^{(1)})}{\partial(c_{01}^{(1)},\dots,c_{0M}^{(1)})} \right| > 0. \tag{4.19}$$

Then the equation (4.5) has a 2π-periodic with respect to θ solution $\Xi_2(t,\varepsilon,\theta)$ belonging to the class $F(m;\varepsilon_0;\theta)$.

We now consider the equation for the vector-function $\Xi_{\nu+2}$ and distinguish in it explicitly the terms which depend on $\Xi_{\nu+1}$, Ξ_ν:

$$\begin{aligned} \varphi(t,\varepsilon)\frac{d\Xi_{\nu+2}}{d\theta} = J_M\Xi_{\nu+2} - \Xi_{\nu+2}J_K + A(t,\varepsilon,\theta)\Xi_{\nu+1} - \Xi_{\nu+1}B(t,\varepsilon,\theta) \\ - \Xi_0 R(t,\varepsilon,\theta)\Xi_\nu - \Xi_\nu R(t,\varepsilon,\theta)\Xi_0 - \sum_{l=1}^{\nu-1}\Xi_l R(t,\varepsilon,\theta)\Xi_{\nu-l}. \end{aligned} \tag{4.20}$$

For $\alpha = \overline{0,\nu+1}$, we have

$$\Xi_\alpha(t,\varepsilon,\theta) = C_\alpha^{(1)}(t,\varepsilon) + \widetilde{\widetilde{\Xi}}_\alpha(t,\varepsilon,\theta), \tag{4.21}$$

where $C_\alpha^{(1)}(t,\varepsilon)$ is the $(M\times K)$-matrix of the form (4.8), and $\widetilde{\widetilde{\Xi}}_\alpha(t,\varepsilon,\theta)$ is the known vector-function belonging to the class $F(m;\varepsilon_0;\theta)$.

We suppose that the matrices $\Xi_0(t,\varepsilon,\theta), \Xi_1(t,\varepsilon,\theta),\dots,\Xi_{\nu-1}(t,\varepsilon,\theta)$ are completely defined, including the matrix $C_{\nu-1}^{(1)}(t,\varepsilon)$, and the matrix $C_\nu^{(1)}(t,\varepsilon)$, $C_{\nu+1}^{(1)}(t,\varepsilon)$ have to be defined.

We write down the conditions of the existence of a 2π-periodic with respect to θ solution of the equation (4.20) as follows:

$$\begin{aligned} V_1\big(A(t,\varepsilon,\theta)\Xi_{\nu+1} - \Xi_{\nu+1}B(t,\varepsilon,\theta) - \Xi_0 R(t,\varepsilon,\theta)\Xi_\nu\big) \\ - V_1\Big(\Xi_\nu R(t,\varepsilon,\theta)\Xi_0 + \sum_{l=1}^{\nu-1}\Xi_l R(t,\varepsilon,\theta)\Xi_{\nu-l}\Big) = 0. \end{aligned} \tag{4.22}$$

Represent the matrix $\widetilde{\Xi}_{\nu+1}$ as
$$\widetilde{\Xi}_{\nu+1} = \widetilde{\Xi}^{(*)}_{\nu+1} + \widetilde{\Xi}^{(**)}_{\nu+1}, \tag{4.23}$$
where $\widetilde{\Xi}^{(*)}_{\nu+1}$ is a 2π-periodic with respect to θ solution of the equation
$$\varphi(t,\varepsilon)\,\frac{d\Xi_{\nu+1}}{d\theta} = J_M\Xi_{\nu+1} - \Xi_{\nu+1}J_K + A(t,\varepsilon,\theta)C^{(1)}_\nu(t,\varepsilon) - C^{(1)}_\nu(t,\varepsilon)B(t,\varepsilon,\theta) \tag{4.24}$$
and $\widetilde{\Xi}^{(**)}_{\nu+1}$ is a 2π-periodic with respect to θ solution of the equation
$$\varphi(t,\varepsilon)\,\frac{d\Xi_{\nu+1}}{d\theta} = J_M\Xi_{\nu+1} - \Xi_{\nu+1}J_K + A(t,\varepsilon,\theta)\widetilde{\Xi}_\nu - \widetilde{\Xi}_\nu B(t,\varepsilon,\theta) - \sum_{l=1}^{\nu-1}\Xi_l R(t,\varepsilon,\theta)\Xi_{\nu-1-l}.$$

The condition of the existence of a 2π-periodic with respect to θ solution of the equation (4.24) has the form
$$V_1(A(t,\varepsilon,\theta)C^{(1)}_\nu - C^{(1)}_\nu B(t,\varepsilon,\theta)) = 0.$$
By (4.12), this equality holds for any matrix C_ν of the kind
$$C_\nu(t,\varepsilon) = \begin{pmatrix} c_{\nu1}(t,\varepsilon) & 0 & \cdots & 0 & 0 & \cdots & 0\\ c_{\nu2}(t,\varepsilon) & c_{\nu1}(t,\varepsilon) & \cdots & 0 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ c_{\nu M}(t,\varepsilon) & c_{\nu,M-1}(t,\varepsilon) & \cdots & c_{\nu1}(t,\varepsilon) & 0 & \cdots & 0\end{pmatrix}.$$
Therefore the equation (4.24) has a 2π-periodic with respect to θ solution of the kind
$$\widetilde{\Xi}^{(1)}_{\nu+1} = L_1\big(A(t,\varepsilon,\theta)C^{(1)}_\nu - C^{(1)}_\nu B(t,\varepsilon,\theta)\big).$$
Taking into account (4.21) and (4.23), the condition (4.22) can be rewritten as
$$V_1\big(A(t,\varepsilon,\theta)C^{(1)}_{\nu+1} - C^{(1)}_{\nu+1}B(t,\varepsilon,\theta)\big) + V_1\Big(A(t,\varepsilon,\theta)(\widetilde{\Xi}^{(*)}_{\nu+1} + \widetilde{\Xi}^{(**)}_{\nu+1}) - (\widetilde{\Xi}^{(*)}_{\nu+1} + \widetilde{\Xi}^{(**)}_{\nu+1})B(t,\varepsilon,\theta)\Big)$$
$$- V_1\big(\Xi_0 R(t,\varepsilon,\theta)\Xi_\nu + \Xi_\nu R(t,\varepsilon,\theta)\Xi_0\big) + V_1^*(t,\varepsilon) = 0, \tag{4.25}$$
where $V_1^*(t,\varepsilon)$ is the known M-vector belonging to the class $S(m;\varepsilon_0)$.

Based on (4.12), (4.21) and (4.23), we can rewrite (4.25) as
$$V_1\Big(A(t,\varepsilon,\theta)L_1\big(A(t,\varepsilon,\theta)C^{(1)}_\nu - C^{(1)}_\nu B(t,\varepsilon,\theta)\big) - L_1\big(A(t,\varepsilon,\theta)C^{(1)}_\nu - C^{(1)}_\nu B(t,\varepsilon,\theta)\big)B(t,\varepsilon,\theta)\Big)$$
$$- V_1\big(L_1(F)R(t,\varepsilon,\theta)C^{(1)}_\nu + C^{(1)}_\nu R(t,\varepsilon,\theta)L_1(F)\big)$$
$$- V_1\big(C_0 R(t,\varepsilon,\theta)C^{(1)}_\nu + C^{(1)}_\nu R(t,\varepsilon,\theta)C_0\big) + Z^{(1)}(t,\varepsilon) = 0, \tag{4.26}$$
where $Z^{(1)}(t,\varepsilon)$ is the known M-vector belonging to the class $S(m;\varepsilon_0)$.

It is not difficult to establish the validity of the relations
$$(XRY)_{\alpha\beta} = \begin{cases} \displaystyle\sum_{j=1}^{\alpha} x_j \sum_{l=1}^{M+1-\beta} r_{\alpha+1-j,l+\beta-1}y_l, & \text{if } \beta \le M,\\ 0, & \text{if } \beta > M,\end{cases}$$
where X, Y are the $(M\times K)$-matrices of the kind (4.8). It follows that in a scalar form the equation (4.26) can be written as
$$\sum_{l=1}^{M} \frac{\partial\Phi^{(1)}_j(t,\varepsilon,c^{(1)}_{01},\ldots,c^{(1)}_{0M})}{\partial c^{(1)}_{0l}}\,c^{(1)}_{\nu l} = z^{(1)}_j(t,\varepsilon), \quad j = \overline{1,M}, \tag{4.27}$$

where $u_j^{(1)}(t,\varepsilon)$ are the known functions belonging to the class $S(m;\varepsilon_0)$. By the condition (4.19), the system (4.27) has a unique solution $c_{\nu 1}^{(1)}(t,\varepsilon),\dots,c_{\nu M}^{(1)}(t,\varepsilon)$ belonging to the class $S(m;\varepsilon_0)$.

Thus, all the matrices $\Xi_\nu(t,\varepsilon,\theta)$ $(\nu=\overline{0,2q-1})$ are completely defined and belong to the class $F(m;\varepsilon_0;\theta)$. Therefore, by (4.2), the matrix $\Xi(t,\varepsilon,\theta,\mu)$ is also completely defined $\forall\,\mu\in(0,1)$ and belongs to the class $F(m;\varepsilon_0;\theta)$.

Lemma 4.1. *Let the equation* (3.8) *satisfy the following conditions:*

(1) $M<K$*;*

(2) $V_1(F(t,\varepsilon,\theta))\equiv 0$*;*

(3) *the equalities* (4.10)*,* (4.11) *hold;*

(4) *the system* (4.18) *has a solution satisfying the condition* (4.19)*.*

Then there exists $\mu_1\in(0,1)$ *such that for any* $\mu\in(0,\mu_1)$ *there exists a transformation of the form*

$$X=\Xi(t,\varepsilon,\theta,\mu)+\Phi(t,\varepsilon,\theta,\mu)Y\Psi(t,\varepsilon,\theta,\mu), \tag{4.28}$$

where the matrix $\Xi(t,\varepsilon,\theta,\mu)$ *is defined by the equality* (4.2) *and the elements of the* $(M\times M)$*-matrix* Φ *and those of the* $(K\times K)$*-matrix* Ψ *belong to the class* $F(m;\varepsilon_0;\theta)$ $\forall\,\mu\in(0,\mu_1)$*, which reduces the equation* (3.8) *to the form*

$$\begin{aligned}\frac{dY}{dt}&=J_MY-YJ_K+\Big(\sum_{l=1}^{q}U_{l1}(t,\varepsilon)\mu^l\Big)Y-Y\Big(\sum_{l=1}^{q}U_{l2}(t,\varepsilon)\mu^l\Big)\\&\quad+\varepsilon\big(U_1(t,\varepsilon,\theta,\mu)Y-YU_2(t,\varepsilon,\theta,\mu)\big)+\mu^{q+1}\big(W_1(t,\varepsilon,\theta,\mu)Y-YW_2(t,\varepsilon,\theta,\mu)\big)\\&\quad+\varepsilon H_1(t,\varepsilon,\theta,\mu)+\mu^{2q}H_2(t,\varepsilon,\theta,\mu)+\mu YR_1(t,\varepsilon,\theta,\mu)Y,\end{aligned} \tag{4.29}$$

where the elements of matrices U_{l1}*,* U_{l2} $(l=\overline{1,q})$ *belong to the class* $S(m;\varepsilon_0)$*, and the elements of matrices* U_1*,* U_2*,* W_1*,* W_2*,* H_1*,* H_2*,* R_1 *of the corresponding dimensions belong to the class* $F(m-1;\varepsilon_0;\theta)$*.*

Proof. Substituting
$$X=\Xi(t,\varepsilon,\theta,\mu)+\widetilde{X}$$
in (3.8), where $\widetilde{X}$ is a new unknown matrix, we obtain

$$\begin{aligned}\frac{d\widetilde{X}}{dt}&=J_M\widetilde{X}-\widetilde{X}J_K+\varepsilon H_3(t,\varepsilon,\theta,\mu)+\mu^{2q}H_4(t,\varepsilon,\theta,\mu)\\&\quad+\Big(\sum_{l=1}^{q}P_l(t,\varepsilon,\theta)\mu^l\Big)\widetilde{X}-\widetilde{X}\Big(\sum_{l=1}^{q}Q_l(t,\varepsilon,\theta)\mu^l\Big)\\&\quad+\mu^{q+1}(W_1^*(t,\varepsilon,\theta,\mu)\widetilde{X}-\widetilde{X}W_2^*(t,\varepsilon,\theta,\mu))+\mu^2\widetilde{X}R(t,\varepsilon,\theta)\widetilde{X}.\end{aligned} \tag{4.30}$$

By Lemma 1 from [1], using the substitution of the kind

$$\widetilde{X}=\Big(E_M+\sum_{l=1}^{q}\Phi_l(t,\varepsilon,\theta)\mu^l\Big)Y\Big(E_K+\sum_{l=1}^{q}\Psi_l(t,\varepsilon,\theta)\mu^l\Big),$$

where E_M,E_K are the identity matrices of dimensions M and K, respectively, the elements of the $(M\times M)$-matrices Φ_l and those of $(K\times K)$-matrices Ψ_l $(l=\overline{1,q})$ belong to the class $F(m;\varepsilon_0;\theta)$, we reduce the equation (4.30) to the form (4.29). ☐

We now consider the case $M=K$. The condition II (2) ensures the existence of a 2π-periodic with respect to θ solution of the equation (4.3), which is of the form

$$\Xi_0(t,\varepsilon,\theta)=C_0^{(2)}(t,\varepsilon)+L_2(F(t,\varepsilon,\theta))$$

with

$$C_0^{(2)}(t,\varepsilon)=\begin{pmatrix} c_{01}^{(2)}(t,\varepsilon) & 0 & \cdots & 0 \\ c_{02}^{(2)}(t,\varepsilon) & c_{01}^{(2)}(t,\varepsilon) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ c_{0M}^{(2)}(t,\varepsilon) & c_{0,M-1}^{(2)}(t,\varepsilon) & \cdots & c_{01}^{(2)}(t,\varepsilon) \end{pmatrix}, \tag{4.31}$$

where the linear matrix-operator $L_2(F)$ can be constructed similarly to the operator $L_1(F)$. The matrix $C_0^{(2)}$ is defined from the equation

$$V_2\big(A(t,\varepsilon,\theta)C_0^{(2)}-C_0^{(2)}B(t,\varepsilon,\theta)\big)=V_2\Big(L_2(F(t,\varepsilon,\theta))B(t,\varepsilon,\theta)-A(t,\varepsilon,\theta)L_2(F(t,\varepsilon,\theta))\Big). \tag{4.32}$$

In scalar form, the condition (4.32) can be written as a triangular with respect to $C_{01}^{(2)},\dots,C_{0M}^{(2)}$ system of linear algebraic equations:

$$\sum_{l=1}^{j} g_{jl}^{(2)}(t,\varepsilon)c_{0l}^{(2)}=h_j^{(2)}(t,\varepsilon),\ \ j=\overline{1,M},$$

where $g_{jl}^{(2)}(t,\varepsilon),h_j^{(2)}(t,\varepsilon)\in S(m;\varepsilon_0)$ and $g_{jj}^{(2)}(t,\varepsilon)=\Gamma_0(a_{1M}(t,\varepsilon,\theta)-b_{1M}(t,\varepsilon,\theta))$ $(j=\overline{1,M})$ are the know functions.

Suppose that

$$g_{jl}^{(2)}(t,\varepsilon)\equiv 0\ \ (j,l=\overline{1,M},\ \ l\le j), \tag{4.33}$$

$$h_j^{(2)}(t,\varepsilon)\equiv 0\ \ (j=\overline{1,M}). \tag{4.34}$$

Then

$$V_2(A(t,\varepsilon,\theta)C_0-C_0B(t,\varepsilon,\theta))=0$$

for any C_0 of the kind (4.31), and

$$V_2\Big(L_2(F(t,\varepsilon,\theta))B(t,\varepsilon,\theta)-A(t,\varepsilon,\theta)L_2(F(t,\varepsilon,\theta))\Big)=0.$$

Therefore the equation (4.32) is satisfied for any $C_0^{(2)}$ of the kind (4.31).

Similarly to the case $M<K$, we define the matrix $C_0^{(2)}(t,\varepsilon)$ from the equation

$$\begin{aligned} V_2\Big(AL_2(AC_0^{(2)}-C_0^{(2)}B)-L_2(AC_0^{(2)}-C_0^{(2)}B)B\Big)& \\ -V_2\big(L_2(F)RC_0^{(2)}+C_0^{(2)}RL_2(F)\big)-V_2(C_0^{(2)}RC_0^{(2)})+U^{(2)}&=0, \end{aligned} \tag{4.35}$$

where $U^{(2)}=U^{(2)}(t,\varepsilon)$ is the known M-vector, which does not depend on $C_0^{(2)}$.

In scalar form, the equation (4.35) can be written as a nonlinear with respect to $c_{01}^{(2)},\dots,c_{0M}^{(2)}$ system of algebraic equations

$$\Phi_j^{(2)}(t,\varepsilon,c_{01}^{(2)},\dots,c_{0M}^{(2)})=0,\ \ j=\overline{1,M}, \tag{4.36}$$

with quadratic nonlinearities.

Suppose that the system (4.36) has a solution $c_{01}^{(2)},\dots,c_{0M}^{(2)}$ such that

$$\inf_G\left|\det\frac{\partial(\Phi_1^{(2)},\dots,\Phi_M^{(2)})}{\partial(c_{01}^{(2)},\dots,c_{0M}^{(2)})}\right|>0. \tag{4.37}$$

Lemma 4.2. *Let the equation* (3.8) *satisfy the following conditions:*

(1) $M=K$*;*

(2) $V_2(F(t,\varepsilon,\theta)) \equiv 0$;

(3) *the equalities* (4.33), (4.34) *hold;*

(4) *the system* (4.36) *has a solution satisfying the condition* (4.37).

Then there exists $\mu_2 \in (0,1)$ *such that for any* $\mu \in (0,\mu_2)$ *there exists a transformation of the form* (4.28), *where the matrix* $\Xi(t,\varepsilon,\theta,\mu)$ *is defined by* (4.2) *and the elements of the* $(M\times M)$*-matrix* Φ *and those of the* $(K\times K)$*-matrix* Ψ *belong to the class* $F(m;\varepsilon_0;\theta)$ $\forall\,\mu\in(0,\mu_2)$, *which reduces the equation* (3.8) *to the form* (4.29).

Proof of Lemma 4.2 is similar to that of Lemma 4.1.

Finally, we consider the case $M > K$.

The condition III (2) ensures the existence of a 2π-periodic with respect to θ solution of the equation (4.3), which has the form

$$\Xi_0(t,\varepsilon,\theta) = C_0^{(3)}(t,\varepsilon) + L_3(F(t,\varepsilon,\theta))$$

with

$$C_0^{(3)}(t,\varepsilon) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \cdots & 0 \\ c_{01}^{(3)}(t,\varepsilon) & 0 & \cdots & 0 \\ c_{02}^{(3)}(t,\varepsilon) & c_{01}^{(3)}(t,\varepsilon) & \cdots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ c_{0K}^{(3)}(t,\varepsilon) & c_{0,K-1}^{(3)}(t,\varepsilon) & \cdots & c_{01}^{(3)}(t,\varepsilon) \end{pmatrix}, \tag{4.38}$$

where the linear matrix-operator $L_3(F)$ is constructed similarly to the operator $L_1(F)$. The matrix $C_0^{(3)}$ is defined from the equation

$$V_3\big(A(t,\varepsilon,\theta)C_0^{(3)} - C_0^{(3)}B(t,\varepsilon,\theta)\big) = V_3\Big(L_3(F(t,\varepsilon,\theta))B(t,\varepsilon,\theta) - A(t,\varepsilon,\theta)L_3(F(t,\varepsilon,\theta))\Big). \tag{4.39}$$

In scalar form, the condition (4.39) can be written as a triangular with respect to $c_{01}^{(3)},\dots,c_{0K}^{(3)}$ system of linear algebraic equations:

$$\sum_{l=1}^{j} g_{jl}^{(3)}(t,\varepsilon)c_{0l}^{(3)} = h_j^{(3)}(t,\varepsilon), \quad j=\overline{1,K},$$

where $g_{jl}^{(3)}(t,\varepsilon), h_j^{(3)}(t,\varepsilon) \in S(m;\varepsilon_0)$ and $g_{jj}^{(3)}(t,\varepsilon) = \Gamma_0(a_{1M}(t,\varepsilon,\theta))$ $(j=\overline{1,K})$ are the known functions.

Suppose that

$$g_{jl}^{(3)}(t,\varepsilon) \equiv 0 \ \ (j,l=\overline{1,K}, \ \ l\le j), \tag{4.40}$$

$$h_j^{(3)}(t,\varepsilon) \equiv 0 \ \ (j=\overline{1,K}). \tag{4.41}$$

Then

$$V_3\big(A(t,\varepsilon,\theta)C_0 - C_0B(t,\varepsilon,\theta)\big) = 0$$

for any C_0 of the kind (4.38) and

$$V_3\Big(L_3(F(t,\varepsilon,\theta))B(t,\varepsilon,\theta) - A(t,\varepsilon,\theta)L_3(F(t,\varepsilon,\theta))\Big) = 0.$$

Therefore the equation (4.39) is satisfied for any $C_0^{(3)}$ of the kind (4.38).

Define the matrix $C_0^{(3)}(t,\varepsilon)$ from the equation

$$V_3\big(AL_3(AC_0^{(3)}-C_0^{(3)}B)-L_3(AC_0^{(3)}-C_0^{(3)}B)B\big) \\ -V_3\big(L_3(F)RC_0^{(3)}+C_0^{(3)}RL_3(F)\big)-V_3(C_0^{(3)}RC_0^{(3)})+U^{(3)}=0, \quad (4.42)$$

where $U^{(3)}=U^{(3)}(t,\varepsilon)$ is the known M-vector, which does not depend on $C_0^{(3)}$.

In scalar form, the equation (4.42) can be written as a nonlinear with respect to $c_{01}^{(3)},\dots,c_{0K}^{(3)}$ system of algebraic equations

$$\Phi_j^{(3)}(t,\varepsilon,c_{01}^{(3)},\dots,c_{0K}^{(3)})=0, \ \ j=\overline{1,K}, \quad (4.43)$$

with quadratic nonlinearities.

Suppose that the system (4.43) has a solution $c_{01}^{(3)},\dots,c_{0K}^{(3)}$ such that

$$\inf_G\left|\det\frac{\partial(\Phi_1^{(3)},\dots,\Phi_K^{(3)})}{\partial(c_{01}^{(3)},\dots,c_{0K}^{(2)})}\right|>0. \quad (4.44)$$

Lemma 4.3. *Let the equation* (3.8) *satisfy the following conditions:*

(1) $M>K$;

(2) $V_3(F(t,\varepsilon,\theta))\equiv 0$;

(3) *the equalities* (4.40), (4.41) *hold;*

(4) *the system* (4.43) *has a solution, which satisfy the condition* (4.44).

Then there exists $\mu_3\in(0,1)$ *such that for any* $\mu\in(0,\mu_3)$ *there exists a transformation of the form* (4.28), *where the matrix* $\Xi(t,\varepsilon,\theta,\mu)$ *is defined by* (4.2) *and the elements of the* $(M\times M)$*-matrix* Φ *and those of the* $(K\times K)$*-matrix* Ψ *belong to the class* $F(m;\varepsilon_0;\theta)$ $\forall\,\mu\in(0,\mu_3)$, *which reduces the equation* (3.8) *to the form* (4.29).

Proof of Lemma 4.3 is similar to that of Lemma 4.1, too.

Introduce the matrices

$$\widetilde U_1(t,\varepsilon,\mu)=\sum_{l=1}^q U_{l1}(t,\varepsilon)\mu^l, \quad \widetilde U_2(t,\varepsilon,\mu)=\sum_{l=1}^q U_{l2}(t,\varepsilon)\mu^l,$$

where U_{l1},U_{l2} $(l=\overline{1,q})$ are defined in Lemma 4.1.

Lemma 4.4. *Let the equation* (4.29) *satisfy the following conditions:*

(1) *eigenvalues* $\lambda_{1j}(t,\varepsilon,\mu)$ $(j=\overline{1,M})$ *of the matrix* $J_M+\widetilde U_1(t,\varepsilon,\mu)$ *and* $\lambda_{2s}(t,\varepsilon,\mu)$ $(s=\overline{1,K})$ *of the matrix* $J_K+\widetilde U_2(t,\varepsilon,\mu)$ *are such that*

$$\inf_G\big|\operatorname{Re}\big(\lambda_{1j}(t,\varepsilon,\mu)-\lambda_{2s}(t,\varepsilon,\mu)\big)\big|\ge\gamma_0\mu^{q_0} \ \ (\gamma_0>0, \ \ 0<q_0\le q; \ \ j=\overline{1,M}; \ \ s=\overline{1,K});$$

(2) *there exist a* $(M\times M)$*-matrix* $P_1(t,\varepsilon,\mu)$ *and a* $(K\times K)$*-matrix* $P_2(t,\varepsilon,\mu)$ *such that*

   (a) *all the elements of these matrices belong to the class* $S(m;\varepsilon_0)\subset F(m;\varepsilon_0;\theta)$;

   (b) $\|P_j^{-1}(t,\varepsilon,\mu)\|^*_{F(m\varepsilon_0,\theta)}\le M_1\mu^{-\alpha}$, $M_1\in(0,+\infty)$, $\alpha\in[0,q]$, $j=1,2$;

   (c) $P_1^{-1}(J_M+\widetilde U_1)P_1=\Lambda_1(t,\varepsilon,\mu)$, $P_2(J_K+\widetilde U_2)P_2^{-1}=\Lambda_2(t,\varepsilon,\mu)$, *where* $\Lambda_1=\operatorname{diag}(\lambda_{11},\dots,\lambda_{1M})$, $\Lambda_2=\operatorname{diag}(\lambda_{21},\dots,\lambda_{2K})$;

(3) $q>q_0+\alpha-1/2$.

Then there exist $\mu_4 \in (0,1)$ and $K_4 \in (0,+\infty)$ such that for any $\mu \in (0,\mu_4)$ the matrix differential equation (4.29) *has a particular solution $Y(t,\varepsilon,\theta,\mu)$ all elements of which belong to the class $F(m-1;\varepsilon_1(\mu);\theta)$, where $\varepsilon_1(\mu)=\min(\varepsilon_0, K_4\mu^{2q_0+2\alpha-1})$.*

Proof of Lemma 4.4 is completely analogous to that of Lemma 3 in [1].

The following Lemma is an immediate consequence of the above ones.

Lemma 4.5. *Let the equation* (3.8) *satisfy all conditions of Lemma* 4.1 (*in case $M<K$*)*, or Lemma* 4.2 (*in case $M=K$*)*, or Lemma* 4.3 (*in case $M>K$*)*, and the equation* (4.29)*, obtained from* (3.8) *by means of the transformation* (4.28)*, satisfy all conditions of Lemma* 4.4*. Then there exist $\mu_5 \in (0,1)$ and $K_5 \in (0,+\infty)$ such that for any $\mu \in (0,\mu_5)$ the equation* (3.8) *has a particular solution belonging to the class $F(m-1;\varepsilon_2(\mu);\theta)$, where $\varepsilon_2(\mu)=K_5\mu^{2q_0+2\alpha-1}$ and q_0,α are defined in Lemma* 4.4.

5 The basic result

Based on the above reasoning in Section 3 and Lemma 4.5 we obtain the following result.

Theorem. *Let each of the equations* (3.6) *satisfy all conditions of Lemma* 4.5*. Then there exist $\mu_6 \in (0,1)$ and $K_6 \in (0,+\infty)$ such that for any $\mu \in (0,\mu_6)$ there exists a transformation of the form* (3.2) *with coefficients from the class $F(m-1;\varepsilon_3(\mu);\theta)$, where $\varepsilon_3(\mu)=K_6\mu^{2q_0+2\alpha-1}$ (q_0, α are defined in Lemma* 4.4)*, which reduces the system* (3.1) *to the block-diagonal form* (3.3)*. The matrices D_{N_1}, D_{N_2} are defined by* (3.7).

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Author's address:

I. I. Mechnikov Odessa National University, 2 Dvoryanskaya St., Odessa 65082, Ukraine.
E-mail: `sergas1959@gmail.com`