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ON ONE UPPER ESTIMATE FOR THE FIRST EIGENVALUE OF A STURM–LIOUVILLE PROBLEM WITH DIRICHLET BOUNDARY CONDITIONS AND A WEIGHTED INTEGRAL CONDITION

Abstract. We consider a Sturm–Liouville problem on the interval $(0,1)$ with Dirichlet boundary conditions and a weighted integral condition on the potential which may have singularities of different orders at the end-points of the interval $(0,1)$. One upper estimate for the first eigenvalue for some values of parameters in the integral condition is obtained.[1]

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Key words and phrases. Sturm–Liouville problem, first eigenvalue, Dirichlet boundary conditions, weighted integral condition, minimization of the functional, variational principle, boundary-value problem.

რეზიუმე. $(0,1)$ შუალედში განხილულია შტურმ-ლიუვილის ამოცანა დირიხლეს სასაზღვრო პირობებითა და პოტენციალზე დადებული წონიანი ინტეგრალური შეზღუდვით; ამასთან დასაშვებია, რომ პოტენციალს გააჩნდეს სინგულარობები $(0,1)$ შუალედის ბოლო წერტილებში. ინტეგრალურ შეზღუდვაში მონაწილე პარამეტრთა ზოგიერთი მნიშვნელობისთვის დადგენილია პირველი საკუთრივი მნიშვნელობის ზემოდან შეფასება.

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1 Introduction

We consider a problem whose origin was the Lagrange problem of finding the form of the firmest column of the given volume. The Lagrange problem was the source for different extremal eigenvalue problems for second-order differential equations with integral conditions on the potential.

We develop the methods used in Yu. V. Egorov and V. A. Kondratiev's works (see, e.g., [1]) devoted to estimation of eigenvalues for Sturm–Liouville problems. The Sturm–Liouville problem for the equation $y'' + \lambda Q(x)y = 0$ with Dirichlet boundary conditions and a non-negative summable on $[0,1]$ function Q satisfying the condition $\int_0^1 Q^\gamma(x)\,dx = 1$ as $\gamma \in \mathbb{R}$, $\gamma \neq 0$, was considered by Yu. V. Egorov and V. A. Kondratiev in [1]. The Sturm–Liouville problem for the equation $y'' - Q(x)y + \lambda y = 0$ with Dirichlet boundary conditions and a real Lebesgue integrable on $(0,1)$ function Q satisfying the condition $\int_0^1 Q^\gamma(x)\,dx = 1$ as $\gamma \geqslant 1$, was considered by V. A. Vinokurov, V. A. Sadovnichii in [2]. In the present article we consider a problem of that kind in case the integral condition contains a weight function. Some results devoted to the Sturm–Liouville problems with weighted integral conditions can be found in [6]– [9].

Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1.1}$$

$$y(0) = y(1) = 0, \tag{1.2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all real–valued measurable on $(0,1)$ functions with non–negative values such that the following integral condition holds:

$$\int_0^1 x^\alpha (1-x)^\beta Q^\gamma(x)\,dx = 1, \quad \alpha,\beta,\gamma \in \mathbb{R}, \quad \gamma \neq 0. \tag{1.3}$$

A function y is a *solution* to problem $(1.1), (1.2)$ if it is absolutely continuous on the segment $[0,1]$, satisfies (1.2), its derivative y' is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1.1) holds almost everywhere in the interval $(0,1)$.

We give estimates for

$$M_{\alpha,\beta,\gamma} = \sup_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q).$$

For any function $Q \in T_{\alpha,\beta,\gamma}$, by H_Q we denote the closure of the set $C_0^\infty(0,1)$ with respect to the norm

$$\|y\|_{H_Q} = \left(\int_0^1 {y'}^2\,dx + \int_0^1 Q(x)y^2\,dx \right)^{\frac{1}{2}}.$$

For any function $Q \in T_{\alpha,\beta,\gamma}$, we can prove (see, e.g., [3,6]) that

$$\lambda_1(Q) = \inf_{y \in H_Q \setminus \{0\}} R[Q,y], \quad \text{where} \quad R[Q,y] = \frac{\int_0^1 ({y'}^2 - Q(x)y^2)\,dx}{\int_0^1 y^2\,dx}.$$

2 One upper estimate for the first eigenvalue for $\gamma < 0$

Theorem 2.1. *If $\gamma < 0$ and $\alpha, \beta > 3\gamma - 1$, then $M_{\alpha,\beta,\gamma} < \pi^2$. If $\gamma < -1$, $\alpha,\beta > -1$, then there exist a function $Q_* \in T_{\alpha,\beta,\gamma}$ and a positive on the interval $(0,1)$ function $u \in H_{Q_*}$ such that $M_{\alpha,\beta,\gamma} = R[Q_*, u]$, moreover, u satisfies the equation*

$$u'' + mu = -x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}} u^{\frac{\gamma+1}{\gamma-1}}$$

and the integral condition

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}} u^{\frac{2\gamma}{\gamma-1}}\,dx = 1.$$

Proof. Suppose that $\gamma < 0$. For any $Q \in T_{\alpha,\beta,\gamma}$ and $y \in H_Q$, by the Hölder inequality we have

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx \leqslant \left(\int_0^1 x^{\alpha}(1-x)^{\beta}Q^{\gamma}(x)\,dx\right)^{\frac{1}{1-\gamma}}\left(\int_0^1 Q(x)y^2\,dx\right)^{\frac{\gamma}{\gamma-1}}.$$

Then

$$\int_0^1 Q(x)y^2\,dx \geqslant \left(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\right)^{\frac{\gamma-1}{\gamma}} \tag{2.1}$$

and

$$\inf_{y\in H_Q\setminus\{0\}} R[Q,y] \leqslant \inf_{y\in H_Q\setminus\{0\}} G[y],$$

where

$$G[y] = \frac{\int_0^1 y'^2\,dx - \left(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\right)^{\frac{\gamma-1}{\gamma}}}{\int_0^1 y^2\,dx}.$$

Consider the function

$$u_\varepsilon(x) = \begin{cases} 0, & 0 < x < \varepsilon, \\ 1, & \varepsilon \leqslant x \leqslant 1-\varepsilon, \\ 0, & 1-\varepsilon < x < 1, \end{cases}$$

where $0 < \varepsilon < \frac{1}{3}$. By the average processing for $\rho = \frac{\varepsilon}{2}$ we obtain the function

$$u_{\varepsilon_\rho}(x) = \int_{-\infty}^{+\infty} \omega_\rho(x-y)u_\varepsilon(y)\,dy = \int_{-\infty}^{+\infty} \omega_\rho(y-x)u_\varepsilon(y)\,dy = \int_{-\rho}^{\rho} \omega_\rho(z)u_\varepsilon(z+x)\,dz.$$

For the function $y_\varepsilon(x) = u_{\varepsilon\rho}(x)\cdot\sin\pi x$ of $C_0^\infty(0,1)$ it is true that for any $Q \in T_{\alpha,\beta,\gamma}$ the function y_ε belongs to H_Q and

$$\|y_\varepsilon(x) - \sin\pi x\|_{H_0^1(0,1)} \longrightarrow 0 \ \text{ as } \ \varepsilon \to 0.$$

For $\gamma < 0$, $\alpha,\beta > 3\gamma - 1$, the integral $\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}(\sin\pi x)^{\frac{2\gamma}{\gamma-1}}\,dx$ converges. Then for any $Q \in T_{\alpha,\beta,\gamma}$ we have

$$\inf_{y\in H_Q\setminus\{0\}} R[Q,y] \leqslant \inf_{y\in H_Q\setminus\{0\}} G[y] \leqslant \lim_{\varepsilon\to 0} G[y_\varepsilon] = G[\sin\pi x] < \pi^2$$

and $M_{\alpha,\beta,\gamma} < \pi^2$.

Let us show the method of finding sharp estimates for $M_{\alpha,\beta,\gamma}$ for $\gamma < -1$, $\alpha,\beta > -1$. For any function $y \in H_0^1(0,1)$, the inequalities $y^2 < Cx$ and $y^2 < C(1-x)$ hold, where $C = \int_0^1 y'^2\,dx$. If the integral $\int_0^1 Q(x)x(1-x)\,dx$ converges, then $\int_0^1 Q(x)y^2\,dx$ also converges. Consequently, for $\gamma < 0$, $\alpha,\beta > 2\gamma - 1$, the sets of functions of H_Q and $H_0^1(0,1)$ coincide.

Let us prove that for $\gamma < 0$, $\alpha,\beta > 2\gamma - 1$ the functional G is bounded from below in $H_0^1(0,1)$. By the Hölder inequality, for $x \in (0,\frac{1}{2})$ we have

$$y^2(x) = \left(\int_0^x y'(t)\,dt\right)^2 \leqslant x\int_0^x y'^2(t)\,dt \leqslant x\int_0^{\frac{1}{2}} y'^2(t)\,dt$$

and for $x \in (\frac{1}{2},1)$ we have

$$y^2(x) = \left(-\int_x^1 y'(t)\,dt\right)^2 \leqslant (1-x)\int_x^1 y'^2(t)\,dt \leqslant (1-x)\int_{\frac{1}{2}}^1 y'^2(t)\,dt.$$

Then

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx \leqslant \int_0^{\frac12} x^{\frac{\alpha-\gamma}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}\Bigg(\int_0^{\frac12} y'^2(t)\,dt\Bigg)^{\frac{\gamma}{\gamma-1}}dx$$

$$+\int_{\frac12}^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta-\gamma}{1-\gamma}}\Bigg(\int_{\frac12}^1 y'^2(t)\,dt\Bigg)^{\frac{\gamma}{\gamma-1}}dx \leqslant \Bigg(\int_0^1 y'^2(t)\,dt\Bigg)^{\frac{\gamma}{\gamma-1}}(C_1+C_2),$$

where

$$C_1=\int_0^{\frac12} x^{\frac{\alpha-\gamma}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}\,dx,\quad C_2=\int_{\frac12}^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta-\gamma}{1-\gamma}}\,dx.$$

Note that for $\gamma<0$, $\alpha,\beta>2\gamma-1$, the integrals $\int_0^{\frac12} x^{\frac{\alpha-\gamma}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}\,dx$ and $\int_{\frac12}^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta-\gamma}{1-\gamma}}\,dx$ converge. Then for any $y\in H_0^1(0,1)$, we have

$$G[y]\geqslant \pi^2\big(1-(C_1+C_2)^{\frac{\gamma-1}{\gamma}}\big).$$

Thus, the functional G is bounded from below in $H_0^1(0,1)$ and

$$m=\inf_{y\in H_0^1(0,1)\setminus\{0\}} G[y]. \tag{2.2}$$

For any function $Q\in T_{\alpha,\beta,\gamma}$,

$$\lambda_1(Q)=\inf_{y\in H_Q\setminus\{0\}} R[Q,y]\leqslant \inf_{y\in H_Q\setminus\{0\}} G[y]=\inf_{y\in H_0^1(0,1)\setminus\{0\}} G[y]=m.$$

Then

$$M_{\alpha,\beta,\gamma}=\sup_{Q\in T_{\alpha,\beta,\gamma}}\lambda_1(Q)\leqslant \inf_{y\in H_0^1(0,1)\setminus\{0\}} G[y]=m.$$

Consequently, $M_{\alpha,\beta,\gamma}\leqslant m$.

Let us prove that for $\gamma<-1$, $\alpha,\beta>-1$ there exist a function $Q_*\in T_{\alpha,\beta,\gamma}$ and a positive on the interval $(0,1)$ function $u\in H_{Q_*}$ such that $M_{\alpha,\beta,\gamma}=R[Q_*,u]=m$.

Put

$$\Gamma_*=\Bigg\{y\in H_0^1(0,1)\ \Big|\ \int_0^1 y^2\,dx=1\Bigg\}$$

and

$$I[y]=\int_0^1 {y'}^2\,dx-\Bigg(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\Bigg)^{\frac{\gamma-1}{\gamma}}.$$

Lemma 2.1. *There exists a function $u_*\in\Gamma_*$ such that $I[u_*]=m$, where m is defined by* (2.2).

Proof. Let $\{\widetilde{y}_k\}$ be a minimizing sequence of the functional G in $H_0^1(0,1)$. Then $y_k=\frac{\widetilde{y}_k}{C_k^{1/2}}$, where $C_k=\int_0^1 \widetilde{y}_k^2\,dx$, is a minimizing sequence of the functional I in Γ_*, i.e., $I[y_k]\to m$ as $k\to\infty$. Then

$$m=\inf_{y\in H_0^1(0,1)\setminus\{0\}} G[y]=\inf_{y\in\Gamma_*} I[y].$$

Let us show that for $\alpha,\beta > -1$, the sequence $\{y_k\}$ is bounded in $H_0^1(0,1)$. Since $m = \inf\limits_{y\in\Gamma_*} I[y]$, for all sufficiently large values of k we have

$$I[y_k] = \int_0^1 {y_k'}^2\,dx - \Big(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y_k|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{\gamma}} < m+1.$$

For $\alpha,\beta \geqslant 0$, by the Hölder inequality, we have

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y_k|^{\frac{2\gamma}{\gamma-1}}\,dx \leqslant \Big(\int_0^1 x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}y_k^2\,dx\Big)^{\frac{\gamma}{\gamma-1}} \leqslant \Big(\int_0^1 y_k^2\,dx\Big)^{\frac{\gamma}{\gamma-1}} = 1$$

and

$$\int_0^1 {y_k'}^2\,dx = I[y_k] + \Big(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y_k|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{\gamma}} \leqslant m+2.$$

For $\alpha,\beta < 0$, by the Hölder inequality, we have

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y_k|^{\frac{2\gamma}{\gamma-1}}\,dx \leqslant \Big(\int_0^1 (x^{-p}(1-x)^{-p})^{1-\gamma}\,dx\Big)^{\frac{1}{1-\gamma}}$$
$$\times\Big(\int_0^1 x^{-\frac{\alpha}{\gamma}}x^{p\cdot\frac{\gamma-1}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}(1-x)^{p\cdot\frac{\gamma-1}{\gamma}}y_k^2\,dx\Big)^{\frac{\gamma}{\gamma-1}} \leqslant M\cdot\Big(\int_0^1 y_k^2\,dx\Big)^{\frac{\gamma}{\gamma-1}} = M,$$

where $M = \big(\int_0^1 (x^{-p}(1-x)^{-p})^{1-\gamma}\,dx\big)^{\frac{1}{1-\gamma}}$ and p is a number such that $-\frac{\alpha}{\gamma}+p\cdot\frac{\gamma-1}{\gamma} > 0$, $-\frac{\beta}{\gamma}+p\cdot\frac{\gamma-1}{\gamma} > 0$, $-p(1-\gamma) > -1$. Consequently, p satisfies the inequalities

$$\frac{\alpha}{\gamma-1} < p < \frac{1}{1-\gamma}\,, \quad \frac{\beta}{\gamma-1} < p < \frac{1}{1-\gamma}\,,$$

which hold for $\alpha,\beta > -1$. The proofs for the cases $\alpha \geqslant 0 > \beta > -1$ and $\beta \geqslant 0 > \alpha > -1$ are similar.

Since for $\alpha,\beta > -1$ the sequence $\{y_k\}$ is bounded in $H_0^1(0,1)$, it contains a subsequence $\{z_k\}$ which converges weakly in $H_0^1(0,1)$ to some function u_*, moreover,

$$\|u_*\|^2_{H_0^1(0,1)} \leqslant \max\big\{m+3, m+2+M^{\frac{\gamma-1}{\gamma}}\big\}.$$

Since the space $H_0^1(0,1)$ is compactly embedded in the space $C[0,1]$, there exists a subsequence $\{s_k\}$ of $\{z_k\}$ which converges in $C[0,1]$. Since the space $C[0,1]$ is embedded in $L_2(0,1)$, the sequence $\{s_k\}$ converges in $L_2(0,1)$ to the function u_*. Consequently, for the functional G we have

$$\int_0^1 s_k^2\,dx \longrightarrow \int_0^1 u_*^2\,dx \ \text{ as } \ k\to\infty$$

and

$$\int_0^1 u_*^2\,dx = 1. \tag{2.3}$$

Since for $\alpha,\beta > -1$ the sequence $\{s_k\}$ is bounded in $H_0^1(0,1)$, by the definition of the norm $\|s_k\|_{H_0^1(0,1)}$ the sequence $\{s_k'\}$ is bounded in $L_2(0,1)$. Then there exists a subsequence $\{w_k\}$ of $\{s_k\}$ such that the sequence $\{w_k'\}$ converges weakly to the function u_*' in $L_2(0,1)$. Then ([10, p. 217])

$$\|u_*'\|^2_{L_2(0,1)} \leqslant \varliminf_{k\to\infty} \|w_k'\|^2_{L_2(0,1)} = A.$$

Thus, we have

$$\|u_*'\|^2_{L_2(0,1)} \leqslant A. \tag{2.4}$$

Let $\{v_k\}$ be a subsequence of $\{w_k\}$ such that

$$\lim_{k\to\infty} \int_0^1 {v_k'}^2\,dx = \varliminf_{k\to\infty} \int_0^1 {w_k'}^2\,dx = A.$$

Since m is a limit of the sequence $\{I[v_k]\}$, $m - A$ is a limit of the sequence

$$\biggl\{ -\Bigl(\int_0^1 x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |v_k|^{\frac{2\gamma}{\gamma-1}}\,dx \Bigr)^{\frac{\gamma-1}{\gamma}} \biggr\}.$$

Then, for any $\varepsilon > 0$, there exists a number K such that for any $k \geqslant K$ the inequality

$$-\Bigl(\int_0^1 x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |v_k|^{\frac{2\gamma}{\gamma-1}}\,dx \Bigr)^{\frac{\gamma-1}{\gamma}} < m - A + \varepsilon$$

holds. Then

$$\Bigl(\int_0^1 x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |v_k|^{\frac{2\gamma}{\gamma-1}}\,dx \Bigr)^{\frac{\gamma-1}{\gamma}} > A - m - \varepsilon$$

and

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |v_k|^{\frac{2\gamma}{\gamma-1}}\,dx > (A - m - \varepsilon)^{\frac{\gamma}{\gamma-1}}. \tag{2.5}$$

Let us use the Lebesgue theorem. For the sequence $\{x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |r_k|^{\frac{2\gamma}{\gamma-1}}\}$, we have

$$x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |r_k|^{\frac{2\gamma}{\gamma-1}} \longrightarrow x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |u_*|^{\frac{2\gamma}{\gamma-1}} \text{ as } k \to \infty \text{ almost everywhere on } [0,1].$$

We have proved the existence of a constant $V = \max\{1, M\}$ such that for any sufficiently large value of k we have

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |r_k|^{\frac{2\gamma}{\gamma-1}}\,dx \leqslant V.$$

Then

$$x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |u_*|^{\frac{2\gamma}{\gamma-1}} \in L_1(0,1)$$

and

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |r_k|^{\frac{2\gamma}{\gamma-1}}\,dx \longrightarrow \int_0^1 x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |u_*|^{\frac{2\gamma}{\gamma-1}}\,dx \text{ as } k \to \infty.$$

If for any $k \geqslant K$ inequality (2.5) holds and

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |r_k|^{\frac{2\gamma}{\gamma-1}}\,dx \longrightarrow \int_0^1 x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |u_*|^{\frac{2\gamma}{\gamma-1}}\,dx \text{ as } k \to \infty,$$

then we have

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} |u_*|^{\frac{2\gamma}{\gamma-1}}\,dx \geqslant (A - m - \varepsilon)^{\frac{\gamma}{\gamma-1}}.$$

Since ε may be sufficiently small, we obtain

$$\Big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u_*|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{\gamma}} \geqslant A-m$$

and

$$-\Big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u_*|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{\gamma}} \leqslant m-A. \tag{2.6}$$

By virtue of (2.4) and (2.6), we obtain

$$I[u_*]\leqslant m. \tag{2.7}$$

Since $m=\inf\limits_{y\in\Gamma_*} I[y]$, we have $I[u_*]=m$. By (2.3), we obtain $u_*\in\Gamma_*$. □

Let us consider the set

$$\Gamma=\Big\{y\in H_0^1(0,1)\mid \int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx=1\Big\}.$$

The function $u=Cu_*$, where

$$C=\Big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u_*|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{1-\gamma}{2\gamma}},$$

is non-negative on $[0,1]$ and belongs to Γ. Then $G[u]=G[u_*]=I[u_*]=m$.

Let us fix the argument u of the functional G and fix some variation $z\in H_0^1(0,1)$ of the argument u and let us consider a set of functions $u+tz$, where t is an arbitrary parameter. On the functions $u+tz$ the functional G turns to the function of $t\in\mathbb{R}$:

$$g(t)=\frac{\int_0^1(u'(x)+tz'(x))^2\,dx-\big(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u(x)+tz(x)|^{\frac{2\gamma}{\gamma-1}}\,dx\big)^{\frac{\gamma-1}{\gamma}}}{\int_0^1(u(x)+tz(x))^2\,dx}.$$

Since the functional G reaches an extremum at $y=u$ and for $\gamma<-1$ the function $g(t)$ is differentiable at zero, we have $g'(0)=0$. Since $u\in\Gamma$ and $G[u]=m$, we obtain

$$\int\limits_0^1 u'z'\,dx-\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u|^{\frac{\gamma+1}{\gamma-1}}\operatorname{sgn}u\,z\,dx=m\int\limits_0^1 uz\,dx. \tag{2.8}$$

For $\gamma<-1$, $\alpha,\beta>-1$, equality (2.8) holds for any function $z\in H_0^1(0,1)$, because by virtue of the Hölder inequality, we have

$$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u|^{\frac{\gamma+1}{\gamma-1}}|z|\,dx$$
$$\leqslant\Big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma+1}{2\gamma}}\Big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|z|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{2\gamma}}.$$

If $z\in C_0^\infty(0,1)$, then u' has a generalized derivative equal to

$$u''=-x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u|^{\frac{\gamma+1}{\gamma-1}}\operatorname{sgn}u-mu.$$

Since $G[y] = G[|y|]$, we can assume that the sequence $\{y_k\}$ is non-negative and $u \geqslant 0$. Similarly, to the case $\alpha = \beta = 0$ we can prove (see, e.g., [3]) that the function u is convex upward. Thus on the interval $(0,1)$ we have $u(x) > 0$.

Since $u \in AC[0,1]$, for $\gamma < -1$, the function $x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u|^{\frac{\gamma+1}{\gamma-1}} \operatorname{sgn} u$ is continuous on the segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and $u'' \in L_p(\rho, 1-\rho)$. Let v be a generalized derivative of u of second order. The Corollary 2.6.1 of Theorem 2.6.1 (see [12, p. 41]) implies that if $u, v \in L_p(\rho, 1-\rho)$, $p \geqslant 1$, then the function u is continuously differentiable on $[\rho, 1-\rho]$ and almost everywhere on it has the classical derivative of the second order $u'' = v$. Thus,

$$u'' + x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u|^{\frac{2}{\gamma-1}}u + mu = 0 \text{ for } x \in [\rho, 1-\rho]. \tag{2.9}$$

Since the number ρ may be sufficiently small and the function u is continuous and positive on $(0,1)$, the function u'' is also continuous on $(0,1)$ and equality (2.9) holds everywhere on $(0,1)$.

On $(0,1)$, let us consider the function

$$Q_*(x) = x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2}{\gamma-1}}.$$

Since $Q_*(x)$ satisfies the integral condition (1.3):

$$\int_0^1 x^{\alpha}(1-x)^{\beta}Q_*^{\gamma}(x)\,dx = \int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2\gamma}{\gamma-1}}\,dx = 1,$$

the function u belongs to H_{Q_*}.

Since u satisfies equation (2.9) and conditions (1.2), for $Q = Q_*$ it satisfies equation (1.1) and conditions (1.2). Therefore, since u is continuous on $[0,1]$ and its derivative u' is continuous on $(0,1)$, the function u is the first eigenfunction of problem (1.1)–(1.3) with $Q = Q_*$ and the first eigenvalue $\lambda_1(Q_*) = m$.

Then

$$\inf_{y \in H_{Q_*} \setminus \{0\}} R[Q_*, y] = R[Q_*, u] = G[u] = m$$

and

$$M_{\alpha,\beta,\gamma} = \sup_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q) \geqslant \lambda_1(Q_*) = \inf_{y \in H_{Q_*} \setminus \{0\}} R[Q_*, y] = R[Q_*, u] = G[u] = m.$$

Consequently, we obtain $M_{\alpha,\beta,\gamma} = m$. □

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