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THE BOUNDARY VALUE PROBLEMS FOR THE BI-LAPLACE–BELTRAMI EQUATION

Abstract. The purpose of the present paper is to investigate the boundary value problems for the bi-Laplace–Beltrami equation $\Delta^2_{\mathscr{C}}\varphi = f$ on a smooth hypersurface $\mathscr{C}$ with the boundary $\Gamma = \partial\mathscr{C}$. The unique solvability of the BVP is proved on the basis of Green's formula and Lax–Milgram Lemma.

We also prove the invertibility of the perturbed operator in the Bessel potential spaces $\Delta^2_{\mathscr{C}} + \mathscr{H}\, I : \mathbb{H}^{s+2}_p(\mathscr{S}) \to \mathbb{H}^{s-2}_p(\mathscr{S})$ for a smooth closed hypersurface $\mathscr{S}$ without boundary for arbitrary $1 < p < \infty$ and $-\infty < s < \infty$, provided $\mathscr{H}$ is a smooth function, has non-negative real part $\operatorname{Re}\mathscr{H}(t) \geqslant 0$ for all $t \in \mathscr{S}$ and non-trivial support $\operatorname{mes}\operatorname{supp}\operatorname{Re}\mathscr{H} \neq 0$.

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რეზიუმე. სტატიის მიზანია გამოვიკვლიოთ სასაზღვრო ამოცანა ბი–ლაპლას–ბელტრამის $\Delta^2_{\mathscr{C}}\varphi = f$ განტოლებისთვის, როგორც ჩვეულებრივი, ასევე შერეული სასაზღვრო პირობებით გლუვ $\mathscr{C}$ ჰიპერზედაპირზე, რომლის საზღვარია $\Gamma = \partial\mathscr{C}$. მოცემული ამოცანის ამოხსნადობა და ამონახსნის ერთადერთობა დამტკიცებულია გრინის ფორმულისა და ლაქს–მილგრამის ლემის საშუალებით.

აგრეთვე დამტკიცებულია $\Delta^2_{\mathscr{C}} + \mathscr{H}\, I \ : \ \mathbb{H}^{s+2}_p(\mathscr{S}) \to \mathbb{H}^{s-2}_p(\mathscr{S})$ შეშფოთებული ოპერატორის შებრუნებადობა ბესელის პოტენციალთა სივრცეებში ჩაკეტილი გლუვი ჰიპერზედაპირისთვის $\mathscr{S}$ საზღვრის გარეშე, $1 < p < \infty$ და $-\infty < s < \infty$ პარამეტრებისთვის. ასევე დამტკიცებულია, რომ $\mathscr{H}$ არის გლუვი ფუნქცია, აქვს არაუარყოფითი $\operatorname{Re}\mathscr{H}(t) \geqslant 0$ ნამდვილი ნაწილი ყველა $t \in \mathscr{S}$-სთვის და $\operatorname{mes}\operatorname{supp}\operatorname{Re}\mathscr{H} \neq 0$.

1 Introduction

Let $\mathscr{C} \subset \mathscr{S}$ be a smooth subsurface of a closed hypersurface $\mathscr{S}$ in the Euclidean space $\mathbb{R}^n$ (see Section 2 for details) and $\Gamma = \partial\mathscr{C} \neq \varnothing$ be its smooth boundary. Let $\mathscr{D}_j := \partial_j - \nu_j\partial_{\boldsymbol{\nu}}$, $j = 1, \dots, n$, be Günter's tangential derivatives, and $\boldsymbol{\Delta}^2 := \sum\limits_{j,k=1}^{n} \mathscr{D}_j^2\mathscr{D}_k^2$ be the bi-Laplace–Beltrami operator restricted to the surface $\mathscr{C}$.

The purpose of the present paper is to investigate the boundary value problems (BVPs) for the bi-Laplace–Beltrami equation

$$\begin{cases} \Delta^2_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ (B_0 u)^+(s) = g(s), & \text{on } \Gamma, \\ (B_1 u)^+(s) = h(s), & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where the boundary operators can be chosen as follows:

$$\begin{gathered} B_0 = I \ \text{ and } \ B_1 = \partial_{\boldsymbol{\nu}_\Gamma}, \ \text{ or } \ B_1 = \Delta_{\mathscr{C}}, \\ B_0 = \partial_{\boldsymbol{\nu}_\Gamma} \ \text{ and } \ B_1 = \Delta_{\mathscr{C}}, \ \text{ or } \ B_1 = \partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}}. \end{gathered} \tag{1.2}$$

The BVP

$$\begin{cases} \Delta^2_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ u^+(\tau) = 0, \quad (\partial_{\boldsymbol{\nu}_\Gamma} u)^+(\tau) = 0, & \tau \in \Gamma, \end{cases}$$

is called **a clamped surface equation** and is considered in the weak classical setting

$$u \in \mathbb{H}^2(\mathscr{C}), \ \ f \in \widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C}).$$

The BVP

$$\begin{cases} \Delta^2_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ u^+(\tau) = g(\tau), \quad (\Delta_{\mathscr{C}} u)^+ + a\partial_{\boldsymbol{\nu}_\Gamma} u)^+(\tau) = h(\tau), & \tau \in \Gamma, \end{cases}$$

with **Steklov Boundary Conditions** is considered in the weak classical setting

$$u \in \mathbb{H}^2(\mathscr{C}), \ \ f \in \widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C}), \ \ g \in \mathbb{H}^{3/2}(\Gamma), \ \ h \in \mathbb{H}^{-3/2}(\Gamma).$$

Here a is a real-valued constant.

The BVP

$$\begin{cases} \Delta^2_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ u^+(\tau) = g(\tau), \quad (\Delta_{\mathscr{C}} u)^+ = h(\tau), & \tau \in \Gamma \end{cases}$$

with **Navier Boundary Conditions** is considered in the weak classical setting

$$u \in \mathbb{H}^2(\mathscr{C}), \ \ f \in \widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C}), \ \ g \in \mathbb{H}^{3/2}(\Gamma), \ \ h \in \mathbb{H}^{-1/2}(\Gamma).$$

First we consider in detail the case

$$\begin{cases} \Delta^2 u(t) = f(t), & t \in \mathscr{C}, \\ (\partial_{\boldsymbol{\nu}_\Gamma} u)^+(s) = g(s), & \text{on } \Gamma, \\ (\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}} u)^+(s) = h(s), & \text{on } \Gamma, \end{cases} \tag{1.3}$$

in the weak classical setting

$$u \in \mathbb{H}^2(\mathscr{C}), \ \ f \in \widetilde{\mathbb{H}}^{-2}_\Gamma\mathscr{C}), \ \ g \in \mathbb{H}^{1/2}(\Gamma), \ \ h \in \mathbb{H}^{-3/2}(\Gamma), \tag{1.4}$$

where

$$\widetilde{\mathbb{H}}^{-2}_\Gamma(\Omega) := \Big\{ f \in \widetilde{\mathbb{H}}^{-2}(\Omega) \ \big| \ \ (f, \varphi)_{\mathbb{L}^2(\Omega)} = 0, \ \ \varphi \in \mathbb{C}^\infty_0(\Omega) \Big\}. \tag{1.5}$$

Remark 1.1. Let us comment on the condition $f \in \widetilde{\mathbb{H}}_{\Gamma}^{-2}(\mathscr{C})$ in (1.4).

As is shown in [13, p. 196], the condition $f \in \widetilde{\mathbb{H}}^{-2}(\mathscr{C})$ does not ensure the uniqueness of a solution to the BVP (1.3). The right-hand side f needs additional constraint that it belongs to the subspace $\widetilde{\mathbb{H}}_0^{-2}(\Omega) \subset \widetilde{\mathbb{H}}^{-2}(\Omega)$ which is the orthogonal complement to the subspace $\widetilde{\mathbb{H}}_{\Gamma}^{-2}(\Omega)$ of those distributions from $\widetilde{\mathbb{H}}^{-2}(\Omega)$ which are supported only on the boundary $\Gamma = \partial\Omega$ of the domain (see (1.5)).

Another cases in (1.2) are considered analogously.

We will prove the unique solvability of the BVP (1.3) in the classical setting (1.4) by applying the Lax–Milgram Lemma.

We also consider the following BVP with the mixed boundary conditions: Let $\mathscr{C} \subset \mathscr{S}$ be a smooth subsurface of a closed hypersurface $\mathscr{S}$ in the Euclidean space $\mathbb{R}^n$ (see Section 2 for details) and its smooth boundary $\partial\mathscr{C} = \Gamma = \Gamma_1 \cup \Gamma_2 \neq \varnothing$ be decomposed into two non-intersecting connected parts. Consider the mixed BVP for the bi-Laplase-Beltrami equation

$$\begin{cases} \Delta^2 u(t) = f(t), & t \in \mathscr{C}, \\ (u)^+(s) = g_1(s), & \text{on } \Gamma_1, \\ (\partial_{\boldsymbol{\nu}_\Gamma} u)^+(s) = g_2(s), & \text{on } \Gamma_2, \\ (\Delta_{\mathscr{C}} u)^+(s) = h_1(s), & \text{on } \Gamma_1, \\ (\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} u)^+(s) = h_2(s), & \text{on } \Gamma_2, \end{cases} \tag{1.6}$$

in the weak classical setting

$$u \in \mathbb{H}^2(\mathscr{C}), \quad f \in \widetilde{\mathbb{H}}_{\Gamma}^{-2}(\mathscr{C}), \quad g_1 \in \mathbb{H}^{3/2}(\Gamma_1), \quad g_2 \in \mathbb{H}^{1/2}(\Gamma_2), \quad h_1 \in \mathbb{H}^{-1/2}(\Gamma_1), \quad h_2 \in \mathbb{H}^{-3/2}(\Gamma_2). \tag{1.7}$$

The following are the main theorems of the present paper. The proofs are exposed in Sections 3 and 4, below.

Prior formulating the theorems let us introduce the Hilbert spaces with detached constants $\mathbb{H}_{\#}^2(\mathscr{S}) := \mathbb{H}^2(\mathscr{S}) \setminus \{const\}$. Another description of the space $\mathbb{H}_{\#}^2(\mathscr{S})$ is that it consists of all functions $\varphi \in \mathbb{H}^2(\mathscr{S})$, which have the zero mean value, $(\varphi, 1)_{\mathscr{S}} = 0$.

Theorem 1.1. *The boundary value problem* (1.3) *in the weak classical setting* (1.4) *has a unique solution in the space* $\mathbb{H}_{\#}^2(\mathscr{C})$.

Theorem 1.2. *The mixed type boundary value problem* (1.6) *in the weak classical setting* (1.7) *has a unique solution in the space* $\mathbb{H}_{\#}^2(\mathscr{C})$.

The Bi-Laplace–Beltrami operator $\Delta^2 = \Delta \times \Delta$ is a model of a fourth-order operator. The BVPs on hypersurfaces arise in a variety of situations and have many practical applications. They appear in various problems of linear elasticity, for example, when looking for small displacements of a plate, whereas the Laplacian describes the behavior of a membrane.

A hypersurface $\mathscr{S}$ in $\mathbb{R}^n$ has the natural structure of an $(n-1)$-dimensional Riemannian manifold and the aforementioned partial differential equations (PDEs) are not the immediate analogues of the ones corresponding to the flat, Euclidean case, since they have to take into consideration geometric characteristics of $\mathscr{S}$ such as curvature. Inherently, these PDEs are originally written in local coordinates, intrinsic to the manifold structure of $\mathscr{S}$.

Another problem we encounter in considering BVPs (1.1) is the existence of a fundamental solution for the bi-Laplace–Beltrami operator. An essential difference between the differential operators on hypersurfaces and the Euclidean space $\mathbb{R}^n$ lies in the existence of the fundamental solution: In $\mathbb{R}^n$, a fundamental solution exists for all partial differential operators with constant coefficients if it is not trivially zero. On a hypersurface, the bi-Laplace–Beltrami operator has no fundamental solution because it has a non-trivial kernel, constants, in all Bessel potential spaces. Therefore we consider the bi-Laplace–Beltrami operator in the Hilbert spaces with detached constants $\Delta_{\mathscr{C}}^2 : \mathbb{H}_{\#}^2(\mathscr{S}) \to \mathbb{H}^{-2}(\mathscr{S})$ and prove that it is an invertible operator. The established invertibility implies the existence of a certain fundamental solution, which can be used to define the volume (Newtonian), single layer and double layer potentials.

2 Auxiliary material

We commence with the definition of a hypersurface. There exist other equivalent definitions, but they are most convenient for us. Equivalence of these definitions and some other properties of hypersurfaces are discussed, e.g., in [7, 8].

Definition 2.1. A subset $\mathscr{S} \subset \mathbb{R}^n$ of the Euclidean space is said to be a **hypersurface** if it has a covering $\mathscr{S} = \bigcup\limits_{j=1}^{M} \mathscr{S}_j$ and coordinate mappings

$$\Theta_j : \omega_j \longrightarrow \mathscr{S}_j := \Theta_j(\omega_j) \subset \mathbb{R}^n, \ \ \omega_j \subset \mathbb{R}^{n-1}, \ \ j = 1, \dots, M, \tag{2.1}$$

such that the corresponding differentials

$$D\Theta_j(p) := \operatorname{matr}\big[\partial_1\Theta_j(p), \dots, \partial_{n-1}\Theta_j(p)\big]$$

have the full rank

$$\operatorname{rank} D\Theta_j(p) = n-1, \ \ \forall\, p \in Y_j, \ \ k = 1, \dots, n, \ \ j = 1, \dots, M,$$

i.e., all points of ω_j are regular for Θ_j for all $j = 1, \dots, M$.

Such a mapping is called an **immersion** as well.

Here and in what follows, $\operatorname{matr}[x_1, \dots, x_k]$ refers to the matrix with the listed vectors $x_1, \dots, x_k$ as columns.

A hypersurface is called **smooth** if the corresponding coordinate diffeomorphisms Θ_j in (2.1) are smooth (C^∞-smooth). Similarly is defined a μ**-smooth** hypersurface.

The next definition of a hypersurface is called **implicit**.

Definition 2.2. Let $k \geqslant 1$ and $\omega \subset \mathbb{R}^n$ be a compact domain. An implicit C^k-smooth hypersurface in $\mathbb{R}^n$ is defined as the set

$$\mathscr{S} = \big\{\mathscr{x} \in \omega : \ \ \Psi_{\mathscr{S}}(\mathscr{x}) = 0\big\},$$

where $\Psi_{\mathscr{S}} : \omega \to \mathbb{R}$ is a C^k-mapping, which has the non-vanishing gradient $\nabla\,\Psi(\mathscr{x}) \neq 0$.

The most important role in the calculus of tangential differential operators that we are going to apply belongs to the unit normal vector field $\boldsymbol{\nu}(y)$, $t \in \mathscr{C}$. The **unit normal vector field** to the surface $\mathscr{C}$, known also as the **Gauß mapping**, is defined by the vector product of the covariant basis

$$\boldsymbol{\nu}(\mathscr{x}) := \pm \frac{\boldsymbol{g}_1(\mathscr{X}) \wedge \cdots \wedge \boldsymbol{g}_{n-1}(\mathscr{X})}{|\boldsymbol{g}_1(\mathscr{X}) \wedge \cdots \wedge \boldsymbol{g}_{n-1}(\mathscr{X})|}\,, \ \ \ \mathscr{X} \in \mathscr{C}.$$

The system of tangential vectors $\{\boldsymbol{g}_k\}_{k=1}^{n-1}$ to $\mathscr{C}$ is, by the definition, linearly independent and is known as the **covariant basis**. There exists the unique system $\{\boldsymbol{g}^k\}_{k=1}^{n-1}$ biorthogonal to it, i.e., the **contravariant basis**

$$\langle \boldsymbol{g}_j, \boldsymbol{g}^k \rangle = \delta_{jk}, \ \ j, k = 1, \dots, n-1.$$

The contravariant basis is defined by the formula

$$\boldsymbol{g}^k = \frac{1}{\det G_{\mathscr{S}}}\, \boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_{k-1} \wedge \boldsymbol{\nu} \wedge \boldsymbol{g}_{k+1} \wedge \cdots \wedge \boldsymbol{g}_{n-1}, \ \ k = 1, \dots, n-1,$$

where

$$G_{\mathscr{S}}(\mathscr{x}) := \big[\big\langle \boldsymbol{g}_k(\mathscr{x}), \boldsymbol{g}_m(\mathscr{x}) \big\rangle\big]_{n-1 \times n-1}, \ \ p \in \mathscr{S},$$

is the **Gram matrix**.

Günter's derivatives are the simplest examples of tangential differential operators

$$\mathscr{D}_j := \partial_j - \nu_j \partial_\nu = \partial_j - \nu_j \sum_{k=1}^{n} \nu_k \partial_k.$$

The surface divergence $\mathbf{div}_{\mathscr{S}}$ and the surface gradient $\nabla_{\mathscr{S}}$ are defined as follows:

$$\mathbf{div}_{\mathscr{S}}\mathbf{U} = \sum_{k=1}^{n} \partial_k U_k, \quad \nabla_{\mathscr{S}}\varphi := (\mathscr{D}_1\varphi, \dots, \mathscr{D}_n\varphi_n)^\top, \quad \mathbf{U} := (U_1, \dots, U_n)^\top,$$

and the surface Laplace–Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}}$ is their superposition

$$\boldsymbol{\Delta}_{\mathscr{S}}\psi = \mathbf{div}_{\mathscr{S}}\nabla_{\mathscr{S}}\psi = \sum_{j=1}^{n} \mathscr{D}_j\psi. \tag{2.2}$$

In contrast to the classical differential geometry, the **surface gradient**, the **surface divergence** and the surface Laplace–Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}}$ are defined by Günter's derivatives much simpler, with the help of only normal vector field ν, while definitions in the classical differential geometry are based on the **Christoffel symbols** Γ^j_{km}, the covariant and the contravariant $G^{-1} := [g^{jk}]$ Riemann metric tensors and are rather complicated.

It is well known that $\mathbf{div}_{\mathscr{S}}$ is the negative dual to the surface gradient

$$\langle \mathbf{div}_{\mathscr{S}}\boldsymbol{V}, f\rangle := -\langle \boldsymbol{V}, \nabla_{\mathscr{S}} f\rangle, \ \ \forall\, \boldsymbol{V} \in \mathscr{V}(\mathscr{S}), \ \ \forall\, f \in C^1(\mathscr{S}).$$

Let $\mathscr{M}$ be a non-trivial, $\operatorname{mes}\mathscr{M} \neq \varnothing$, smooth closed hypersurface, $s \in \mathbb{R}$ and $1 < p < \infty$. For the definitions of Bessel's potential $\mathbb{H}^s_p(\mathscr{M})$ and Sobolev–Slobodeckii $\mathbb{W}^s_p(\mathscr{M})$ spaces for a closed smooth manifold $\mathscr{M}$ we refer to [22] (see also [6, 12, 13]). For $p = 2$, the Sobolev–Slobodetski $\mathbb{W}^s(\mathscr{M}) := \mathbb{W}^s_2(\mathscr{M})$ and the Bessel potential $\mathbb{H}^s(\mathscr{M}) := \mathbb{H}^s_2(\mathscr{M})$ spaces coincide (i.e., the norms are equivalent).

Let $\mathscr{C}$ be a subsurface of a smooth closed surface $\mathscr{M}$, $\mathscr{C} \subset \mathscr{M}$, with the smooth boundary $\Gamma := \partial\mathscr{C}$. The space $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$ is defined as the subspace of those functions $\varphi \in \mathbb{H}^s_p(\mathscr{M})$, which are supported in the closure of the subsurface, $\operatorname{supp}\varphi \subset \overline{\mathscr{C}}$, whereas $\mathbb{H}^s_p(\mathscr{C})$ denotes the quotient space $\mathbb{H}^s_p(\mathscr{C}) = \mathbb{H}^s_p(\mathscr{M})/\widetilde{\mathbb{H}}^s_p(\mathscr{C}^c)$ and $\mathscr{C}^c := \mathscr{M} \setminus \overline{\mathscr{C}}$ is the complementary subsurface to $\mathscr{C}$. The space $\mathbb{H}^s_p(\mathscr{C})$ can be identified with the space of distributions φ on $\mathscr{C}$ which have an extension to a distribution $\ell\varphi \in \mathbb{H}^s_p(\mathscr{M})$. Therefore $r_{\mathscr{C}}\mathbb{H}^s_p(\mathscr{M}) = \mathbb{H}^s_p(\mathscr{C})$, where $r_{\mathscr{C}}$ denotes the restriction operator of functions (distributions) from the surface $\mathscr{M}$ to the subsurface $\mathscr{C}$.

The spaces $\widetilde{\mathbb{W}}^s_p(\mathscr{C})$ and $\mathbb{W}^s_p(\mathscr{C})$ are defined similarly (see [22] and also [6, 12, 13]).

By $\mathbb{X}^s_p(\mathscr{C})$ we denote one of the spaces: $\mathbb{H}^s_p(\mathscr{C})$ or $\mathbb{W}^s_p(\mathscr{C})$, and by $\widetilde{\mathbb{X}}^s_p(\mathscr{C})$ one of the spaces: $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$ and $\widetilde{\mathbb{W}}^s_p(\mathscr{C})$ (if $\mathscr{C}$ is open).

The bi-Laplace–Beltrami operator has the finite dimensional kernel $\dim\operatorname{Ker}\Delta_{\mathscr{C}} \leq \infty$, and its kernel consists only of constants. Hence the space $\mathbb{X}^s(\mathscr{C})$ decomposes into the direct sum

$$\mathbb{X}^s_p(\mathscr{C}) = \mathbb{X}^s_{p,\#}(\mathscr{C}) + \{const\},$$

where

$$\mathbb{X}^s_{p,\#}(\mathscr{C}) := \big\{\varphi \in \mathbb{X}^s_2(\mathscr{C}) : \ (\varphi, 1) = 0\big\} \tag{2.3}$$

is the space without constants.

Lemma 2.1. *The bi-Laplace–Beltrami operator* $\Delta^2_{\mathscr{S}}\varphi := (\mathbf{div}_{\mathscr{S}}\nabla_{\mathscr{S}})^2\varphi : \mathbb{H}^2(\mathscr{S}) \to \mathbb{H}^{-2}(\mathscr{S})$ *is elliptic, self-adjoint* $(\Delta^2_{\mathscr{S}})^* = \Delta^2_{\mathscr{S}}$*, non-negative*

$$(\Delta^2_{\mathscr{S}}\varphi, \varphi) = (\Delta_{\mathscr{S}}\varphi, \Delta_{\mathscr{S}}\varphi) = \big\|\Delta_{\mathscr{S}}\varphi\big|\mathbb{L}_2(\mathscr{S})\big\|^2 \geq 0, \ \ \varphi \in \mathbb{H}^2(\mathscr{S})$$

and the homogenous equation has only a constant solution

$$(\Delta^2_{\mathscr{S}}\varphi, \varphi) = 0, \ \ \textit{only for} \ \ \varphi = const. \tag{2.4}$$

Proof. $\Delta^2_{\mathscr{S}}$ is elliptic and self-adjoint since $\Delta_{\mathscr{S}}$ is elliptic and self-adjoint (see [7]).

Due to (2.2) and (2.4), we get

$$0 = (\Delta^2_{\mathscr{S}}\varphi, \varphi) = (\Delta_{\mathscr{S}}\varphi, \Delta_{\mathscr{S}}\varphi) = \big\|\Delta_{\mathscr{S}}\varphi\big|\mathbb{L}_2(\mathscr{S})\big\|^2$$

which gives $\Delta_{\mathscr{S}}\varphi = 0$ and, consequently, $\varphi = const$ (see [7]).

Corollary 2.1. *The space $\mathbb{X}^s(\mathscr{C})$ decomposes into the direct sum*

$$\mathbb{X}^s(\mathscr{C}) = \mathbb{X}^s_{\#}(\mathscr{C}) + \{const\},$$

where $\mathbb{X}^s_{\#}(\mathscr{C})$ is the space with detached constants and the operator $\Delta^2_{\mathscr{S}}$ is invertible between the spaces with detached constants (see (2.3))

$$\Delta^2_{\mathscr{S}} : \mathbb{X}^{s+2}_{\#}(\mathscr{S}) \longrightarrow \mathbb{X}^{s-2}_{\#}(\mathscr{S}). \tag{2.5}$$

Therefore $\Delta^2_{\mathscr{S}}$ has the fundamental solution in the setting (2.5).

Proof. The boundedness in (2.5) follows from that of the operator

$$\Delta_{\mathscr{S}} : \mathbb{X}^{s+1}_{\#}(\mathscr{S}) \longrightarrow \mathbb{X}^{s-1}_{\#}(\mathscr{S})$$

proved in [10].

Since $\Delta^2_{\mathscr{S}}$ has the trivial kernel in the setting (2.5) and is self-adjoint (see the foregoing Lemma 2.1), it has the trivial co-kernel as well and is invertible. □

Corollary 2.2. *For the bi-Laplace–Beltrami operator $\Delta^2_{\mathscr{C}}$ on the open hypersurface $\mathscr{C}$ the following* I *and* II *Green's formulae are valid:*

$$(\Delta^2_{\mathscr{C}}\varphi, \psi)_{\mathscr{C}} - (\Delta_{\mathscr{C}}\varphi, \Delta_{\mathscr{C}}\psi)_{\mathscr{C}} = -((\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}}\varphi)^+, \psi^+)_\Gamma + ((\Delta_{\mathscr{C}}\varphi)^+, (\partial_{\boldsymbol{\nu}_\Gamma}\psi)^+)_\Gamma, \tag{2.6}$$
$$(\Delta^2_{\mathscr{C}}\varphi, \psi)_{\mathscr{C}} + ((\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}}\varphi)^+, \psi^+)_\Gamma - ((\Delta_{\mathscr{C}}\varphi)^+, (\partial_{\boldsymbol{\nu}_\Gamma}\psi)^+)_\Gamma$$
$$= (\varphi, \Delta^2_{\mathscr{C}}\psi)_{\mathscr{C}} + (\varphi^+, (\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}}\psi)^+)_\Gamma - ((\partial_{\boldsymbol{\nu}_\Gamma}\varphi)^+, (\Delta_{\mathscr{C}}\psi)^+)_\Gamma$$

for arbitrary $\varphi, \psi \in \mathbb{X}^2(\mathscr{C})$ (see [4]).

Lemma 2.2 (see [14] (Lax–Milgram))**.** *Let $\mathfrak{B}$ be a Banach space, $A(\varphi, \psi)$ be a continuous, bilinear form*

$$A(\,\cdot\,,\,\cdot\,) : \mathfrak{B} \times \mathfrak{B} \longrightarrow \mathbb{R}$$

and positive definite

$$A(\varphi, \varphi) \geq C\big\|\varphi|\mathfrak{B}\big\|^2, \ \ \forall\, \varphi \in \mathfrak{B}, \ \ C > 0.$$

Let $L(\,\cdot\,) : \mathfrak{B} \to \mathbb{R}$ be a continuous linear functional.

A linear equation

$$A(\varphi, \psi) = L(\psi)$$

has a unique solution $\varphi \in \mathfrak{B}$ for an arbitrary $\psi \in \mathfrak{B}$.

3 The solvability of BVPs for the bi-Laplace–Beltrami equation

Let again $\mathscr{C} \subset \mathscr{S}$ be a smooth subsurface of a closed hypersurface $\mathscr{S}$ and $\Gamma = \partial\mathscr{C} \neq \varnothing$ be its smooth boundary.

To prove the forthcoming theorem about the unique solvability we will need more properties of the trace operators (called retractions) and their inverses, called co-retractions (see [22, § 2.7]).

To keep the exposition simpler we recall a very particular case of Lemma 4.8 from [6], which we apply to the present investigation.

Lemma 3.1. *Let $s > 0$, $s \notin \mathbb{N}$, $p = 2$, $\boldsymbol{B}(D)$ be a normal differential operator of the third order defined in the vicinity of the boundary $\Gamma = \partial\mathscr{C}$ and $\boldsymbol{A}(D)$ be a normal differential operator of the fourth order defined on the surface $\mathscr{C}$. Then there exists a continuous linear operator*

$$\mathscr{B} : \mathbb{H}^s(\Gamma) \otimes \mathbb{H}^{s-1}(\Gamma) \otimes \mathbb{H}^{s-2}(\Gamma) \otimes \mathbb{H}^{s-3}(\Gamma) \longrightarrow \mathbb{H}^{s+\frac{1}{2}}(\mathscr{C})$$

such that

$$(\mathscr{B}\Phi)^+ = \varphi_0, \quad (\boldsymbol{B}_1(D)\mathscr{B}\Phi)^+ = \varphi_1, \quad (\boldsymbol{B}_2(D)\mathscr{B}\Phi)^+ = \varphi_2,$$
$$(\boldsymbol{B}_3(D)\mathscr{B}\Phi)^+ = \varphi_3, \quad \boldsymbol{A}(D)\mathscr{B}\Phi \in \widetilde{\mathbb{H}}^{s-4+\frac{1}{2}}(\mathscr{C})$$

for an arbitrary quadruple of the functions $\Phi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3)^\top$, *where* $\varphi_0 \in \mathbb{H}^s(\Gamma)$, $\varphi_1 \in \mathbb{H}^{s-1}(\Gamma)$, $\varphi_2 \in \mathbb{H}^{s-2}(\Gamma)$ *and* $\varphi_3 \in \mathbb{H}^{s-3}(\Gamma)$.

Corollary 3.1. *Let* u *be a solution of the equation* $\Delta^2_{\mathscr{C}} u = f$. *Then it has the traces* $u^+ \in \mathbb{H}^{\frac{3}{2}}$, $(\partial_{\boldsymbol{\nu}_\Gamma} u)^+ \in \mathbb{H}^{\frac{1}{2}}$, $(\Delta_{\mathscr{C}} u)^+ \in \mathbb{H}^{-\frac{1}{2}}$, $(\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} u)^+ \in \mathbb{H}^{-\frac{3}{2}}$.

Proof. The existence of the traces $u^+ \in \mathbb{H}^{\frac{3}{2}}$, $(\partial_{\boldsymbol{\nu}_\Gamma} u)^+ \in \mathbb{H}^{\frac{1}{2}}$ is a direct consequence of the general trace theorem (see [22] for details). Let us prove the existence of the rest traces. Concerning the existence of the trace $(\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} \varphi)^+$ in (1.3) for a solution $u \in \mathbb{H}^2(\mathscr{C})$ is not guaranteed by the general trace theorem, but, according to Lemma 3.1, there exists a function $\psi \in \mathbb{H}^2(\mathscr{C})$ such that $(\partial_{\boldsymbol{\nu}_\Gamma} \psi)^+ = 0$. Then the first Green's formula (2.6) ensures the existence of the trace. Indeed, by setting $\varphi = u$ and inserting the data $\Delta^2_{\mathscr{C}} \varphi = f(t)$ into the first Green's formula (2.6), we get

$$-((\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} u)^+, \psi^+)_\Gamma = (f, \psi)_{\mathscr{C}} - (\Delta_{\mathscr{C}} u, \Delta_{\mathscr{C}} \psi)_{\mathscr{C}}. \tag{3.1}$$

The scalar product $(\Delta_{\mathscr{C}} u, \Delta_{\mathscr{C}} \psi)$ in the right-hand side of equality (3.1) is correctly defined and defines correct duality in the left-hand side of the equality. Since $\psi^+ \in \mathbb{H}^{3/2}(\Gamma)$ is arbitrary, by the duality argument this implies that $(\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} u)^+$ should be in the dual space, i.e., in $\mathbb{H}^{-3/2}(\Gamma)$.

Let us now prove the existence of the trace $(\Delta_{\mathscr{C}} \varphi)^+$. Taking an arbitrary $\psi \in \mathbb{H}^2(\mathscr{C})$ and rewriting the first Green's formula (2.6) in the form

$$((\Delta_{\mathscr{C}} u)^+, (\partial_{\boldsymbol{\nu}_\Gamma} \psi)^+)_\Gamma = (f, \psi)_{\mathscr{C}} - (\Delta_{\mathscr{C}} u, \Delta_{\mathscr{C}} \psi)_{\mathscr{C}} + ((\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} u)^+, \psi^+)_\Gamma, \tag{3.2}$$

we note that the right-hand side of equality (3.2) is correctly determined and defines correct duality in the left-hand side. Since $(\partial_{\boldsymbol{\nu}_\Gamma} \psi)^+ \in \mathbb{H}^{1/2}(\Gamma)$ is arbitrary, by the duality argument this implies that $(\Delta_{\mathscr{C}} u)^+$ should be in the dual space, i.e., in $\mathbb{H}^{-1/2}(\Gamma)$. □

Proof of Theorem 1.1. We commence with the reduction of the BVP (1.3) to an equivalent one with the homogeneous condition and apply Lemma 3.1: there exists a function $\Phi \in \mathbb{H}^2(\mathscr{C})$ such that $(\partial_{\boldsymbol{\nu}_\Gamma} \Phi)^+(t) = g(t)$ for $t \in \Gamma$ and $\Delta^2_{\mathscr{C}} \Phi \in \widetilde{\mathbb{H}}^{-2}_0(\overline{\mathscr{C}})$.

For a new unknown function $v := u - \Phi$ we have the following equivalent reformulation of the BVP (1.3):

$$\begin{cases} \Delta^2_{\mathscr{C}} v(t) = f_0(t), & t \in \mathscr{C}, \\ (\partial_{\boldsymbol{\nu}_\Gamma} v)^+(s) = 0, & \text{on } \Gamma, \\ (\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} v)^+(s)\rangle = h_0(s), & \text{on } \Gamma, \end{cases} \tag{3.3}$$

where

$$f_0 := f + \Delta^2_{\mathscr{C}} \Phi \in \widetilde{\mathbb{H}}^{-2}_0(\mathscr{C}), \quad h_0 := h + (\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} \Phi)^+ \in \mathbb{H}^{-3/2}(\Gamma), \quad v^+ \in \widetilde{\mathbb{H}}^{3/2}(\Gamma).$$

By inserting the data from the reformulated boundary value problem (3.3) into the first Green's identity (2.6), where $\varphi = \psi = v$, we get

$$(\Delta_{\mathscr{C}} v, \Delta_{\mathscr{C}} v)_{\mathscr{C}} = (\Delta^2_{\mathscr{C}} v, v)_{\mathscr{C}} + ((\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} v)^+, v^+)_\Gamma - ((\Delta_{\mathscr{C}} v)^+, (\partial_{\boldsymbol{\nu}_\Gamma} v)^+)_\Gamma = (f_0, v)_{\mathscr{C}} + (h_0, v^+)_\Gamma. \tag{3.4}$$

In the left-hand side of equality (3.4) we have a symmetric bilinear form, which is positive definite:

$$(\Delta_{\mathscr{S}} \varphi, \Delta_{\mathscr{S}} \varphi) = \left\| \Delta_{\mathscr{S}} \varphi \mid \mathbb{L}_2(\mathscr{S}) \right\|^2 \geq 0, \quad \varphi \in \mathbb{H}^2_\#(\mathscr{S}).$$

$(h_0, v^+)_\Gamma$ and $(f_0, v)_{\mathscr{C}}$ from equality (3.4) are the correctly defined continuous functionals, since $h_0 \in \mathbb{H}^{-3/2}(\Gamma)$, $f_0 \in \widetilde{\mathbb{H}}^{-2}(\mathscr{C})$, while their counterparts in the functional belong to the dual spaces $v^+ \in \widetilde{\mathbb{H}}^{3/2}(\Gamma)$ and $v \in \widetilde{\mathbb{H}}^2(\Gamma, \mathscr{C}) \subset \mathbb{H}^2(\mathscr{C})$.

The Lax–Milgram Lemma 2.2 accomplishes the proof. □

4 The solvability of mixed BVPs for the bi-Laplace–Beltrami equation

Proof of Theorem 1.2. We commence with the reduction of the BVP (1.6) to an equivalent one with the homogeneous conditions. Towards this end, we extend the boundary data $g_1 \in \mathbb{H}^{3/2}(\Gamma_1)$, $g_2 \in \mathbb{H}^{1/2}(\Gamma_2)$ and $h_1 \in \mathbb{H}^{-1/2}(\Gamma_1)$ up to some functions $\widetilde{g_1} \in \mathbb{H}^{3/2}(\Gamma)$, $\widetilde{g_2} \in \mathbb{H}^{1/2}(\Gamma)$ and $\widetilde{h_1} \in \mathbb{H}^{-1/2}(\Gamma)$ on the entire boundary Γ and apply Lemma 3.1: there exists a function $\Phi \in \mathbb{H}^2(\mathscr{C})$ such that

$$\Phi^+ = \widetilde{g_1}, \ \ (\partial_{\boldsymbol{\nu}_\Gamma}\Phi)^+ = \widetilde{g_2}, \ \ (\Delta_{\mathscr{C}}\Phi)^+ = h_1, \ \text{ and } \ \Delta^2_{\mathscr{C}}\Phi \in \widetilde{\mathbb{H}}^{-2}_0(\overline{\mathscr{C}}).$$

For a new unknown function $v := u - \Phi$ we have the following equivalent reformulation of the BVP (1.6):

$$\begin{cases} \Delta^2 v(t) = f_0(t), & t \in \mathscr{C}, \\ (v)^+(s) = 0, & \text{on } \Gamma_1, \\ (\partial_{\boldsymbol{\nu}_\Gamma} v)^+(s) = 0, & \text{on } \Gamma_2, \\ (\Delta_{\mathscr{C}} v)^+(s) = 0, & \text{on } \Gamma_1, \\ (\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}} v)^+(s) = h_0(s), & \text{on } \Gamma_2, \end{cases} \tag{4.1}$$

where

$$\begin{gathered} f_0 := f + \Delta^2_{\mathscr{C}}\Phi \in \widetilde{\mathbb{H}}^{-2}_0(\mathscr{C}), \quad h_0 := h_2 + (\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}}\Phi)^+ \in \mathbb{H}^{-3/2}(\Gamma_2), \\ v^+ \in \widetilde{\mathbb{H}}^{3/2}(\Gamma_2), \quad (\partial_{\boldsymbol{\nu}_\Gamma} v)^+ \in \widetilde{\mathbb{H}}^{1/2}(\Gamma_1), \quad (\Delta_{\mathscr{C}} v)^+ \in \widetilde{\mathbb{H}}^{-1/2}(\Gamma_2) \end{gathered} \tag{4.2}$$

To justify the last inclusion $v^+ \in \widetilde{\mathbb{H}}^{3/2}(\Gamma_2)$, $(\partial_{\boldsymbol{\nu}_\Gamma} v)^+ \in \widetilde{\mathbb{H}}^{1/2}(\Gamma_1)$ and $(\Delta_{\mathscr{C}} v)^+ \in \widetilde{\mathbb{H}}^{-1/2}(\Gamma_2)$, note that, due to our construction, the traces of a solution vanish: $v^+ \,|_{\Gamma_1} = 0$, $(\partial_{\boldsymbol{\nu}_\Gamma} v)^+ \,|_{\Gamma_2} = 0$ and $(\Delta_{\mathscr{C}} v)^+ \,|_{\Gamma_1} = 0$. By inserting the data from the reformulated boundary value problem (4.1) into the first Green's identity (2.6), where $\varphi = \psi = v$, we get

$$\begin{aligned} (\Delta_{\mathscr{C}} v, \Delta_{\mathscr{C}} v)_{\mathscr{C}} = (\Delta^2_{\mathscr{C}} v, v)_{\mathscr{C}} + ((\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}} v)^+, v^+)_{\Gamma_1} + ((\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}} v)^+, v^+)_{\Gamma_2} \\ - ((\Delta_{\mathscr{C}} v)^+, (\partial_{\boldsymbol{\nu}_\Gamma} v)^+)_{\Gamma_1} - ((\Delta_{\mathscr{C}} v)^+, (\partial_{\boldsymbol{\nu}_\Gamma} v)^+)_{\Gamma_2} = (f_0, v)_{\mathscr{C}} + (h_0, v^+)_{\Gamma_2} \end{aligned} \tag{4.3}$$

In the left-hand side of equality (4.3) we have a symmetric bilinear form, which is positive definite:

$$(\Delta_{\mathscr{S}}\varphi, \Delta_{\mathscr{S}}\varphi) = \|\Delta_{\mathscr{S}}\varphi \mid \mathbb{L}_2(\mathscr{S})\|^2 \geq 0, \ \ \varphi \in \mathbb{H}^2_{\#}(\mathscr{S}),$$

$(h_0, v^+)_{\Gamma_2}$ and $(f_0, v)_{\mathscr{C}}$ from equality (4.3) are the correctly defined continuous functionals, since $h_0 \in \mathbb{H}^{-3/2}(\Gamma_2)$, $f_0 \in \widetilde{\mathbb{H}}^{-2}(\mathscr{C})$, while their counterparts in the functional belong to the dual spaces $v^+ \in \widetilde{\mathbb{H}}^{3/2}(\Gamma_2)$ and $v \in \widetilde{\mathbb{H}}^2(\Gamma, \mathscr{C}) \subset \mathbb{H}^2(\mathscr{C})$.

The Lax–Milgram Lemma 2.2 accomplishes the proof. □

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