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APPROXIMATING SOLUTION OF DISTRIBUTED DELAY DIFFERENTIAL EQUATION USING GAMMA SERIES OF DELAY DENSITY FUNCTION

Abstract. The linear chain trick can be used to solve differential equations with distributed delays of gamma type. In this paper we show that other densities of delay can be expressed as a sum of gamma densities, which can then be used to find approximate solution of differential equation with distributed delay.[1]

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Key words and phrases. Distributed delay, gamma series, Laguerre polynomials, linear chain trick.

რეზიუმე. ნაშრომში ჯაჭვური კანონის გამოყენებით ამოხსნილია γ-ტიპის დაგვიანებულ არგუმენტიანი განტოლება. ნაჩვენებია, რომ დაგვიანების სიმკვრივე შეიძლება გამოისახოს როგორც γ-ტიპის სიმკვრივეთა ჯამი. აღნიშნული გამოსახვა შეიძლება გამოყენებულ იქნას დაგვიანებულ არგუმენტიანი განტოლების მიახლოებითი ამონახსნის მოსაძებნად.

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1 Introduction

In this paper we are looking for an approximate solution of a differential equation with distributed delay, i.e.,

$$\begin{aligned} \dot{x}(t) &= f\Big(t, x(t), \int_0^\infty x(t-s)g(s)\,\mathrm{d}s\Big), \\ x(t) &= \phi(t), \ \ t \le t_0, \end{aligned} \tag{1.1}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a Lipschitz function (to ensure the existence and uniqueness of the solution), ϕ is an initial function (we usually need it to be continuous and bounded on its domain) and $g : [0,\infty) \to [0,\infty)$ is a weight function which describes how past states of x are affecting present rate of change. We can presume that g is normed, that is, $\int_0^\infty g(s)\,\mathrm{d}s = 1$. This means that g is a density of some nonnegative random variable which we interpret as a delay.

Problem (1.1) is a generalization of a differential equation with a constant delay, i.e.,

$$\begin{aligned} \dot{x}(t) &= f\big(t, x(t), x(t-\tau)\big), \\ x(t) &= \phi(t), \ \ t \in [t_0-\tau, t_0], \end{aligned} \tag{1.2}$$

where f, ϕ are the same as in (1.2) and $\tau > 0$ is a constant delay. We can find a solution of (1.2) by the method of steps. However, the method of steps can be used to transform (1.1) to an ordinary differential equation only if $0 \notin \operatorname{supp}(g)$. This restriction may be quite problematic, not often describing the modelled phenomena well.

Another possible way to solve (1.1) is the use of the Laplace transform. This is not a versatile method, since it entails several nontrivial steps, such as finding the Laplace transform of g, solving an algebraic equation and, finally, applying an inverse Laplace transform on a possibly complicated function.

In a special case, where g is a density of gamma distribution, that is,

$$g_a^p(t) = \begin{cases} \dfrac{a^p t^{p-1}\mathrm{e}^{-at}}{\Gamma(p)}, & t \ge 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1.3}$$

where $a > 0$, $p \in \mathbb{N}$ and $\Gamma(t)$ denotes the gamma function at t, we can transform (1.1) to a system of ordinary differential equations. This process is called the linear chain trick and is in detail explained in [3]. We will briefly describe it for the case of a scalar equation with one distributed delay of gamma type, but it can be easily generalized to the case of a vector equation or multiple distributed delays of gamma type.

Consider (1.1) with $g = g_a^p$. We can introduce new variables $y_1, y_2, \dots, y_p$,

$$y_k = \int_0^\infty x(t-s)g_a^k(s)\,\mathrm{d}s, \ \ k = 1, 2, \dots, p. \tag{1.4}$$

Since

$$\begin{aligned} \dot{g}_a^p &= a(g_a^{p-1} - g_a^p), \ \ p > 1, \\ \dot{g}_a^1 &= -a g_a^1, \end{aligned} \tag{1.5}$$

new variables y_k satisfy the system of ordinary differential equations

$$\begin{aligned} \dot{y}_p(t) &= a\big(y_{p-1}(t) - y_p(t)\big) \\ \dot{y}_{p-1}(t) &= a\big(y_{p-2}(t) - y_{p-1}(t)\big) \\ &\ \vdots \\ \dot{y}_2(t) &= a\big(y_1(t) - y_2(t)\big) \\ \dot{y}_1(t) &= a\big(x(t) - y_1(t)\big). \end{aligned} \tag{1.6}$$

Together with the original equation

$$\dot{x}(t) = f\big(t, x(t), y_p(t)\big), \tag{1.7}$$

we obtain a system of $p+1$ ordinary differential equations. The initial values are given by

$$\begin{aligned} x(t_0) &= \phi(t_0), \\ y_k(t_0) &= \int_0^\infty \phi(t_0 - s) g_a^k(s)\,\mathrm{d}s, \quad k = 1, 2, \ldots, p. \end{aligned} \tag{1.8}$$

System (1.6) is in itself an autonomous linear system with constant coefficients, therefore if (1.1) is autonomous or linear (with constant coefficients), the same is true for the new system.

2 The main result

The linear chain trick can only be used for gamma densities. However, if we could express other densities of nonnegative random variables as a sum of gamma densities, we could apply linear chain trick on each element of the sum. In other words, we are interested in describing the linear span of $\{g_a^p,\ a > 0,\ p \in \mathbb{N}\}$.

The method of expanding density of a nonnegative random variable into a sum of gamma densities is described in [1]. We perform similar construction for $a = 1$. The choice of the value of parameter a is not important at the moment, so the optimal value is to be discussed.

Consider the space $L^2_\gamma(\mathbb{R}_0^+)$, i.e., the linear space of real functions $f : [0, \infty) \to \mathbb{R}$ satisfying

$$\int_0^\infty \mathrm{e}^{-x} f^2(x)\,\mathrm{d}x < \infty. \tag{2.1}$$

This is a Hilbert space with the inner product $\langle f, g\rangle_\gamma$ given by

$$\langle f, g\rangle_\gamma = \int_0^\infty \mathrm{e}^{-x} f(x) g(x)\,\mathrm{d}x. \tag{2.2}$$

Lemma 2.1. *The set of Laguerre polynomials*

$$\Big\{L_n(x) = \frac{\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,(x^n \mathrm{e}^{-x}),\ n \in \mathbb{N}\Big\} = \Big\{L_n(x) = \sum_{j=0}^n \binom{n}{j}\frac{(-1)^j}{j!}\,x^j,\ n \in \mathbb{N}\Big\} \tag{2.3}$$

is a complete orthonormal set in $L^2_\gamma(\mathbb{R}_0^+)$.

Proof. See [2]. □

Let f be a density of a nonnegative random variable. We want to express it as a series

$$f(x) = \mathrm{e}^{-x}\sum_{k=0}^\infty a_k L_k(x). \tag{2.4}$$

For $n \in \mathbb{N}$ (using orthonormality of Laguerre polynomials),

$$\int_0^\infty f(x) L_n(x)\,\mathrm{d}x = \int_0^\infty L_n(x)\mathrm{e}^{-x}\sum_{k=0}^\infty a_k L_k(x)\,\mathrm{d}x = a_n \tag{2.5}$$

holds. This is well-defined if the first n raw moments of the density function f are finite. Series (2.4) is, in fact, a sum of gamma densities, since

$$\begin{aligned} \mathrm{e}^{-x}\sum_{k=0}^{\infty} a_k L_k(x) &= \sum_{k=0}^{\infty}\Big(\mathrm{e}^{-x} a_k \sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j}{j!}x^j\Big) \\ &= \sum_{k=0}^{\infty}\Big(\mathrm{e}^{-x} a_k \sum_{j=0}^{k}\binom{k}{j}(-1)^j g_1^{j+1}(x)\Big) = \sum_{k=0}^{\infty}\alpha_k g_1^{k+1}(x), \end{aligned} \tag{2.6}$$

where α_k contains all coefficients at the corresponding gamma density g_1^{k+1}. We call series (2.4) the gamma series of the function f.

We have assumed that (2.4) held. Regarding that, there arises the important question for what densities f the corresponding gamma series converges to the original function f.

Theorem 2.1. *Let f be a density of a nonnegative random variable with all of the raw moments finite and let there exist $x_0 > 0$ and constants $c > 0$, $\delta > 0$ such that for all $x \geq x_0$, the inequality $f(x) \leq c\mathrm{e}^{-x\frac{1+\delta}{2}}$ is satisfied. Then the gamma series of f converges to f in the sense of*

$$\lim_{n\to\infty}\int_0^{\infty}\mathrm{e}^{x}\Big(f(x)-\mathrm{e}^{-x}\sum_{k=0}^{n}a_k L_k(x)\Big)^2\,\mathrm{d}x = 0. \tag{2.7}$$

Proof. Denote $h(x) = \mathrm{e}^{x} f(x)$. The gamma series of h is $\sum_{k=0}^{\infty} a_k L_k(x)$. The Laguerre polynomials are a complete orthonormal set in $L^2_\gamma(\mathbb{R}^+_0)$, therefore $h \in L^2_\gamma(\mathbb{R}^+_0)$ must hold. To prove this, we calculate its norm in $L^2_\gamma(\mathbb{R}^+_0)$, i.e.,

$$\int_0^{\infty}\mathrm{e}^{-x}h^2(x)\,\mathrm{d}(x) = \int_0^{\infty}\mathrm{e}^{x}f^2(x)\,\mathrm{d}x = \underbrace{\int_0^{x_0}\mathrm{e}^{x}f^2(x)\,\mathrm{d}x}_{I_1} + \underbrace{\int_{x_0}^{\infty}\mathrm{e}^{x}f^2(x)\,\mathrm{d}x}_{I_2}. \tag{2.8}$$

The first term I_1 is an integral of a bounded function over a finite interval, so $I_1 < \infty$. Using the theorem's assumptions, we can show $I_2 < \infty$ as well, since

$$I_2 \leq c^2\int_{x_0}^{\infty}\mathrm{e}^{x-x(1+\delta)}\,\mathrm{d}x = \frac{c^2}{\delta}\mathrm{e}^{x_0} < \infty. \tag{2.9}$$

This means that $h \in L^2_\gamma(\mathbb{R}^+_0)$ and the rest of the theorem is a consequence of the Fourier series theory, specifically the Riesz–Fischer theorem. □

Corollary 2.1. *Any density of a nonnegative random variable with a compact support can be expressed as a gamma series.*

This result can be used to find an approximate solution of equation (1.1):

- Find the first n terms of gamma series of g.
- Apply the linear chain trick to each term.
- Solve (analyze) the resulting system of ordinary differential equations.

Remark 2.1. To find the first n terms in the gamma series of a function we need only the first n raw moments. This is useful in the case where delays are measured experimentally and we need to estimate the probability density function, since we can use sample raw moments instead of theoretical ones and thus obtain an estimation in the form of a gamma series.

Table 1. The first n coefficients of gamma series (2.6) of hat distribution (3.2).

	$n=1$	$n=3$	$n=5$	$n=10$	$n=20$
α_0	1	0.083308	−0.591735	−0.154871	0.152195
α_1	0	2.333400	5.291708	1.046663	−0.978461
α_2		−1.916725	−7.011521	11.170907	6.456673
α_3		0.500017	4.766992	−40.563154	71.573716
α_4			−1.719577	71.475719	−598.149055
α_5			0.261133	−80.029644	2354.843929
α_6				60.821780	−6267.150889
α_7				−31.534065	12544.162081
α_8				10.745646	−19803.452473
α_9				−2.178752	25272.614335
α_{10}				0.1997700	−26411.951554
α_{11}					22735.938215
α_{12}					−16129.161624
α_{13}					9390.531536
α_{14}					−4446.376297
α_{15}					1686.572512
α_{16}					−500.587669
α_{17}					112.058428
α_{18}					−17.798574
α_{19}					1.788478
α_{20}					−0.085504
Σ	1	1	1	1	1

3 Example

Consider the initial value problem

$$\begin{aligned} \dot{x}(t) &= -2\int_0^\infty x(t-s)g_h(s)\,\mathrm{d}s, \\ x(t) &= 1, \quad t \le 0, \end{aligned} \tag{3.1}$$

where g_h is the probability density function of the hat distribution, to be specific,

$$g_h(t) = \begin{cases} t, & t \in [0,1], \\ 2-t, & t \in [1,2], \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

We will compare an approximate solution $\widehat{x}_n$ obtained by expanding g_h into the gamma series with the first n terms and another approximate solution x_h, where x_h is obtained by a discretization

$$x_h(t+h) = x(t) - 2h\sum_{k=0}^{200} x(t-hk)g_h(hk) \tag{3.3}$$

with step size $h = 0.01$.

To illustrate the method, we compute the approximate solution of problem (3.1) for different values of n, in particular, for $n = 1, 3, 5, 10, 20$.

First, we compute the first n coefficients of gamma series of the hat distribution by numerically integrating (2.5) and then sum the results according to (2.6). Numerical values of coefficients α_k, $k = 0, 1, \ldots, n$, are given in Table 1. Notice that the sum of coefficients for each n is 1.

Instead of the initial value problem (3.1), we can solve the problem

$$\dot{\widehat{x}}(t) = -2\int_0^\infty \Big(\widehat{x}(t-s)\sum_{k=0}^{n} g_1^{k+1}(s)\Big)\,\mathrm{d}s = -2\sum_{k=0}^{n}\alpha_k\int_0^\infty \widehat{x}(t-s)g_1^{k+1}(s)\,\mathrm{d}s, \qquad (3.4)$$
$$x(t) = 1, \quad t \le 0.$$

Using the linear chain trick, we obtain a system of $n+1$ ordinary differential equations with constant coefficients

$$\begin{pmatrix}\dot{\widehat{x}}(t)\\ \dot{y}_1(t)\\ \dot{y}_2(t)\\ \vdots\\ \dot{y}_n(t)\end{pmatrix} = \begin{pmatrix}0 & -2\alpha_0 & -2\alpha_1 & \cdots & -2\alpha_n\\ 1 & -1 & 0 & \cdots & 0\\ 0 & 1 & -1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -1\end{pmatrix}\begin{pmatrix}\widehat{x}(t)\\ y_1(t)\\ y_2(t)\\ \vdots\\ y_n(t)\end{pmatrix}. \qquad (3.5)$$

Initial values are $\widehat{x}(0) = y_1(0) = \cdots = y_n(0) = 1$.

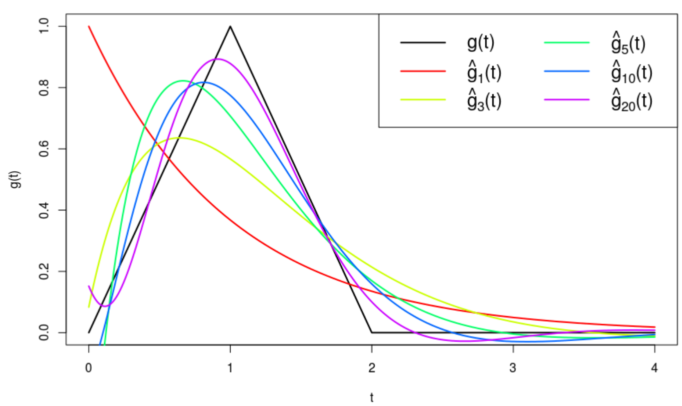

Figure 1. Approximation of the hat probability density function g_h by the first n terms of its gamma series.

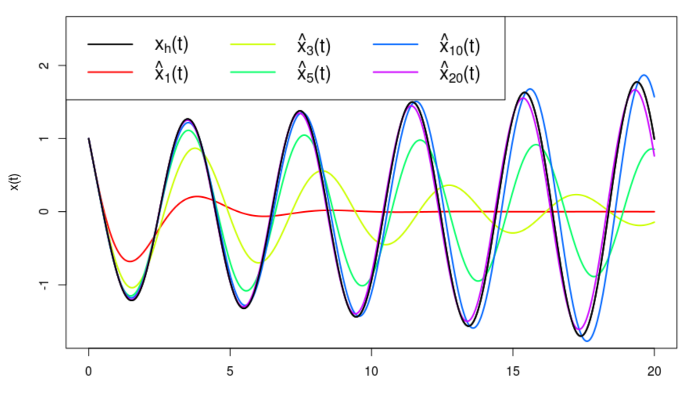

Figure 2. Approximate solutions $\widehat{x}_n$ of (3.1) obtained by approximating g_h by the first n terms of its gamma series and approximate solution x_h computed by discretization.

We denote by $\widehat{x}_n$ the solution of (3.4) obtained by using the gamma series of order n. Solutions x_h and $\widehat{x}_n$ are computed by using R. Approximation of the hat density is given in Figure 1 and the corresponding solutions are given in Figure 2. Since we do not know the exact solution, we do not know how precise our solutions are. To our knowledge, there is no distributed delay differential equation (except for a delay of gamma type) with a known exact solution that could by used as a test case.

References

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