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ON THE BEHAVIOR OF SOLUTIONS TO SECOND-ORDER DIFFERENTIAL EQUATION WITH GENERAL POWER-LAW NONLINEARITY

Abstract. The second-order differential equation with general power-law nonlinearity with continuous potential bounded by positive constants is considered. The behavior of solutions to the equation is studied with respect to the values of nonlinearity. The necessary and sufficient conditions for the existence of a finite right-side boundary of the domain or horizontal asymptote are obtained. The distance to the right-side boundary of the domain and the limits of solutions with horizontal asymptotes near their boundaries are estimated. The continuous dependence of the right-side boundary of the domain and horizontal asymptotes on initial data is proved.[1]

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რეზიუმე. განხილულია მეორე რიგის დიფერენციალური განტოლება ზოგადი ხარისხობრივი არაწრფივობით და დადებითი მუდმივებით შემოსაზღვრული უწყვეტი პოტენციალით. ნაპოვნია სასრულ შუალედზე განსაზღვრული ვერტიკალური და ჰორიზონტალური ასიმპტოტების მქონე ამონახსნების არსებობის აუცილებელი და საკმარისი პირობები და დადგენილია მათი ასიმპტოტური შეფასებები.

[1]Reported on Conference "Differential Equation and Applications", September 4–7, 2017, Brno

Consider the second-order Emden–Fowler type nonlinear equation

$$y'' = p(x, y, y')|y|^{k_0}|y'|^{k_1}\operatorname{sgn}(yy'), \quad k_0 > 0, \quad k_1 > 0, \quad k_0, k_1 \in \mathbb{R}, \tag{0.1}$$

with positive continuous in x and Lipschitz continuous in u, v function $p(x, u, v)$.

The asymptotic behavior of solutions to (0.1) in the case $k_1 = 0$ is described in [5]. Using the methods described in [1] by I. V. Astashova, the behavior of decreasing solutions to (0.1) near the right domain boundary is investigated with respect to the values k_0 and k_1.

In the case $p = p(x)$, the asymptotic behavior of solutions to (0.1) is obtained by V. M. Evtukhov [6]. Using the methods described in [2–4] by I. V. Astashova, the behavior of positive increasing solutions to (0.1) near the right endpoint of their domains is investigated with respect to the values k_0 and k_1.

1 Preliminary results

Consider the behavior of solutions according to initial data.

Lemma 1.1. *Suppose $k_0 > 0$, $k_1 > 0$. Let $p(x, u, v)$ be a positive continuous in x and Lipschitz continuous in u, v function. Then all maximally extended solutions to equation* (0.1) *can be divided into the following five types according to their behavior:*

0. *Constant solutions;*
1. *Increasing positive solutions;*
2. *Increasing negative solutions;*
3. *Increasing solutions negative near the left boundary of the domain and positive near the right boundary of the domain;*
4. *Decreasing solutions positive near the left boundary of the domain and negative near the right boundary of the domain.*

Proof. Let us show first that if there is a point x_0 such that $y'(x_0) = 0$, then $y(x) \equiv y(x_0)$. Indeed, from equation (0.1) we derive that $y''(x_0) = 0$ and since $y_0(x) \equiv y(x_0)$ is a solution to (0.1), by the theorem of the existence and uniqueness, $y(x) \equiv y_0(x) \equiv y(x_0)$.

Thus, every solution with an extremum at some point is a constant solution (type 0), and therefore every non-constant solution is either increasing or decreasing on its domain.

Consider increasing solutions. Assume that at some point x_0 we have $y(x_0) > 0$ and $y'(x_0) > 0$. Then, according to the equation, $\operatorname{sgn} y'' = \operatorname{sgn} y$, and therefore $y''(x) > 0$ and $y'(x)$ is positive and increasing, while $y(x) > 0$. This implies $y(x) > 0$, $y'(x) > 0$ and $y''(x) > 0$ for all $x > x_0$, so the solution is positive and increasing on its domain. Consider now $x < x_0$. Since $y'(x)$ is positive on the whole domain of the solution, either there is a point $\widetilde{x}$ such that $y(\widetilde{x}) = 0$ or $y(x) > 0$ (also $y'(x) > 0$, and therefore $y''(x) > 0$) for all $x < x_0$. Consider the first case. Since the first derivative of the solution is positive, $y'(x) > 0$ and $y(x) < 0$ (therefore, $y''(x) < 0$) for all $x < \widetilde{x}$. Thus, $y(x)$ is an increasing solution negative near the left boundary of the domain and positive near the right one.

Assume now that at some point x_0 we have $y(x_0) < 0$, $y'(x_0) > 0$. According to the equation, $\operatorname{sgn} y'' = \operatorname{sgn} y$, and therefore $y''(x) < 0$, $y'(x) > 0$ and $y(x) < 0$ for all $x < x_0$. Consider $x > x_0$: since $y'(x) > 0$, either the solution $y(x)$ is negative and increasing on the whole domain or there exists a point $\widetilde{x}$ such that $y(\widetilde{x}) = 0$. In the second case, for $x > \widetilde{x}$ we have $y(x) > 0$, $y'(x) > 0$, and thus $y(x)$ is an increasing solution, negative near the left boundary of domain and positive near the right one.

Consider decreasing solutions. Suppose at some point x_0 we have $y(x_0) > 0$ and $y'(x_0) < 0$. According to the equation, $\operatorname{sgn} y'' = -\operatorname{sgn} y$, and therefore $y''(x) < 0$ and $y'(x)$ is negative and decreasing, while $y(x) > 0$. Thus, $y'(x) < y'(x_0)$ and

$$y(x) < y(x_0) + y'(x_0)(x - x_0) = -|y'(x_0)|x + \big(y(x_0) - y'(x_0)x_0\big),$$

while $y(x)$ is positive. Since $y(x)$ is estimated from above by a linear function, it cannot be positive on its whole domain and therefore there exists a point $\widetilde{x}$ such that $y(\widetilde{x}) = 0$. Note that $y'(\widetilde{x})$ is negative and therefore in some neighbourhood $(\widetilde{x}, \widetilde{x}+\varepsilon)$, $\varepsilon > 0$, the solution $y(x)$ and its derivative $y'(x)$ are both negative and, due to equation (0.1), we have $y''(x) > 0$. Then for all $x > \widetilde{x}$, the solution is decreasing, and since its derivative is of a constant (negative) sign, we have $y(x) < 0$, $y'(x) < 0$, $y''(x) > 0$ for all $x > \widetilde{x}$ and $y(x) > 0$, $y'(x) < 0$, $y''(x) > 0$ at $x < \widetilde{x}$. Thus, $y(x)$ is a decreasing solution, positive near the left boundary of the domain and negative near the right one. □

Lemma 1.2. *Suppose $k_0 > 0$, $k_1 > 0$, $k_1 \neq 2$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying the inequalities*

$$0 < m \leq p(x,u,v) \leq M < +\infty. \tag{1.1}$$

Then for any solution $y(x)$ to equation (0.1)*, strictly monotonous and having a constant sign on $[x_1,x_2]$, the following inequalities hold:*

$$\begin{aligned} m\big(|y(x_2)|^{k_0+1} - |y(x_1)|^{k_0+1}\big)\operatorname{sgn}(yy') &\leq \frac{k_0+1}{2-k_1}\big(|y'(x_2)|^{2-k_1} - |y'(x_1)|^{2-k_1}\big)\operatorname{sgn} y \\ &\leq M\big(|y(x_2)|^{k_0+1} - |y(x_1)|^{k_0+1}\big)\operatorname{sgn}(yy'). \end{aligned} \tag{1.2}$$

Proof. Due to inequalities (1.1) and equation (0.1), we can estimate the absolute value of the second derivative as

$$m|y|^{k_0}|y'|^{k_1} \leq |y''| = |p(x,y,y')|y|^{k_0}|y'|^{k_1}\operatorname{sgn}(yy')| \leq M|y|^{k_0}|y'|^{k_1}.$$

Then

$$m|y|^{k_0}|y'| \leq |y''|\,|y'|^{1-k_1} \leq M|y|^{k_0}|y'|$$

and by integrating these inequalities on (x_1,x_2), we obtain

$$\begin{aligned} &\frac{m}{k_0+1}\big(|y(x_2)|^{k_0+1} - |y(x_1)|^{k_0+1}\big)\operatorname{sgn}(yy') \\ &\quad\leq \frac{1}{2-k_1}\big(|y'|^{2-k_1} - |y'(x_1)|^{2-k_1}\big)\operatorname{sgn}(y'y'') \leq \frac{M}{k_0+1}\big(|y(x_2)|^{k_0+1} - |y(x_1)|^{k_0+1}\big)\operatorname{sgn}(yy'), \end{aligned}$$

where $\operatorname{sgn}(yy')$ and $\operatorname{sgn}(y'y'')$ are constant and can be taken at any point from $[x_1,x_2]$. Therefore if $\operatorname{sgn} y' \neq 0$,

$$\begin{aligned} &m\big(|y(x_2)|^{k_0+1} - |y(x_1)|^{k_0+1}\big)\operatorname{sgn}(yy') \\ &\quad\leq \frac{k_0+1}{2-k_1}\big(|y'(x_2)|^{2-k_1} - |y'(x_1)|^{2-k_1}\big)\operatorname{sgn} y \leq M\big(|y(x_2)|^{k_0+1} - |y(x_1)|^{k_0+1}\big)\operatorname{sgn}(yy'). \end{aligned}$$ □

2 Increasing solutions

Theorem 2.1. *Suppose $k_0 > 0$, $k_1 > 0$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1)*. Let $y(x)$ be a maximally extended solution to* (0.1) *with $y(x_0) \geq 0$ and $y'(x_0) > 0$ at some point x_0. Then the existence of a finite point $x^* > x_0$ such that $\lim\limits_{x\to x^*-0} y'(x) = +\infty$ is equivalent to the condition $k_0+k_1 > 1$. Moreover, there exists a positive constant $\xi = \xi(m,k_0)$ such that*

$$x^* - x_0 < \xi(y'(x_0))^{-\frac{k_0+k_1-1}{k_0+1}}.$$

Proof. Consider the case $k_0+k_1 > 1$.

Denote $y_1 = y'(x_0) > 0$. According to Lemma 1.1, the solution $y(x)$ with positive initial data tends to infinity along with its derivative. This implies that for any $i \in \mathbb{N}$ there exists a point $x_i > x_{i-1}$ such that $y'(x_i) = 2y'(x_{i-1}) = 2^i y_1$. Let us estimate the difference $x_{i+1} - x_i$.

For $x \in [x_i, x_{i+1}]$, the inequalities

$$y'(x) \geq y_1, \quad y(x) - y(x_i) \geq y_1(x - x_i)$$

hold, and since $y(x_i) \geq y(x_0) \geq 0$, we have $y(x) \geq y_1(x - x_i)$, hence

$$y^{k_0}(x) \geq (y_1(x - x_i))^{k_0} \text{ and } (y'(x))^{k_1} \geq y_1^{k_1},$$
$$y''(x) = p(x, y, y')|y|^{k_0}|y'|^{k_1} \operatorname{sgn}(yy') \geq m y_1^{k_0+k_1}(x - x_i)^{k_0}.$$

Integrating this inequality on the segment $[x_i, x_{i+1}]$, we obtain

$$y'(x_{i+1}) - y'(x_i) \geq \frac{m}{k_0 + 1} y_1^{k_0+k_1}(x_{i+1} - x_i)^{k_0+1},$$

which means

$$2^i y_1 \geq \frac{m}{k_0 + 1} y_1^{k_0+k_1}(x_{i+1} - x_i)^{k_0+1},$$
$$(x_{i+1} - x_i)^{k_0+1} \leq 2^i \frac{k_0 + 1}{m} y_1^{-(k_0+k_1-1)},$$
$$x_{i+1} - x_i \leq 2^{\frac{i}{k_0+1}} \Big(\frac{k_0 + 1}{m}\Big)^{\frac{1}{k_0+1}} y_1^{-\frac{k_0+k_1-1}{k_0+1}}.$$

Thus, the distance $x_{i+1} - x_i$ is estimated from above by the term of a converging series multiplied by a positive constant. This implies that there exists a limit

$$x^* = \lim_{n\to+\infty} \sum_{i=0}^{n}(x_{i+1} - x_i) + x_0 = \lim_{n\to+\infty} x_n,$$

and since a solution to (0.1) is continuous, $\lim\limits_{x\to x^*-0} y'(x) = +\infty$. Moreover,

$$x^* - x_0 = \sum_{i=0}^{+\infty}(x_{i+1} - x_i) \leq \sum_{i=0}^{+\infty} 2^{\frac{i}{k_0+1}} \Big(\frac{k_0 + 1}{m}\Big)^{\frac{1}{k_0+1}} y_1^{-\frac{k_0+k_1-1}{k_0+1}},$$
$$x^* - x_0 \leq \Big(\frac{k_0 + 1}{m}\Big)^{\frac{1}{k_0+1}} y_1^{-\frac{k_0+k_1-1}{k_0+1}} \sum_{i=0}^{+\infty} 2^{\frac{i}{k_0+1}},$$

which implies

$$x^* - x_0 < \xi \,(y'(x_0))^{-\frac{k_0+k_1-1}{k_0+1}}$$

for

$$\xi = \xi(m, k_0) = \Big(\frac{k_0 + 1}{m}\Big)^{\frac{1}{k_0+1}} (1 - 2^{\frac{1}{k_0+1}})^{-1} > 0.$$

For the case $k_0 + k_1 \leq 1$, we can apply the following

Theorem (K. Dulina, T. Korchemkina [5])**.** *Suppose $k > 0$, $k \neq 1$. Let the function $P(x, u, v)$ be continuous in x, Lipschitz continuous in u, v. Let there exist the constants $u_0 > 0$, $v_0 > 0$ and $\alpha \leq 1 - k$ such that for $u > u_0$, $v > v_0$ the inequality $P(x, u, v) \leq C|v|^{-\alpha}$ holds. Then any non-extensible solution $y(x)$ to equation*

$$y'' - P(x, y, y')|y|^k \operatorname{sgn} y = 0$$

with initial data $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ can be extended on $(x_0, +\infty)$ and

$$\lim_{x\to+\infty} y(x) = \lim_{x\to+\infty} y(x) = +\infty.$$

Indeed, here we have $P(x, u, v) = p(x, u, v)|v|^{k_1} \leq M v^{k_1}$, so, the above theorem holds if $k_1 \leq 1-k_0$, i.e., $k_0 + k_1 \leq 1$. □

Remark. It is sufficient that $p(x, u, v) \geq m$ for the solution to have a finite right-side boundary x^* of its domain.

Note that after the substitution $y(x) \mapsto -y(-x)$ we obtain an equation of the same type as (0.1), so the following statement is also true.

Theorem 2.2. *Suppose $k_0 > 0$, $k_1 > 0$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Let $y(x)$ be a maximally extended solution to* (0.1) *with $y(x_0) \le 0$ and $y'(x_0) > 0$ at some point x_0. Then the existence of a finite point $x_* < x_0$ such that $\lim\limits_{x \to x_*+0} y'(x) = -\infty$ is equivalent to the condition $k_0 + k_1 > 1$. Moreover, there exists a positive constant $\xi = \xi(m, k_0)$ such that*

$$x_0 - x_* < \xi \left(y'(x_0)\right)^{-\frac{k_0+k_1-1}{k_0+1}}.$$

It follows from [5, Theorem 3.4] that in the case $k_1 > 2$ all positive increasing solutions are *the black hole* solutions [7], i.e., $\lim\limits_{x \to x^*-0} y(x) < \infty$.

Applying now Lemma 1.2 for $x_1 = x_0$, $x_2 = x$ and considering inequalities (1.2) as $x \to x^* - 0$, we obtain the following estimates for the limit $\lim\limits_{x \to x^*-0} y(x)$.

Theorem 2.3. *Suppose $k_1 > 2$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Let $y(x)$ be a maximally extended solution to* (0.1) *with $y(x_0) \ge 0$ and $y'(x_0) > 0$ at some point x_0. Then for the right-side boundary of the domain x^* which existence is stated in Theorem* 2.1*, the limit $\lim\limits_{x \to x^*-0} y(x) = y^*$ is finite and*

$$\frac{k_0+1}{2-k_1}\frac{1}{M}\left(y'(x_0)\right)^{2-k_1} \le (y^*)^{k_0+1} - y_0^{k_0+1} \le \frac{k_0+1}{2-k_1}\,\frac{1}{m}\left(y'(x_0)\right)^{2-k_1}.$$

Analogously, we obtain the similar statement for the limit $\lim\limits_{x \to x_*-0} y(x)$.

Theorem 2.4. *Suppose $k_1 > 2$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Let $y(x)$ be a maximally extended solution to* (0.1) *with $y(x_0) \le 0$ and $y'(x_0) > 0$ at some point x_0. Then for the left-side boundary of the domain x_* which existence is stated in Theorem* 2.2*, the limit $\lim\limits_{x \to x_*-0} y(x) = y_*$ is finite and*

$$\frac{k_0+1}{2-k_1}\,\frac{1}{M}\left(y'(x_0)\right)^{2-k_1} \le |y_*|^{k_0+1} - |y_0|^{k_0+1} \le \frac{k_0+1}{2-k_1}\,\frac{1}{m}\left(y'(x_0)\right)^{2-k_1}.$$

3 Decreasing solutions

Consider now decreasing solutions. Let us prove that every solution of such type has two horizontal asymptotes.

Theorem 3.1. *Suppose $k_0 > 0$, $k_1 \in (0,2)$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Then any solution $y(x)$ to equation* (0.1) *with initial data $y(x_0) \le 0$, $y'(x_0) < 0$ is defined on the whole axis and there exists a finite negative value $y_+ < y(x_0)$ such that $\lim\limits_{x \to +\infty} y(x) = y_+$. Moreover,*

$$\frac{k_0+1}{2-k_1}\,\frac{1}{M}\,|y'(x_0)|^{2-k_1} \le |y_+|^{k_0+1} - |y(x_0)|^{k_0+1} \le \frac{k_0+1}{2-k_1}\,\frac{1}{m}\,|y'(x_0)|^{2-k_1}.$$

Proof. According to the proof of Lemma 1.1, for any $x > x_0$, we have $y(x) < 0$, $y'(x) < 0$ and therefore $y''(x) > 0$. This implies that $y'(x) \to 0$ as $x \to \widetilde{x}$, where $\widetilde{x} > x_0$ is a right domain boundary of $y(x)$.

Denote $y_1 = |y'(x_0)| = -y'(x_0)$. While $y'(x) \ne 0$, from Lemma 1.2 with $x_1 = x_0$ and $x_2 = x > x_0$ we derive

$$\frac{k_0+1}{2-k_1}\,\frac{|y'(x_0)|^{2-k_1} - |y'(x)|^{2-k_1}}{M} \le |y(x)|^{k_0+1} - |y(x_0)|^{k_0+1} \le \frac{k_0+1}{2-k_1}\,\frac{|y'(x_0)|^{2-k_1} - |y'(x)|^{2-k_1}}{m}.$$

Denote $Y = \lim_{x \to \widetilde{x}} y(x)$, then considering the above inequalities at $x \to \widetilde{x}$, we obtain

$$\frac{k_0+1}{2-k_1} \frac{|y'(x_0)|^{2-k_1}}{M} \le |Y|^{k_0+1} - |y(x_0)|^{k_0+1} \le \frac{k_0+1}{2-k_1} \frac{|y'(x_0)|^{2-k_1}}{m},$$

which implies $|Y| < +\infty$.

Consider now $\widetilde{x}$ in correspondence with k_1. Let $x^* > x_0$, $x^* \le +\infty$ be the closest to x_0 point such that $\lim_{x \to x^*} y'(x) = 0$.

From equation (0.1), on the interval (x_0, x^*), we derive

$$y''|y'|^{-k_1} = p(x,y,y')|y|^{k_0} \operatorname{sgn}(yy'),$$

and since at $x > x_0$ we have $y(x) < 0$, $y'(x) < 0$, therefore

$$y''(-y')^{-k_1} = p(x,y,y')|y|^{k_0},$$

and for $k_1 \neq 1$,

$$\frac{1}{1-k_1}\left(|y'(x_0)|^{1-k_1} - |y'|^{1-k_1}\right) = \int_{x_0}^{x} p(x,y,y')|y|^{k_0}\,dx.$$

In the case $k_1 \in (1,2)$, we get

$$\frac{1}{1-k_1}\left(|y'(x_0)|^{1-k_1} - |y'|^{1-k_1}\right) \le \int_{x_0}^{x} M|Y|^{k_0}\,dx = M|Y|^{k_0}(x-x_0),$$

$$x - x_0 \ge \frac{1}{M|Y|^{k_0}(k_1-1)}\left(|y'(x)|^{1-k_1} - |y'(x_0)|^{1-k_1}\right).$$

Since $y'(x) \to 0$ as $x \to x^*$ and $1-k_1 < 0$, the right part of the above inequality tends to infinity as $x \to x^*$, which implies $x^* = +\infty$, and therefore the solution $y(x)$ is defined on $(x_0, +\infty)$, $y_+ = Y$ and the theorem for the case $k_1 \in (1,2)$ is proved.

Analogously, in the case $k_1 = 1$, we obtain

$$x - x_0 \ge \frac{1}{M|Y|^{k_0}}\left(\ln|y'(x_0)| - \ln|y'|\right).$$

Since $y'(x) \to 0$ as $x \to x^*$, the right part of the above inequality tends to infinity as $x \to x^*$, which implies $x^* = +\infty$, and therefore the solution $y(x)$ is defined on $(x_0, +\infty)$, $y_+ = Y$ and hence the theorem for the case $k_1 = 1$ is also proved.

In the case $k_1 \in (0,1)$, we denote $\widetilde{x}_0 = x_0$ if $y(x_0) \neq 0$ and otherwise $\widetilde{x}_0 = x_0 + \varepsilon$, where $\varepsilon > 0$ is such that $y(x) < 0$ and $y'(x) < 0$ on $(x_0, x_0+\varepsilon)$. Then $|y(x)|^{k_0} \ge |y(\widetilde{x}_0)|^{k_0}$ on $(\widetilde{x}_0, x^*)$, and analogously we obtain the estimate

$$\frac{1}{1-k_1}\left(|y'(\widetilde{x}_0)|^{1-k_1} - |y'|^{1-k_1}\right) \ge \int_{\widetilde{x}_0}^{x} m|y(\widetilde{x}_0)|^{k_0}\,dx = m|y(\widetilde{x}_0)|^{k_0}(x-\widetilde{x}_0),$$

$$x - \widetilde{x}_0 \le \frac{1}{m|y(\widetilde{x}_0)|^{k_0}(1-k_1)}\left(|y'(\widetilde{x}_0)|^{1-k_1} - |y'(x)|^{1-k_1}\right).$$

Since $y'(x) \to 0$ as $x \to x^*$ and $1-k_1 > 0$, the right part of the above inequality tends to a constant value $\frac{|y'(\widetilde{x}_0)|^{1-k_1}}{m|y(\widetilde{x}_0)|^{k_0}(1-k_1)}$ as $x \to x^*$, which implies $x^* < +\infty$, and therefore the solution $y(x)$ is unique only on (x_0, x^*). Note that even though the uniqueness of solutions is not satisfied, there is only one possible way to extend the solution $y(x)$ to the right. Thus, $y(x) < 0$, is decreasing on (x_0, x^*) and is equal to a constant on $[x^*, +\infty)$. This implies $y_+ = \lim_{x \to +\infty} y(x) = y(x^*) = Y$ and the theorem is proved. □

Since the substitution $y(x) \mapsto -y(-x)$ gives an equation of the same type as (0.1), the following statement is also true.

Theorem 3.2. *Suppose $k_0 > 0$, $k_1 \in (0,2)$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Then any solution $y(x)$ to equation* (0.1) *with initial data $y(x_0) \geq 0$, $y'(x_0) < 0$ is defined on the whole axis and there exists a finite positive value $y_- > y(x_0)$ such that $\lim\limits_{x\to-\infty} y(x) = y_-$. Moreover,*

$$\frac{k_0+1}{2-k_1}\,\frac{1}{M}\,|y'(x_0)|^{2-k_1} \leq |y_-|^{k_0+1} - |y(x_0)|^{k_0+1} \leq \frac{k_0+1}{2-k_1}\,\frac{1}{m}\,|y'(x_0)|^{2-k_1}\,.$$

Definition ([8])**.** $y(x)$ is a *white hole* solution to equation (0.1) if there exists a finite point $\widetilde{x}$ such that $\lim\limits_{x\to\widetilde{x}} y'(x) = 0$, but $\lim\limits_{x\to\widetilde{x}} y(x) \neq 0$.

Thus, all decreasing solutions to equation (0.1) in the case $k_1 \in (1,2)$ are the white hole solutions.

Lemma 3.1. *Suppose $k_0 > 0$, $k_1 \in (0,2)$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Then any decreasing solution $y(x)$ to equation* (0.1) *is defined on the whole axis and there exist a finite positive value y_- and a finite negative value y_+ such that $\lim\limits_{x\to\pm\infty} y(x) = y_\pm$. Moreover,*

$$\left(\frac{m}{M}\right)^{\frac{1}{k_0+1}} \leq \left|\frac{y_+}{y_-}\right| \leq \left(\frac{M}{m}\right)^{\frac{1}{k_0+1}}.$$

Proof. Indeed, let x_0 be a zero of a decreasing solution $y(x)$ to equation (0.1). Then the limits $y_\pm = \lim\limits_{x\to\pm\infty} y(x)$ are finite and the estimates from Theorems 3.1 and 3.2 take the form

$$\begin{aligned}\frac{k_0+1}{2-k_1}\,\frac{1}{M}\,|y'(x_0)|^{2-k_1} &\leq |y_+|^{k_0+1} \leq \frac{k_0+1}{2-k_1}\,\frac{1}{m}\,|y'(x_0)|^{2-k_1},\\ \frac{k_0+1}{2-k_1}\,\frac{1}{M}\,|y'(x_0)|^{2-k_1} &\leq y_-^{k_0+1} \leq \frac{k_0+1}{2-k_1}\,\frac{1}{m}\,|y'(x_0)|^{2-k_1},\end{aligned}$$

hence

$$\frac{m}{M} \leq \left|\frac{y_+}{y_-}\right|^{k_0+1} \leq \frac{M}{m}\,,$$

which implies the statement of the lemma. □

Applying Lemma 3.1 for the case $p(x,u,v) \equiv p_0 = const$, we obtain the following

Corollary. *Suppose $k_0 > 0$, $k_1 \in (0,2)$, $p(x,u,v) \equiv p_0 = const$. Then any solution $y(x)$ to* (0.1) *satisfying at some point x_0 the condition $y'(x_0) < 0$ is defined on the whole axis and the limits $y_\pm = \lim\limits_{x\to\pm\infty} y(x)$ are finite and satisfying the equality $y_- = -y_+$.*

Theorem 3.3. *Suppose $k_0 > 0$, $k_1 \geq 2$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Then any solution $y(x)$ to equation* (0.1) *with initial data $y(x_0) \leq 0$, $y'(x_0) < 0$ is unbounded and defined on the whole axis.*

Proof. Let us prove the theorem for $x > x_0$. Consider first the case $k_1 > 2$.

According to the proof of Lemma 1.1, for any $x > x_0$ we have $y(x) < 0$, $y'(x) < 0$ and, therefore, $y''(x) > 0$. This implies that $y'(x) \to 0$ as $x \to \widetilde{x}$, where $\widetilde{x} > x_0$ is the right domain boundary of $y(x)$.

Denote $y_1 = |y'(x_0)| = -y'(x_0)$. While $y'(x) \neq 0$, from Lemma 1.2 with $x_1 = x_0$ and $x_2 = x > x_0$ we derive

$$m\big(|y(x)|^{k_0+1} - |y(x_0)|^{k_0+1}\big) \leq \frac{k_0+1}{k_1-2}\left(|y'(x)|^{2-k_1} - y_1^{2-k_1}\right) \leq M\big(|y(x)|^{k_0+1} - |y(x_0)|^{k_0+1}\big).$$

Denote $Y = \lim_{x\to\widetilde{x}} y(x)$, then considering the above inequalities at $x \to \widetilde{x}$, we obtain

$$\frac{k_0+1}{k_1-2}\,\frac{|y'(x)|^{2-k_1}-y_1^{2-k_1}}{M} \le |Y|^{k_0+1} - |y(x_0)|^{k_0+1} \le \frac{k_0+1}{k_1-2}\,\frac{|y'(x)|^{2-k_1}-y_1^{2-k_1}}{m},$$

and since $y'(x) \to 0$ as $x \to \widetilde{x}$ and $2-k_1<0$, it follows that $|Y| = +\infty$.

Analogously, for $k_1 = 2$, we obtain

$$\frac{k_0+1}{M}\big(\ln y_1 - \ln|y'(x)|\big) \le |Y|^{k_0+1} - |y(x_0)|^{k_0+1} \le \frac{k_0+1}{m}\big(\ln y_1 - \ln|y'(x)|\big),$$

and since $y'(x) \to 0$ as $x \to \widetilde{x}$, it follows that $|Y| = +\infty$.

Consider now $\widetilde{x}$ in correspondence with k_1. Let $x^* > x_0$, $x^* \le +\infty$ be the closest to x_0 point such that $\lim_{x\to x^*} y'(x) = 0$.

From equation (0.1), on the interval (x_0, x^*), we derive

$$y''|y'|^{-k_1} = p(x,y,y')|y|^{k_0}\operatorname{sgn}(yy'),$$

and since at $x > x_0$ there is $y(x) < 0$, $y'(x) < 0$, we have

$$y''(-y')^{-k_1} = p(x,y,y')|y|^{k_0},$$

$$\frac{1}{1-k_1}\big(|y'(x_0)|^{1-k_1} - |y'|^{1-k_1}\big) = \int_{x_0}^{x} p(x,y,y')|y|^{k_0}\,dx,$$

therefore

$$\frac{1}{1-k_1}\big(|y'(x_0)|^{1-k_1} - |y'|^{1-k_1}\big) \le \int_{x_0}^{x} M|Y|^{k_0}\,dx = M|Y|^{k_0}(x-x_0)$$

and

$$x - x_0 \ge \frac{1}{M|Y|^{k_0}(k_1-1)}\big(|y'(x)|^{1-k_1} - |y'(x_0)|^{1-k_1}\big).$$

Since $y'(x) \to 0$ as $x \to x^*$ and $1-k_1 < 0$, the right part of the above inequality tends to infinity as $x \to x^*$, which implies $x^* = +\infty$ and, therefore, the solution $y(x)$ is defined on $(x_0, +\infty)$, $y_+ = Y$ and the theorem is proved. □

4 Continuous dependence of boundaries of domain or horizontal asymptotes of solutions on initial data

Consider first continuous dependence of the right-side boundary of the domain on initial data.

Theorem 4.1. *Suppose $k_0 > 0$, $k_1 > 0$, $k_0 + k_1 > 1$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequality $p(x,u,v) \ge m > 0$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any x_0, $\widetilde{x}_0$, y_0, z_0, y_1, z_1 satisfying $|\widetilde{x}_0 - x_0| < \delta$, $|z_0 - y_0| < \delta$, $|z_1 - y_1| < \delta$, $y_0 \ge 0$, $y_1 > 0$, $z_0 \ge 0$, $z_1 > 0$, the maximally extended solutions $y(x)$ and $z(x)$ to equation* (0.1) *with the initial data*

$$\begin{cases} y(x_0) = y_0, \\ y'(x_0) = y_1 \end{cases} \tag{4.1}$$

and

$$\begin{cases} y(\widetilde{x}_0) = z_0, \\ y'(\widetilde{x}_0) = z_1, \end{cases} \tag{4.2}$$

respectively, have finite right-side boundaries of the domains $x_1^* > x_0$ and $x_2^* > \widetilde{x}_0$, respectively, and $|x_2^* - x_1^*| < \varepsilon$.

Proof. From Theorem 2.1 it follows that $y'(x) \to +\infty$ as $x \to x_1^* - 0$, there exists a point x_1 such that $\widetilde{y}_1 = y'(x_1)$ satisfies

$$\widetilde{y}_1 > \left(\frac{\varepsilon}{2\xi}\right)^{-\frac{k_0+1}{k_0+k_1-1}}, \quad \xi \widetilde{y}_1^{-\frac{k_0+k_1-1}{k_0+1}} < \frac{\varepsilon}{2},$$

where ξ is a constant from Theorem 2.1. Then

$$x_1^* - x_1 < \xi (y'(x_1))^{-\frac{k_0+k_1-1}{k_0+1}} < \frac{\varepsilon}{2}.$$

For any $\varepsilon > 0$, there exists $\widetilde{\delta} > 0$ such that if $|\widetilde{z}_1 - \widetilde{y}_1| < \widetilde{\delta}$, then $\xi \widetilde{z}_1^{-\frac{k_0+k_1-1}{k_0+1}} < \frac{\varepsilon}{2}$. Also for every $\widetilde{\delta} > 0$ there exists $\delta > 0$ such that for any x_0, $\widetilde{x}_0$, y_0, z_0, y_1, z_1 satisfying $|\widetilde{x}_0 - x_0| < \delta$, $|z_0 - y_0| < \delta$, $|z_1 - y_1| < \delta$, $y_0 \geq 0$, $y_1 > 0$, $z_0 \geq 0$, $z_1 > 0$ the inequality $|z'(x_1) - y'(x_1)| < \widetilde{\delta}$ holds. Then from Theorem 2.1 we derive that the solution $z(x)$ with initial data (4.2) has a finite right-side boundary of the domain x_2^* and

$$x_2^* - x_1 < \xi (z'(x_1))^{-\frac{k_0+k_1-1}{k_0+1}} < \frac{\varepsilon}{2}.$$

Thus, for any ε, there exists $\delta > 0$ such that

$$|x_2^* - x_1^*| \leq |x_2^* - x_1| + |x_1 - x_1^*| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon. \qquad \square$$

Analogously, continuous dependence of the left-side boundary of the domain on the initial data is obtained.

Theorem 4.2. *Suppose $k_0 > 0$, $k_1 > 0$, $k_0 + k_1 > 1$. Let $p(x, u, v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying the inequality $p(x, u, v) \geq m > 0$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any x_0, $\widetilde{x}_0$, y_0, z_0, y_1, z_1 satisfying $|\widetilde{x}_0 - x_0| < \delta$, $|z_0 - y_0| < \delta$, $|z_1 - y_1| < \delta$, $y_0 \leq 0$, $y_1 > 0$, $z_0 \leq 0$, $z_1 > 0$, the maximally extended solutions $y(x)$ and $z(x)$ to equation* (0.1) *with initial data* (4.1) *and* (4.2)*, respectively, have finite left-side boundaries of domains $x_{1*} < x_0$ and $x_{2*} < \widetilde{x}_0$, respectively, and $|x_{2*} - x_{1*}| < \varepsilon$.*

Analogously, with the help of the estimates from Theorems 3.1 and 3.2 the following results on the continuous dependence of solutions' limits on the initial data are obtained.

Theorem 4.3. *Suppose $k_0 > 0$, $k_1 \in (0, 2)$. Let $p(x, u, v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1)*. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any x_0, $\widetilde{x}_0$, y_0, z_0, y_1, z_1 satisfying $|\widetilde{x}_0 - x_0| < \delta$, $|z_0 - y_0| < \delta$, $|z_1 - y_1| < \delta$, $y_0 \leq 0$, $y_1 < 0$, $z_0 \leq 0$, $z_1 < 0$, the maximally extended solutions $y(x)$ and $z(x)$ to equation* (0.1) *with initial data* (4.1) *and* (4.2)*, respectively, have finite limits $y_+ < y(x_0)$ and $z_+ < z(\widetilde{x}_0)$, respectively, as $x \to +\infty$, and $|y_+ - z_+| < \varepsilon$.*

Theorem 4.4. *Suppose $k_0 > 0$, $k_1 \in (0, 2)$. Let $p(x, u, v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1)*. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any x_0, $\widetilde{x}_0$, y_0, z_0, y_1, z_1 satisfying $|\widetilde{x}_0 - x_0| < \delta$, $|z_0 - y_0| < \delta$, $|z_1 - y_1| < \delta$, $y_0 \geq 0$, $y_1 < 0$, $z_0 \geq 0$, $z_1 < 0$, the maximally extended solutions $y(x)$ and $z(x)$ to equation* (0.1) *with initial data* (4.1) *and* (4.2)*, respectively, have finite limits $y_- > y(x_0)$ and $z_- > z(\widetilde{x}_0)$, respectively, as $x \to -\infty$, and $|x_- - z_-| < \varepsilon$.*

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