Memoirs on Differential Equations and Mathematical Physics

Volume 72, 2017, 91–101

Zdeněk Opluštil

ON THE SOLVABILITY OF A NONLOCAL BOUNDARY VALUE PROBLEM FOR THE FIRST ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. We study a nonlocal boundary value problem for nonlinear functional differential equations. New effective conditions are found for the solvability and unique solvability of the problem under consideration. General results are applied to differential equations with deviating arguments.*

2010 Mathematics Subject Classification. 34K10.

Key words and phrases. Boundary value problem, nonlinear functional differential equation, solvability, unique solvability.

რეზიუმე. ნაშრომში შესწავლილია არალოკალური სასაზღვრო ამოცანა არაწრფივი ფუნქციონალურ-დიფერენციალური განტოლებებისთვის. დადგენილია განხილული ამოცანის ამოხსნადობის და ცალსახად ამოხსნადობის ახალი ეფექტური პირობები. ზოგადი შედეგები გამოყენებულია გადახრილ-არგუმენტებიანი დიფერენციალური განტოლებებისთვის.

*Reported on Conference "Differential Equation and Applications", September 4-7, 2017, Brno

Introduction

On the interval $[a,b]$, we consider the functional differential equation

$$u'(t) = F(u)(t), \tag{0.1}$$

where $F : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a continuous (in general) nonlinear operator. As usually, by a solution of this equation we understand an absolutely continuous function $u : [a,b] \to \mathbb{R}$ satisfying equality (0.1) almost everywhere on $[a,b]$. Along with equation (0.1), we consider the nonlocal boundary condition

$$h(u) = \varphi(u), \tag{0.2}$$

where $\varphi : C([a,b];\mathbb{R}) \to \mathbb{R}$ is a continuous (in general) nonlinear functional and $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ is a (nonzero) linear bounded functional.

The following notation is used in the sequel.

- $\mathbb{R}$ is the set of all real numbers. $\mathbb{R}_+ = [0,+\infty[$.
- $C([a,b];\mathbb{R})$ is the Banach space of continuous functions $v : [a,b] \to \mathbb{R}$ with the norm $\|v\|_C = \max\{|v(t)| : t \in [a,b]\}$.
- $AC([a,b];\mathbb{R})$ is the set of absolutely continuous functions $v : [a,b] \to \mathbb{R}$.
- $L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,\mathrm{d}s$.
- $L([a,b];\mathbb{R}_+) = \big\{p \in L([a,b];\mathbb{R}) : p(t) \geq 0 \text{ for almost all } t \in [a,b]\big\}$.
- $\mathcal{L}_{ab}$ is the set of linear operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ for which there exists a function $\eta \in L([a,b];\mathbb{R}_+)$ such that
$$|\ell(v)(t)| \leq \eta(t)\|v\|_C \ \text{ for a.e. } t \in [a,b] \text{ and all } v \in C([a,b];\mathbb{R}).$$
- P_{ab} is the set of so-called *positive* operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.
- F_{ab} is the set of linear bounded functionals $h : C([a,b];\mathbb{R}) \to \mathbb{R}$.
- PF_{ab} is the set of so-called *positive* functionals $h \in F_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $\mathbb{R}_+$.
- $\mathcal{B}^i_{hc} = \{u \in C([a,b];\mathbb{R}) : h(u)\operatorname{sgn}\big((2-i)u(a)+(i-1)u(b)\big) \leq c\}$, where $h \in F_{ab}$, $c \in \mathbb{R}$, $i = 1,2$.
- $K([a,b] \times A; B)$, where $A, B \subseteq \mathbb{R}$, is the set of function $f : [a,b] \times A \to B$ satisfying the Carathéodory conditions, i.e., $f(\,\cdot\,,x) : [a,b] \to B$ is a measurable function for all $x \in A$, $f(t,\cdot\,) : A \to B$ is a continuous function for almost every $t \in [a,b]$, and for every $r > 0$, there exists $q_r \in L([a,b];\mathbb{R}_+)$ such that
$$|f(t,x)| \leq q_r(t) \ \text{ for a.e. } t \in [a,b] \text{ and all } x \in A,\ |x| \leq r.$$

As usual, throughout the paper we suppose the following assumptions on a nonlinear operator F and a functional φ:

$$\begin{gathered} F : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R}) \ \text{ is a continuous operator such that the relation} \\ \sup\big\{|F(v)(\cdot)| : v \in C([a,b];\mathbb{R}),\ \|v\|_C \leq r\big\} \in L([a,b];\mathbb{R}_+) \ \text{ holds for every } r > 0. \end{gathered} \tag{H1}$$

and

$$\begin{gathered} \varphi : C([a,b];\mathbb{R}) \to \mathbb{R} \ \text{ is a continuous functional such that the condition} \\ \sup\big\{|\varphi(v)| : v \in C([a,b];\mathbb{R}),\ \|v\|_C \leq r\big\} < +\infty \ \text{ holds for every } r > 0. \end{gathered} \tag{H2}$$

The solvability of boundary value problems for functional differential equations is being studied intensively. There are many interesting results in the literature (see, e.g., [1–6, 8] and references therein). But in the case, where a nonlocal boundary condition is considered, there are still many open problems.

In this paper, we generalize the results stated in [4] in such a way that the boundary condition (0.2) is considered as a nonlocal perturbation of the two point condition

$$u(a) + \lambda u(b) = \varphi(u). \tag{0.3}$$

where $\lambda \in \mathbb{R}_+$. Consequently, in what follows, we consider the linear functional h in the form

$$h(v) \stackrel{\text{def}}{=} v(a) + \lambda v(b) - h_0(v) + h_1(v) \;\text{ for }\; v \in C([a,b];\mathbb{R}), \tag{0.4}$$

where $\lambda \in \mathbb{R}_+$ and $h_0, h_1 \in PF_{ab}$. There is no loss of generality to assume h like the above one, because an arbitrary linear functional h can be represented in this form.

One can see that a particular case of equation (0.1) is, for example, the differential equation with deviating arguments

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\sigma(t)) + f\big(t, u(t), u(\mu(t))\big), \tag{0.5}$$

where $p, g \in L([a,b];\mathbb{R}_+)$, $\tau, \sigma, \mu : [a,b] \to [a,b]$ are measurable functions, and $f \in K([a,b] \times \mathbb{R}^2;\mathbb{R})$. We mention that the conditions for the solvability and unique solvability of boundary value problems for this equation are presented in Section 2.

On the other hand, the boundary condition (0.2) covers, for example, the Cauchy problem, antiperiodic problem, condition (0.3) and an integral condition of the form $\int\limits_a^b u(s)\,\mathrm{d}s = c$.

The statements formulated below generalize some results stated in [7] concerning the linear case, as well as, some results presented in [4] concerning problem (0.1), (0.3).

1 Main results

In this section, new effective conditions are found for the solvability and unique solvability of problem (0.1), (0.2).

Theorem 1.1. *Let $c \in \mathbb{R}_+$, h be defined by* (0.4)*, where $\lambda \in\,]0,1]$ and*

$$h_0(1) < \lambda. \tag{1.1}$$

Let, moreover,

$$\varphi(v)\operatorname{sgn} v(b) \le c \;\text{ for }\; v \in C([a,b];\mathbb{R}) \tag{1.2}$$

and there exist

$$\ell_0, \ell_1 \in P_{ab} \tag{1.3}$$

such that on the set $\mathcal{B}^2_{hc}([a,b];\mathbb{R})$ the inequality

$$\big(F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)\big)\operatorname{sgn} v(t) \ge -q(t, \|v\|_C) \;\text{ for a.e. }\; t \in [a,b] \tag{1.4}$$

holds, where the function $q \in K([a,b] \times \mathbb{R}_+;\mathbb{R}_+)$ satisfies

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)\,\mathrm{d}s = 0. \tag{1.5}$$

If, in addition,

$$(1 + h_1(1))\|\ell_0(1)\|_L + \lambda\|\ell_1(1)\|_L < \lambda - h_0(1), \tag{1.6}$$

then problem (0.1), (0.2) *has at least one solution.*

Remark 1.1. Let the operator $\psi : L([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ be defined by the formula

$$\psi(z)(t) \overset{\text{def}}{=} z(a+b-t) \ \text{ for a.e. } t \in [a,b] \text{ and all } z \in L([a,b];\mathbb{R}).$$

Let, moreover, $\lambda \in [1,+\infty[$, ω be the restriction of ψ to the space $C([a,b];\mathbb{R})$, and

$$\begin{aligned}
\widehat{F}(z)(t) &\overset{\text{def}}{=} -\psi(F(\omega(z)))(t) \ \text{ for a.e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}),\\
\widehat{h}(z) &\overset{\text{def}}{=} z(a) + \frac{1}{\lambda}\, z(b) - \frac{1}{\lambda}\, h_0(\omega(z)) + \frac{1}{\lambda}\, h_1(\omega(z)) \ \text{ for } \ z \in C([a,b];\mathbb{R}),\\
\widehat{\varphi}(z) &\overset{\text{def}}{=} \frac{1}{\lambda}\, \varphi(\omega(z)) \ \text{ for } \ z \in C([a,b];\mathbb{R}).
\end{aligned}$$

It is not difficult to verify that if u is a solution of problem (0.1), (0.2), then the function $v \overset{\text{def}}{=} \omega(u)$ is a solution of the problem

$$v'(t) = \widehat{F}(v)(t), \quad \widehat{h}(v) = \widehat{\varphi}(v), \tag{1.7}$$

and vice versa, if v is a solution of problem (1.7), then the function $u \overset{\text{def}}{=} \omega(v)$ is a solution of problem (0.1), (0.2).

Using the transformation described in the previous remark, we can immediately derive from Theorem 1.1 the following statement.

Theorem 1.2. *Let $c \in \mathbb{R}_+$, h be defined by* (0.4)*, where $\lambda \in [1,+\infty[$ and*

$$h_0(1) < 1. \tag{1.8}$$

Let, moreover, the condition

$$\varphi(v)\operatorname{sgn} v(a) \le c \ \text{ for } \ v \in C([a,b];\mathbb{R}) \tag{1.9}$$

be fulfilled and there exist $\ell_0, \ell_1 \in P_{ab}$ such that on the set $\mathcal{B}^1_{hc}([a,b];\mathbb{R})$ the inequality

$$\big(F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)\big)\operatorname{sgn} v(t) \le q(t, \|v\|_C) \ \text{ for a.e. } \ t \in [a,b]$$

hold, where the function $q \in K([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$ satisfies (1.5)*. If, in addition,*

$$\|\ell_0(1)\|_L + (\lambda + h_1(1))\|\ell_1(1)\|_L < 1 - h_0(1), \tag{1.10}$$

then problem (0.1), (0.2) *has at least one solution.*

The next theorems deal with the unique solvability of problem (0.1), (0.2).

Theorem 1.3. *Let h be defined by* (0.4)*, where $\lambda \in [0,1[$ and $h_0(1)$ satisfies* (1.1)*. Let, moreover, the condition*

$$(\varphi(v) - \varphi(w))\operatorname{sgn}(v(b) - w(b)) \le 0 \tag{1.11}$$

hold for every $v, w \in C([a,b];\mathbb{R})$ and there exist $\ell_0, \ell_1 \in P_{ab}$ such that on the set $\mathcal{B}^2_{hc}([a,b];\mathbb{R})$ with $c = |\varphi(0)|$ the inequality

$$\big(F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)\big)\operatorname{sgn}(v(t) - w(t)) \ge 0 \tag{1.12}$$

is fulfilled for a.e. $t \in [a,b]$. If, in addition, condition (1.6) *is satisfied, then problem* (0.1), (0.2) *is uniquely solvable.*

Theorem 1.4. *Let h be defined by* (0.4)*, where $\lambda \ge 1$ and $h_0(1)$ satisfies* (1.8)*. Let, moreover, the condition*

$$(\varphi(v) - \varphi(w))\operatorname{sgn}(v(a) - w(a)) \le 0 \tag{1.13}$$

hold for every $v, w \in C([a,b];\mathbb{R})$ and there exist $\ell_0, \ell_1 \in P_{ab}$ such that, on the set $\mathcal{B}^1_{hc}([a,b];\mathbb{R})$ with $c = |\varphi(0)|$, the inequality

$$\big(F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)\big)\operatorname{sgn}(v(t) - w(t)) \le 0 \tag{1.14}$$

is fulfilled for a.e. $t \in [a,b]$. If, in addition, condition (1.10) *is satisfied, then problem* (0.1), (0.2) *is uniquely solvable.*

2 Corollaries for nonlinear delay differential equations

In this section, corollaries of the main theorems are presented. We formulate the conditions guaranteeing the solvability and the unique solvability of the problem

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\sigma(t)) + f(t, u(t), u(\mu(t))), \tag{0.5}$$
$$\varphi(u) = h(u), \tag{0.2}$$

where a linear functional h is considered by formula (0.4).

Corollary 2.1. *Let $c \in \mathbb{R}_+$ and h be defined by* (0.4)*, where $\lambda \in [0,1[$ and $h_0(1)$ satisfies* (1.1)*. Let, moreover,* (1.2) *and*

$$f(t,x,y)\operatorname{sgn} x \geq -q(t) \ \text{ for a.e. } t \in [a,b] \text{ and all } x, y \in \mathbb{R}$$

be satisfied, where $q \in L([a,b];\mathbb{R}_+)$. If, in addition,

$$(1 + h_1(1)) \int_a^b p(s)\,\mathrm{d}s + \lambda \int_a^b g(s)\,\mathrm{d}s < \lambda - h_0(1), \tag{2.1}$$

then problem (0.5), (0.2) *has at least one solution.*

Corollary 2.2. *Let $c \in \mathbb{R}_+$ and h be defined by* (0.4)*, where $\lambda \geq 1$ and $h_0(1)$ satisfies* (1.8)*. Let, moreover,* (1.9) *and*

$$f(t,x,y)\operatorname{sgn} x \leq q(t) \ \text{ for a.e. } t \in [a,b] \text{ and all } x, y \in \mathbb{R}$$

be satisfied, where $q \in L([a,b];\mathbb{R}_+)$. If, in addition,

$$\int_a^b p(s)\,\mathrm{d}s + (\lambda + h_1(1)) \int_a^b g(s)\,\mathrm{d}s < 1 - h_0(1), \tag{2.2}$$

then problem (0.5), (0.2) *has at least one solution.*

Corollary 2.3. *Let h be defined by* (0.4)*, where $\lambda \in [0,1[$ and $h_0(1)$ satisfies* (1.1)*. Let, moreover, conditions* (2.1) *and*

$$\big[f(t,x_1,y_1) - f(t,x_2,y_2)\big]\operatorname{sgn}(x_1 - x_2) \geq 0 \ \text{ for a.e. } t \in [a,b] \text{ and all } x_1, x_2, y_1, y_2 \in \mathbb{R}$$

hold. If, in addition, condition (1.11) *is fulfilled for every $v, w \in C([a,b];\mathbb{R})$, then problem* (0.5), (0.2) *is uniquely solvable.*

Corollary 2.4. *Let h be defined by* (0.4)*, where $\lambda \geq 1$ and $h_0(1)$ satisfies* (1.8)*. Let, moreover, conditions* (2.2) *and*

$$\big[f(t,x_1,y_1) - f(t,x_2,y_2)\big]\operatorname{sgn}(x_1 - x_2) \leq 0 \ \text{ for a.e. } t \in [a,b] \text{ and all } x_1, x_2, y_1, y_2 \in \mathbb{R}$$

hold. If, in addition, condition (1.13) *is fulfilled for every $v, w \in C([a,b];\mathbb{R})$, then problem* (0.5), (0.2) *is uniquely solvable.*

3 Auxiliary propositions

We use the lemma on a priory estimate stated in [6] to prove main results of the paper. It can be formulated as follows.

Lemma 3.1 ([6, Corollary 2]). *Let there exist a positive number ρ and an operator $\ell \in \mathcal{L}_{ab}$ such that the homogeneous problem*

$$u'(t) = \ell(u)(t), \quad h(u) = 0 \tag{3.1}$$

has only the trivial solution, and for every $\delta \in]0,1[$ an arbitrary function $u \in AC([a,b];\mathbb{R})$ satisfyings the relation

$$u'(t) = \ell(u)(t) + \delta\big[F(u)(t) - \ell(u)(t)\big] \text{ for a.e. } t \in [a,b], \quad h(u) = \delta\varphi(u) \tag{3.2}$$

admits the estimate

$$\|u\|_C \le \rho. \tag{3.3}$$

Then problem (0.1), (0.2) *has at least one solution.*

Definition 3.1. Let $h \in F_{ab}$. We say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\mathcal{U}(h)$, if there exists $r > 0$ such that for arbitrary $q^* \in L([a,b];\mathbb{R}_+)$ and $c \in \mathbb{R}_+$ every function $u \in AC([a,b];\mathbb{R})$ satisfying the inequalities

$$h(u)\operatorname{sgn} u(b) \le c, \tag{3.4}$$

$$-(u'(t) - \ell(u)(t))\operatorname{sgn} u(t) \le q^*(t) \text{ for a.e. } t \in [a,b] \tag{3.5}$$

admits the estimate

$$\|u\|_C \le r(c + \|q^*\|_L). \tag{3.6}$$

Lemma 3.2. *Let $c \in \mathbb{R}_+$ and* (1.2) *hold. Let, moreover, there exists $\ell \in \mathcal{U}(h)$ such that on the set $\mathcal{B}^2_{hc}([a,b];\mathbb{R})$ the inequality*

$$-(F(v)(t) - \ell(v)(t))\operatorname{sgn} v(t) \le q(t, \|v\|_C) \text{ for a.e. } t \in [a,b] \tag{3.7}$$

is fulfilled, where the function $q \in K([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$ satisfies (1.5). *Then problem* (0.1), (0.2) *has at least one solution.*

Proof. Since $\ell \in \mathcal{U}(h)$, it is not difficult to show that the homogeneous problem (3.1) has only the trivial solution.

Assume that a function $u \in AC([a,b];\mathbb{R})$ satisfies (3.2) with some $\delta \in]0,1[$. By virtue of (1.2), inequality (3.4) is fulfilled, i.e., $u \in \mathcal{B}^2_{hc}([a,b];\mathbb{R})$. Moreover, from relations (3.2) and (3.7) we obtain that (3.5) holds with $q^* \equiv q(\,\cdot\,, \|u\|_C)$. Therefore, in view of (3.4), (3.5) and the assumption $\ell \in \mathcal{U}(h)$, there exist $r > 0$ such that estimate (3.6) holds.

On the other hand, according to (1.5), there exists $\rho > 2rc$ such that

$$\frac{1}{x}\int_a^b q(s,x)\,\mathrm{d}s < \frac{1}{2r} \text{ for } x > \rho.$$

The last inequality, together with (3.6), yields that estimate (3.3) is satisfied. Since ρ depends neither on u nor on δ, it follows from Lemma 3.1 that problem (0.1), (0.2) has at least one solution. □

4 Proofs of main theorems

Proof of Theorem 1.1. Put $\ell = \ell_0 - \ell_1$, where ℓ_0, $\ell_1 \in P_{ab}$ are such that condition (1.6) holds. Firstly, we show that ℓ belongs to the set $\mathcal{U}(h)$.

Let $c \in \mathbb{R}_+$, $q^* \in L([a,b];\mathbb{R}_+)$, and $u \in AC([a,b];\mathbb{R})$ satisfy (3.4) and (3.5). We prove that estimate (3.6) holds, where the number r depends only on $\|\ell_0(1)\|_L$, $\|\ell_1(1)\|_L$, λ, $h_0(1)$, and $h_1(1)$.

It is obvious that

$$u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \widetilde{q}(t) \text{ for a.e. } t \in [a,b], \tag{4.1}$$

where

$$\widetilde{q}(t) = u'(t) - \ell(u)(t) \text{ for a.e. } t \in [a,b].$$

Hence, in view of (0.4), (3.4) and (3.5), we get

$$\big(u(a)+\lambda u(b)-h_0(u)+h_1(u)\big)\operatorname{sgn} u(b)\le c \tag{4.2}$$

and

$$-\widetilde{q}(t)\operatorname{sgn} u(t)\le q^*(t) \ \text{ for a.e. } \ t\in[a,b]. \tag{4.3}$$

First suppose that the function u does not change its sign. Then from (4.2) it follows that

$$|u(a)|+\lambda|u(b)|-h_0(|u|)+h_1(|u|)\le c \ \text{ if } \ u(b)\neq 0. \tag{4.4}$$

Put

$$M_0=\max\{|u(t)|:\ t\in[a,b]\} \tag{4.5}$$

and choose $t_{M_0}\in[a,b]$ such that

$$|u(t_{M_0})|=M_0. \tag{4.6}$$

Clearly, $M_0\ge 0$ and, in view of (1.3), (4.3) and (4.6), from relation (4.1) we get

$$-|u(t)|'\le M_0\ell_1(1)(t)+q^*(t) \ \text{ for a.e. } \ t\in[a,b].$$

The integration of the last inequality from t_{M_0} to b with respect to (1.3), (4.4), (4.6), $\lambda\in\,]0,1]$ and $h_0,\ h_1\in PF_{ab}$, results in

$$M_0\big(\lambda-h_0(1)-\lambda\|\ell_1(1)\|_L\big)\le\|q^*\|_L+c. \tag{4.7}$$

Moreover, it follows from condition (1.6) that $\lambda-h_0(1)-\lambda\|\ell_1(1)\|_L>0$ and thus, relations (4.5) and (4.7) yield

$$\|u\|_C\le\big(\|q^*\|_L+c\big)\big(\lambda-h_0(1)-\lambda\|\ell_1(1)\|_L\big)^{-1}.$$

Consequently, estimate (3.6) holds with $r=(\lambda-h_0(1)-\lambda\|\ell_1(1)\|_L)^{-1}$.

Suppose now that the function u changes its sign. Put

$$m=-\min\{u(t):\ t\in[a,b]\},\quad M=\max\{u(t):\ t\in[a,b]\} \tag{4.8}$$

and choose $t_m,t_M\in[a,b]$ such that

$$-m=u(t_m),\quad M=u(t_M). \tag{4.9}$$

Obviously, $m>0$, $M>0$, and either

$$t_m>t_M \tag{4.10}$$

or

$$t_m<t_M. \tag{4.11}$$

Suppose that relation (4.10) holds. Then there exists $a_1\in\,]t_M,t_m[$ such that

$$u(a_1)=0,\quad u(t)>0 \ \text{ for } \ t_M\le t<a_1. \tag{4.12}$$

Let

$$a_2=\sup\big\{t\in[t_m,b]:\ u(s)<0 \ \text{ for } \ t_m\le s\le t\big\}.$$

Obviously,

$$u(t)<0 \ \text{ for } \ t_m\le t<a_2 \ \text{ and if } \ a_2<b \ \text{ then } \ u(a_2)=0. \tag{4.13}$$

Hence, in view of (4.2) and (4.9), we obtain

$$\lambda u(a_2)\ge -M(1+h_1(1))-mh_0(1)-c. \tag{4.14}$$

Integrating (4.1) from t_M to a_1 and from t_m to a_2, with respect to (1.3), (4.3), (4.8), (4.9), (4.12), (4.13), and (4.14), one gets

$$M\le M\int\limits_{t_M}^{a_1}\ell_1(1)(s)\,\mathrm{d}s+m\int\limits_{t_M}^{a_1}\ell_0(1)(s)\,\mathrm{d}s+\int\limits_{t_M}^{a_1}q^*(s)\,\mathrm{d}s$$

and

$$\lambda m - M\big(1 + h_1(1)\big) - m h_0(1) - c \le \lambda M \int\limits_{t_m}^{a_2} \ell_0(1)(s)\,\mathrm{d}s + \lambda m \int\limits_{t_m}^{a_2} \ell_1(1)(s)\,\mathrm{d}s + \lambda \int\limits_{t_m}^{a_2} q^*(s)\,\mathrm{d}s.$$

Hence, we have

$$\begin{gathered} M(1-A) \le mC + \|q^*\|_L, \\ m(\lambda - h_0(1) - \lambda B) \le M(1 + h_1(1) + \lambda D) + \lambda\|q^*\|_L + c, \end{gathered} \tag{4.15}$$

where

$$A = \int\limits_{t_M}^{a_1} \ell_1(1)(s)\,\mathrm{d}s, \quad B = \int\limits_{t_m}^{a_2} \ell_1(1)(s)\,\mathrm{d}s, \quad C = \int\limits_{t_M}^{a_1} \ell_0(1)(s)\,\mathrm{d}s, \quad D = \int\limits_{t_m}^{a_2} \ell_0(1)(s)\,\mathrm{d}s.$$

By virtue of (1.6) and $\lambda \in]0,1]$, it is clear that $\lambda - h_0(1) - \lambda B > 0$ and $1 - A > 0$. Consequently, inequalities (4.15) imply

$$\begin{gathered} 0 < M(1-A)(\lambda - h_0(1) - \lambda B) \le C\big(M(1 + h_1(1) + \lambda D) + \lambda\|q^*\|_L + c\big) + \|q^*\|_L(\lambda - h_0(1) - \lambda B), \\ 0 < m(1-A)(\lambda - h_0(1) - \lambda B) \le (mC + \|q^*\|_L)\big(1 + h_1(1) + \lambda D\big) + (1-A)(\lambda\|q^*\|_L + c). \end{gathered} \tag{4.16}$$

Observe that

$$(1-A)(\lambda - h_0(1) - \lambda B) \ge \lambda - \lambda(A+B) - h_0(1) \ge \lambda - \lambda\|\ell_1(1)\|_L - h_0(1). \tag{4.17}$$

Moreover, from (1.6) and $\lambda \in]0,1]$ we get

$$C(1 + h_1(1) + \lambda D) \le (1 + h_1(1))(C + D) \le (1 + h_1(1))\|\ell_0(1)\|_L. \tag{4.18}$$

In view of inequalities (1.6), (4.17), and (4.18), it follows from (4.16) that

$$\begin{gathered} M \le r_0\big(1 + \lambda + h_1(1) + \lambda\|\ell_0(1)\|_L\big)(c + \|q^*\|_L), \\ m \le r_0\big(1 + \lambda + h_1(1) + \lambda\|\ell_0(1)\|_L\big)(c + \|q^*\|_L), \end{gathered} \tag{4.19}$$

where

$$r_0 = \Big(\lambda - h_0(1) - \lambda\|\ell_1(1)\|_L - (1 + h_1(1))\|\ell_0(1)\|_L\Big)^{-1}. \tag{4.20}$$

Consequently, estimate (3.6) holds, where the number r is given by

$$r = r_0\big(1 + \lambda + h_1(1) + \lambda\|\ell_0(1)\|_L\big). \tag{4.21}$$

Let now (4.11) hold. Then there exists $a_3 \in]t_M, t_m[$ such that

$$u(a_3) = 0, \quad u(t) < 0 \ \text{ for } \ t_m \le t < a_3. \tag{4.22}$$

Put

$$a_4 = \sup\big\{t \in [t_M, b] : u\,(s) > 0 \ \text{ for } \ t_M \le s \le t\big\}.$$

It is clear that $u(t) > 0$ for $t_M \le t < a_4$ and if $a_4 < b$ then $u(a_4) = 0$. Hence, by virtue of (4.2), we obtain

$$\lambda u(a_4) \le m + M h_0(1) + m h_1(1) + c. \tag{4.23}$$

Integrating (4.1) from t_m to a_3 and from t_M to a_4 and taking into account (1.3), (4.3), (4.8), (4.9), (4.22) and (4.23), one gets

$$m \le M \int\limits_{t_m}^{a_3} \ell_0(1)(s)\,\mathrm{d}s + m \int\limits_{t_m}^{a_3} \ell_1(1)(s)\,\mathrm{d}s + \int\limits_{t_m}^{a_3} q^*(s)\,\mathrm{d}s$$

and

$$-m - Mh_0(1) - mh_1(1) - c + \lambda M \le \lambda m \int\limits_{t_M}^{a_4} \ell_0(1)(s)\,\mathrm{d}s + \lambda M \int\limits_{t_M}^{a_4} \ell_1(1)(s)\,\mathrm{d}s + \lambda \int\limits_{t_M}^{a_4} q^*(s)\,\mathrm{d}s.$$

Hence,

$$\begin{gathered} m(1-\widetilde{A}) \le M\widetilde{C} + \|q^*\|_L, \\ M\big(\lambda - h_0(1) - \lambda\widetilde{B}\big) \le m\big(1 + h_1(1) + \lambda\widetilde{D}\big) + c + \lambda\|q^*\|_L, \end{gathered} \tag{4.24}$$

where

$$\widetilde{A} = \int\limits_{t_m}^{a_3} \ell_1(1)(s)\,\mathrm{d}s, \quad \widetilde{B} = \int\limits_{t_M}^{a_4} \ell_1(1)(s)\,\mathrm{d}s, \quad \widetilde{C} = \int\limits_{t_m}^{a_3} \ell_0(1)(s)\,\mathrm{d}s, \quad \widetilde{D} = \int\limits_{t_M}^{a_4} \ell_0(1)(s)\,\mathrm{d}s.$$

In view of $\lambda \in\,]0,1]$ and (1.6), we have $\lambda - h_0(1) - \lambda\widetilde{B} > 0$ and $1 - \widetilde{A} > 0$. Therefore, inequalities (4.24) yield

$$\begin{gathered} 0 < m(1-\widetilde{A})\big(\lambda - h_0(1) - \lambda\widetilde{B}\big) \le m\widetilde{C}(1 + h_1(1) + \lambda\widetilde{D}) + (\|q^*\|_L + c)\big(1 + \lambda + h_1(1) + \|\ell_0(1)\|_L\big), \\ 0 < M(1-\widetilde{A})\big(\lambda - h_0(1) - \lambda\widetilde{B}\big) \le M\widetilde{C}(1 + h_1(1) + \lambda\widetilde{D}) + (\|q^*\|_L + c)\big(1 + \lambda + h_1(1) + \|\ell_0(1)\|_L\big). \end{gathered}$$

Now, analogously as in case (4.10), we show that relations (4.19) hold with r_0 given by (4.20). Consequently, estimate (3.6) is fulfilled, where the number r is defined by (4.21).

We have proved that estimate (3.6) holds in all possible cases and therefore, the operator $\ell = \ell_0 - \ell_1$ belongs to the set $\mathcal{U}(h)$. Therefore, it follows from Lemma 3.2 that problem (0.1), (0.2) has at least one solution. $\square$

Proof of Theorem 1.2. According to Remark 1.1, the assertion of the theorem follows immediately from Theorem 1.1. $\square$

Proof of Theorem 1.3. It follows from assumption (1.11) that inequality (1.2) is fulfilled on the set $C([a,b];\mathbb{R})$, where $c = |\varphi(0)|$. On the other hand, from (1.12) we get that inequality (1.4) holds on the set $\mathcal{B}^2_{hc}([a,b];\mathbb{R})$, where $q \equiv |F(0)|$. Consequently, according to Theorem 1.1, problem (0.1), (0.2) has at least one solution. Moreover, it follows from the proof of Theorem 1.1 that the operator $\ell = \ell_0 - \ell_1$ belongs to the set $\mathcal{U}(h)$.

It remains to prove that problem (0.1), (0.2) has at most one solution. Let u_1, u_2 be solutions of problem (0.1), (0.2). Put

$$u(t) = u_1(t) - u_2(t) \;\text{ for }\; t \in [a,b].$$

From relations (1.11) and (1.12), we get $u_1, u_2 \in \mathcal{B}^2_{hc}([a,b];\mathbb{R})$ with $c \equiv |\varphi(0)|$,

$$h(u)\operatorname{sgn} u(b) \le 0,$$

and

$$-(u'(t) - \ell(u)(t))\operatorname{sgn} u(t) \le 0 \;\text{ for a.e. }\; t \in [a,b].$$

Consequently, the last inequalities, together with $\ell \in \mathcal{U}(h)$, result in $u \equiv 0$, which yields $u_1 \equiv u_2$. $\square$

Proof of Theorem 1.4. The assertion can be proved analogously to Theorem 1.3. We only use Theorem 1.2 instead of Theorem 1.1 and relations (1.13), (1.14) instead of (1.11), (1.12). $\square$

Proofs of Corollaries 2.1–2.4. The assertions of corollaries follow from Theorems 1.1–1.4. $\square$

Acknowledgement

This research was supported by MEYS under the National Sustainability Programme I (Project LO1202).

References

[1] N. V. Azbelev, V. P. Maksimov and L. F. Rakhmatullina, *Introduction to the Theory of Functional-Differential Equations.* (Russian) "Nauka", Moscow, 1991.

[2] R. Hakl, I. Kiguradze and B. Půža, Upper and lower solutions of boundary value problems for functional differential equations and theorems on functional differential inequalities. *Georgian Math. J.* **7** (2000), no. 3, 489–512.

[3] R. Hakl, A. Lomtatidze and J. Šremr, *Some Boundary Value Problems for First Order Scalar Functional Differential Equations.* Folia Facultatis Scientiarum Naturalium Universitatis Masarykianae Brunensis. Mathematica 10. Masaryk University, Brno, 2002.

[4] R. Hakl, A. Lomtatidze and J. Šremr, On an antiperiodic type boundary-value problem for first order nonlinear functional differential equations of non-Volterra's type. (Russian) *Nelīnīĭnī Koliv.* **6** (2003), no. 4, 550–573; translation in *Nonlinear Oscil. (N.Y.)* **6** (2003), no. 4, 535–559.

[5] J. Hale, *Theory of Functional Differential Equations.* Second edition. Applied Mathematical Sciences, Vol. 3. Springer-Verlag, New York–Heidelberg, 1977.

[6] I. Kiguradze and B. Půža, On boundary value problems for functional-differential equations. International Symposium on Differential Equations and Mathematical Physics (Tbilisi, 1997). *Mem. Differential Equations Math. Phys.* **12** (1997), 106–113.

[7] A. Lomtatidze, Z. Opluštil and J. Šremr, On a nonlocal boundary value problem for first order linear functional differential equations. *Mem. Differential Equations Math. Phys.* **41** (2007), 69–85.

[8] Š. Schwabik and M. Tvrdý and O. Vejvoda, *Differential and Integral Equations. Boundary Value Problems and Adjoints.* D. Reidel Publishing Co., Dordrecht–Boston, Mass.–London, 1979.

(Received 09.10.2017)

Author's address:

Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Technická 2, Brno 616 69, Czech Republic.

E-mail: oplustil@fme.vutbr.cz