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SURVEY ON QUALITATIVE THEORY OF DYNAMIC EQUATIONS ON TIME SCALE

Abstract. The qualitative theory of dynamic equation on time scale has received lots of attention from researchers due to its applicability to various fields. One of the most important aspects is that it unifies the theory of difference equations, differential equations, quantum calculus, and several others. In this article, we discuss the development of this theory along with the main emphasis on a second order dynamic equation. The qualitative properties, especially development in the oscillation theory of such equations, are discussed in detail. Several examples are provided for a better understanding of this topic.

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1 Introduction

The theory of time scale or more general measure chain has been introduced by Stefan Hilger (see [41]. The main aim is to unify the theory of discrete and continuous calculus. Since a time scale is any closed subset of the real line, it has been widely applied to the qualitative analysis of the dynamic equations and their related applications (see [15, 47, 62, 63]).

Now, we present some basic definitions, useful theorems and basic facts of time scales.

Definition 1.1 ([15])**.** For $t \in \mathbb{T}$, the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\} \ \text{ and } \ \rho(t) := \sup\{s \in \mathbb{T} : \ s < t\},$$

respectively.

The classification of points of time scale $\mathbb{T}$. For $t \in \mathbb{T}$, t is called the right-scattered point if $t < \sigma(t)$ and the right dense point if $t = \sigma(t)$ for $t < \sup \mathbb{T}$. Similarly, t is called the left-scattered point if $t > \rho(t)$ and the left dense point if $t = \rho(t)$ for $t > \inf \mathbb{T}$.

Remark 1.1. We put $\inf \varnothing = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t), $\sup \varnothing = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\varnothing$ is an empty set.

Definition 1.2 ([15])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limit exists (finite) at all left-dense points in $\mathbb{T}$, which is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

We define $\mathbb{T}^\kappa = \mathbb{T} - \{\xi\}$, if $\mathbb{T}$ has a left-scattered maximum ξ, and $\mathbb{T}^\kappa = \mathbb{T}$, otherwise.

Definition 1.3 ([15])**.** For a function $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define $f^\Delta(t)$, to be a number (provided it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood $\mathcal{Z} = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$ such that

$$\big|[f(\sigma(t)) - f(r)] - f^\Delta(t)[\sigma(t) - r]\big| \le \varepsilon|\sigma(t) - r|, \ \ \forall\, r \in \mathcal{Z}.$$

Thus, we call $f^\Delta(t)$ the Δ or Hilger derivative of f at t.

Theorem 1.1 ([15])**.** *For the functions $g, f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the following statements are true:*

1. *If f is differentiable at t, then f is continuous at t;*
2. *If f is continuous at t and t is right-scattered, then f is Δ-derivative at t and*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)};$$

3. *If t is right-dense, then f is differentiable at t iff*

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists and is of finite value;

4. *If f is differentiable at t, then*

$$f^\sigma = f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t);$$

5. *If f and g both are differentiable at t, then a product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t and*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)),$$

hence, for $t \in \mathbb{T}$ such that $a \leq t \leq b$, $\forall\, a, b \in \mathbb{T}$, we have the following facts:

$$\int_a^b f^{\sigma}(s) g^{\Delta}(s)\, \Delta s = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(s) g(s)\, \Delta s, \tag{1.1}$$

$$\int_a^b f(s) g^{\Delta}(s)\, \Delta s = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(s) g^{\sigma}(s)\, \Delta s; \tag{1.2}$$

6. *If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f(t)}{g(t)}$ is differentiable at t and*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Definition 1.4 ([15])**.** A function $w : \mathbb{T} \to \mathbb{R}$ is regressive if $1 + \mu(t)q(t) \neq 0$, $\forall\, t \in \mathbb{T}$. Denote the collection of all rd-continuous functions $w : \mathbb{T} \to \mathbb{R}$ by $\mathcal{R}$, and

$$\mathcal{R}^{+} = \left\{ w \in \mathcal{R} :\ 1 + \mu(t)w(t) > 0 \text{ for all } t \in \mathbb{T} \right\}.$$

Definition 1.5 ([15])**.** A function $F : \mathbb{T} \to \mathbb{R}$ is called an anti-derivative of $f : \mathbb{T} \to \mathbb{R}$, provided $F^{\Delta}(t) = f(t)$, $\forall\, t \in \mathbb{T}$. Then $\forall\, a, b \in \mathbb{T}$ such that $a \leq b$, Cauchy integral is defined by

$$\int_a^b f(s)\, \Delta s = F(b) - F(a). \tag{1.3}$$

Definition 1.6 ([43])**.** Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ and $|q| < 1$, we have the relation $f^{\Delta}(t) = D_q f(t)$, where

$$D_q f(t) = \begin{cases} \dfrac{f(qt) - f(t)}{t(q-1)}, & t \neq 0, \\ \lim\limits_{n \to \infty} \dfrac{f(q^n) - f(0)}{q^n}, & t = 0, \end{cases}$$

is the q-difference operator.

Remark 1.2. In Definition 1.5, equation (1.3) does not hold for all time scales, for example, in q-calculus (i.e., $\mathbb{T} = \overline{q^{\mathbb{Z}}}$) the following relation is not always true:

$$\int_a^b D_q f(t)\, d_q t = f(b) - f(a)$$

(for more details see [14, p. 12]).

Definition 1.7. If $w \in \mathcal{R}$, then we define an exponential function by

$$e_w(t, s) = \exp\left(\int_s^t \eta_{\mu(\tau)}(w(\tau))\, \Delta\tau \right), \ \ \forall\, t \in \mathbb{T}, \ \ s \in \mathbb{T}^{\kappa},$$

where $\eta_h(z)$ is the cylinder transformation, which is defined by

$$\eta_h(z) = \begin{cases} \dfrac{\log(1 + hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

Theorem 1.2. *Let $\mathbb{T}$ be time scales with $a, \tau \in \mathbb{T}$, and p, q be the positive real numbers such that $p \leq 1$, $p+q > 1$ and let r, s be the non-negative rd-continuous functions on $[a,\tau]_{\mathbb{T}}$ such that*

$$\int_a^\tau \frac{1}{(r(t))^{1/(p+q-1)}}\,\Delta t < \infty.$$

If $y : [a,\tau]_{\mathbb{T}} \to \mathbb{R}^+$ is delta derivative with $y(a) = 0$, then

$$\int_a^\tau s(x)|y(x) + y^\sigma(t)|^p |y^\Delta(x)|^q\,\Delta x \leq L(a,\tau,p,q) \int_a^\tau r(x)|y^\Delta(x)|^{p+q}\,\Delta x, \tag{1.4}$$

where

$$\begin{aligned} L(a,\tau,p,q) &= \sup_{a\leq x\leq\tau}\Big(\mu^p(x)\,\frac{s(x)}{r(x)}\Big) \\ &\quad + 2^p\Big(\frac{q}{p+q}\Big)^{q/(p+q)}\Bigg[\int_a^\tau \frac{(s(x))^{(p+q)/p}}{(r(x))^{q/p}}\bigg(\int_a^x \frac{1}{(r(t))^{1/(p+q-1)}}\,\Delta t\bigg)^{p+q-1}\Delta x\Bigg]^{p/(p+q)}. \end{aligned} \tag{1.5}$$

Lemma 1.1 (Young's inequality [42])**.** *Let $A, B \geq 0$, $\xi > 1$ and $\frac{1}{\xi} + \frac{1}{\eta} = 1$, then $AB \leq \frac{A^\xi}{\xi} + \frac{B^\eta}{\eta}$, where the equality holds iff $B = A^{\xi-1}$.*

Definition 1.8 ([13])**.** A function g belongs to the class of $\mathbb{G}$, denoted by $g \in \mathbb{G}$, if and only if $g \in C'(\mathbb{R},\mathbb{R})$ with $yg(y) > 0$ and $g'(y) > 0$ for all $y \neq 0$. For any positive constant k, the subset $\mathbb{G}_k$ of $\mathbb{G}$ is denoted by

$$\mathbb{G}_k = \big\{g \in \mathbb{G} :\ g'(y) \geq k \text{ for all } y \neq 0\big\}.$$

Remark 1.3. For the later convenience in the notation, for $g \in \mathbb{G}$, we set

$$g'_*(t) = \begin{cases} g'(x(t)), & x^\sigma(t) = x(t), \\ \dfrac{g(x^\sigma(t)) - g(x(t))}{x^\sigma(t) - x(t)}, & x^\sigma(t) \neq x(t). \end{cases}$$

2 Qualitative theory

Oscillation is one of the most important qualitative properties of the solution. So, we focus our attention on this theory. In [13], Anderson studied the oscillatory behaviour of the following second-order dynamic equation:

$$(r(t)x^\Delta(t))^\Delta + f\big(t, x^\sigma(t), x^\Delta(t)\big) = 0, \quad \forall\, t \in [t_0,\infty)_{\mathbb{T}}, \tag{2.1}$$

where $\mathbb{T}$ is a time scale unbounded above with $t_0 \in \mathbb{T}$, the function $r : \mathbb{T} \to (0,\infty)$ is right-dense continuous, and the function $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is right-dense continuous in the first variable. Here, we assume that the functions r and f are sufficiently smooth to ensure that the solution of equation (2.1) always has a solution that is continuable on $[t_0,\infty)_{\mathbb{T}}$. Also, in [13], the author illustrated the new understanding by extending some continuous results from differential equations to dynamic equations on arbitrary time scales (unbounded above), it includes the classical results of difference equations and q-difference equations. For instance, equation (2.1) is studied extensively by Wang [64] in the case $\mathbb{T} = \mathbb{R}$ (see also Guvenilir and Zafer [37]). Related discussions can be found in Nasr [48] and Wong [65], who study the oscillation of

$$(r(t)x'(t))' + p(t)|x(t)|^{\alpha-1}x(t) = f(t),$$

and an extension of this work to the equation

$$x''(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = f(t)$$

can be found in Sun [61]. In the related work [18], Cakmak and Tiryaki considered the oscillation for the forced equation

$$x''(t) + p(t)f(x(\tau(t))) = a(t)$$

(see also Li and Zhu [46], and Wong [66]).

Recently, there has also been a spate of papers on second-order nonlinear dynamic equations on time scales. For a sampling of the work done on second-order equations, one may consult the monograph by Agarwal, Grace, and O'Regan [5]. For a few examples of works since then, Bohner and Tisdell [16] examined oscillation and non-oscillation for

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)x^{\sigma}(t) = f(t).$$

Erbe, Peterson, and Saker [25] studied the unforced delay dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)x(\tau(t)) = 0,$$

and Saker [57] studied the oscillation of the related forced dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)f(x^{\sigma}(t)) = a(t).$$

In [9, 59], the authors considered the oscillation of the neutral delay dynamic equation

$$\Big(r(t)\Big(\big[x(t) + p(t)x(\tau(t))\big]^{\Delta}\Big)^{\gamma}\Big)^{\Delta} + f\big(t, x(t), x(\delta(t))\big) = 0,$$

while Agarwal, O'Regan and Saker [8] discussed the oscillatory behaviour of the nonlinear perturbed dynamic equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + f(t, x(t)) = g(t, x(t), x^{\Delta}(t)).$$

Oscillatory criteria for the forced dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) + q(t)|x(\theta(t))|^{\beta-1}x(\theta(t)) = f(t)$$

were analyzed in [12].

For $a, b \in [t_0, \infty)_{\mathbb{T}}$ with $a < b$, the admissible set

$$\mathbb{J}(a, b) = \Big\{u \in C^{\Delta}_{rd}([a, b]_{\mathbb{T}}) : \ \ u(a) = 0 = u(b), \ \ u \neq 0\Big\}.$$

Lemma 2.1. *If x is a solution of (2.1) such that the product $xx^{\sigma} > 0$ on some interval $[a, b]_{\mathbb{T}} \subset [t_0, \infty)_{\mathbb{T}}$, then we have the strict inequality*

$$\int_a^b \left[(u^{\sigma}(t))^2\, \frac{f(t, x^{\sigma}(t), x^{\Delta}(t))}{x^{\sigma}(t)} - r(t)(u^{\Delta}(t))^2\right] \Delta t < 0 \tag{2.2}$$

for any $u \in \mathbb{J}(a, b)$.

Proof. Define the Riccati substitution w via

$$w^{\Delta}(t) = \frac{r(t)x^{\Delta}(t)}{-x(t)}\,, \ \ t \in [t_0, \infty)_{\mathbb{T}}.$$

Using the delta quotient rule and the fact that x is a solution of (2.1), we have

$$w^{\Delta}(t) = \frac{x(t)}{r(t)x^{\sigma(t)}}\, w^2(t) + \frac{f(t, x^{\sigma}(t), x^{\Delta}(t))}{x^{\sigma}(t)}\,. \tag{2.3}$$

Multiply both sides of (2.3) by $(u^2)^\sigma$,

$$\int_a^b (u^2)^\sigma(t) w^\Delta(t)\,\Delta t$$
$$= \int_a^b \frac{x(t)w^2(t)}{r(t)x^\sigma(t)}\,(u^\sigma)^2(t)\,\Delta t + \int_a^b (u^\sigma)^2(t)\,\frac{f(t,x^\sigma(t),x^\Delta(t))}{x^\sigma(t)}\,\Delta t - \int_a^b (u^2)^\Delta(t)w(t)\,\Delta t$$
$$= \int_a^b \frac{x(t)w^2(t)}{r(t)x^\sigma(t)}\,(u^\sigma)^2(t)\,\Delta t + \int_a^b (u^\sigma)^2(t)\,\frac{f(t,x^\sigma(t),x^\Delta(t))}{x^\sigma(t)}\,\Delta t,$$

so that

$$\int_a^b (u^\sigma)^2(t)\,\frac{f(t,x^\sigma(t),x^\Delta(t))}{x^\sigma(t)}\,\Delta t = -\int_a^b [u(t)+u^\sigma(t)]u^\Delta(t)w(t)\,\Delta t - \int_a^b \frac{x(t)w^2(t)}{r(t)x^\sigma(t)}\,(u^\sigma)^2(t)\,\Delta t.$$

Note that, while suppressing the independent variable t,

$$[u+u^\sigma]u^\Delta w = 2u^\sigma u^\Delta w + u^\Delta w(u-u^\sigma) = 2u^\sigma u^\Delta w + \mu(u^\Delta)^2 r x^\Delta/x,$$

where we have used the simple formula $u^\sigma = u + \mu u^\Delta$ and the Riccati form for w. Thus we have

$$[u+u^\sigma]u^\Delta w = 2u^\sigma u^\Delta w + (u^\Delta)^2 r x^\sigma/x - r(u^\Delta)^2,$$

using the simple formula on x this time. Plugging these conclusions back into the integral equalities above, we obtain

$$\int_a^b (u^\sigma)^2(t)\,\frac{f(t,x^\sigma(t),x^\Delta(t))}{x^\sigma(t)}\,\Delta t$$
$$= -\int_a^b \Big[2u^\sigma(t)u^\Delta(t)w(t) + (u^\Delta)^2(t)r(t)x^\sigma(t)/x(t) - r(t)(u^\Delta)^2(t)\Big]\,\Delta t - \int_a^b \frac{x(t)w^2(t)}{r(t)x^\sigma(t)}\,(u^\sigma)^2(t)\,\Delta t$$
$$= -\int_a^b (u^\sigma)^2(t)\left[w(t)\sqrt{\frac{x(t)}{r(t)x^\sigma(t)}} + \frac{u^\Delta(t)}{u^\sigma(t)}\,\sqrt{\frac{r(t)x^\sigma(t)}{x(t)}}\right]^2 \Delta t + \int_a^b r(t)(u^\Delta)^2(t)\,\Delta t.$$

It follows that

$$\int_a^b \left[(u^\sigma)^2(t)\,\frac{f(t,x^\sigma(t),x^\Delta(t))}{x^\sigma(t)} - r(t)(u^\Delta)^2(t)\right]\Delta t \le 0.$$

If

$$\int_a^b (u^\sigma)^2(t)\left[w(t)\sqrt{\frac{x(t)}{r(t)x^\sigma(t)}} + \frac{u^\Delta(t)}{u^\sigma(t)}\,\sqrt{\frac{r(t)x^\sigma(t)}{x(t)}}\right]^2 \Delta t = 0,$$

then the integrand is zero, videlicet

$$0 = u^\sigma w\sqrt{\frac{x}{rx^\sigma}} + u^\Delta\sqrt{\frac{rx^\sigma}{x}} = \sqrt{rxx^\sigma}\left(\frac{u}{x}\right)^\Delta$$

on $[a,b]_{\mathbb{T}}$. Since $r, x, x^\sigma \ne 0$ on $[a,b]_{\mathbb{T}}$ by assumption, x must be a constant multiple of $u \in \mathbb{J}(a,b)$, this is a contradiction with the nonzero nature of x on $[a,b]_{\mathbb{T}}$. Thus, (2.2) holds. $\square$

Anderson has established some new interval criteria for oscillation of the dynamic equation (2.1) as follows.

Theorem 2.1. *If for some interval $[a,b] \subset [t_0,\infty)_{\mathbb{T}}$ there exists an admissible function $u \in \mathbb{J}(a,b)$ such that for any $x \in C^{\Delta}_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ with $xx^\sigma > 0$ on $[a,b]_{\mathbb{T}}$, we have*

$$\int_a^b \left[(u^\sigma(t))^2 \frac{f(t,x^\sigma(t),x^\Delta(t))}{x^\sigma(t)} - r(t)(u^\Delta(t))^2\right]\Delta t \geq 0, \tag{2.4}$$

where

$$\mathbb{J}(a,b) = \left\{u \in C^{\Delta}_{rd}([a,b]_{\mathbb{T}}):\ u(a) = 0 = u(b),\ u \neq 0\right\},$$

then every solution of equation (2.1) *has at least one generalized zero in* $[a,b]_{\mathbb{T}}$.

Proof. Suppose there exists a solution x of (2.1) such that $xx^\sigma > 0$ on $[a,b]_{\mathbb{T}}$. Then, by assumption, there exists $u \in \mathbb{J}(a,b)$ such that (2.4) holds for this particular solution x. From Lemma 2.1, though, we then have that (2.2) holds, a contradiction of (2.4). □

In 2015, Xun-Huan Deng et al. [21] investigated oscillation of second-order non-linear delay dynamic equation on time scales:

$$\left(r(t)|x^\Delta(t)|^{\gamma-1}x^\Delta(t)\right)^\Delta + p(t)f(x(\tau(t))) = 0, \quad \gamma > 0. \tag{2.5}$$

For their results, they imposed some conditions

(A_1) $p \in C_{rd}(\mathbb{T},(0,\infty))$, $r \in C_{rd}(\mathbb{T},(0,\infty))$ satisfies

$$\int_{t_0}^\infty \left(\frac{1}{r(s)}\right)^{1/\gamma}\Delta s = \infty;$$

(A_2) the delay function $\tau \in C_{rd}(\mathbb{T},\mathbb{T})$ satisfies $\tau(t) \leq t$ and $\lim_{t\to\infty}\tau(t) = \infty$;

(A_3) $f \in C(\mathbb{R},\mathbb{R})$ satisfies $f(y)/(|y|^{\gamma-1}y) \geq K > 0$ for $y \neq 0$, where K is a constant.

Along with the above conditions, they also introduced the auxiliary functions, for $a \in [t_0,\infty)_{\mathbb{T}}$

$$R(t,a) = \int_a^t \left(\frac{1}{r(s)}\right)^{1/\gamma}\Delta s, \quad \eta_1(t,a) = \frac{R(\tau(t),a)}{R(\sigma(t),a)}, \ \ \eta_2(t,a) = \frac{R(t,a)}{R(\sigma(t),a)}. \tag{2.6}$$

Theorem 2.2. *Assume that conditions (A_1)–(A_3) hold. Furthermore, suppose that there exist a function $a(t)$ and a positive Δ-differentiable function $\delta(t)$ such that for sufficiently large $T \in \mathbb{T}$,*

$$\limsup_{t\to\infty}\int_{t_1}^t \left[\psi(s) - \frac{\gamma^\gamma([\psi_1(s)]_+)^{1+\gamma}}{(1+\gamma)^{1+\gamma}(\psi_2(s))^\gamma}\right]\Delta s > \delta(t_1)\Big[\frac{1}{R^\gamma(t_1,T)} + r(t_1)a(t_1)\Big],$$

where $t_1 \in \mathbb{T}$ and $t_1 > T$, $R(t_1,T)$ is given in (2.6). *Then*

1. *every solution $x(t)$ of* (2.5) *is oscillatory for $\gamma \geq 1$;*

2. *every solution $x(t)$ of* (2.5) *oscillates for $0 < \gamma < 1$ and $a(t) = 0$.*

Theorem 2.3. *Assume that conditions (A_1)–(A_3) hold. Furthermore, suppose that there exist a function $a(t)$ and a positive Δ-differentiable function $\delta(t)$ such that for some $H_1 \in \mathcal{R}_1$ and sufficiently large $T \in \mathbb{T}$,*

$$\limsup_{t\to\infty} \frac{1}{H_1(\sigma(t),t_1)} \int_{t_1}^{t} \left[H_1(\sigma(t),\sigma(s))\psi(s) - \frac{\gamma^\gamma\left(\left[H_1(\sigma(t),\sigma(s))\psi_1(s) + H_1^{\Delta_s}(\sigma(t),s)\right]\right)^{1+\gamma}}{(1+\gamma)^{1+\gamma}((H_1(\sigma(t),\sigma(s)))\psi_2(s))^\gamma} \right] \Delta s$$
$$> \delta(t_1)\Big[\frac{1}{R^\gamma(t_1,T)} + r(t_1)a(t_1)\Big],$$

where $t_1 \in \mathbb{T}$ and $t_1 > T$, $R(t_1,T)$ is given in (2.6)*. Then*

1. *every solution $x(t)$ of* (2.5) *is oscillatory for $\gamma \geq 1$;*
2. *every solution $x(t)$ of* (2.5) *oscillates for $0 < \gamma < 1$ and $a(t) = 0$.*

Theorem 2.4. *Assume that conditions (A_1)–(A_3) hold. Furthermore, suppose that there exist a function $a(t)$ and a positive Δ-differentiable function $\delta(t)$ such that for some $H_2 \in \mathcal{R}_2$, it has a non-positive rd-continuous Δ-partial derivative $H_2^{\Delta_s}(t,s)$ with respect to the second variable and satisfies*

$$\limsup_{t\to\infty} \frac{1}{H_2(\sigma(t),t_1)} \int_{t_1}^{t} \left[H_2(\sigma(t),\sigma(s))\psi(s) - \frac{\gamma^\gamma\left(\left[H_2(\sigma(t),\sigma(s))\psi_1(s) + H_2^{\Delta_s}(\sigma(t),s)\right]\right)^{1+\gamma}}{(1+\gamma)^{1+\gamma}((H_2(\sigma(t),\sigma(s)))\psi_2(s))^\gamma} \right] \Delta s$$
$$> \delta(t_1)\Big[\frac{1}{R^\gamma(t_1,T)} + r(t_1)a(t_1)\Big],$$

where $T \in \mathbb{T}$ is a sufficiently large number, $t_1 \in \mathbb{T}$ and $t_1 > T, R(t_1,T)$ is given in (2.6)*. Then*

1. *every solution $x(t)$ of* (2.5) *is oscillatory for $\gamma \geq 1$;*
2. *every solution $x(t)$ of* (2.5) *oscillates for $0 < \gamma < 1$ and $a(t) = 0$.*

In [34], Graef et al. considered the following second-order functional dynamic equation on time scales and established a new oscillation criteria in which they used a Riccati transformation technique as follows:

$$\left(r(t)|x^\Delta(t)|^\gamma x^\Delta(t)\right)^\Delta + F\big(t, x(t), x(\tau(t)), x^\Delta(\tau(t))\big) = 0, \tag{2.7}$$

where $\gamma > 0$ is a constant, r is a positive real-valued rd-continuous function defined on $\mathbb{T}, F : \mathbb{T}\times\mathbb{R}^3 \to \mathbb{R}$ is a continuous function. Along with this, they introduced two cases

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} = \infty \tag{2.8}$$

and

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} < \infty.$$

Recently, there has been increasing interest in obtaining sufficient conditions for oscillation of the solutions of different classes of dynamic equations with or without deviating arguments on time scales. Some similar and particular kind of dynamic equations have been discussed for oscillation by many authors. For example, Agarwal et al. [3] considered the second-order linear delay dynamic equation

$$x^{\Delta\Delta}(t) + q(t)x(\tau(t)) = 0 \ \text{ for } \ t \in \mathbb{T}$$

and established some sufficient conditions for the oscillation. Sahiner [55] considered the second-order nonlinear delay dynamic equation

$$x^{\Delta\Delta}(t) + q(t)f(x(\tau(t))) = 0 \ \text{ for } \ t \in \mathbb{T}$$

and obtained some sufficient conditions for the oscillation by using a Riccati type transformation. Han et al. [38] extended the results in Agarwal et al. [3] to the second-order Emden–Fowler delay dynamic equation

$$x^{\Delta\Delta}(t) + q(t)x^{\gamma}(\tau(t)) = 0 \ \text{ for } \ t \in \mathbb{T},$$

where γ is a quotient of an odd positive integer. Erbe et al. [27] considered the second-order nonlinear delay dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + q(t)f(x(\tau(t))) = 0 \ \text{ for } \ t \in \mathbb{T}$$

and gave some oscillation results that improved the results established by Zhang and Shanliang [67] and Sahiner [55].

Han et al. [40] considered the second-order nonlinear delay dynamic equation

$$\big(r(t)(x^{\Delta}(t))^{\gamma}\big)^{\Delta} + q(t)f(x(\tau(t))) = 0 \ \text{ for } \ t \in \mathbb{T}$$

and established some oscillation results for $\gamma \geq 1$, being an odd positive integer, that improve and extend the results of Saker [56], [58] and Sahiner [55]. Also, in [20], Chen considered the second-order half-linear dynamic equation

$$\big(r(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t)\big)^{\Delta} + q(t)|x(t)|^{\gamma-1}x(t) = 0 \ \text{ for } \ t \in \mathbb{T}$$

and obtained some sufficient conditions for the oscillation that improve and extend the results of Saker [58], Agarwal et al. [3] and Hassan [39].

Lemma 2.2 (Mean Value Theorem on time-scale [36])**.** *If f is a continuous function on $[a,b]_{\mathbb{T}}$ and is Δ-differentiable on $[a,b)_{\mathbb{T}}$, then there exist $\xi, \eta \in [a,b]_{\mathbb{T}}$ such that*

$$f^{\Delta}(\eta)(b-a) \leq f(b) - f(a) \leq f^{\Delta}(\xi)(b-a).$$

In what follows, let $\tau_*(t) := \min\{t, \tau(t)\}$.

Lemma 2.3. *Suppose that the following conditions are satisfied:*

$(C1)$ $u \in C^2_{rd}(I,\mathbb{R})$, *where* $I = [T,\infty) \subset \mathbb{T}$ *for some* $T > 0$*;*

$(C2)$ $u(t) > 0, u^{\Delta}(t) > 0$, *and* $u^{\Delta\Delta}(t) \leq 0$ *for* $t \geq T$*.*

Then, for each $0 < k < 1$, there is $T_k \geq T$ such that

$$u(\tau(t) \geq ku(t)\,\frac{\tau_*(t)}{t} \ \text{ for } \ t \geq T_k.$$

Lemma 2.4. *Assume that* (2.8) *holds,* $r^{\Delta}(t) \geq 0$,

$$\operatorname{sgn} F(t,x,u,v,w) = \operatorname{sgn} x \ \text{ for } \ t \in [t_0,\infty)_{\mathbb{T}} \ \text{ and } \ x,u,v,w \in \mathbb{R},$$

and x is an eventually positive solution of (2.7)*. Then there exists $T \geq t_0$ such that*

$$x^{\Delta}(t) > 0, \ \ x^{\Delta\Delta}(t) < 0 \ \text{ and } \ \big(r(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t)\big)^{\Delta} < 0.$$

Theorem 2.5. *In addition to condition* (2.8)*, assume there are positive functions $\delta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $p \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that*

$$F(t,x,u,v,w)/|x|^{\gamma-1}x \geq p(t) \tag{2.9}$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$, $x \in \mathbb{R}\setminus\{0\}$, and $u,v,w \in \mathbb{R}$, and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left\{ \delta(s)p(s) - \frac{r(s)[(\delta^{\Delta}(s))_+]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)} \right\} \Delta s = \infty. \tag{2.10}$$

Then every solution of equation (2.7) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose, to the contrary, that equation (2.7) has a non-oscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$, say $x(t)>0$ and $x(\tau(t))>0$ on $[t_1,\infty)_{\mathbb{T}}$ for some $t_1\in[t_0,\infty)_{\mathbb{T}}$. From (2.7) and (2.9), we have

$$\big(r(t)|x^{\Delta}(t)|^{\gamma}x^{\Delta}(t)\big)^{\Delta}\le -p(t)x^{\gamma}(t)<0 \tag{2.11}$$

for all $t\ge t_1$, and so $r(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t)$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. As in Lemma 2.4, we should have

$$x^{\Delta}(t)>0 \ \text{ for } \ t\in[t_2,\infty)_{\mathbb{T}} \tag{2.12}$$

for some $t_2\ge t_1$. In view of equations (2.11) and (2.12), we see that

$$\big(r(t)(x^{\Delta}(t))^{\gamma}\big)^{\Delta}\le -p(t)x^{\gamma}(t)<0 \ \text{ for } \ t\in[t_2,\infty)_{\mathbb{T}}.$$

Now, consider the generalized Riccati substitution

$$w(t)=\delta(t)\,\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)} \ \text{ for } \ t\ge t_2. \tag{2.13}$$

Clearly, $w(t)>0$, and

$$\begin{aligned} w^{\delta}(t)&=\big(r(t)(x^{\Delta}(t))^{\gamma}\big)^{\Delta}\,\frac{\delta(t)}{x^{\gamma}(t)}+\big(r(t)(x^{\Delta}(t))^{\gamma}\big)^{\sigma}\Big(\frac{\delta(t)}{x^{\gamma}(t)}\Big)^{\Delta}\\ &\le -\delta(t)p(t)+\big(r(t)(x^{\Delta}(t))^{\gamma}\big)^{\sigma}\left(\frac{\delta^{\Delta}(t)}{x^{\gamma}(\sigma(t))}-\frac{\delta(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\right)\\ &=-\delta(t)p(t)+\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\,w^{\sigma}(t)-\delta(t)\,\frac{(r(t)(x^{\delta}(t))^{\sigma})^{\sigma}(x^{\gamma}(t))^{\delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\\ &=-\delta(t)p(t)+\frac{(\delta^{\Delta}(t))_{+}}{\delta^{\sigma}(t)}\,w^{\sigma}(t)-\delta(t)\,\frac{(r(t)(x^{\delta}(t))^{\sigma})^{\sigma}(x^{\gamma}(t))^{\delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\,. \end{aligned} \tag{2.14}$$

From (2.12), we have

$$(x^{\gamma}(t))^{\Delta}=\gamma\Big\{\int_0^1\big[(1-h)x(t)+hx^{\sigma}(t)\big]^{\gamma-1}\,dh\Big\}x^{\Delta}(t)\ge\begin{cases}\gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t), & 0<\gamma\le 1,\\ \gamma(x(t))^{\gamma-1}x^{\Delta}(t), & \gamma>1.\end{cases} \tag{2.15}$$

If $0<\gamma\le 1$, then (2.14) and (2.15) imply

$$\begin{aligned} w^{\Delta}(t)&\le -\delta(t)p(t)+\frac{(\delta(t))_{+}}{\delta^{\sigma}(t)}\,w^{\sigma}(t)-\delta(t)\,\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}\gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\\ &=-\delta(t)p(t)+\frac{(\delta^{\Delta}(t))_{+}}{\delta^{\sigma}(t)}\,w^{\sigma}(t)-\gamma\delta(t)\,\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}}{x^{\gamma+1}(\sigma(t))}\,\frac{x^{\gamma}(\sigma(t))}{x^{\gamma}(t)}\,x^{\Delta}(t). \end{aligned} \tag{2.16}$$

If $\gamma>1$, (2.14) and (2.15) imply

$$\begin{aligned} w^{\Delta}(t)&\le -\delta(t)p(t)+\frac{(\delta(t))_{+}}{\delta^{\sigma}(t)}\,w^{\sigma}(t)-\delta(t)\,\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}\gamma(x(t))^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\\ &=-\delta(t)p(t)+\frac{(\delta^{\Delta}(t))_{+}}{\delta^{\sigma}(t)}\,w^{\sigma}(t)-\gamma\delta(t)\,\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}}{x^{\gamma+1}(\sigma(t))}\,\frac{x^{\gamma}(\sigma(t))}{x^{\gamma}(t)}\,x^{\Delta}(t). \end{aligned} \tag{2.17}$$

Since $t\le\sigma(t)$ and $x(t)$ is increasing on $[t_2,\infty)_{\mathbb{T}}$, we have $x(t)\le x(\sigma(t))$. Therefore, (2.16) and (2.17) yield

$$w^{\Delta}(t)\le -\delta(t)p(t)+\frac{(\delta^{\Delta}(t))_{+}}{\delta^{\sigma}(t)}\,w^{\sigma}(t)-\gamma\delta(t)\,\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}}{x^{\gamma+1}(\sigma(t))}\,x^{\Delta}(t) \tag{2.18}$$

on $[t_2,\infty)_{\mathbb{T}}$ for $\gamma>0$.

Since $r(t)(x^{\Delta}(t))^{\gamma}$ is decreasing, we have

$$r(t)(x^{\Delta}(t))^{\gamma} \geq \big(r(t)(x^{\Delta}(t))^{\gamma}\big)^{\sigma},$$

so,

$$x^{\Delta}(t) \geq \frac{[(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}]}{r^{1/\gamma}(t)}\,. \tag{2.19}$$

Using (2.19) and (2.18), we obtain

$$w^{\Delta}(t) \leq -\delta(t)p(t) + \frac{(\delta^{\Delta}(t))_{+}}{\delta^{\sigma}(t)}\,w^{\sigma}(t) - \gamma\delta(t)r^{-1/\gamma}(t)\,\frac{[(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}]^{(\gamma+1)/\gamma}}{x^{\gamma+1}(\sigma(t))}\,. \tag{2.20}$$

From (2.13) and (2.20), we conclude that

$$w^{\Delta}(t) \leq -\delta(t)p(t) + \frac{(\delta^{\Delta}(t))_{+}}{\delta^{\sigma}(t)}\,w^{\sigma}(t) - \gamma\delta(t)r^{-1/\gamma}(t)(\delta^{\sigma}(t))^{-(\gamma+1)/\gamma}(w^{\sigma}(t))^{(\gamma+1)/\gamma}.$$

Letting

$$X = \frac{(\gamma\delta(t))^{\gamma/(\gamma+1)}w^{\sigma}(t)}{r^{1/(\gamma+1)}(t)\delta^{\sigma}(t)}\,,\quad \lambda = \frac{\gamma+1}{\gamma}\,,$$

and

$$Y = \frac{r^{\gamma/(\gamma+1)}(t)((\delta^{\Delta}(t)))_{+}}{\lambda^{\gamma}(\gamma\delta(t))^{\gamma/\lambda}}\,, \tag{2.21}$$

it is easy to have

$$w^{\Delta}(t) \leq -\delta(t)p(t) + \frac{r(t)[(\delta^{\Delta}(t))_{+}]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(t)} \;\text{ for }\; t \in [t_2,\infty)_{\mathbb{T}}.$$

Now, integrating (2.21) from t_2 to t, we obtain

$$\int_{t_2}^{t} \left\{\delta(s)p(s) - \frac{r(s)[(\delta^{\Delta}(s))_{+}]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right\}\Delta s \leq -w(t) + w(t_2) \leq w(t_2),$$

which contradicts condition (2.10). Therefore, equation (2.7) is oscillatory. □

For the next Theorem, first we need to consider a condition

$$\operatorname{sgn} F(t,x,u,v,w) = \operatorname{sgn} x \;\text{ for }\; t \in [t_0,\infty)_{\mathbb{T}} \;\text{ and }\; x,u,v,w \in \mathbb{R}. \tag{2.22}$$

Theorem 2.6. *Assume that conditions* (2.7) *and* (2.22) *hold,* $r^{\Delta}(t) \geq 0$*, and there are positive functions* $\delta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *and* $p \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *and a constant* $k \in (0,1)$ *such that*

$$F(t,x,u,v,w)/|u|^{\gamma-1}u \geq p(t)$$

for $t \in [t_0,\infty)_{\mathbb{T}}, x,u \in \mathbb{R}\setminus\{0\}$*, and* $v,w \in \mathbb{R}$*. If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left\{\delta(s)p(s)\Big[\frac{k\tau_*(s)}{s}\Big] - \frac{r(s)[(\delta^{\Delta}(s))_{+}]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right\}\Delta s = \infty,$$

then every solution of equation (2.7) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$*.*

Proof. Suppose that equation (2.7) has a non-oscillatory solution (t), say $x(t) > 0$ and $x(\tau(t)) > 0$ on $[t_1,\infty)_{\mathbb{T}}$ for some $t_1 \in [t_0,\infty)_{\mathbb{T}}$. From Lemma 2.3, there exists $t_2 \geq t_1$ such that

$$x(\tau(t)) \geq \frac{k\tau_*(t)}{t}x(t) \;\text{ for all }\; t \geq t_2.$$

Defining $w(t)$ as in the proof of the above theorem, we have

$$\begin{aligned}
w^{\Delta}(t) &= \big(r(t)(x^{\Delta}(t))^{\gamma}\big)^{\Delta}\,\frac{\delta(t)}{x^{\gamma}(t)} + \big(r(t)(x^{\Delta}(t))^{\gamma}\big)^{\sigma}\Big(\frac{\delta(t)}{x^{\gamma}(t)}\Big)^{\Delta} \\
&= -\delta(t)\,\frac{F(t,x(t)x(\tau(t)),x^{\Delta}(t),x^{\Delta}(\tau(t)))}{x^{\gamma}(t)} + \big(r(t)(x^{\Delta}(t))^{\gamma}\big)^{\sigma}\Big(\frac{\delta^{\Delta}(t)}{x^{\gamma}(\sigma(t))} - \frac{\delta(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)}\Big) \\
&= -\delta(t)\,\frac{F(t,x(t)x(\tau(t)),x^{\Delta}(t),x^{\Delta}(\tau(t)))(x(\tau(t)))^{\gamma}}{(x(\tau(t)))^{\gamma}} \\
&\qquad + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\,w^{\sigma}(t) - \delta(t)\,\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))} \\
&\le -\delta(t)p(t)\,\frac{(x(\tau(t)))^{\gamma}}{x^{\gamma}(t)} + \frac{(\delta^{\Delta}(t))_{+}}{\delta^{\sigma}(t)}\,w^{\sigma}(t) - \delta(t)\,\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\,. \qquad (2.23)
\end{aligned}$$

In view of the above theorem, we conclude from (2.23) that

$$w^{\Delta}(t) \le -\delta(t)p(t)\Big[\frac{k\tau_{*}(t)}{t}\Big]^{\gamma} + \frac{(\delta^{\Delta}(t))_{+}}{\delta^{\sigma}(t)} - \gamma\delta(t)r^{-1/\gamma}(t)(\delta^{\sigma}(t))^{-(\gamma+1)/\gamma}(w^{\sigma}(t))^{(\gamma+1)/\gamma}.$$

The remainder of the proof is similar to that of the above theorem and it is omitted here. □

In [2], Agarwal et al. discussed the oscillatory behaviour of the following second-order dynamic equations on time scale:

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)f(x(\tau(t))) = 0 \qquad (2.24)$$

and

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(t) = 0. \qquad (2.25)$$

In view of the paper of Xun-Huan Deng et al. [21] (see the above), we have the conditions (A_2) and (A_3). Along with this, we introduce one more condition

(A_4): r and p are the positive real-valued rd-continuous functions defined on $[t_0,\infty)_{\mathbb{T}} = [t_0,\infty)\cap\mathbb{T}$.

Along with this, we assume

$$r^{\Delta}(t) \ge 0,\quad \int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty \ \text{ and } \int_{t_0}^{\infty} p(t)\tau(t)\,\Delta t = \infty, \qquad (2.26)$$

and

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} = \infty. \qquad (2.27)$$

Theorem 2.7. *Assume that* (A_2)–(A_4) *and* (2.26) *are satisfied, and let*

$$\sigma(t) > t \ \text{ and } \ \rho(t) < t \ \text{ for all } \ t \in [t_0,\infty)_{\infty}.$$

Assume further that there exist a nonnegative function η and a positive, differentiable function δ such that for some $H \in \mathcal{W}$ and for sufficiently large t_1,

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_1)} \sum_{s=t_1}^{\rho(\rho(t))} \mu(s)H(\sigma(t),\sigma(s))\Big[\delta^{\sigma}(s)\phi(s) - \frac{r(s)\delta^{2}(s)A^{2}(t,s)}{4C(s)\delta^{\sigma}(s)}\Big] = \infty,$$

where ψ, A and C are given as follows:

$$\psi(t) = \frac{Kp(t)\tau(t)}{\sigma(t)}\,,\quad C(t) = \frac{t}{\sigma(t)}$$

and

$$A(t,s) = \frac{\delta^{\sigma}(s)C_1(s)}{\delta(s)} + \frac{H^{\Delta_s}(\sigma(t),s)}{H(\sigma(t),\sigma(s))}\,,\quad C_1(s) = \frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)} + 2\,\frac{s\eta(s)}{\sigma(s)}\,.$$

Then (2.24) *is oscillatory.*

Theorem 2.8. *Assume that* (A_4) *and* (2.27) *are satisfied. Let* $H \in \mathcal{R}$ *be such that* H *has a non-positive rd-continuous* Δ*-partial derivative* $H^{\Delta_s}(t,s)$ *with respect to the second variable and satisfies*

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_0)} \int_{t_0}^{t} \Bigg[H(\sigma(t),\sigma(s)) a^{\gamma}(s)\delta^{\sigma}(s)p(s) - \frac{r(s)\left[(H(\sigma(t),\sigma(s))\frac{\delta^{\Delta}(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t),s))_+\right]^{\gamma+1}\delta^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}((\beta(s)\delta^{\sigma}(s))H(\sigma(t),\sigma(s)))^{\gamma}} \Bigg] \Delta s = \infty,$$

where α, β *and* δ *are given as follows:*

1. $\alpha(t) = \dfrac{R(t)}{R(t)+\mu(t)}$,
2. $\beta(t) = \begin{cases} \alpha(t), & 0<\gamma\le 1, \\ \alpha^{\gamma}(t), & \gamma>1, \end{cases}$
3. δ *is a positive* Δ*-differentiable function.*

Moreover,

$$\left(H(\sigma(t),\sigma(s))\frac{\delta^{\Delta}(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t),s)\right)_+ := \max\left\{0, H(\sigma(t),\sigma(s))\frac{\delta^{\Delta}(s)}{\delta(s)} + H^{\Delta_s}(\sigma(t),s)\right\}.$$

Then (2.25) *is oscillatory.*

In [10], Agarwal et al. dealt with the oscillation of the second order mixed nonlinear neutral dynamic equation with a negative neutral term on time scales:

$$\left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\Delta} + f(t,x(\tau_1(t))) + g(t,x(\tau_2(t))) = 0, \tag{2.28}$$

where

$$z(t) = x(t) - p_1(t)x(\eta_1(t)) + p_2(t)x(\eta_2(t)),$$

subject to the following hypothesis:

(H_1) $\mathbb{T}$ is a time scale unbounded above and $t_0 \in \mathbb{T}$ with $t_0 > 0$. We define the time scale interval $[t_0,\infty)_{\mathbb{T}} = [t_0,\infty)\cap\mathbb{T}$.

(H_2) η_1, τ_1 and $\tau_2 : \mathbb{T}\to\mathbb{T}$ are the rd-continuous functions such that $\eta_1(t)\le t$, $\tau_1(t)\le t$, $\tau_2(t)\ge t$, $\lim_{t\to\infty}\tau_1(t) = \infty = \lim_{t\to\infty}\eta_1(t)$ and $\eta_2 : \mathbb{T}\to\mathbb{T}$ is an injective rd-continuous increasing function such that $\eta_2(t)\ge t$.

(H_3) p_1 and p_2 are the non-negative rd-continuous functions on an arbitrary time scale $\mathbb{T}$, where

$$0 \le p_1(t) \le p_1 < 1 \quad\text{and}\quad 0\le p_2(t)\le p_2.$$

(H_4) r is a positive rd-continuous function such that

$$\int_{t_0}^{\infty} \frac{\Delta s}{r^{1/\gamma}(s)} = \infty.$$

(H_5) $f, g \in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ such that

$$uf(t,u)\ge 0,\quad ug(t,u)\ge 0,\quad f(t,u)\ge q_1(t)u^{\alpha} \;\text{ and }\; g(t,u)\ge q_2(t)u^{\beta} \;\text{ for }\; u\ne 0,$$

where q_1 and q_2 are the non-negative rd-continuous functions on an arbitrary time scale $\mathbb{T}$, α and β are the quotients of odd positive integers.

(H_6) γ is a quotient of odd positive integers. Throughout this paper we assume that

$$d_+(t) = \max\{0, d(t)\}, \quad d_-(t) = \max\{0, -d(t)\},$$

$$A(t) = \begin{cases} b_0^{\alpha-\beta}, & \alpha \geq \beta, \\ b_1^{\alpha-\beta}\Bigg[\displaystyle\int_{t_1}^{t} \frac{\Delta s}{r^{1/\gamma}(s)}\Bigg]^{\alpha-\beta}, & \alpha < \beta, \end{cases} \qquad C(t) = \begin{cases} b_0^{\frac{\beta}{\gamma}-1}, & \dfrac{\beta}{\gamma} \geq 1, \\ b_1^{\frac{\beta}{\gamma}-1}\Bigg[\displaystyle\int_{t_1}^{\sigma(t)} \frac{\Delta s}{r^{1/\gamma}(s)}\Bigg]^{\frac{\beta}{\gamma}-1}, & \alpha < \beta, \end{cases}$$

where b_0 and b_1 are positive constants.

Theorem 2.9. *Assume that (H_1)–(H_6) hold, $\tau_2(t) \geq \eta_2(t)$ for all $t \geq t_0$ and there exist positive real-valued Δ-differentiable functions $R(t)$ and $\delta(t)$ such that for sufficiently large T and t_1, we have*

$$\frac{R(t)}{r^{1/\gamma}(t)\int_{t_1}^{t} \frac{1}{r^{1/\gamma}(s)}\,\Delta s} - R^{\Delta}(t) \leq 0 \tag{2.29}$$

and

$$\limsup_{t\to\infty} \int_T^t \left[\delta(s)\xi(s)\big[q_1(s)L^{\alpha}(s)A(s) + q_2(s)\big] - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}} \frac{r(s)(\delta_+^{\Delta}(s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}\right] \Delta s = \infty,$$

and

$$L(s) = \min\left\{\frac{R(\tau_1(t))}{R(t)}, \frac{R(\eta_2^{-1}(\tau_1(t)))}{R(t)}\right\},$$

$$\xi(t) = \min\left\{\frac{1}{(1+p_2(\tau_1(t)))^{\alpha}}, \frac{1}{(1+p_2(\tau_2(t)))^{\beta}}, \frac{1}{(1+p_2(\eta_2^{-1}\tau_1(t)))^{\alpha}}, \frac{1}{(1+p_2(\eta_2^{-1}\tau_2(t)))^{\alpha}}\right\}.$$

Then, every solution of (2.28) *is almost oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or converges to zero as $t \to \infty$.*

Theorem 2.10. *Assume that (H_1)–(H_6) hold and $\tau_2(t) \geq \eta_2(t)$ for all $t \geq t_0$. Furthermore, suppose that there exist positive real-valued Δ-differentiable functions $R(t)$ and $\delta(t)$ such that equation* (2.29) *is satisfied and for sufficiently large T, we have*

$$\limsup_{t\to\infty} \int_T^t \left[\delta(s)\xi(s)\big[q_1(s)L^{\alpha}(s)A(s) + q_2(s)v^{\beta}(s)\big] - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}} \frac{r(s)(\delta_+^{\Delta}(s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}\right] \Delta s = \infty,$$

where

$$L(s) = \min\left\{\frac{R(\tau_1(t))}{R(t)}, \frac{R(\eta_2^{-1}(\tau_1(t)))}{R(t)}\right\} \text{ and } v(s) = \min\left\{1, \frac{R(\eta_2^{-1}(s))}{R(s)}\right\}.$$

Then every solution of (2.28) *is almost oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or converges to zero as $t \to \infty$.*

Theorem 2.11. *Assume that (H_1)–(H_6) and* (2.29) *hold, $\tau_2(t) \geq \eta_2(t)$ for all $t \geq t_0$ and there exist the functions H, h such that for each fixed t, $H(t,s)$ and $h(t,s)$ are rd-continuous with respect to s on*

$$\mathbb{D} = \big\{(t,s):\ t \geq s \geq t_0\big\}$$

such that

$$H(t,t) = 0, \ \ t \geq t_0, \quad H(t,s) > 0, \ \ t > s \geq t_0, \tag{2.30}$$

and H has a non-positive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ satisfying

$$H^{\Delta_s}(t,s) + H(t,s)\,\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)} = -\frac{h(t,s)}{\delta^{\sigma}(t)}\,(H(t,s))^{\frac{\gamma}{\gamma+1}}. \tag{2.31}$$

Assume that there exists a positive real-valued Δ-differentiable function $\delta(t)$ such that for sufficiently large $T \geq t_1 > t_0$, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Big[\delta(s)\xi(s)H(t,s)\big[q_1(s)L^\alpha(s)A(s)+q_2(s)\big] - \frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}} \frac{r(s)(h_-(s,t))^{\gamma+1}}{\delta^\gamma(s)C^\gamma(s)}\Big]\Delta s = \infty.$$

Then every solution of (2.28) *is almost oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or convergences to zero as* $t\to\infty$.

Theorem 2.12. *Assume that* (H_1)–(H_6) *hold,* $\eta_2(t) \geq \tau_2(t)$ *for all* $t \geq t_0$. *Also, assume that there exist the functions* H, h *and* δ *defined as in Theorem 2.11 and satisfying equations* (2.30), (2.31) *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Big[\delta(s)\xi(s)H(t,s)\big[q_1(s)L^\alpha(s)A(s)+q_2(s)v^\beta(t)\big] - \frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}} \frac{r(s)(h_-(s,t))^{\gamma+1}}{\delta^\gamma(s)C^\gamma(s)}\Big]\Delta s = \infty.$$

Then every solution of (2.28) *is almost oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or convergences to zero as* $t\to\infty$.

In [54], Qiu et al. discussed new oscillation criteria of the following second order dynamic equation:

$$\big(r(t)(x^\Delta(t))^\gamma\big)^\Delta + p(t)(x^\Delta(t))^\gamma + f(t,x(g(t))) = 0 \tag{2.32}$$

on a time scale $\mathbb{T}$ satisfying $\inf\mathbb{T} = t_0$ and $\sup\mathbb{T} = \infty$. They also assumed that

(C_1) $r \in C_{rd}(\mathbb{T},(0,\infty))$;

(C_2) $p \in C_{rd}(\mathbb{T},\mathbb{R}_4)$, where $\mathbb{R}_4 = [0,\infty)_{\mathbb{T}}$;

(C_3) γ is a quotient of odd positive integer;

(C_4) $g \in C(\mathbb{T},\mathbb{T})$ is non-decreasing and $g(t) \geq t$ for $t \in \mathbb{T}$;

(C_5) $f \in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ and there exists a function $q \in C_{rd}(\mathbb{T},\mathbb{R}_+)$ such that

$$uf(t,u) \geq q(t)u^{\gamma+1};$$

(C_6) $-p(t)/r(t)$ is positively regressive, which means $1-\mu(t)p(t)/r(t) > 0$ and

$$\int_{t_0}^\infty \Big(\frac{e_{-p/r}(t,t_0)}{r(t)}\Big)^{1/\gamma}\Delta t = \infty.$$

Let

$$\mathbb{D}_0 = \big\{s\in\mathbb{T}:\ s\geq 0\big\} \text{ and } \mathbb{D} = \big\{(t,s)\in\mathbb{T}^2 : t\geq s\geq 0\big\}.$$

For any function $f(t,s):\ \mathbb{T}^2\to\mathbb{R}$, we denote the partial derivative of f with respect to s by f_2^Δ. Define

$$(\mathcal{A},\mathcal{B}) = \Big\{(A,B):\ A(s)\in C^1_{rd}(\mathbb{D}_0,\mathbb{R}_0\setminus\{0\}),\ B(s)\in C^1_{rd}(\mathbb{D}_0,\mathbb{R}),\ s\in\mathbb{D}_0\Big\}.$$

Theorem 2.13. *Assume that* (C_1)–(C_6) *hold and there exist* $(A,B)\in(\mathcal{A},\mathcal{B})$ *and* H *in* (2.30) *such that, for any* $t_1\in\mathbb{T}$,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^t \Big[H(t,s)\big(A(s)q(s)-B^\Delta(s)\big) - H_2^\Delta(t,s)B^\sigma(s) - \phi_1(s)\Big]\Delta s = \infty,$$

where

$$\phi_1(t)=\begin{cases}-H(t,s)A(s)p(s)(\alpha_1(s))^{\gamma/(1-\gamma)} \\ \qquad +\dfrac{r(s)\big(H_2^\Delta(t,s)A^\sigma(s)+H(t,s)A^\Delta(s)\big)^2}{4\gamma H(t,s)A(s)\alpha_1(s)}, & 0<\gamma<1 \\ \Big[r(s)\big(H_2^\Delta(t,s)A^\sigma(s)+H(t,s)A^\Delta(s)\big)-H(t,s)A(s)p(s)\Big]^2 \\ \qquad \times(4H(t,s)A(s)r(s))^{-1}, & \gamma=1, \\ \dfrac{r(s)\big(H_2^\Delta(t,s)A^\sigma(s)+H(t,s)A^\Delta(s)\big)^2}{4\gamma H(t,s)A(s)\alpha_2(s)}, & \gamma>1,\ \ p=0, \\ \min\Big\{\frac{r(s)\big(H_2^\Delta(t,s)A^\sigma(s)+H(t,s)A^\Delta(s)\big)^2}{4\gamma H(t,s)A(s)\alpha_2(s)},\frac{\gamma-1}{(H(t,s)A(s)p(s))^{1/\gamma-1}} \\ \qquad \times\Big(r(s)\big(H_2^\Delta(t,s)A^\sigma(s)+H(t,s)A^\Delta(s)\big)(\gamma\alpha_2(s))^{-1}\Big)^{\gamma/(\gamma-1)}\Big\}, & \gamma>1,\ \ p>0.\end{cases}$$

Then equation (2.32) *is oscillatory.*

Theorem 2.14. *Assume that* (C_1)*–*(C_6) *hold and there exists* $A\in C_{rd}^1(\mathbb{D}_0,\mathbb{R}\setminus\{0\})$ *such that, for any* $t_1\in\mathbb{T}$,

$$\limsup_{t\to\infty}\int_{t_1}^{t}\big[A(s)q(s)-\phi_3(s)\big]\,\Delta s=\infty,$$

where

$$\phi_3(t)=\begin{cases}-A(s)p(s)(\alpha_1(s))^{\gamma/(1+\gamma)}+\dfrac{r(s)(A^\Delta(s))^2}{4\gamma A(s)\alpha_1(s)}, & 0<\gamma<1, \\ \dfrac{\big[A^\Delta(s)r(s)-A(s)p(s)\big]^2}{(4A(s)r(s))}, & \gamma=1, \\ \dfrac{r(s)(A^\Delta(s))^2}{4\gamma A(s)\alpha_2(s)}, & \gamma>1,\ \ p=0, \\ \min\Big\{\dfrac{r(s)(A^\Delta(s))^2}{4\gamma A(s)\alpha_2(s)},\dfrac{\gamma-1}{(A(s)p(s))^{1/\gamma-1}}\Big(\dfrac{A^\Delta(s)r(s)}{\gamma\alpha_2(s)}\Big)^{\gamma/(\gamma-1)}\Big\}, & \gamma>1,\ \ p>0.\end{cases}$$

Then equation (2.32) *is oscillatory.*

In [29], Grace et al. studied the oscillation criteria for the dynamic equations on time scales

$$\big(a(t)(x^\Delta(t))^\alpha\big)^\Delta+q(t)x^\beta(t)=0, \tag{2.33}$$

for the forced dynamic equation

$$\big(a(t)(x^\Delta(t))^\alpha\big)^\Delta+q(t)x^\beta(t)=e(t),$$

and for the forced-perturbed dynamic equation

$$\big(a(t)(x^\Delta(t))^\alpha\big)^\Delta+q(t)x^\beta(t)=e(t)+p(t)x^\gamma(t),$$

where α, β and γ are ratios of positive odd integers, a, p and q are real-valued, positive and rd-continuous functions and e is a real-valued and rd-continuous function on a time scale $\mathbb{T}\subset\mathbb{R}$ with $\sup\mathbb{T}=\infty$.

In order to achieve the goal, they introduced some assumptions:

$$\int_t^\infty a^{-1/\alpha}(s)\Delta s < \infty \tag{2.34}$$

and

$$0 < Q(t) := \int_t^\infty q(s)\,\Delta s < \infty. \tag{2.35}$$

Let

$$H(t,c) = \left[Q(t) + c\int_t^\infty a^{-1/\alpha}(s)(Q^\sigma(s))^{(1+\alpha)/\alpha}\,\Delta s\right]^{1/\alpha}, \tag{2.36}$$

where c is any positive constant.

Theorem 2.15. *Let $\beta > \alpha$, and let conditions* (2.34) *and* (2.35) *hold. If*

$$\limsup_{t\to\infty}\int_{t_1}^t a^{-1/\alpha}(s)H^\sigma(s,c)\,\Delta s, \quad t_1 \in [t_0,\infty)_{\mathbb{T}},$$

for any positive constant c, then equation (2.33) *is oscillatory.*

Theorem 2.16. *Let $\beta = \alpha$, and let conditions* (2.34) *and* (2.35) *hold. If*

$$\limsup_{t\to\infty}\left(\int_{t_0}^t a^{-1/\alpha}(s)\,\Delta s\right)H(t,\alpha) < 1,$$

where H is as in (2.36)*, then equation* (2.33) *is oscillatory.*

Theorem 2.17. *Let $\beta < \alpha$, and let conditions* (2.34) *and* (2.35) *hold. If for every constant $c > 0$,*

$$\limsup_{t\to\infty} Q^\delta(t)\left(\int_{t_0}^t a^{-1/\alpha}(s)\,\Delta s\right)\left[Q(t) + c\int_t^\infty a^{-1/\alpha}(s)(Q^\delta(s))^{(1+\alpha)/\alpha}Q^\delta(s)\,\Delta s\right]^{1/\alpha} = \infty,$$

where $\delta = \frac{1}{\beta} - \frac{1}{\alpha}$, then equation (2.33) *is oscillatory.*

In 2009 Grace et al. [30] discussed various different kind of new oscillation criteria for the second order dynamic equation (2.33). In their discussion, they assumed $a(t) > 0$,

$$\int_{t_0}^\infty a^{-1/\alpha}(s)\,\Delta s < \infty \tag{2.37}$$

and

$$\int_{t_0}^\infty a^{-1/\alpha}(s)\,\Delta s = \infty. \tag{2.38}$$

The problem of determining the non-oscillation and oscillation of all solutions of the second order linear equations, e.g., equation (2.33) with $\alpha = \beta = 1$, nonlinear equations, e.g., equation (2.33) with $\alpha = 1$ and $\beta \neq 1$, half-linear equations, e.g., equation (2.33) with $\alpha = \beta$ has been a very active area of research in the last two decades (for recent contributions we refer the reader to [1, 4–7, 22, 24, 27] and the references cited therein). There is no result concerning the oscillation of the nonlinear equation (2.33) with $\alpha \neq \beta$ on time scales.

In that paper, they intended to employ the generalized Riccati transformation technique to establish several new oscillation criteria for the nonlinear equation (2.33) with $\alpha \neq \beta$ and $\alpha \neq \beta$ on time scales. The results of this paper not only extend the known results appeared in the literature, but also improve and unify these criteria.

Theorem 2.18. *Let condition* (2.37) *hold. If there exists a positive non-decreasing delta differentiable function ξ such that for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$*

$$\limsup_{t\to\infty} \int_{t_1}^{\infty} \left[\xi(s)q(s) - \delta_1(s)\eta^{\alpha}(s)\xi^{\Delta}(s)\right] \Delta s = \infty \tag{2.39}$$

and

$$\int_{t_1}^{\infty} \left(\frac{1}{a(s)} \int_{t_1}^{t} \theta^{\beta}(u)q(u)\,\Delta u\right)^{1/\alpha} \Delta s = \infty,$$

where

$$\delta_1(t) = \begin{cases} c_1, & c_1 \text{ is any positive constant, if } \beta > \alpha, \\ 1, & \text{if } \beta = \alpha, \\ c_2\eta^{\beta-\alpha}(t), & c_2 \text{ is any positive constant, if } \beta < \alpha, \end{cases} \tag{2.40}$$

$$\eta(t) = \left(\int_{t_1}^{t} a^{-1/\alpha}(s)\,\Delta s\right)^{-1}, \quad \theta(t) = \int_{t}^{\infty} a^{-1/\alpha}(s)\,\Delta s,$$

then equation (2.33) *is oscillatory.*

Theorem 2.19. *Let condition* (2.38) *hold. If there exists a positive nondecreasing delta differentiable function ξ such that for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$ condition* (2.39) *holds, then equation* (2.33) *is oscillatory.*

Theorem 2.20. *Let conditions* (2.37) *and* (2.40) *hold. If there exists a nondecreasing positive delta differentiable function ξ such that for $t_1 \in [t_0, \infty)_{\mathbb{T}}$*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\xi(s)q(s) - \frac{(\alpha/\beta)^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{a(s)(\xi^{\Delta}(s))^{\alpha+1}}{(\xi(s)\delta_2(s))^{\alpha}}\right] \Delta s = \infty, \tag{2.41}$$

where

$$\delta_2(t) = \begin{cases} c_1, & c_1 \text{ is any positive constant, if } \beta > \alpha, \\ 1, & \text{if } \beta = \alpha, \\ c_2\eta^{\beta-\alpha}(t), & c_2 \text{ is any positive constant, if } \beta < \alpha, \end{cases}$$

and the functions θ and η are as in Theorem 2.18, then equation (2.33) *is oscillatory.*

Theorem 2.21. *Let condition* (2.38) *hold. If there exists a nondecreasing positive delta differentiable function ξ such that for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$ condition* (2.41) *holds, then equation* (2.33) *is oscillatory.*

Next theorem presents the following oscillation result for (2.33) when $\alpha \geq 1$.

Theorem 2.22. *Let $\alpha \geq 1$ and conditions* (2.37) *and* (2.40) *hold. If there exists a positive delta differentiable function ξ such that for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\xi(s)q(s) - \left(\frac{a^{1/\alpha}(s)}{4\beta\delta_3(s)} \left(\frac{(\xi^{\Delta}(s))^2}{\xi(s)} (\eta^{\alpha}(s))^{\alpha-1}\right)\right)\right] \Delta s = \infty, \tag{2.42}$$

where $\delta_3(t) = \delta_2^{\alpha}(t)$ and η and δ_2 are as in Theorem 2.18, then equation (2.33) *is oscillatory.*

Theorem 2.23. *Let condition* (2.39) *hold. If there exists a positive delta differentiable function ξ such that for any $t_1 \in [t_0, \infty)_{\mathbb{T}}$ condition* (2.42) *holds, then equation* (2.33) *is oscillatory.*

In [28], Grace et al. concerned with the oscillatory behavior of solutions of the second-order nonlinear dynamic equations of the form

$$(a(t)x^{\Delta}(t))^{\Delta} + f(t, x^{\sigma}(t)) = 0, \quad t \geq t_0, \tag{2.43}$$

subject to the following hypotheses:

(i) a is a positive real-valued rd-continuous function satisfying condition as follows

$$\int_{t_0}^{\infty} \frac{\Delta s}{a(s)} = \infty, \tag{2.44}$$

(ii) $f : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous satisfying

$$\operatorname{sgn} f(t, x) = \operatorname{sgn}(x) \text{ and } f(t, x) \leq f(t, y), \quad x \leq y, \quad t \geq t_0.$$

There are some results that are obtained for special cases of equation (2.43), e.g., when $\alpha = 1$ and $f(t, x) = q(t)x$, or $a = 1$ and $f(t, x) = q(t)f(x)$, where f satisfies the condition $|f(x)/x| \geq k > 0$ for $x \neq 0$, or $f'(x) \geq f(x)/x > 0$ for $x \neq 0$ (see [23,24,26]). In [38], Han et al. considered the second-order Emden–Fowler dynamic equation on time scales:

$$x^{\Delta\Delta}(t) + q(t)x^{\Delta}(t) = 0,$$

where α is the ratio of odd integers, and they used the Riccati transformation technique to obtain several oscillation criteria for this equation. For the continuous case, (2.43) can be rewritten as

$$(a(t)x'(t))' + f(t, x(t)) = 0.$$

Lemma 2.5. *Suppose $|y^{\Delta}|$ is one sign on $[t_0, \infty)$ and $\alpha > 0$. Then*

$$\frac{|y|^{\Delta}}{(|y|^{\sigma})^{\alpha}} \leq \frac{(|y|^{1-\alpha})^{\Delta}}{(1-\alpha)} \leq \frac{|y|^{\Delta}}{|y|^{\alpha}} \quad \text{on } [t_0, \infty).$$

Definition 2.1. Equation (2.43) (or the function f) is said to be strongly super-linear if there exists a constant $\beta > 1$ such that

$$\frac{|f(t, x)|}{|x|^{\beta}} \leq \frac{|f(t, y)|}{|y|^{\beta}} \text{ for } |x| \leq |y|, \quad xy > 0, \quad t \geq t_0, \tag{2.45}$$

and it is said to be strongly sub-linear if there exists a constant $\gamma \in (0, 1)$ such that

$$\frac{|f(t, x)|}{|x|^{\gamma}} \geq \frac{|f(t, y)|}{|y|^{\gamma}} \text{ for } |x| \leq |y|, \quad xy > 0, \quad t \geq t_0. \tag{2.46}$$

If equation (2.45) holds with $\beta = 1$, then equation (2.43) is called super-linear and if (2.46) holds with $\gamma = 1$, then (2.43) is called sub-linear.

Lemma 2.6. *Condition* (2) *implies that*

$$|f(t, x)| \leq |f(t, y)| \text{ for } |x| \leq |y|, \quad xy \geq 0, \quad t \geq t_0. \tag{2.47}$$

The following lemma will be used in the coming results.

Lemma 2.7. *Suppose x solves* (2.43) *and is of one sign on $[t_0, \infty)$. Let $u, v, t \geq t_0$. Then*

$$|x|^{\Delta}(v) = x^{\Delta}(v) \operatorname{sgn}(v)$$

and

$$|x|(t) = |x|(u) + a(v)|x|^{\Delta}(v) \int_{u}^{t} \frac{\Delta s}{a(s)} - \int_{u}^{t} \frac{1}{a(s)} \int_{v}^{s} \left|f(\tau, x(\sigma(\tau)))\right| \Delta\tau \, \Delta s. \tag{2.48}$$

Next, we define a notation

$$A(t) = \int_{t_0}^{t} \frac{\Delta s}{a(s)} \quad \text{for} \quad t \geq t_0.$$

The following simple consequence of Lemma 2.7 will be used in our main results.

Lemma 2.8. *Assume that equation* (2.44) *holds. Suppose x solves* (2.43) *and is of one sign on* $[t_0, \infty)$. *Then on* $[t_0, \infty)$,

$$|x|^{\Delta} \geq 0, \quad \textit{hence } |x| \textit{ is increasing.} \tag{2.49}$$

Moreover, pick any $t_1 > t_0$ and let

$$\overline{c} = x(t_0) \quad \textit{and} \quad c^* = \left\{ \frac{|x(t_0)|}{A(t_1)} + a(t_0)|x|^{\Delta}(t_0) \right\} \operatorname{sgn}(x(t_0)).$$

Then

$$|x| \geq \overline{c} \quad \textit{on } [t_0, \infty), \quad \textit{where } \ \overline{c}x > 0, \tag{2.50}$$

and

$$|x| \leq |c^* A| \quad \textit{on } [t_0, \infty), \quad \textit{where } \ c^* A x > 0.$$

Theorem 2.24. *Assume condition* (2.44) *holds. If*

$$\int_{t_0}^{\infty} |f(\tau, c)| \, \Delta\tau = \infty \quad \textit{for all} \quad c \neq 0, \tag{2.51}$$

then equation (2.43) *is oscillatory.*

Proof. Differentiating (2.48) with respect to t and then letting $t = t_0$ and using (2.50) and (2.47), we find, for all $v \geq t_0$,

$$|x|^{\Delta}(t_0) \geq \frac{1}{a(t_0)} \int_{t_0}^{v} \big|f(\tau, x(\sigma(\tau)))\big| \, \Delta\tau \geq \frac{1}{a(t_0)} \int_{t_0}^{v} |f(\tau, \overline{c})| \, \Delta\tau,$$

which contradicts (2.51) and completes the proof. □

Theorem 2.25. *Assume condition* (2.44) *holds. If*

$$\int_{t_0}^{\infty} \frac{1}{a(s)} \int_{s}^{\infty} |f(\tau, c)| \, \Delta\tau \, \Delta s = \infty \quad \textit{for all} \quad c \neq 0, \tag{2.52}$$

then all bounded solutions of equation (2.43) *are oscillatory.*

Proof. Let x be a bounded non-oscillatory solution of (2.43) such that x is of one sign on $[t_0, \infty)$. Using (2.50) and (2.47), we obtain

$$\big|f(\tau, x(\sigma(\tau)))\big| \geq |f(\tau, \overline{c})| \quad \text{for all} \quad \tau \geq t_0.$$

Thus, using (2.48) with $v \geq t \geq t_0$, together with (2.49), we find

$$|x(t)| \geq \int_{t_0}^{t} \frac{1}{a(s)} \int_{s}^{v} \big|f(\tau, x(\sigma(\tau)))\big| \, \Delta\tau \, \Delta s \geq \int_{t_0}^{t} \frac{1}{a(s)} \int_{s}^{v} |f(\tau, \overline{c})| \, \Delta\tau \, \Delta s,$$

which, since x is bounded, contradicts (2.52) and completes the proof. □

For the upcoming oscillatory results, we will use the following notation:

$$A(t) = \int_{t_0}^{t} \frac{\Delta s}{a(s)} \quad \text{for} \quad t \geq t_0$$

and

$$\widehat{A}(t) = \int_{t_0}^{\infty} \frac{\Delta s}{a(s)} \quad \text{for} \quad t \geq t_0.$$

Theorem 2.26. *Assume condition* (2.44) *holds. Suppose* (2.43) *is super-linear. If*

$$\limsup_{t\to\infty} \left\{ A(t) \int_{t}^{\infty} |f(\tau, c)| \, \Delta\tau \right\} > |c| \quad \textit{for all} \quad c \neq 0, \tag{2.53}$$

then equation (2.43) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of (2.43) such that x is of one sign on $[t_0, \infty)$. Using (2.50) and (2.45) (with $\beta = 1$), we obtain

$$\frac{|f(\tau, x(\sigma(\tau)))|}{|x(\sigma(\tau))|} \geq \frac{|f(\tau, \overline{c})|}{|\overline{c}|} \quad \text{for all} \quad \tau \geq t_0.$$

Thus, using (2.48) with $v \geq t \geq t_0$, along with (2.49), we find

$$|x(t)| \geq \int_{t_0}^{t} \frac{t}{a(s)} \int_{s}^{v} \left| f(\tau, x(\sigma(\tau))) \right| \Delta\tau \, \Delta s \geq \int_{t_0}^{t} \frac{1}{a(s)} \int_{s}^{v} \frac{|f(\tau, \overline{c})|}{|\overline{c}|} |x(\sigma(\tau))| \, \Delta\tau \, \Delta s,$$

and hence

$$|\overline{c}| \geq A(t) \int_{t}^{v} |f(\tau, \overline{c})| \, \Delta\tau,$$

which contradicts (2.53) and completes the proof. $\square$

Next, we present the following result for the strongly superlinear equation (2.43).

Theorem 2.27. *Assume that condition* (2.44) *holds. Suppose* (2.43) *is strongly super-linear. If*

$$\int_{t_0}^{\infty} \frac{1}{a(s)} \int_{s}^{\infty} |f(\tau, c)| \, \Delta\tau \, \Delta s = \infty \quad \textit{for all} \quad c \neq 0, \tag{2.54}$$

then equation (2.43) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of (2.43) such that x is of one sign on $[t_0, \infty)$. Using (2.50) and (2.45) (with $\beta = 1$), we obtain

$$\frac{|f(\tau, x(\sigma(\tau)))|}{|x(\sigma(\tau))|^{\beta}} \geq \frac{|f(\tau, \overline{c})|}{|\overline{c}|^{\beta}} \quad \text{for all} \quad \tau \geq t_0.$$

Thus, differentiating (2.48) with respect to t and using (2.49) and Lemma 2.5 (the inequality on the left-hand side), we find, for $v \geq t$,

$$\begin{aligned} |x|^{\Delta}(t) &\geq \frac{1}{a(t)} \int_{t}^{v} \left| f(\tau, x(\sigma(\tau))) \right| \Delta\tau \geq \frac{1}{a(t)} \int_{t}^{v} \frac{|f(\tau, \overline{c})|}{|\overline{c}|^{\beta}} |x(\sigma(\tau))| \, \Delta\tau \\ &\geq \frac{1}{a(t)} \int_{t}^{v} \frac{|f(\tau, \overline{c})|}{|\overline{c}|^{\beta}} \, \Delta\tau \, |x(\sigma(\tau))|^{\beta} \geq \frac{1}{a(t)} \int_{t}^{v} \frac{|f(\tau, \overline{c})|}{|\overline{c}|} \, \Delta\tau \, \frac{(\beta - 1)|x(\sigma(\tau))|^{\beta}}{-(|x|^{1-\beta})^{\Delta}(t)}, \end{aligned}$$

and hence

$$-\left(|x|^{1-\beta}\right)^{\Delta}(t) \geq \frac{\beta-1}{|\overline{c}|a(t)} \int\limits_{t}^{v} |f(\tau,\overline{c})|\,\Delta\tau.$$

Integrating this inequality from t_0 to $t \geq t_0$, we obtain

$$|x(t_0)|^{1-\beta} \geq |x(t)|^{1-\beta} + \frac{(\beta-1)}{|\overline{c}|^{\beta}} \int\limits_{t_0}^{t} \frac{1}{a(s)} \int\limits_{s}^{v} |f(\tau,\overline{c})|\,\Delta\tau\,\Delta s \geq \frac{(\beta-1)}{|\overline{c}|^{\beta}} \int\limits_{t_0}^{t} \frac{1}{a(s)} \int\limits_{s}^{v} |f(\tau,\overline{c})|\,\Delta\tau\,\Delta s,$$

which contradicts (2.54) and completes the proof. □

Theorem 2.28. *Assume condition* (2.44) *holds. Suppose equation* (2.43) *is strongly sub-linear. If*

$$\int\limits_{t_0}^{\infty} |f(\tau,cA(\tau))|\,\Delta\tau = \infty \ \text{ for all } \ c \neq 0, \tag{2.55}$$

then equation (2.43) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of (2.43) such that x is of one sign on $[t_0,\infty)$. Using (2.8) and (2.46) (with $0<\gamma<1$), we obtain

$$\frac{|f(\tau,x(\tau)|}{|x(\tau)|^{\gamma}} \geq \frac{|f(\tau,c^*A(\tau))|}{|c^*A(\tau)|^{\gamma}} \ \text{ for } \ \tau \geq t_1.$$

Thus, using (2.48) with $u=t_0$ and $v \geq t \geq t_1$, we find

$$\begin{aligned}|x(t)| &\geq \int\limits_{t_0}^{t} \frac{1}{a(s)} \int\limits_{s}^{v} \big|f(\tau,x(\sigma(\tau)))\big|\,\Delta\tau\,\Delta s \geq \int\limits_{t_0}^{t} \frac{1}{a(t)} \int\limits_{t}^{v} \big|f(\tau,x(\sigma(\tau))\big|\,\Delta\tau\,\Delta s \\ &= A(t) \int\limits_{t}^{v} \big|f(\tau,x(\sigma(\tau)))\big|\,\Delta\tau \geq A(t) \int\limits_{t}^{v} |f(\tau,x(\tau))|\,\Delta\tau \geq A(t) \int\limits_{t}^{v} \frac{|f(\tau,c^*A(\tau))|}{|c^*A(\tau)|^{\gamma}} |x(\tau)|^{\gamma}\,\Delta\tau\end{aligned}$$

(where we have used (2.49) and (2.47) in the second last inequality), and hence

$$\Big|\frac{x(t)}{A(t)}\Big| \geq z(t), \ \text{ where } \ z(t) := |c^*|^{-\gamma} \int\limits_{t}^{v} \big|f(\tau,c^*A(\tau))\big| \,\Big|\frac{x(\tau)}{A(\tau)}\,\Delta\tau\Big|.$$

Thus, using Lemma 2.5 (the inequality on the right-hand side),

$$\begin{aligned}-|z|^{\Delta}(\tau) = -z^{\Delta}(\tau) &= |c^*|^{-\gamma}\big|f(\tau,c^*A(\tau))\big| \,\Big|\frac{x(\tau)}{A(\tau)}\Big|^{\gamma} \\ &\geq |c^*|^{-\gamma}\big|f(\tau,c^*A(\tau))\big|\,|z(\tau)|^{\gamma} \geq |c^*|^{-\gamma}\big|f(\tau,c^*A(\tau))\big| \,\frac{(1-\gamma)(-|z|^{\Delta}(\tau))}{-(|z|^{1-\gamma})^{\Delta}(\tau)}\,,\end{aligned}$$

and hence

$$-\left(|z|^{1-\gamma}\right)^{\Delta}(\tau) \geq \frac{(1-\gamma)}{|c^*|^{\gamma}}\,\big|f(\tau,c^*A(\tau)\big|.$$

Integrating this inequality from t_1 to $t \geq t_1$, we obtain

$$|z(t_1)|^{1-\gamma} \geq |z(t)|^{1-\gamma} + \frac{(1-\gamma)}{|c^*|^{\gamma}} \int\limits_{t_1}^{t} \big|f(\tau,c^*A(\tau))\big|\,\Delta\tau \geq \frac{(1-\gamma)}{|c^*|^{\gamma}} \int\limits_{t_1}^{t} \big|f(\tau,c^*A(\tau))\big|\,\Delta\tau,$$

which contradicts (2.55) and completes the proof. □

3 Important applications

The results obtained above for equation (2.43) have been applied to the second-order Emden–Fowler dynamic equation on time scales:

$$(a(t)x^{\Delta}(t))^{\Delta} + q(t)(x^{\alpha}(t))^{\alpha} = 0, \tag{3.1}$$

where a and q are the nonnegative rd-continuous functions and a is the ratio of positive odd integers.

Theorem 3.1. *Let condition* (2.44) *hold and define $A(t)$. Equation* (3.1) *is oscillatory if one of the following conditions holds:*

- $\int\limits_{t_0}^{\infty} q(\tau)\,\Delta\tau = \infty$ *if* $\alpha > 0$*;*
- $\limsup\limits_{t\to\infty} \left\{ A(t) \int\limits_{t}^{\infty} q(\tau)\,\Delta\tau \right\} > c$ *for any* $c > 0$ *and for* $\alpha \geq 1$*;*
- $\int\limits_{t_0}^{\infty} \frac{1}{a(s)} \int\limits_{s}^{\infty} q(\tau)\,\Delta\tau\,\Delta s = \infty$ *if* $\alpha > 1$*;*
- $\int\limits_{t_0}^{\infty} (A(\tau))^{\alpha} q(\tau)\,\Delta\tau = \infty$ *if* $0 < \alpha < 1$*.*

Remark 3.1. From the results of this paper, we can obtain some oscillation criteria for equation (2.43) on different types of time scales. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $x^{\Delta}(t) = x'(t)$ for $t \in \mathbb{T}$. In this case, the results of this paper are the same as those in [45]. If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ and $x^{\Delta}(t) = \Delta x(t) = x(t+1) - x(t)$. In this case, the results of this paper are the discrete analogues of those in [45]. If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, then $\sigma(t) = t + h$, and $x^{\Delta}(t) = (x(t+h) - x(t))/h$. The reformulation of our results is easy and we left that to the readers. We may employ other types of time scales, e.g. $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, $\mathbb{T} = \mathbb{N}_0^2$.

In [31], Grace et al. discussed the oscillatory behaviour of the following second-order dynamic equation on time scales:

$$\left(a(t)(x^{\Delta}(t))^{\alpha}\right)^{\Delta} + q(t)x^{\beta}(t) = 0, \tag{3.2}$$

where α and β are the ratios of positive odd integers, a and q are real-valued, positive and rd-continuous functions on a time scale $\mathbb{T} \subset \mathbb{R}$ with $\sup \mathbb{T} = \infty$.

In this paper, the authors have obtained some comparison results which were applied to neutral dynamic equation of the form

$$\left(a(t)\Big(\big[x(t) + p(t)x[x(\tau(t))]^{\Delta}\big]\Big)^{\alpha}\right)^{\Delta} + q(t)x^{\beta}(t) = 0. \tag{3.3}$$

In order to achieve the oscillatory result, they assumed some conditions as follows:

$$\int^{\infty} a^{-1/\alpha}(s)\,\Delta s = \infty. \tag{3.4}$$

First, we derive a lemma.

Lemma 3.1. *Assume condition* (3.4) *holds. If the inequality*

$$\left(a(t)(x^{\Delta}(t))^{\alpha}\right)^{\Delta} + q(t)x^{\beta}(t) \leq 0, \tag{3.5}$$

where

$$\int^{\infty} a(t)\,\Delta t < \infty,$$

has an eventually positive solution, then (3.2) *also has an eventually positive solution.*

Proof. Let $x(t)$ be an eventually positive solution of inequality (3.5). It is easy to see that $x^{\Delta}(t)>0$ is eventual, i.e., there exists $t_0\ge 0$ such that $x^{\Delta}(t)>0$, for $t\ge t_0$. Now, let

$$y(t)=a(t)(x^{\Delta}(t))^{\alpha},\ \ t\ge t_0. \tag{3.6}$$

Then

$$x^{\Delta}(t)=\Big(\frac{y(t)}{a(t)}\Big)^{1/\alpha}:=\Psi(a;y)(t)>0\ \text{ for }\ t\ge t_0. \tag{3.7}$$

Integrating this equality from t_0 to $t\ge t_0$, we have

$$x(t)\ge x(t)-x(t_0)=\int\limits_{t_0}^{t}\Psi(a;y)(s)\,\Delta s:=Y(a;y)(t). \tag{3.8}$$

From (3.5)–(3.8), we get

$$y^{\Delta}(t)+q(t)((Y(a;y))(t))^{\beta}\le 0\ \text{ for }\ t\ge t_0. \tag{3.9}$$

Integrating this inequality from $v\ge t$ and letting $v\to\infty$, we have

$$y(t)\ge\int\limits_{t}^{\infty}q(s)((Y(a;y))(s))^{\beta}\,\Delta s. \tag{3.10}$$

Now, we define a sequence of successive approximations $\{u_n(t)\}$ as follows:

$$\begin{cases} u_0(t)=y(t)\\ u_{n+1}(t)=\displaystyle\int\limits_{t}^{\infty}q(s)((Y(a;y))(s))^{\beta}\,\Delta s,\ \ n=0,1,\dots\,. \end{cases} \tag{3.11}$$

One can easily prove that

$$0<u_n(t)\le y(t),\ \ u_{n+1}(t)\le u_n(t),\ \ n=0,1,\dots,\ \ \text{and}\ \ t\ge t_0.$$

Now, let

$$u(t)=\lim_{t\to\infty}u_n(t)>0.$$

Since $0<u(t)\le u_n(t)\le y(t)$ for all $n\ge 0$ and $Y(a;u_n)\le Y(a;u)$, the convergence of the sequence in (3.11) is uniform with respect to n. Taking the limit on the both sides of (3.11), we obtain

$$u(t)=\int\limits_{t}^{\infty}q(s)((Y(a;y))(s))^{\beta}\,\Delta s\ \text{ for }\ t\ge t_0.$$

Therefore,

$$u^{\Delta}(t)=-q(s)((Y(a;y))(s))^{\beta}\ \text{ for }\ t\ge t_0. \tag{3.12}$$

Define

$$v(t)=(Y(a;y))(t)\ \text{ for }\ t\ge t_0.$$

Then $v(t)>0$. Thus, for $t\ge t_0$,

$$v^{\Delta}(t)=\Psi(a;u)(t)=\Big(\frac{u(t)}{a(t)}\Big)^{1/\alpha}$$

and

$$a(t)(v^{\Delta}(t))^{\alpha}=u(t). \tag{3.13}$$

From (3.12) and (3.13), we get

$$\big(a(t)(v^{\Delta}(t))^{\alpha}\big)^{\Delta}+q(t)v^{\beta}(t)=0.$$

This shows that (3.2) has a positive solution $v(t)$.

Similarly, we can show that if the inequality

$$\left(a(t)(x^{\Delta}(t))^{\alpha}\right)^{\Delta}+q(t)x^{\beta}(t)\geq 0$$

has an eventually negative solution, then (3.2) also has an eventually negative solution. This completes the proof. □

From Lemma 3.1, one can easily obtain the following comparison results.

Theorem 3.2. *Assume that* (3.4) *holds. If* $0<Q(t)\leq q(t)$ *for* $t\geq t_0\geq 0$ *and the equation*

$$\left(a(t)(z^{\Delta}(t))^{\alpha}\right)^{\Delta}+Q(t)z^{\beta}(t)=0$$

is oscillatory, then (3.2) *is also oscillatory.*

Proof. Immediately follows from Lemma 3.1. □

Theorem 3.3. *Assume that* (3.4) *holds. If* $b(t)\geq a(t)$ *for* $t\geq t_0$ *and the equation*

$$\left(b(t)(z^{\Delta}(t))^{\alpha}\right)^{\Delta}+q(t)z^{\beta}(t)=0$$

is oscillatory, then (3.2) *is oscillatory.*

Proof. Suppose that (3.2) is non-oscillatory. Without loss of generality, we may assume that (3.2) has a positive solution $x(t)$ for $t\geq t_0$. Using the same arguments as in Lemma 3.1, we can show that (3.8) holds. Thus

$$x(t)\geq Y(a;y)(t)\geq Y(b;y)(t).$$

So,

$$y^{\delta}(t)+q(t)(Y(b;y)(t))^{\beta}\leq 0 \;\text{ for }\; t\geq t_0.$$

The rest of the proof is similar to that of Lemma 3.1 and hence omitted. □

Next, consider another dynamic equation

$$\left(a(t)(x^{\Delta}(t))^{\alpha}\right)^{\Delta}+q(t)f(x(t))=0, \tag{3.14}$$

where α, a and q are as in (3.2), $f:\mathbb{R}\to\mathbb{R}$ is continuous and satisfies

$$x(f(x))-x^{\beta}\geq 0 \;\text{ for }\; x\neq 0. \tag{3.15}$$

The following comparison result can be obtained.

Theorem 3.4. *Assume that* (3.4) *and* (3.15) *hold. If* (3.2) *is oscillatory, then* (3.14) *is oscillatory.*

Proof. Without loss of generality, suppose that (3.14) has an eventually positive solution x. By conditions (3.15) and (3.6), we have

$$\left(a(t)(x^{\Delta}(t))^{\alpha}\right)^{\Delta}+q(t)x^{\beta}(t)\leq 0.$$

Thus, by Lemma 3.1, we see that (3.2) has a positive solution, which is a contradiction to the fact that (3.2) is oscillatory. This completes the proof. □

Next, as an application to the above results, we consider the neutral dynamic equation (3.3), where $p(t)$ is the rd-continuous function with $0\leq p(t)\leq 1$ and $\tau:\mathbb{T}\to\mathbb{T}$ satisfies $\tau(t)\leq t$ for $t\in\mathbb{T}$ and $\lim\limits_{t\to\infty}\tau(t)=\infty$.

The next oscillatory result is as follows.

Theorem 3.5. *Assume that condition* (3.4) *holds. If the equation*

$$\big(a(t)(x^{\Delta}(t))^{\alpha}\big)^{\Delta}+q(t)(1-p(t))^{\beta}x^{\beta}(t)=0 \tag{3.16}$$

is oscillatory, then (3.3) *is also oscillatory.*

Proof. Without loss of generality, suppose that x is a non-oscillatory solution of equation (3.3), say, $x(t)>0$ for $t\geq t_0\geq 0$. Let

$$y(t)=x(t)+p(t)x[\tau(t)],\quad t\geq t_0.$$

Then

$$\big(a(t)(y^{\Delta}(t))^{\alpha}\big)^{\Delta}+q(t)x^{\beta}(t)=0. \tag{3.17}$$

It is easy to see that $y^{\Delta}(t)>0$ and $x(t)\leq y(t)$ and $x(\tau(t))\leq y(\tau(t))\leq y(t)$ for $t\geq t_1$ for some $t_1\geq t_0$. Thus, we have

$$x(t)=y(t)-p(t)x[\tau(t)]\geq y(t)-p(t)y(t):=(1-p(t))y(t). \tag{3.18}$$

Using (3.18) and (3.17), we obtain

$$\big(a(t)(y^{\Delta}(t))^{\alpha}\big)^{\Delta}+q(t)(1-p(t))^{\beta}y^{\beta}(t)\leq 0 \ \text{ for } \ t\geq t_1.$$

By Lemma 3.1, we see that (3.16) has an eventually positive solution, which is a contradiction. This completes the proof. □

In [35], Grace et al. discussed the oscillatory behaviour of the equation

$$x^{\Delta}(t)=e(t)-\int\limits_0^t k(t,s)f(s,x(s))\,\Delta s,\quad t\geq 0, \tag{3.19}$$

and the Volterra integral equation

$$x(t)=e(t)-\int\limits_0^t k(t,s)f(s,x(s))\,\Delta s,\quad t\geq 0, \tag{3.20}$$

where $e:[0,\infty)_{\mathbb{T}}\to\mathbb{R}$ is rd-continuous, $k(t,\cdot\,):[0,\infty)_{\mathbb{T}}\to[0,\infty)$ is rd-continuous for each fixed $t\in\mathbb{T}$, $k(\,\cdot\,,s):[0,\infty)_{\mathbb{T}}\to[0,\infty)$ is rd-continuous for each fixed $s\in\mathbb{T}$, $f(\,\cdot\,,x):[0,\infty)_{\mathbb{T}}\to\infty$ is rd-continuous for each $x\in\mathbb{R}$, and $f(t,\cdot\,):[0,\infty)_{\mathbb{T}}\to\mathbb{R}$ is continuous for each $t\in\mathbb{T}$.

Some oscillation theorems for Volterra integro-differential and integral equations in the continuous case can be found in [52,53], the oscillation problem for the integral and integro-dynamic equations on time scales is a fairly new topic. To the best of our knowledge, the only study regarding the integro-dynamic equation (3.19) has been recently carried out in [32]. Therefore, the objective in this paper is to make further contributions to the subject by studying the oscillation problem for the equations of form (3.19) and (3.20).

Some necessary assumptions and lemmas are given as follows.

Lemma 3.2 ([5])**.** *If X and Y are non-negative real numbers, then*

$$X^{\lambda}-(1-\lambda)Y^{\lambda}-\lambda XY^{\lambda-1}\leq 0 \ \textit{for} \ 0<\lambda<1.$$

The equality holds if and only if $X=Y$.

Throughout the work in [35], the authors assumed that the following hypothesis (H) holds.

(H) There exist rd-continuous functions $a,q,m:[0,\infty)_{\mathbb{T}}\to(0,\infty)$ and a real number λ, $0<\lambda\leq 1$, such that

$$k(t,s)\leq a(t)q(s) \ \text{ for all } t\geq s$$

and

$$0<xf(t,x)\leq m(t)|x|^{\lambda+1} \ \text{ for } \ x\neq 0 \ \text{ and } \ t\geq 0.$$

In what follows, denote

$$h_{\pm}(t) = e(t) \pm (1-\lambda)\lambda^{\lambda/(1-\lambda)} a(t) \int_0^t p^{\lambda/(\lambda-1)}(s) m^{1/(1-\lambda)}(s) q^{1/(1-\lambda)}(s)\,\Delta s, \quad 0<\lambda<1, \tag{3.21}$$

where $p : [0,\infty)_{\mathbb{T}} \to (0,\infty)$ is a given rd-continuous function.

We first give the sufficient conditions under which every non-oscillatory solution of equation (3.19) satisfies $x(t) = O(t), t \to \infty$.

Theorem 3.6. *Let $0<\lambda<1$ and (H) hold, and let $h_{\pm}$ be defined by (3.21). Assume that*

$$\int_0^\infty a(s)\,\Delta s < \infty \tag{3.22}$$

and

$$\int_0^\infty s p(s)\,\Delta s < \infty. \tag{3.23}$$

If

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t h_{\pm}(s)\,\Delta s < \infty, \quad \liminf_{t\to\infty} \frac{1}{t}\, h_{\pm}(s)\,\Delta s > -\infty, \tag{3.24}$$

then every non-oscillatory solution $x(t)$ of equation (3.19) satisfies

$$\limsup_{t\to\infty} \frac{|x(t)|}{t} < \infty. \tag{3.25}$$

Proof. Let $x(t)$ be a solution of equation (3.19), we may assume that $x(t)>0$ for all $t \geq t_1$ for some $t_1 > 0$. Let

$$k_1 := \min\left\{f(t,x(t)) : t \in [0,t_1]_{\mathbb{T}}\right\} \leq 0 \;\text{ and }\; k_2 := -k_1 \int_0^{t_1} q(s)\,\Delta s \geq 0.$$

In view of the condition (H), we can then write

$$\begin{aligned}
x^{\Delta}(t) &= e(t) - \int_0^{t_1} k(t,s) f(s,x(s))\,\Delta s - \int_{t_1}^t k(t,s) f(s,x(s))\,\Delta s \\
&\leq e(t) - k_1 a(t) \int_0^{t_1} q(s)\,\Delta s + a(t) \int_{t_1}^t q(s) m(s) x^{\lambda}(s)\,\Delta s \\
&= e(t) + k_2 a(t) + a(t) \int_{t_1}^t \big(q(s)m(s)x^{\lambda}(s) - p(s)x(s)\big)\,\Delta s + a(s) \int_{t_1}^t p(s)x(s)\,\Delta s.
\end{aligned}$$

Applying the above lemma to $q(s)m(s)x^{\lambda}(s) - p(s)x(s)$ with

$$X = (qm)^{1/\lambda} x, \quad Y = \Big(\frac{p(qm)^{-1/\lambda}}{\lambda}\Big)^{1/(\lambda-1)},$$

we have

$$q(s)m(s)x^{\lambda}(s) - p(s)(x(s)) \leq (1-\lambda)\lambda^{\lambda/(\lambda-1)} p^{\lambda/(\lambda-1)}(s) m^{1/(1-\lambda)}(s) q^{1/(1-\lambda)}(s).$$

Thus,

$$x^{\Delta}(s) \leq h_{\pm}(t) + k_2 a(t) + a(t) \int_{t_1}^{t} p(s)x(s)\,\Delta s. \tag{3.26}$$

Integrating equation (3.26) from t_1 to t, and using equation (3.22) and the fact that $a(t)$ is bounded, say by c_1, we get

$$x(t) \leq x(t_1) + \int_{t_1}^{t} h_{+}(s)\,\Delta s + k_2 \int_{t_1}^{t} a(s)\,\Delta s + c_1 \int_{t_1}^{t} \int_{t_1}^{t} p(s)x(s)\,\Delta s\,\Delta r.$$

If we now employ [17, Lemma 3], to the interchange of the order of this integration, then it follows that

$$z(t) \leq x(t_1) + \int_{t_1}^{t} h_{+}(s)\,\Delta s + k_2 \int_{t_1}^{t} a(s)\,\Delta s + c_1 t \int_{t_1}^{t} p(s)x(s)\,\Delta s, \tag{3.27}$$

and so,

$$\frac{x(t)}{t} \leq c_2 + c_1 \int_{t_1}^{t} sp(s)\,\frac{x(s)}{s}\,\Delta s, \quad t \geq t_1.$$

Now, in view of (3.22) and (3.24), $c_2 > 0$ is an upper bound for

$$\frac{1}{t}\,x(t_1) + \frac{1}{t} \int_{t_1}^{t} h_{+}(s)\,\Delta s + \frac{k_2}{t} \int_{t_1}^{t} a(s)\,\Delta s.$$

Applying Gronwall's inequality to the above inequality, then using equation (3.23), we have

$$\limsup_{t\to\infty} \frac{x(t)}{t} < \infty. \tag{3.28}$$

If $x(t)$ is eventually negative, we can obtain that $y = -x$ and that y satisfies equation (3.19) with $e(t)$ replaced by $-e(t)$ and $f(t,x)$ replaced by $-f(t,x)$. It follows in the similar manner that

$$\limsup_{t\to\infty} \frac{-x(t)}{t} < \infty.$$

Thus, from the last two relations, we conclude that equation (3.25) holds. $\square$

Theorem 3.7. *Let* $0 < \lambda < 1$ *and* (H) *hold, and let* $h_{\pm}$ *be defined by* (3.21). *Assume that* (3.22) *and* (3.24) *are satisfied and that*

$$\limsup_{t\to\infty} \int_{0}^{\infty} sp(s)\,\Delta s < \infty.$$

If

$$\limsup_{t\to\infty} h_{+}(s)\,\Delta s = \infty, \quad \liminf_{t\to\infty} h_{-}(s)\,\Delta s = -\infty, \tag{3.29}$$

then equation (3.19) *is oscillatory.*

Proof. Suppose on the contrary that there is a non-oscillatory solution $x(t)$ of equation (3.19), which is eventually positive, say $x(t) > 0$ for $t \geq t_1$ for some $t_1 > 0$. The proof when $x(t)$ is eventually negative is similar. Proceeding as in the proof of Theorem 3.6, we arrive at (3.27). Therefore,

$$x(t) \leq x(t_1) + \int_{t_1}^{t} h_{+}(s)\,\Delta s + k_2 \int_{t_1}^{t} a(s)\,\Delta s + c_1 t \int_{t_1}^{t} sp(s)\,\frac{x(s)}{s}\,\Delta s. \tag{3.30}$$

On the other hand, (3.28) implies (3.23), and so the conclusion of equation (3.30) holds. This, together with equation (3.22), shows that the last two integrals of the above relations are bounded. Finally, taking lim sup as $t \to \infty$ and using (3.29) in (3.30), the result contradicts the fact that $x(t)$ is eventually positive. □

Similar to the sub-linear case, one can easily prove the following theorems for the integro-dynamic equation (3.19) when $\lambda = 1$. Now, assume the following:

$$\limsup_{t\to\infty} \int_0^t e(s)\,\Delta s = \infty, \quad \liminf_{t\to\infty} \int_0^t e(s)\,\Delta s = -\infty. \tag{3.31}$$

Theorem 3.8. *Let $\lambda = 1$ and (H) hold. In addition to (3.22), assume that*

$$\limsup_{t\to\infty} \int_0^t sm(s)q(s)\,\Delta s < \infty. \tag{3.32}$$

If

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t e(s)\,\Delta s < \infty, \quad \liminf_{t\to\infty} \frac{1}{t}\int_0^t e(s)\,\Delta s > -\infty, \tag{3.33}$$

then every non-oscillatory solution of equation (3.19) satisfies (3.25).

Theorem 3.9. *Let $\lambda = 1$ and (H) hold. Assume that (3.22), (3.31) and (3.33) are satisfied. If*

$$\limsup_{t\to\infty} t\int_0^t sm(s)q(s)\,\Delta s < \infty,$$

then equation (3.19) is oscillatory.

Theorem 3.10. *Let $0 < \lambda < 1$, (H) and (3.23) hold, and let $h_\pm$ be defined by (3.21). If*

$$\limsup_{t\to\infty} \frac{a(t)}{t} < \infty, \quad \limsup_{t\to\infty} \frac{h_+(t)}{t} < \infty, \quad \liminf_{t\to\infty} \frac{h_-(t)}{t} > -\infty, \tag{3.34}$$

then every non-oscillatory solution $x(t)$ of equation (3.20) satisfies (3.25).

Proof. Let $x(t)$ be an eventually positive solution of (3.20), say $x(t) > 0$ for $t \ge t_1$ for some $t_1 > 0$. Proceeding as in the proof of Theorem 3.6, similarly to (3.26), we arrive at

$$x(t) \le h_+(t) + k_2 a(t) + a(t)\int_{t_2}^t p(s)x(s)\,\Delta s$$

and hence

$$\frac{x(t)}{t} \le k_3 + k_2k_4 \int_{t_1}^t sps(s)\,\frac{x(s)}{s}\,\Delta s, \quad t \ge t_1, \tag{3.35}$$

where, in view of (3.34), k_3 and k_4 are, respectively, the upper bounds for $h_+(t)/t$ and $a(t)/t$. An application of the Gronwall inequality to (3.35) gives (3.25). The proof is similar when $x(t)$ is eventually negative. □

Theorem 3.11. *Let $0 < \lambda < 1$, (H) and (3.23) hold, and let $h_\pm$ be defined by (3.21). Assume that*

$$\limsup_{t\to\infty} a(t) < \infty, \quad \limsup_{t\to\infty} \frac{h_+(t)}{t} < \infty, \quad \liminf_{t\to\infty} \frac{h_-(t)}{t} > -\infty. \tag{3.36}$$

If

$$\limsup_{t\to\infty} h_+(t) = \infty, \quad \liminf_{t\to\infty} h_-(t) = -\infty, \tag{3.37}$$

then equation (3.20) *is oscillatory.*

Proof. Suppose on the contrary that there is a non-oscillatory solution $x(t)$ of equation (3.20), which is eventually positive, say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq 0$. As in the proof of the above theorem, we have

$$x(t) \leq h_+(t) + k_2 a(t) + a(t) \int_{t_1}^{t} sp(s)\, \frac{x(s)}{s}\, \Delta s, \quad t \geq t_1. \tag{3.38}$$

Note that since $a(t)$ is bounded, condition (3.36) implies (3.34), and hence the conclusion of the above theorem holds. In view of (3.21), (3.37) and the fact that $\frac{x(t)}{t}$ is bounded, by taking lim inf on both sides of (3.38) as $t \to \infty$, we obtain a contradiction with $x(t)$ being eventually positive. The proof is similar when $x(t)$ is eventually negative. $\square$

Again, similar to sub-linear case, we have the following results for the Volterra integral equation (3.20) when $\lambda = 1$.

Theorem 3.12. *Let $\lambda = 1$ and* (H) *hold. Assume that* (3.32) *holds. If*

$$\limsup_{t\to\infty} \frac{a(t)}{t} < \infty, \quad \limsup_{t\to\infty} \frac{e(t)}{t} < \infty, \quad \liminf_{t\to\infty} \frac{e(t)}{t} > -\infty,$$

then every non-oscillatory solution $x(t)$ of equation (3.20) *satisfies* (3.25).

Assume

$$\limsup_{t\to\infty} e(t) = \infty \text{ and } \liminf_{t\to\infty} e(t) = -\infty. \tag{3.39}$$

Theorem 3.13. *Let $\lambda = 1$ and* (H) *hold. Assume that* (3.32) *and* (3.39) *are satisfied. If*

$$\limsup_{t\to\infty} a(t) < \infty, \quad \limsup_{t\to\infty} \frac{e(t)}{t} < \infty, \quad \liminf_{t\to\infty} \frac{e(t)}{t} > -\infty,$$

then equation (3.20) *is oscillatory.*

Motivated by the results from [35] by Grace et al., in their paper [5], Agarwal et al. concerned with the asymptotic behavior of non-oscillatory solutions of the second-order integro-dynamic equation on time scales of the form

$$\big(r(t)(x^\Delta(t))^\alpha\big)^\Delta + \int_0^t a(t,s) F(s, x(s))\, \Delta s = 0, \tag{3.40}$$

and the oscillatory behavior of the second-order forced integro-dynamic equation

$$(r(t)(x^\Delta(t)))^\Delta + \int_0^t a(t,s) F(s, x(s))\, \Delta s = e(t). \tag{3.41}$$

Here, $\mathbb{T} \subset \mathbb{R}_+ = [0, \infty)$ is an arbitrary time-scale with $0 \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$.

Furthermore, throughout the paper they assumed:

1. $e, r : \mathbb{T} \to \mathbb{R}$ and $a : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ are rd-continuous and $r(t) > 0$, and $a(t,s) \geq 0$ for $t > s$, α is the ratio of positive odd integer and

$$\sup_{t \geq t_0} \int_0^{t_0} a(t,s)\, \Delta s := k < \infty, \quad t_0 \geq 0; \tag{3.42}$$

2. $F : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is continuous and assume that there exist continuous functions $f_1, f_2 : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ such that $F(t,x) = f_1(t,x) - f_2(t,x)$ for $t \geq 0$;

3. there exit constants β and γ, being the ratios of positive odd integer, and the functions $p_i \in C_{rd}(\mathbb{T}, (0,\infty))$, $i = 1,2$, such that

$$xf_1(t,x) \geq p_1(t)x^{\beta+1} \ \text{ for } \ x \neq 0 \ \text{ and } \ t \geq 0,$$
$$xf_2(t,x) \geq p_2(t)x^{\gamma+1} \ \text{ for } \ x \neq 0 \ \text{ and } \ t \geq 0.$$

Now, define

$$R(t,t_0) = \int_{t_0}^{t} \left(\frac{s}{r(s)}\right)^{1/\alpha} \Delta s, \quad t > t_0 \geq 0.$$

Note that due to the monotonicity,

$$\lim_{t\to\infty} R(t,t_0) \neq 0. \tag{3.43}$$

Lemma 3.3. *If X and Y are non-negative real numbers, then*

$$X^\lambda + (\lambda-1)Y^\lambda - \lambda XY^{\lambda-1} \geq 0 \ \textit{ for } \ \lambda > 1, \tag{3.44}$$
$$X^\lambda - (1-\lambda)Y^\lambda - \lambda XY^{\lambda-1} \leq 0 \ \textit{ for } \ \lambda < 1, \tag{3.45}$$

where the equality holds if and only if $X = Y$.

4 Oscillatory results for (3.40)

Theorem 4.1. *Let the conditions* (i)–(iii) *hold with $\gamma = 1$ and $\beta > 1$ and suppose*

$$\lim_{t\to\infty} \frac{1}{R(t,t_0)} \int_{t_0}^{t} \left(\frac{1}{r(v)} \int_{t_0}^{v}\int_{t_0}^{u} a(u,v) p_1^{1/(1-\beta)}(s) p_2^{1/(\beta-1)}(s)\, \Delta s\, \Delta u\right)^{1/\alpha} \Delta u < \infty \tag{4.1}$$

for some $t_0 \geq 0$. If x is a non-oscillatory solution of (3.40)*, then*

$$x(t) = O(R(t,t_0)) \ \textit{ as } \ t \to \infty. \tag{4.2}$$

Proof. Let x be a non-oscillatory solution of equation (3.40). Hence x is either eventually positive or eventually negative. First assume x is eventually positive, say $x(t) > 0$ for $t \geq t_0$ for some $t_1 > 0$. Using the conditions (ii) and (iii) with $\beta > 1$ and $\gamma = 1$ in equation (3.40), for $t \geq t_1$, we obtain

$$\left(r(t)(x^\Delta(t))^\alpha\right)^\Delta \leq -\int_0^{t_1} a(t,s)F(s,x(s))\,\Delta s + \int_{t_1}^{t} a(t,s)\left[p_2(s)x(s) - p_1(s)x^\beta(s)\right]^{1/(\beta-1)} \Delta s. \tag{4.3}$$

If we apply (3.44) with $\lambda = \beta$, $X = p_1^{1/\beta}x$, and $Y = (\frac{1}{\beta}\, p_2 p_1^{-1/\beta})^{1/(\beta-1)}$, we have

$$p_2(t)x(t) - P_1(t)x^\beta(t) \leq (\beta-1)\beta^{\beta/(1-\beta)} p_1^{1/(1-\beta)}(t) p_2^{\beta/(\beta-1)}(t), \quad t \geq t_1. \tag{4.4}$$

Substituting (4.4) into (4.3), we get

$$\left(r(t)(x^\Delta(t))^\alpha\right)^\Delta \leq -\int_{t_1}^{t} a(t,s)F(s,x(s))\,\Delta s + (\beta-1)\beta^{\beta/(1-\beta)} \int_{t_1}^{t} a(t,s) p_1^{1/(1-\beta)}(s) p_2^{\beta/(\beta-1)}(s)\,\Delta s \tag{4.5}$$

for all $t \geq t_1 \geq 0$. Let

$$m := \max\left\{|F(t,x(t))| : \ t \in [0,t_1]_{\mathbb{T}}\right\}.$$

By using assumption (i), we have

$$\left| -\int_{t_1}^{t} a(t,s)F(s,x(s))\,\Delta s\right| \le \int_{0}^{t_1} a(t,s)|F(s,x(s))|\,\Delta s \le mk := b. \tag{4.6}$$

Hence from (4.5) and (4.6), we obtain

$$\left(r(t)(x^{\Delta}(t))^{\alpha}\right)^{\Delta} \le b + (\beta-1)\beta^{\beta/(1-\beta)} \int_{0}^{t_1} a(t,s)p_1^{1/(1-\beta)}(s)p_2^{\beta/(\beta-1)}(s)\,\Delta s.$$

Integrating this inequality from t_1 to t we arrive to

$$\begin{aligned}(x^{\Delta}(t))^{\alpha} \le{}& \frac{r(t_1)|(x^{\Delta}(t_1))^{\alpha}|}{r(t)} \\ &+ b\,\frac{(t-t_1)}{r(t)} + \frac{(\beta-1)\beta^{\beta/(1-\beta)}}{r(t)} \int_{t_1}^{t}\int_{t_1}^{u} a(u,s)p_1^{1/(1-\beta)}(s)p_2^{\beta/(\beta-1)}(s)\ \Delta s\,\Delta u,\end{aligned}$$

or

$$(x^{\Delta}(t))^{\alpha} \le \frac{c_0 t}{r(t)} + \frac{(\beta-1)\beta^{\beta/(1-\beta)}}{r(t)} \int_{t_1}^{t}\int_{t_1}^{t} a(u,s)p_1^{1/(1-\beta)}(s)p_2^{\beta/(\beta-1)}(s)\ \Delta s\,\Delta u,$$

where

$$c_0 = \frac{r(t_1)|(x^{\Delta}(t_1))^{\alpha}|}{t_1} + b.$$

By employing a well-known inequality

$$(a_1+b_1)^{\lambda} \le \sigma_{\lambda}(a_1^{\lambda}+b_1^{\lambda}) \ \text{ for } \ a_1 \ge 0,\ \ b_1 \ge 0 \ \text{ and } \ \lambda > 0,$$

where $\sigma_{\lambda} = 1$ if $\lambda < 1$ and $\sigma_{\lambda} = 2^{\lambda-1}$ if $\lambda \ge 1$, we see that there exist the positive constants c_1 and c_2 depending on α such that

$$x^{\Delta}(t) \le c_1\Big(\frac{t}{r(t)}\Big)^{1/\alpha} + c_2\Big(\frac{1}{r(t)}\int_{t_1}^{t}\int_{t_1}^{u} a(u,s)p_1^{1/(1-\beta)}(s)p_2^{\beta/(\beta-1)}(s)\ \Delta s\,\Delta u\Big)^{1/\alpha}.$$

Integrating this inequality from T_1 to $t \ge t_1$, we obtain

$$\begin{aligned}|x(t)| &\le |x(t_1)| + c_1R(t,t_1) + c_2\int_{t_1}^{t}\left(\frac{1}{r(v)}\int_{t_1}^{v}\int_{t_1}^{u} a(u,s)p_1^{1/(1-\beta)}(s)p_2^{\beta/(\beta-1)}(s)\ \Delta s\,\Delta u\right)^{1/\alpha}\Delta v \\ &\le |x(t_1)| + c_1R(t,t_0) + c_2\int_{t_0}^{t}\left(\frac{1}{r(v)}\int_{t_0}^{v}\int_{t_0}^{u} a(u,s)p_1^{1/(1-\beta)}(s)p_2^{\beta/(\beta-1)}(s)\ \Delta s\,\Delta u\right)^{1/\alpha}\Delta v.\end{aligned}$$

Dividing the above relation by $R(t,t_0)$ and using (3.43) and (4.1), we see that (4.2) holds. The proof is similar if x is eventually negative. □

Theorem 4.2. *Let the conditions* (i)–(ii) *hold with $f_2 = 0$ and $xf_1(t,x) > 0$ for $x \ne 0$ and $t \ge 0$. If x is a non-oscillatory solution of equation* (3.40)*, then equation* (4.2) *holds.*

Proof. Let $x(t)$ be a non-oscillatory solution of equation (3.40) with $f_2 = 0$. First, assume x is eventually positive, say $x(t) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. From (3.40), we find that

$$\left(r(t)(x^{\Delta}(t))^{\alpha}\right)^{\Delta} = -\int_{0}^{t} a(t,s)f_1(s,x(s))\,\Delta s \le \int_{0}^{t_1} a(t,s)f_1(s,x(s))\,\Delta s.$$

Using (3.42) in the above inequality, we obtain $(r(t)(x^\Delta(t))^\alpha)^\Delta \le b$. The rest of the proof is similar to that of the above theorem and hence is omitted. □

Theorem 4.3. *Let the conditions* (i)–(iii) *hold with* $\beta > 1$ *and* $\gamma < 1$ *and assume that there exists a positive rd-continuous function* $\xi : \mathbb{T} \to \mathbb{T}$ *such that*

$$\lim_{t\to\infty} \frac{1}{R(t,t_0)} \int_{t_0}^{t} \left(\frac{1}{r(v)} \int_{t_0}^{v} \int_{t_0}^{u} a(u,v) \times \left[c_1 \xi^{\beta/(\beta-1)}(s) p_1^{1/(1-\beta)}(s) + c_2 \xi^{\gamma/(\gamma-1)}(s) p_2^{1/(1-\gamma)}(s) \right] \Delta s\, \Delta u \right)^{1/\alpha} \Delta v < \infty$$

for some $t_0 \ge 0$, *where* $c_1 = (\beta-1)\beta^{\beta/(1-\beta)}$ *and* $c_2 = (1-\gamma)\gamma^{\gamma/(1-\gamma)}$. *If* x *is a non-oscillatory solution of equation* (3.40), *then* (4.2) *holds.*

Proof. Let x be a non-oscillatory solution of equation (3.40). First, assume x is eventually positive, say $x(t) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Using (ii) and (iii) in equation (3.40), we obtain

$$\left(r(t)(x^\Delta(t))^\alpha\right)^\Delta \le -\int_0^t a(t,s) F(s,x(s))\, \Delta s + \int_{t_1}^{t} a(t,s) \left[\xi(s)x(s) - p_1(s)x^\beta(s)\right] \Delta s + \int_{t_1}^{t} a(t,s)\left[p_2(sx^\gamma(s) - \xi(s)x(s))\right] \Delta s.$$

As in the proof of the above theorems, one can easily show that

$$\left(r(t)(x^\Delta(t))^\alpha\right)^\Delta \le -\int_0^t a(t,s) F(s,x(s))\, \Delta s + \int_{t_1}^{t} a(t,s) \left[(\beta-1)\beta^{\beta/(1-\beta)} \xi^{\beta/(\beta-1)}(s) p_1^{1/(1-\beta)}(s) + (1-\gamma)\gamma^{\gamma/(1-\gamma)} \xi^{\gamma/(1-\gamma)}(s) p_2^{1/(1-\gamma)}(s) \right] \Delta s.$$

The rest of the proof is similar to that of Theorem 4.1 and hence is omitted. □

Theorem 4.4. *Let the conditions* (i)–(iii) *hold with* $\beta > 1$ *and* $\gamma < 1$ *and suppose that there exists a positive rd-continuous function* $\xi : \mathbb{T} \to \mathbb{T}$ *such that*

$$\lim_{t\to\infty} \frac{1}{R(t,t_0)} \int_{t_0}^{t} \left(\frac{1}{r(v)} \int_{t_0}^{v} \int_{t_0}^{u} a(u,v) \xi^{\beta/(\beta-1)}(s) p_1^{1/(1-\beta)}(s)\, \Delta s\, \Delta u \right)^{1/\alpha} \Delta v < \infty$$

and

$$\lim_{t\to\infty} \frac{1}{R(t,t_0)} \int_{t_0}^{t} \left(\frac{1}{r(v)} \int_{t_0}^{v} \int_{t_0}^{u} a(u,v) \xi^{\gamma/(\gamma-1)}(s) p_1^{1/(1-\gamma)}(s)\, \Delta s\, \Delta u \right)^{1/\alpha} \Delta v < \infty$$

for some $t_0 \ge 0$. *If* x *is a non-oscillatory solution of equation* (3.40), *then equation* (4.2) *holds.*

For the cases where both f_1 and f_2 are super-linear ($\beta > \gamma > 1$) or else sub-linear ($1 > \beta > \gamma > 0$), we have the following result.

Theorem 4.5. *Let the conditions* (i)–(iii) *hold with* $\beta > \gamma$ *and assume*

$$\lim_{t\to\infty} \frac{1}{R(t,t_0)} \int_{t_0}^{t} \left(\frac{1}{r(v)} \int_{t_0}^{v} \int_{t_0}^{u} a(u,v) p_1^{\gamma/(\gamma-\beta)}(s) p_2^{\beta/(\beta-\gamma)}(s)\, \Delta s\, \Delta u \right)^{1/\alpha} \Delta v < \infty$$

for some $t_0 \ge 0$. *If* x *is a non-oscillatory solution of equation* (3.40), *then* (4.2) *holds.*

Proof. Let x be a non-oscillatory solution of (3.40). First, assume x is eventually positive, say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Using the conditions (ii) and (iii) in equation (3.40), we get

$$\big(r(t)(x^\Delta(t))^\alpha\big)^\Delta \leq -\int_0^{t_1} a(t,s)F(s,x(s))\,\Delta s + \int_{t_1}^{t} a(t,s)\big[p_2(s)x^\gamma(s) - p_1(s)x^\beta(s)\big]\,\Delta s.$$

By applying Young's inequality with

$$n = \frac{\beta}{\gamma}, \quad X = x^\gamma(s), \quad Y = \frac{\gamma p_2(s)}{\beta p_1(s)}, \quad m = \frac{m}{\beta-\gamma},$$

we obtain

$$\begin{aligned} p_2(s)x^\gamma(s) - p_1(s)x^\beta(s) &= \frac{\beta}{\gamma}\,p_1(s)\Big[x^\gamma(s)\,\frac{\gamma p_2(s)}{\beta p_1(s)} - \frac{\gamma}{\beta}\,(x^\gamma(s))^{\beta/\gamma}\Big] \\ &= \frac{\beta}{\gamma}\,p_1(s)\Big[XY - \frac{1}{n}\,X^n\Big] \leq \frac{\beta}{\gamma}\,p_1(s)\Big(\frac{1}{m}\,Y^m\Big) = \Big(\frac{\beta-\gamma}{\gamma}\Big)\Big[\frac{\gamma}{\beta}\,p_2(s)\Big]^{\beta/(\beta-\gamma)}(p_1(s))^{\gamma/(\gamma-\beta)}. \end{aligned}$$

The rest of the proof is similar to that of Theorem 4.1 and hence is omitted. □

5 Oscillatory results for (3.41)

In order to achieve the results, some necessary assumptions are given as follows:

(I) $e, r : \mathbb{T} \to \mathbb{R}$ and $a : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ are rd-continuous, $r(t) > 0$ and $a(t,s) \geq 0$ for $t > s$ and there exist rd-continuous functions $k, m : \mathbb{T} \to \mathbb{R}^+$ such that

$$a(t,s) \leq k(t)m(s), \quad t \geq s, \tag{5.1}$$

with

$$k_1 := \sup_{t\geq 0} k(t) < \infty, \quad k_2 = \sup_{t\geq 0} \int_0^t m(s)\,\Delta s < \infty.$$

In this case, condition (3.42) is satisfied with $k = k_1k_2$.

(II) $F : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous and assume that there exists an rd-continuous function $q : \mathbb{T} \to (0,\infty)$ and a real number β with $0 < \beta \leq 1$ such that

$$xF(t,x) \leq q(t)x^{\beta+1} \ \text{ for } \ x \neq 0 \ \text{ and } \ t \geq 0.$$

In what follows,

$$g_\pm(t) = e(t) \pm k_1k_1(1-\beta)\beta^{\beta/(1-\beta)} \int_0^t p^{\beta/(\beta-1)}(s)q^{1/(1-\beta)}(s)m^{1/(1-\beta)}(s)\,\Delta s,$$

where $0 < \beta < 1$, $p \in C_{rd}(\mathbb{T},(0,\infty))$.

First, we give some sufficient conditions under which non-oscillatory solutions x of equation (3.41) satisfy

$$x(t) = O(t) \ \text{ as } \ t \to \infty.$$

Theorem 5.1. *Let $0 < \beta < 1$ and the conditions* (I) *and* (II) *hold, assume the function $1/r(t)$ is bounded, and for some $t_0 \geq 0$,*

$$\int_{t_0}^{\infty} \frac{s}{r(s)}\,\Delta s < \infty. \tag{5.2}$$

Let $p \in C_{rd}(\mathbb{T},(0,\infty))$ such that

$$\int_{t_0}^{\infty} sp(s)\,\Delta s < \infty. \tag{5.3}$$

If

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \frac{1}{r(u)} \int_{t_0}^{u} g_-(s,p)\,\Delta s\,\Delta u < \infty,\\
&\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \frac{1}{r(u)} \int_{t_0}^{u} g_+(s,p)\,\Delta s\,\Delta u > -\infty,
\end{aligned} \tag{5.4}$$

then every non-oscillatory solution $x(t)$ of (3.41) satisfies

$$\limsup_{t\to\infty} \frac{|x(t)|}{t} < \infty.$$

Proof. Let x be a non-oscillatory solution of (3.41). First, assume x is eventually positive, say $x(t) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Using condition (5.1) in (3.41), we have

$$\big(r(t)(x^{\Delta}(t))\big)^{\Delta} \le e(t) - \int_{0}^{t_1} a(t,s)F(s,x(s))\,\Delta s + \int_{t_1}^{t} a(t,s)q(s)x^{\beta}(s)\,\Delta s \tag{5.5}$$

for $t \ge t_1$. Let

$$c := \max_{0\le t\le t_1} |F(t,x(t))| < \infty.$$

By assumption (5.1), we obtain

$$\left| -\int_{0}^{t_1} a(t,s)F(s,x(s))\,\Delta s\right| \le c\int_{0}^{t_1} a(t,s)\,\Delta s \le ck_1k_2 =: b, \quad t \ge t_1.$$

Hence from (5.5), we have

$$\big(r(t)(x^{\Delta}(t))\big)^{\Delta} \le e(t) + b + k_1\int_{t_1}^{t} \big[m(s)q(s)x^{\beta}(s) - p(s)x(s)\big]\,\Delta s + k_1\int_{t_1}^{t} p(s)x(s)\,\Delta s, \quad t \ge t_1.$$

Applying (3.45), we obtain

$$\lambda = \beta, \quad X = (qm)^{1/\beta}x, \quad Y = \Big(\frac{1}{\beta}\,p(mq)^{-1/\beta}\Big)^{\frac{1}{\beta-1}},$$

we get

$$m(s)q(s)x^{\beta}(s) - p(s)x(s) \le (1-\beta)\beta^{\beta/(1-\beta)}p^{\beta/(\beta-1)}(s)m^{1/(1-\beta)}(s)q^{1/(1-\beta)}(s).$$

Thus, we obtain

$$r(t)x^{\Delta}(t) \le r(t_1)x^{\Delta}(t_1) + \int_{t_1}^{t} g_+(s,p)\,\Delta s + b(t-t_1) + k_1\int_{t_1}^{t}\int_{t_1}^{u} p(s)x(s)\;\Delta s\,\Delta u$$

for $t \ge t_1$. Hence, we obtain

$$r(t)x^{\Delta}(t) \le r(t_1)x^{\Delta}(t_1) + \int_{t_1}^{t} g_+(s,p)\,\Delta s + b(t-t_1) + k_1 t\int_{t_1}^{t} p(s)x(s)\,\Delta s, \quad t \ge t_1,$$

and so,

$$x^{\Delta}(t) \le \frac{r(t_1)x^{\Delta}(t_1)}{r(t)} + \frac{1}{r(t)}\int_{t_1}^{t} g_+(s,p)\,\Delta s + \frac{1}{r(t)}\,b(t-t_1) + \frac{1}{r(t)}\,k_1 t\int_{t_1}^{t} p(s)x(s)\,\Delta s, \quad t \ge t_1.$$

Integrating this inequality from t_1 to t and using (5.2) and the fact that the function $\frac{t}{r(t)}$ is bounded for $t \ge t_1$, say by k_3, we see that

$$\begin{aligned} x(t) &\le x(t_1) + r(t_1)x^{\Delta}(t_1)\int_{t_1}^{t} \frac{1}{r(t)}\,\Delta s \\ &\quad + \int_{t_1}^{t} \frac{1}{r(u)}\int_{t_1}^{u} g_+(s)\,\Delta s\,\Delta u + b\int_{t_1}^{t} \frac{s}{r(s)}\,\Delta s + k_1k_3\int_{t_1}^{t}\int_{t_1}^{u} p(s)x(s)\,\Delta s\,\Delta u, \quad t \ge t_1. \end{aligned}$$

Once again, we have

$$\begin{aligned} x(t) &\le x(t_1) + r(t_1)x^{\Delta}(t_1)\int_{t_1}^{t} \frac{1}{r(t)}\,\Delta s \\ &\quad + \int_{t_1}^{t} \frac{1}{r(u)}\int_{t_1}^{u} g_+(s)\,\Delta s\,\Delta u + b\int_{t_1}^{t} \frac{s}{r(s)}\,\Delta s + k_1k_3 t\int_{t_1}^{t} p(s)x(s)\,\Delta s, \quad t \ge t_1, \end{aligned}$$

and so,

$$\frac{x(t)}{t} \le c_1 + c_2\int_{t_1}^{t} sp(s)\Big(\frac{x(s)}{s}\Big)\,\Delta s, \quad t \ge t_1. \tag{5.6}$$

Noting (5.2) and (5.4), $c_2 = k_1k_3$ and c_1 is an upper bound for

$$\frac{1}{t}\left[x(t_1) + r(t_1)x^{\Delta}(t_1)\int_{t_1}^{t} \frac{1}{r(s)}\,\Delta s + \int_{t_1}^{t} \frac{1}{r(u)}\int_{t_1}^{u} g_+(s)\,\Delta s\,\Delta u + b\int_{t_1}^{t} \frac{s}{r(s)}\,\Delta s\right]$$

for $t \ge t_1$, and applying Gronwall's inequality to inequality (5.6) and then using condition (5.3), we have

$$\limsup_{t\to\infty} \frac{x(t)}{t} < \infty.$$

Similarly, we can do for the eventually negative solution x for equation (3.41). □

Theorem 5.2. *Let* $0 < \beta < 1$, *and conditions* (I), (II), (5.2), (5.3) *and* (5.4) *hold, assume the function* $t/r(t)$ *is bounded, and there is a function* $p \in C_{rd}(\mathbb{T},(0,\infty))$ *such that* (5.3) *holds. If for every* $0 < M < 1$,

$$\limsup_{t\to\infty}\left[Mt + \int_{t_0}^{u} g_-(s,p)\,\Delta s\,\Delta u\right] = \infty, \quad \liminf_{t\to\infty}\left[Mt + \int_{t_0}^{u} g_+(s,p)\,\Delta s\,\Delta u\right] = -\infty, \tag{5.7}$$

then equation (3.41) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of equation (3.41), say $x(t) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. The proof when $x(t)$ is eventually negative is similar. Proceeding as in the proof of Theorem 4.1, we arrive at (4.1). Therefore,

$$\begin{aligned} x(t) \le x(t_1) + r(t_1)x^{\Delta}(t_1) \int_{t_1}^{t} \frac{1}{r(s)}\,\Delta s \\ + \int_{t_1}^{t} \frac{1}{r(u)} \int_{t_1}^{u} g_{+(s,p)}\,\Delta s\,\Delta u + b\int_{t_1}^{\infty} \frac{s}{r(s)}\,\Delta s + k_1k_3t\int_{t_1}^{\infty} sp(s)\Big(\frac{x(s)}{s}\Big)\,\Delta s, \quad t \ge t_1. \end{aligned}$$

Clearly, the conclusion of Theorem 4.1 holds. This, together with (5.2), imply that

$$x(t) \le M_1 + mt + \int_{t_1}^{t} \frac{1}{r(u)} \int_{t_1}^{u} g_{+}(s,p)\,\Delta s\,\Delta u, \tag{5.8}$$

where M_1 and M are the positive real numbers. Note that we make $M < 1$ possible by increasing the size of t_1. Finally, taking $\liminf$ in (5.8) as $t \to \infty$ and using (5.7) result in a contradiction with the fact that $x(t)$ is eventually positive. □

Moreover, Negi et al. [50] revealed an open problem given in the paper [35] by Grace et al.

As a future direction on the problem, we may ask for weakening condition (H). It should also be noted that the problem is open in the super-linear case $\lambda > 1$.

Motivated by the above, we establish some sufficient conditions for oscillation of the following, more general, p-Laplacian ($p > 1$) dynamic equation with $z(t_*) = 0$ for fixed $t_* \in \mathbb{T}$:

$$(r(t)\varPhi_p(z(t)))^{\Delta} = e(t) - \int_{t_*}^{t} k(t,\tau) f(\tau, \varPhi_p(z(\tau)))\,\Delta\tau, \tag{5.9}$$

on a time scale $\mathbb{T}$ such that $\sup(\mathbb{T}) = \infty$, where $\varPhi_p(z) = z|z|^{p-1}$; $[t_*,\infty)_{\mathbb{T}}$ is a time-scale interval or half-line; $r : [t_*,\infty)_{\mathbb{T}} \to \mathbb{R}^+$ is rd-continuous on $\mathbb{T}$; $e : [t_*,\infty)_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous on $\mathbb{T}$; $k(t,\cdot) : [t_*,\infty)_{\mathbb{T}} \to \mathbb{R}^+$ and $k(\cdot,s) : [t_*,\infty)_{\mathbb{T}} \to \mathbb{R}^+$ are rd-continuous at each $t \in \mathbb{T}$ and for fixed $s \in \mathbb{T}$, respectively. $f(\cdot,z^*) : [t_*,\infty)_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous for each $z^* \in \mathbb{R}$; $f(t,\cdot) : [t_*,\infty)_{\mathbb{T}} \to \mathbb{R}$ is a rd-continuous function at each $t \in \mathbb{T}$.

In [50], Negi et al. conferred some new oscillation criteria of the first-order p-Laplacian nonlinear dynamic equations on time scales. Moreover, the Kamenev- and Philos-type oscillation criteria were discussed. Consequently, the idea of techniques proposed to obtain results in his paper is implemented to improve and extend the results in the literature. Further, some outcomes are demonstrated through some interesting examples.

They provided three new oscillatory results for equation (5.9) under certain conditions. Equation (5.9) is a half-linear dynamic equation which appears in the various real world problems and phenomena, for instance, the turbulent flow of a polytropic gas in a porus medium and the study of non-newtonian fluid theory.

We introduce some auxiliary assumptions which will help in our investigations.

1. $zf(t,\varPhi_p(z)) > 0$ for $z \ne 0$ and $k(t,s) > 0$ for all $s \le t$, for all $s,t \in \mathbb{T}$;
2. there exists $M : [t_*,\infty)_{\mathbb{T}^\kappa} \to \mathbb{R}$ such that $M(t_*) = 0$ and $e(t) = M^{\Delta}(t)$ for all $t \ge t_*$.

Theorem 5.3. *If conditions* (1), (2) *hold and there exists a delta-derivative function* $S : [t_*,\infty)_{\mathbb{T}} \to \mathbb{R}^+$ *such that* $S^{\Delta}(t) \ge 0$ *with* $\frac{M(t)}{S(t)} \to 0$ *as* $t \to \infty$ *and*

$$\int_{t_*}^{\infty} \frac{M(\sigma(\tau))S^{\Delta}(\tau)}{S(\sigma(\tau))S(\tau)}\,\Delta\tau = 0, \tag{5.10}$$

then equation (5.9) *has an oscillatory solution on* $[t_*,\infty)_{\mathbb{T}}$.

Proof. Let us assume that equation (5.9) has a non-oscillatory solution. Then, without loss of generality, assume that $z(t) > 0$ on $[t_*, \infty)_{\mathbb{T}}$ for $t_* \in \mathbb{T}$. Hence, by the assumption, there exists a delta-derivative function S and then we define the following function:

$$W(t) = \frac{M(t) - r(t)\Phi_p(z(t))}{S(t)}, \quad t_* \leq t. \tag{5.11}$$

Differentiating equation (5.11) with respect to t and using equation (5.9), we obtain

$$\begin{aligned} W^{\Delta}(t) &= \frac{\big(M(t) - r(t)\Phi_p(z(t))\big)^{\Delta} S(t)}{S(t)S(\sigma(t))} \\ &\geq \frac{\big[M(\sigma(t)) - r(\sigma(t))\Phi_p(z(\sigma(t)))\big] S^{\Delta}(t)}{S(t)S(\sigma(t))} - \frac{M(\sigma(t))S^{\Delta}(t)}{S(\sigma(t))S(t)}. \end{aligned} \tag{5.12}$$

Integrating equation (5.12) from t_* to t and using equation (5.10) we get the following relation:

$$\begin{aligned} -\int_{t_*}^{t} \frac{M(\sigma(\tau))S^{\Delta}(\tau)}{S(\sigma(\tau))S(\tau)}\,\Delta\tau &\leq W(t) \leq \frac{M(t)}{S(t)} \\ \longrightarrow 0 &\leq W(t) \leq 0 \longleftarrow \quad \text{as } t \to \infty. \end{aligned} \tag{5.13}$$

Clearly, $W(t) = 0$, which implies $r(t)\Phi_p(z(t)) = M(t)$, and putting this value into the original equation (5.9) and using $e(t) = M^{\Delta}(t)$, we have that the right-hand side (integral part) of equation (5.9) is zero, which is impossible because k and f are positive functions. Hence, we get a contradiction. A similar argument will be hold when $z(t) < 0$ on $[t_*, \infty)_{\mathbb{T}}$. This completes the proof. □

Corollary 5.1. *If conditions* (1), (2) *hold and* $M(t) \to 0$ *as* $t \to \infty$, *then equation* (5.9) *has an oscillatory solution on* $[t_*, \infty)_{\mathbb{T}}$.

Theorem 5.4. *If conditions* (1), (2) *hold and there exists a delta-derivative function* $S : [t_*, \infty)_{\mathbb{T}} \to \mathbb{R}^+$ *such that* $S^{\Delta}(t) \geq 0$ *with the conditions*

$$\liminf_{t\to\infty} \frac{1}{r(t)} \left[M(t) + S(t) \int_{t_*}^{t} \frac{M(\sigma(\tau))S^{\Delta}(\tau)}{S(\sigma(\tau))S(\tau)}\,\Delta\tau\right] = -\infty \tag{5.14}$$

and

$$\limsup_{t\to\infty} \frac{1}{r(t)} \left[M(t) + S(t) \int_{t_*}^{t} \frac{M(\sigma(\tau))S^{\Delta}(\tau)}{S(\sigma(\tau))S(\tau)}\,\Delta\tau\right] = +\infty,$$

then equation (5.9) *oscillates on* $[t_*, \infty)_{\mathbb{T}}$.

Proof. Following the proof of Theorem 5.3, integrating equation (5.12) from t_* to t, we obtain

$$\Phi_p(z(t)) \leq \frac{1}{r(t)} \left[M(t) + S(t) \int_{t_*}^{t} \frac{M(\sigma(\tau))S^{\Delta}(\tau)}{S(\sigma(\tau))S(\tau)}\,\Delta\tau\right]. \tag{5.15}$$

Now, if we take $\liminf\limits_{t\to\infty}$ on both sides of equation (5.15), we get a contradiction as $\Phi_p(z(t))$ is positive, but due to equation (5.14), the right-hand side of the later inequality approaches $-\infty$ as $t \to \infty$. □

Theorem 5.5. *Assume condition* (1) *holds and the relations*

$$\limsup_{t\to\infty} \frac{1}{r(t)} \int_{t_*}^{t} e(\tau)\,\Delta\tau = \infty$$

and

$$\liminf_{t\to\infty} \frac{1}{r(t)} \int_{t_*}^{t} e(\tau)\,\Delta\tau = -\infty \tag{5.16}$$

are satisfied, then every solution of equation (5.9) *oscillates on* $[t_*, \infty)_{\mathbb{T}}$.

Proof. On the contrary, we assume that the solution of equation (5.9) is non-oscillatory, so it is necessary to assume that it is either positive or negative over the half-time scale interval. That is, there exists $t_* \in \mathbb{T}$ such that $z(t) > 0$ on $[t_*, \infty)_{\mathbb{T}}$. Indeed, from equation (5.9) and condition (1), we have

$$(r(t)\Phi_p(z(t)))^{\Delta}(t) \le e(t) \ \text{ for } \ t_* \le t. \tag{5.17}$$

Integrating the later inequality from t_* to t, we obtain

$$\Phi_p(z(t))(t) \le r(t_*)\Phi_p(z(t_*)) + \frac{1}{r(t)} \int_{t_*}^{t} e(\tau)\,\Delta\tau.$$

Since $z(t_*) = 0$, this implies that

$$\Phi_p(z(t))(t) \le \frac{1}{r(t)} \int_{t_*}^{t} e(\tau)\,\Delta\tau.$$

Finally, we take $\liminf\limits_{t\to\infty}$ on both sides of the latter inequality, and then we get a contradiction that $z(t) > 0$, but we have one of the conditions of equation (5.16). Thus, we obtain the required result. $\square$

They imposed the above techniques to find oscillatory solutions of the first-order integro-dynamic equation (3.19) for $z(t_*) = 0$. Therefore, keeping Theorems 5.3, 5.4 and 5.5 in mind, we immediately obtain the following results.

Theorem 5.6. *If conditions* (1), (2) *hold and there exists a delta-derivative function* $S : [t_*, \infty)_{\mathbb{T}} \to \mathbb{R}^+$ *such that* $S^{\Delta}(t) \ge 0$ *with* $\frac{M(t)}{S(t)} \to 0$ *as* $t \to \infty$ *and*

$$\int_{t_*}^{\infty} \frac{M(\sigma(\tau))S^{\Delta}(\tau)}{S(\sigma(\tau))S(\tau)}\,\Delta\tau = 0,$$

then equation (3.19) *oscillates on* $[t_*, \infty)_{\mathbb{T}}$.

Proof. Following the steps of the proof of Theorem 5.3 and using equation (5.13), we must have

$$W(t) = \frac{M(t) - z(t)}{S(t)}, \ \ t_* \le t. \tag{5.18}$$

Consequently, from equations (5.18), (5.12) and (5.13) (see Theorem 5.3), we have $M(t) = z(t)$, which gives a contradiction. $\square$

Theorem 5.7. *If conditions* (1), (2) *hold and there exists a delta-derivative function* $S : [t_*, \infty)_{\mathbb{T}} \to \mathbb{R}^+$ *such that* $S^{\Delta}(t) \ge 0$, *and the relations*

$$\liminf_{t\to\infty} \left[M(t) + S(t) \int_{t_*}^{t} \frac{M(\sigma(\tau))S^{\Delta}(\tau)}{S(\sigma(\tau))S(\tau)}\,\Delta\tau \right] = -\infty$$

and

$$\limsup_{t\to\infty} \left[M(t) + S(t) \int_{t_*}^{t} \frac{M(\sigma(\tau))S^{\Delta}(\tau)}{S(\sigma(\tau))S(\tau)}\,\Delta\tau \right] = +\infty$$

hold, then equation (3.19) *oscillates on* $[t_*, \infty)_{\mathbb{T}}$.

Theorem 5.8. *If condition* (1) *holds, as well as the conditions*

$$\limsup_{t\to\infty} \frac{1}{r(t)} \int\limits_{t_*}^{t} e(\tau)\,\Delta\tau = \infty$$

and

$$\liminf_{t\to\infty} \frac{1}{r(t)} \int\limits_{t_*}^{t} e(\tau)\,\Delta\tau = -\infty$$

hold, then equation (3.19) *oscillates on* $[t_*,\infty)_{\mathbb{T}}$*.*

Remark 5.1. In [35], Grace and Zafar discussed the oscillatory behaviour of solutions of equation (3.19) with an assumption:

- **H:** $k(t,s) \le a(t)q(s)$ for $s \le t$ and $0 < zf(t,z) < m(t)|z|^{\lambda+1}$, where $m,a,q : [t_*,\infty)_{\mathbb{T}} \to \mathbb{R}^+$ are rd-continuous functions.

They only discussed the case $0 < \lambda \le 1$, whereas the problem is still open for the case $\lambda > 1$. Therefore, we look at the conditions given in [35], and then take a quick look at the conditions proposed in the present paper, we comply that the proposed results hold for all real values of λ. That is, with the condition **H** in hand, we can obtain some oscillatory results for equation (3.19) with the extreme value $\lambda > 1$ (see Theorems 5.6–5.8).

Furthermore, we also discuss the oscillatory solutions of the second-order forced integro-dynamic equation on time scales which was considered by Agarwal et al. [5], see (3.41).

We now give two oscillatory theorems for equation (3.41) on $[0,\infty)_{\mathbb{T}}$, fixed $0 \in \mathbb{T}$ with $z^{\Delta}(0) = 0$. The proofs are omitted here, since they are similar to the previous results.

Theorem 5.9. *Assume that conditions* (1), (2) *hold and there exists a delta-derivative function* $S : [t_*,\infty)_{\mathbb{T}} \to \mathbb{R}^+$ *such that* $S^{\Delta}(t) \ge 0$, *and the relations*

$$\liminf_{t\to\infty} \int\limits_{0}^{t} \frac{1}{r(s)} \left[M(s) + S(s) \int\limits_{0}^{s} \frac{M(\sigma(\tau))S^{\Delta}(\tau)}{S(\sigma(\tau))S(\tau)}\,\Delta\tau \right] \Delta s = -\infty$$

and

$$\limsup_{t\to\infty} \int\limits_{0}^{t} \frac{1}{r(s)} \left[M(s) + S(s) \int\limits_{0}^{s} \frac{M(\sigma(\tau))S^{\Delta}(\tau)}{S(\sigma(\tau))S(\tau)}\,\Delta\tau \right] \Delta s = +\infty$$

hold, then equation (3.41) *oscillates on* $[t_*,\infty)_{\mathbb{T}}$*.*

Theorem 5.10. *Assume that conditions* (1) *and* $r(t_*) = 0$ *hold. Moreover, the following erlations*

$$\limsup_{t\to\infty} \int\limits_{t_*}^{t} \left(\frac{1}{r(s)} \int\limits_{t_*}^{s} e(\tau)\,\Delta\tau \right) \Delta s = \infty$$

and

$$\liminf_{t\to\infty} \int\limits_{t_*}^{t} \left(\frac{1}{r(s)} \int\limits_{t_*}^{s} e(\tau)\,\Delta\tau \right) \Delta s = -\infty$$

are satisfied, then every solution of equation (3.41) *oscillates on* $[t_*,\infty)_{\mathbb{T}}$*.*

Finally, we mention the last result of our paper for a case $r(t_*) \ne 0$.

Theorem 5.11. *Assume that the conditions in Theorem 5.10 are satisfied. Moreover, if*

$$\int\limits_{t_*}^{\infty} \frac{1}{r(\tau)}\,\Delta\tau < \infty$$

holds, then equation (3.41) *oscillates on* $[t_*,\infty)_{\mathbb{T}}$*.*

6 Kamenev- and Philos-type criteria

Theorem 6.1. *Assume conditions* (1), (2) *hold. Also, there exist a number* $\mathcal{L} > 0$ *and a delta-derivative function* $S : [t_*, \infty)_{\mathbb{T}} \to \mathbb{R}^+$ *such that* $S^{\Delta}(t) \geq 0$ *with the following conditions:*

$$\limsup_{t\to\infty} \int_{t_*}^{t} \frac{M(\sigma(\tau))}{S(\sigma(\tau))} \left[((t-\tau)^{\mathcal{L}})^{\Delta_\tau} - (t-\tau)^{\mathcal{L}} \frac{S^{\Delta}(\tau)}{S(\tau)} \right] \Delta\tau = \infty$$

and

$$\liminf_{t\to\infty} \int_{t_*}^{t} \frac{M(\sigma(\tau))}{S(\sigma(\tau))} \left[((t-\tau)^{\mathcal{L}})^{\Delta_\tau} - (t-\tau)^{\mathcal{L}} \frac{S^{\Delta}(\tau)}{S(\tau)} \right] \Delta\tau = -\infty.$$

Then equation (5.9) *has an oscillatory solution on* $[t_*, \infty)_{\mathbb{T}}$.

Proof. The partial proof of this theorem follows from the proof of Theorem 5.3. Then, further, there exists $\mathcal{L} > 0$ and multiplying equation (5.12) by $(t-\tau)^{\mathcal{L}}$ for $\tau < t$, and then integrating from t_* to t, we obtain

$$\int_{t_*}^{t} (t-\tau)^{\mathcal{L}} W^{\Delta}(\tau)\, \Delta\tau \geq -\int_{t_*}^{t} (t-\tau)^{\mathcal{L}} \frac{M(\sigma(\tau)) S^{\Delta}(\tau)}{S(\sigma(\tau)) S(\tau)}\, \Delta\tau. \tag{6.1}$$

Now, first we expand the left-hand side of the above inequality as follows:

$$\int_{t_*}^{t} (t-\tau)^{\mathcal{L}} W^{\Delta}(\tau)\, \Delta\tau = -(t-t_*)^{\mathcal{L}} W(t_*) - \int_{t_*}^{t} ((t-\tau)^{\mathcal{L}})^{\Delta_\tau} W(\sigma(\tau))\, \Delta\tau.$$

Since

$$W(t) = \frac{M(t) - (r(t)\Phi_p(z(t)))}{S(t)},$$

we get

$$\begin{aligned}\int_{t_*}^{t} (t-\tau)^{\mathcal{L}} W^{\Delta}(\tau)\, \Delta\tau &= -(t-t_*)^{\mathcal{L}} W(t_*) - \int_{t_*}^{t} ((t-\tau)^{\mathcal{L}})^{\Delta_\tau} \frac{M(\tau) - (r(\tau)\Phi_p(z(\tau)))}{S(\tau)}\, \Delta\tau \\ &= -(t-t_*)^{\mathcal{L}} W(t_*) - \int_{t_*}^{t} ((t-\tau)^{\mathcal{L}})^{\Delta_\tau} \frac{M(\sigma(\tau))}{S(\tau)}\, \Delta\tau + \int_{t_*}^{t} ((t-\tau)^{\mathcal{L}})^{\Delta_\tau} \frac{r(\tau)\Phi_p(z(\tau))}{S(\tau)}\, \Delta\tau.\end{aligned}$$

By using the fact that $((t-\tau)^{\mathcal{L}})^{\Delta_\tau} \leq -\mathcal{L}(t-\sigma\tau)^{\mathcal{L}-1} \leq 0$ given in [57] by Saker, $r(t) > 0$, $S(t) > 0$ and $z(t) > 0$, we obtain

$$\int_{t_*}^{t} (t-\tau)^{\mathcal{L}} W^{\Delta}(\tau)\, \Delta\tau \leq -(t-t_*)^{\mathcal{L}} W(t_*) - \int_{t_*}^{t} ((t-\tau)^{\mathcal{L}})^{\Delta_\tau} \frac{M(\sigma(\tau))}{S(\tau)}\, \Delta\tau. \tag{6.2}$$

It follows from equations (6.1) and (6.2) that

$$\int_{t_*}^{t} \frac{M(\sigma(\tau))}{S(\tau)} \left[((t-\tau)^{\mathcal{L}})^{\Delta_\tau} - (t-\tau)^{\mathcal{L}} \frac{S^{\Delta}(\tau)}{S(\tau)} \right] \Delta\tau \leq -(t-t_*)^{\mathcal{L}} W(t_*),$$

which implies that

$$\liminf_{t\to\infty} \frac{1}{t^{\mathcal{L}}} \int_{t_*}^{t} \frac{M(\sigma(\tau))}{S(\tau)} \left[((t-\tau)^{\mathcal{L}})^{\Delta_\tau} - (t-\tau)^{\mathcal{L}} \frac{S^{\Delta}(\tau)}{S(\tau)} \right] \Delta\tau \leq -\liminf_{t\to\infty} \left(1 - \frac{t_*}{t}\right)^{\mathcal{L}} W(t_*) < \infty.$$

Thus, we arrive at a contradiction with the fact that the left-hand side of the above inequality is $-\infty$. $\square$

Theorem 6.2. *If condition* (1) *holds and there exists* $\mathcal{L} > 0$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^{\mathcal{L}}} \int_{t_*}^{t} (t-\tau)^{\mathcal{L}} e(\tau)\,\Delta\tau = \infty$$

and

$$\liminf_{t\to\infty} \frac{1}{t^{\mathcal{L}}} \int_{t_*}^{t} (t-\tau)^{\mathcal{L}} e(\tau)\,\Delta\tau = -\infty,$$

then equation (5.9) *has an oscillatory solution on* $[t_*, \infty)_{\mathbb{T}}$*.*

Proof. We will prove this theorem by contradiction. So, let us assume that equation (5.9) has a non-oscillatory solution, which implies that there exists $t_* \in \mathbb{T}$ such that the solution "$z(t)$" of equation (5.9) is either $z(t) > 0$ or $z(t) < 0$ on $[t_*, \infty)_{\mathbb{T}}$. Without loss of generality, let $z(t)$ be a positive solution of equation (5.9). Now, since there exists $\mathcal{L} > 0$ and multiplying equation (5.17) by $(t-\tau)^{\mathcal{L}}$, and then integrating from t_* to t, we get

$$\int_{t_*}^{t} (t-\tau)^{\mathcal{L}} (r(\tau)\Phi_p(z(\tau)))^{\Delta}(\tau)\,\Delta\tau \le \int_{t_*}^{t} (t-\tau)^{\mathcal{L}} e(\tau)\,\Delta\tau. \tag{6.3}$$

From Theorem 6.1, it is easy to observe that

$$\int_{t_*}^{t} (t-\tau)^{\mathcal{L}} (r(\tau)\Phi_p(z(\tau)))^{\Delta}(\tau)\,\Delta\tau \ge -(t-t_*)^{\mathcal{L}} r(t_*)\Phi_p(z(t_*)). \tag{6.4}$$

From equations (6.3) and (6.4), we have

$$-(t-t_*)^{\mathcal{L}} r(t_*)\Phi_p(z(t_*)) \le \int_{t_*}^{t} (t-\tau)^{\mathcal{L}} e(\tau)\,\Delta\tau. \tag{6.5}$$

Finally, dividing equation (6.5) by $t^{\mathcal{L}}$ and then taking $\liminf\limits_{t\to\infty}$ on both sides, we obtain

$$\liminf_{t\to\infty} \left\{ -\Big(1-\frac{t_*}{t}\Big)^{\mathcal{L}} r(t_*)\Phi_p(z(t_*)) \right\} \le \liminf_{t\to\infty} \frac{1}{t^{\mathcal{L}}} \int_{t_*}^{t} (t-\tau)^{\mathcal{L}} e(\tau)\,\Delta\tau,$$

thus, we get a contradiction because the left-hand side is finite, but the right-hand side is $-\infty$. Therefore, the proof is completed. □

7 Philos-type criteria

In this section, we investigate the well-known Philos-type oscillation criteria. These criteria generalize the previous known Kamenev-type criteria. For these criteria, we require some preliminaries.

Let us define $\mathbb{D} := \{(t,s) :\ t_* \le s \le t\}$, then there exists a *rd*-continuous function $\mathbb{P} : \mathbb{D} \to \mathbb{R}$ with the conditions

$$\begin{cases} \mathbb{P}(t,t) = 0, \\ \mathbb{P}(t,s) > 0, \\ \mathbb{P}^{\Delta_s}(t,s) \le 0 \ \text{ for } \ t_* \le s < t < \infty, \end{cases} \tag{7.1}$$

where $\mathbb{P}^{\Delta_s}(t,s)$ is a partial derivative of the function $\mathbb{P}(t,s)$ with respect to s.

Theorem 7.1. *Let conditions* (1), (2) *and the condition of equality in* (7.1) *hold. Moreover, assume that there exists a delta-derivative function* $S : [t_*, \infty)_{\mathbb{T}} \to \mathbb{R}^+$ *such that* $S^\Delta(t) \geq 0$ *and*

$$\limsup_{t\to\infty} \frac{1}{\mathbb{P}(t,t_*)} \int_{t_*}^{t} \left[\frac{\mathbb{P}^{\Delta_\tau}(t,\tau)M(\tau)}{S(\tau)} - \frac{\mathbb{P}(t,\sigma(\tau))M(\sigma(\tau))S^\Delta(\tau)}{S(\sigma(\tau))S(\tau)} \right] \Delta\tau = \infty \tag{7.2}$$

and

$$\liminf_{t\to\infty} \frac{1}{\mathbb{P}(t,t_*)} \int_{t_*}^{t} \left[\frac{\mathbb{P}^{\Delta_\tau}(t,\tau)M(\tau)}{S(\tau)} - \frac{\mathbb{P}(t,\sigma(\tau))M(\sigma(\tau))S^\Delta(\tau)}{S(\sigma(\tau))S(\tau)} \right] \Delta\tau = -\infty,$$

then equation (5.9) *has an oscillatory solution on* $[t_*, \infty)_{\mathbb{T}}$.

Proof. Assume that equation (5.9) has a non-oscillatory solution, meaning that there exists $t_* \in \mathbb{T}$ such that equation (5.9) has either positive or negative solution on $[t_*, \infty)_{\mathbb{T}}$. Thus, without loss of generality, we assume that $z(t)$ is a positive solution of equation (5.9). Now, multiplying equation (5.12) by $\mathbb{P}(t,\sigma(\tau))$ for $\tau \leq t$, after integrating from t_* to t, we obtain

$$\int_{t_*}^{t} \mathbb{P}(t,\sigma(\tau))W^\Delta(\tau)\,\Delta\tau \geq -\int_{t_*}^{t} \frac{\mathbb{P}(t,\sigma(\tau))M(\sigma(\tau))S^\Delta(\tau)}{S(\sigma(\tau))S(\tau)}\,\Delta\tau. \tag{7.3}$$

First, to expand the left-hand side of equation (7.3) with equation (7.1) in hand, we obtain

$$\begin{aligned} \int_{t_*}^{t} \mathbb{P}(t,\sigma(\tau))W^\Delta(\tau)\,\Delta\tau &= -\mathbb{P}(t,t_*)W(t_*) - \int_{t_*}^{t} \mathbb{P}^{\Delta_\tau}(t,\tau)W(\tau)\,\Delta\tau \\ &= -\mathbb{P}(t,t_*)W(t_*) - \int_{t_*}^{t} \mathbb{P}^{\Delta_\tau}(t,\tau)\,\frac{M(\tau)}{S(\tau)}\,\Delta\tau + \int_{t_*}^{t} \mathbb{P}^{\Delta_\tau}(t,\tau)r(\tau)\Phi_p(z(\tau))\,\Delta\tau. \end{aligned} \tag{7.4}$$

Since $\mathbb{P}^{\Delta_\tau}(t,\tau) \leq 0$, we have

$$\int_{t_*}^{t} \mathbb{P}(t,\sigma(\tau))W^\Delta(\tau)\,\Delta\tau \leq -\mathbb{P}(t,t_*)W(t_*) - \int_{t_*}^{t} \mathbb{P}^{\Delta_\tau}(t,\tau)\,\frac{M(\tau)}{S(\tau)}\,\Delta\tau. \tag{7.5}$$

Equations (7.3) and (7.5) yield the inequality

$$\frac{1}{\mathbb{P}(t,t_*)} \int_{t_*}^{t} \left[\mathbb{P}^{\Delta_\tau}(t,\tau)\,\frac{M(\tau)}{S(\tau)} - \frac{\mathbb{P}(t,\sigma(\tau))M(\sigma(\tau))S^\Delta(\tau)}{S(\sigma(\tau))S(\tau)} \right] \Delta\tau \leq -W(t_*),$$

and then taking $\liminf\limits_{t\to\infty}$ on both side and using equation (7.2), we arrive at a contradiction. Hence, the proof is completed. □

Theorem 7.2. *Suppose conditions* (1) *and* (7.1) *hold. Moreover, the conditions*

$$\limsup_{t\to\infty} \frac{1}{\mathbb{P}(t,t_*)} \int_{t_*}^{t} \mathbb{P}(t,\sigma(\tau))e(\tau)\,\Delta\tau = \infty$$

and

$$\liminf_{t\to\infty} \frac{1}{\mathbb{P}(t,t_*)} \int_{t_*}^{t} \mathbb{P}(t,\sigma(\tau))e(\tau)\Delta\tau\,\Delta\tau = -\infty \tag{7.6}$$

are satisfied, then equation (5.9) *oscillates on* $[t_*, \infty)_{\mathbb{T}}$.

Proof. Assume that equation (5.9) has a non-oscillatory solution, which implies that there exists $t_* \in \mathbb{T}$ such that equation (5.9) has either positive or negative solution on $[t_*, \infty)_{\mathbb{T}}$. Thus, without loss of generality, we assume that $z(t)$ is a positive solution of equation (5.9). Now, multiplying equation (5.17) by $\mathbb{P}(t, \sigma(\tau))$ for $t_* \leq \tau \leq t$, and integrating from t_* to t, we conclude that

$$\int_{t_*}^{t} \mathbb{P}(t, \sigma(\tau))(r(\tau)\Phi_p(z(\tau)))^{\Delta} \, \Delta\tau \leq \int_{t_*}^{t} \mathbb{P}(t, \sigma(\tau))e(\tau) \, \Delta\tau. \tag{7.7}$$

Observing equations (7.4), (7.5), we have

$$\begin{aligned}
&\int_{t_*}^{t} \mathbb{P}(t, \sigma(\tau))(r(t)\Phi_p(z(t)))^{\Delta}(\tau) \, \Delta\tau \\
&\qquad = -\mathbb{P}(t, t_*)r(t_*)\Phi_p(z(t_*)) - \int_{t_*}^{t} \mathbb{P}^{\Delta_\tau}(t, \tau)r(\tau)\Phi_p(z(\tau)) \, \Delta\tau \geq -\mathbb{P}(t, t_*)r(t_*)\Phi_p(z(t_*)).
\end{aligned} \tag{7.8}$$

It follows from equations (7.7) and (7.8) that

$$-\Phi_p(z(t_*)) \leq \frac{1}{\mathbb{P}(t, t_*)} \int_{t_*}^{t} \mathbb{P}(t, \sigma(\tau))e(\tau) \, \Delta\tau,$$

so, now taking $\liminf\limits_{t\to\infty}$ on both sides and using equation (7.6), we get a contradiction. Therefore, we have the oscillatory solutions of equation (5.9) on $[t_*, \infty)_{\mathbb{T}}$. □

In [51], Negi et al. discussed the oscillatory behaviour of the following second-order dynamic equation on time scales:

$$y^{\Delta\Delta}(t) + a(t)y^{\Delta}(t) + y(t) + \mathcal{K}(y(t-h)) + f(\mathcal{W}(y^{\Delta}(t-h))) = 0, \quad \forall\, t \in \mathbb{T}, \tag{7.9}$$

where the functions $a : \mathbb{T} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, and the time scale $\mathbb{T}$ satisfies $t - h < t$ and $t - h \in \mathbb{T}$, $\forall\, t \in \mathbb{T}$, for some positive real constant h. One can easily see that for some $h > 0$, in [44], Kubyshkin and Moryakova considered the second-order differential-difference equation of delay type

$$\ddot{x}(t) + A\dot{x}(t) + x(t) + \mathcal{K}(x(t-h)) + \mathcal{W}(\dot{x}(t-h)) = 0, \tag{7.10}$$

which can be achieved by taking $\mathbb{T} = \mathbb{R}$, $a(t) = A > 0$ and $f(x) = x$, $\forall\, x \in \mathbb{R}$ in equation (7.9). Here, the real constants $A, h > 0$, and the functions $\mathcal{K}, \mathcal{W} : \mathbb{R} \to \mathbb{R}$ are defined by $\mathcal{K}(x(t)) = k_1x(t) + k_2x^2(t) + \cdots$, and $\mathcal{W}(x(t)) = w_1x(t) + w_2x^2(t) + \cdots$, $\forall\, k_i, w_j \in \mathbb{R}$, respectively. Equation (7.9) is very general in nature, and techniques from time scales calculus can analyze it. Equation (7.9) covers not only differential equations (i.e., $\mathbb{T} = \mathbb{R}$) and difference equations (i.e., $\mathbb{T} = \mathbb{Z}$), but also covers more general time scales $h\mathbb{Z} = \{hn : \ n \in \mathbb{Z}\}$ for $h > 0$, $\mathbb{T} = \bigcup\limits_{k\in\mathbb{Z}} [k(a+b), k(a+b)+a]$ for $a, b > 0$, and $\mathbb{T} = \bigcup\limits_{m\in\mathbb{Z}} \{m + \frac{1}{n} : \ n \in \mathbb{N}\} \cup \mathbb{Z}$, etc. Throughout the paper [51], they obtained some sufficient conditions of the oscillation for the dynamic equation (7.9). To the best of their knowledge, no work has been done regarding the oscillatory behaviors of (7.9) so far.

In their paper, firstly, they dealt with two functions $\mathcal{K}(y(t-h))$ and $\mathcal{W}(y^{\Delta}(t-h))$, which play an important role in our analytical findings. As we see from the assumption

$$|F(t,u)| \geq p(t)|u|^{\gamma}, \quad |G(t,u,v)| \leq q(t)|u|^{\gamma}, \ \ \forall\, u \in \mathbb{R}\setminus\{0\}, \ \ v \in \mathbb{R}, \ \ t \in \mathbb{T},$$

the absolute value of functions F and G are related to the absolute value of the unknown function $u(t)$ by the functions $p(t)$ and $q(t)$, respectively. In the equation

$$\Big(r(t)\Big(\big[y(t) + p(t)y(t-\tau)\big]^{\Delta}\Big)^{\gamma}\Big)^{\Delta} + f(t, y(t-\delta)),$$

Saker [59] assumed that the continuous function $f : \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is such that $uf(t,u) > 0$, $\forall\, u \neq 0$ and $|f(t,u)| \geq q(t)|u|^{\gamma}$, where a nonnegative function $q(t)$ is defined on $\mathbb{T}$. In [19], Chen considered the following equation

$$((x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)(x^{\Delta}(t))^{\gamma} + q(t)f(x^{\sigma}(t)) = 0, \tag{7.11}$$

where the function $f \in C(\mathbb{R},\mathbb{R})$ satisfies $xf(x) > 0$ and $f(x) \geq Lx$ for $\forall x \neq 0$, L is positive real constant. The author established the sufficient conditions of Kamenev-type as well as Philos-type oscillation criteria through employing the Riccati transformation. Moreover, in [33], Graef and Hill investigated the non-oscillation solutions of the higher order nonlinear delay dynamic equation on time scales:

$$(a(t)x^{\Delta}(t))^{\Delta^{n-1}} + q(t)f(x(g(t))) = r(t), \quad \forall\, t \in \mathbb{T},$$

and established the sufficient conditions of the non-oscillation, in which they considered the function $f \in C(\mathbb{R},\mathbb{R})$ being such that for $\gamma > 0$, $|f(x(t))| \leq |x(t)|^{\gamma} + B$ for all $x(t)$, $\forall\, t \in \mathbb{T}$, where A, B are non-negative real constants.

In order to establish some oscillation criteria for (7.9), we need $|\mathcal{K}y(t)| \geq p(t)|y(t)|$ for $y(t) \neq 0$ such that $y(t)\mathcal{K}(y(t)) > 0$. Moreover, there exists a function $f \in C(\mathbb{R},\mathbb{R})$ such that $|f(\mathcal{W}(y^{\Delta}(t)))| \geq M(t)|y^{\Delta}(t)|$, as well as $y(t)f(\mathcal{W}(y^{\Delta}(t))) > 0$, $\forall\, y(t) \neq 0$ in $\mathbb{R}$, where $M(t)$ is a nonnegative rd-continuous function defined on $\mathbb{T}$. Now we choose the real coefficients w_j, k_i such that

$$k_i, w_j = \begin{cases} 0 & \text{if } i,\ j \text{ are even natural numbers,} \\ +ve & \text{if } i,\ j \text{ are odd natural numbers,} \end{cases}$$

and $\mathcal{W}, \mathcal{K}$ are defined in (7.10), then we obtain the following relation:

$$|\mathcal{K}(y(t))| = \left|k_1 y(t) + k_3 y^3(t) + \cdots\right| = k_1|y(t)|\left|1 + \frac{k_3}{k_1}y^2(t) + \cdots\right| \geq k_1|y(t)|, \tag{7.12}$$

such that $y(t)\mathcal{K}(y(t)) > 0\ \ \forall\, y(t) \in \mathbb{R}$, $t \in \mathbb{T}$. Similarly, we immediately obtain an inequality

$$|\mathcal{W}(y^{\Delta}(t))| \geq w_1|y^{\Delta}(t)|, \quad \forall\, t \in \mathbb{T}. \tag{7.13}$$

Let us now consider a function $f \in C(\mathbb{R},\mathbb{R})$ for $y(t) \in \mathbb{R}$ such that

$$f(\mathcal{W}y^{\Delta}(t)) = q(t)\,\mathrm{sgn}(y(t))|\mathcal{W}(y^{\Delta}(t))|, \quad \forall\, t \in \mathbb{T},$$

then, from (7.13), we obtain,

$$|f(\mathcal{W}y^{\Delta}(t))| = \left|q(t)\,\mathrm{sgn}(y(t))|\mathcal{W}(y^{\Delta}(t))|\right| \geq q(t)w_1|y^{\Delta}(t)|, \quad \forall\, y(t) \in \mathbb{R}, \quad \forall\, t \in \mathbb{T},$$

where $q(t)$ is nonnegative rd-continuous defined function on $\mathbb{T}$. Thus, we can find such a function $f \in C(\mathbb{R},\mathbb{R})$ which satisfies

$$|f(\mathcal{W}(y^{\Delta}(t)))| \geq q(t)w_1|y^{\Delta}(t)|, \tag{7.14}$$

and $y(t)f(\mathcal{W}(y^{\Delta}(t))) > 0$ for $y(t) \neq 0$, $\forall\, t \in \mathbb{T}$ and $q(t)$ is rd-continuous defined on $\mathbb{T}$. In equation (7.14), the absolute value of f is related to the absolute value of $y^{\Delta}(t)$, $\forall\, t \in \mathbb{T}$.

For simplicity, throughout this paper, we denote $[a,\infty)_{\mathbb{T}} = [a,\infty) \cap \mathbb{T}$. In addition, we also need the following assumptions:

($\mathbf{O_1}$) Assume $a, p : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ are the positive rd-continuous functions such that $0 < p(t) \leq k_1 < \infty$ and

$$q(t) := p(t)\frac{\mu(t)}{w_1};$$

($\mathbf{O_2}$) $\displaystyle\int_{t_0}^{\infty} \frac{1}{e_{z(t)}(t,s_0)}\,\Delta t = \infty$, where $z(t) := \dfrac{a(t)}{1 - a(t)\mu(t)} > 0$, $\forall\, t \in \mathbb{T}$.

Lemma 7.1. *Let $y(t)$ be a non-oscillate solution of (7.9) and assume that (O_1), (O_2) and relations (7.12) and (7.14) hold, then there exists $s_0 \geq 0$; $s_0 > t_0$ such that*

$$y(t) > 0, \;\; y^{\Delta}(t) > 0 \;\; \text{and} \;\; y^{\Delta\Delta}(t) < 0 \tag{7.15}$$

and

$$y(t-h) > 0, \quad y^{\Delta}(t-h) > 0 \;\; \text{and} \;\; y^{\Delta\Delta}(t-h) < 0 \;\; \text{on} \; [s_0, \infty)_{\mathbb{T}}.$$

Lemma 7.2. *If (7.15) holds, then for $t \neq s_0$, we have*

$$0 < \mathcal{G}(t) \leq \frac{y(t)}{y^{\sigma}(t)} \leq 1, \tag{7.16}$$

where

$$\mathcal{G}(t) := \frac{t - s_0}{t - s_0 + \mu(t)}\,.$$

Lemma 7.3. *If (7.15) and (7.16) hold, then for $2s_0 \leq t$, we have*

$$\frac{t}{2}\,\frac{\mathcal{G}(t)w(t)}{\delta(t)} \leq (t - s_0)\,\frac{\mathcal{G}(t)w(t)}{\delta(t)} \leq \frac{y(t-h)}{y^{\sigma}(t)} \leq 1, \tag{7.17}$$

where $w(t) = \delta(t)\,\frac{y^{\Delta}(t)}{y(t)}$ is a Riccati transformation function.

8 Oscillatory results

Theorem 8.1. *Assume that (O_1), (O_2) and relations (7.12), (7.14) hold. If there exists a function $\delta(t) > 0$ such that*

$$\limsup_{t\to\infty} \int_{2s_0}^{t} \mathcal{F}(s)\,\Delta s = \infty, \tag{8.1}$$

where $s_0 \geq 0$, $t_0 \leq 2s_0 < t$,

$$\mathcal{F}(t) := \left(\delta^{\sigma}(t)\mathcal{G}(t) - \frac{\left(\delta^{\Delta}(t) - \mathcal{G}(t)\delta^{\sigma}(t)(a(t) + p(t)\frac{s}{2})\right)^2}{4\delta^{\sigma}(t)\mathcal{G}(t)}\right),$$

then equation (7.9) oscillates on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume to the contrary that (7.9) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (7.9). Then, without loss of generality, we assume that $y(t)$ is an eventually positive solution i.e., there exists $s_0 \geq t_0$ such that $y(t) > 0$, $\forall\, t \in [s_0, \infty)_{\mathbb{T}}$. A similar argument holds for the case where $y(t)$ is an eventually negative solution. We define a Riccati transformation function such that

$$w(t) = \delta(t)\,\frac{y^{\Delta}(t)}{y(t)}\,, \;\; t_0 \leq 2s_0 < t. \tag{8.2}$$

Considering the Δ-derivative of equation (8.2) with respect to t, we have

$$\begin{aligned} w^{\Delta}(t) &= \delta^{\Delta}(t)\,\frac{y^{\Delta}(t)}{y(t)} + \delta^{\sigma}(t)\Big(\frac{y^{\Delta\Delta}(t)y(t) - (y^{\Delta}(t))^2}{y(t)y^{\sigma}(t)}\Big) \\ &= w(t)\,\frac{\delta^{\Delta}(t)}{\delta(t)} - w^2(t)\,\frac{\delta^{\sigma}(t)y(t)}{y^{\sigma}(t)\delta^2(t)} + \frac{\delta^{\sigma}(t)}{y^{\sigma}(t)}\,y^{\Delta\Delta}(t). \end{aligned} \tag{8.3}$$

From (7.16) and (8.3), we obtain

$$w^{\Delta}(t) \leq w(t)\,\frac{\delta^{\Delta}(t)}{\delta(t)} - w^2(t)\mathcal{G}(t)\,\frac{\delta^{\sigma}(t)}{\delta^2(t)} + \frac{\delta^{\sigma}(t)}{y^{\sigma}(t)}\,y^{\Delta\Delta}(t). \tag{8.4}$$

To solve the right-hand side of equation (8.4), we use equation (7.16) and the relation $y^{\Delta}(t) > 0$, we obtain

$$\begin{aligned}\frac{\delta^{\sigma}(t)}{y^{\sigma}(t)}\, y^{\Delta\Delta}(t) &\le -\delta^{\sigma}(t)\bigg(\frac{y(t)}{y^{\sigma}(t)} + p(t)\,\frac{y^{\sigma}(t-h)}{y^{\sigma}(t)} + a(t)\,\frac{y^{\Delta}(t)}{y^{\sigma}(t)}\bigg)\\ &\le -\delta^{\sigma}(t)\mathcal{G}(t) - \delta^{\sigma}(t)a(t)\,\frac{\mathcal{G}(t)}{\delta(t)}\,w(t) - \delta^{\sigma}(t)p(t)\,\frac{y(t-h)}{y^{\sigma}(t)}\,. \end{aligned} \tag{8.5}$$

From (8.5) and (7.17), we get

$$\frac{\delta^{\sigma}(t)}{y^{\sigma}(t)}\, y^{\Delta\Delta}(t) \le -\delta^{\sigma}(t)\mathcal{G}(t) - \delta^{\sigma}(t)a(t)\,\frac{\mathcal{G}(t)}{\delta(t)}\,w(t) - p(t)\delta^{\sigma}(t)\frac{s}{2}\,\frac{\mathcal{G}(t)}{\delta(t)}\,w(t). \tag{8.6}$$

Substituting (8.6) into (8.4), we arrive at

$$\begin{aligned} w^{\Delta}(t) &\le w(t)\,\frac{\delta^{\Delta}(t)}{\delta(t)} - w^2(t)\mathcal{G}(t)\,\frac{\delta^{\sigma}(t)}{\delta^2(t)} - \delta^{\sigma}(t)\mathcal{G}(t) - \delta^{\sigma}(t)a(t)\,\frac{\mathcal{G}(t)}{\delta(t)}\,w(t) - p(t)\delta^{\sigma}(t)\frac{s}{2}\,\frac{\mathcal{G}(t)}{\delta(t)}\,w(t)\\ &= -\delta^{\sigma}(t)\mathcal{G}(t) + \frac{1}{\delta(t)}\left(\delta^{\Delta}(t) - \mathcal{G}(t)\delta^{\sigma}(t)\Big(a(t) + p(t)\,\frac{t}{2}\Big)\right)w(t) - \frac{\delta^{\sigma}(t)}{\delta^2(t)}\,\mathcal{G}(t)w^2(t)\\ &= -\delta^{\sigma}(t)\mathcal{G}(t) - \left(w(t)\,\frac{\sqrt{\delta^{\sigma}(t)\mathcal{G}(t)}}{\delta(t)} - \frac{\delta^{\Delta}(t) - \mathcal{G}(t)\delta^{\sigma}(t)(a(t) + p(t)\frac{s}{2})}{2\sqrt{\delta^{\sigma}(t)\mathcal{G}(t)}}\right)^2\\ &\qquad + \frac{\big(\delta^{\Delta}(t) - \mathcal{G}(t)\delta^{\sigma}(t)(a(t) + p(t)\frac{s}{2})\big)^2}{4\delta^{\sigma}(s)\mathcal{G}(s)}\,. \end{aligned} \tag{8.7}$$

From equations (8.1) and (8.7), we arrive at

$$w^{\Delta}(t) \le -\mathcal{F}(t) \;\text{ for }\; 2s_0 < t. \tag{8.8}$$

Integrating equation (8.8) from $2s_0$ to t, we have

$$\int_{2s_0}^{t} \mathcal{F}(s)\,\Delta s \le w(2s_0) < \infty. \tag{8.9}$$

For sufficient large t, we derive a contradiction to (8.1), as the left-hand side of (8.9) is finite, which completes the proof of our theorem. □

From Theorem 8.1, we may also obtain some results concerning the oscillation behavior of solutions of equation (7.9).

Corollary 8.1. *Assume that* (O_1), (O_2) *and relations* (7.12), (7.14) *hold. Moreover, if there exists a* Δ*-derivative function* $\delta(t) > 0$ *and* $s_0 \ge 0$, $t_0 \le 2s_0 < t$, *respectively, satisfying the conditions*

$$\limsup_{t\to\infty} \int_{2s_0}^{t} \delta^{\sigma}(s)\mathcal{G}(s)\,\Delta s = \infty$$

and

$$\limsup_{t\to\infty} \int_{2s_0}^{t} \frac{\big(\delta^{\Delta}(s) - \mathcal{G}(s)\delta^{\sigma}(s)(a(s) + p(s)\frac{s}{2})\big)^2}{4\delta^{\sigma}(s)\mathcal{G}(s)}\,\Delta s < \infty,$$

then equation (7.9) *oscillates on* $[t_0, \infty)_{\mathbb{T}}$.

The next result immediately follows from Theorem 8.1 by different choices of $\delta(t)$. In particular, we take $\delta(t)$ as a positive constant (say $C > 0$) and establish the following

Corollary 8.2. *Assume that* (O_1), (O_2) *and relations* (7.12), (7.14) *hold. Moreover, if there exists* $s_0 \geq 0$ *such that* $t_0 \leq 2s_0 < t$, *satisfying the condition*

$$\limsup_{t\to\infty} \int\limits_{2s_0}^{t} \frac{\mathcal{G}(s)}{4} \Big(4 - \Big(a(s) + p(s)\,\frac{s}{2}\Big)^2\Big)\,\Delta s = \infty,$$

then equation (7.9) *oscillates on* $[t_0,\infty)_{\mathbb{T}}$.

We introduce one more condition $\mu(t)\,\frac{\delta^\Delta(t)}{\delta^\sigma(t)} < 1$ to obtain a new oscillations criterion for equation (7.9).

Theorem 8.2. *Assume that* (O_1), (O_2) *and relations* (7.12), (7.14) *hold. Moreover, if there exist a* Δ*-derivative function* $\delta(t) > 0$ *and* $s_0 \geq 0$ *such that* $\mu(t)\,\frac{\delta^\Delta(t)}{\delta^\sigma(t)} < 1$ *and* $t_0 \leq 2s_0 < t$, *respectively, satisfying the condition*

$$\limsup_{t\to\infty} \int\limits_{2s_0}^{t} \frac{1}{1 - \frac{\delta^\Delta(s)}{\delta^\sigma(s)}\,\mu(s)} \Big[\mathcal{G}(s)\delta(s) - \frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\Big]\,\Delta s = \infty, \tag{8.10}$$

where

$$A(t) := \frac{\delta^\Delta(t)}{\delta^\sigma(t)} - \mathcal{G}(t)\Big(a(t) + p(t)\,\frac{t}{2}\Big), \tag{8.11}$$

then equation (7.9) *oscillates on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume to the contrary that (7.9) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (7.9). Then, without loss of generality, we assume that $y(t)$ is an eventually positive solution of (7.9) i.e., there exists $t_0 \leq s_0$ such that $y(t) > 0$, $\forall\, t \in [s_0,\infty)_{\mathbb{T}}$. A similar argument holds for the case if $y(t)$ is eventually negative. Now, Δ-differentiating equation (8.2) with respect to t, we have

$$w^\Delta(t) = \delta(t)\Big(\frac{y^\Delta(t)}{y(t)}\Big)^\Delta + \delta^\Delta(t)\Big(\frac{y^\Delta(t)}{y(t)}\Big)^\sigma. \tag{8.12}$$

From equations (8.2), (7.16) and (8.12), we obtain

$$\begin{aligned} w^\Delta(t) &\leq \frac{\delta^\Delta(t)}{\delta^\sigma(t)}\,w^\sigma(t) - \frac{\mathcal{G}(t)}{\delta(t)}\,w^2(t) + \delta(t)\,\frac{y^{\Delta\Delta}(t)}{y^\sigma(t)} \\ &= \frac{\delta^\Delta(t)}{\delta^\sigma(t)}\,(w(t) + \mu(t)w^\Delta(t)) - \frac{\mathcal{G}(t)}{\delta(t)}\,w^2(t) + \delta(t)\,\frac{y^{\Delta\Delta}(t)}{y^\sigma(t)}\,. \end{aligned} \tag{8.13}$$

To solve the right-hand side of equation (8.13), we replace $\delta^\sigma(t)$ by $\delta(t)$ in (8.6) and we obtain

$$\delta(t)\,\frac{y^{\Delta\Delta}}{y^\sigma(t)} \leq -\mathcal{G}(t)\delta(t) - a(t)\mathcal{G}(t)w(t) - p(t)\,\frac{t}{2}\,\mathcal{G}(t)w(t). \tag{8.14}$$

Substituting (8.14) into (8.13), we arrive at

$$w^\Delta(t) \leq \frac{\delta^\Delta(t)}{\delta^\sigma(t)}\,(w(t) + \mu(t)w^\Delta(t)) - \frac{\mathcal{G}(t)}{\delta(t)}\,w^2(t) - \mathcal{G}(t)\delta(t) - a(t)\mathcal{G}(t)w(t) - p(t)\,\frac{t}{2}\,\mathcal{G}(t)w(t),$$

which is equivalent to

$$\begin{aligned} &\Big(1 - \frac{\delta^\Delta(t)}{\delta^\sigma(t)}\,\mu(t)\Big)w^\Delta(t) \leq -\mathcal{G}(t)\delta(t) + A(t)w(t) - \frac{\mathcal{G}(t)}{\delta(t)}\,w^2(t) \\ &\quad = -\mathcal{G}(t)\delta(t) + \frac{\delta(t)A^2(t)}{4\mathcal{G}(t)} - \Big(\sqrt{\frac{\mathcal{G}(t)}{\delta(t)}}\,w(t) - \frac{A(t)}{2}\,\sqrt{\frac{\delta(t)}{\mathcal{G}(t)}}\Big)^2 \leq -\mathcal{G}(t)\delta(t) + \frac{\delta(t)A^2(t)}{4\mathcal{G}(t)}\,. \end{aligned} \tag{8.15}$$

Since

$$\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\,\mu(t) < 1,$$

dividing (8.15) by $1 - \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\,\mu(t)$, we arrive at

$$w^{\Delta}(t) \le -\frac{1}{1 - \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\,\mu(t)}\left[\mathcal{G}(t)\delta(t) - \frac{\delta(t)A^2(t)}{4\mathcal{G}(t)}\right]. \tag{8.16}$$

Integrating equation (8.16) from $2s_0$ to t, we get

$$\int_{2s_0}^{t} \frac{1}{1 - \frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\,\mu(s)}\left[\mathcal{G}(s)\delta(s) - \frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\right]\Delta s \le w(2s_0) - w(t) < w(2s_0) < \infty.$$

For sufficiently large t, we derive a contradiction to (8.10), as the left-hand side of the above relation is finite, which completes the proof of our theorem. □

Theorem 8.3. *Assume that* (O_1), (O_2) *and relations* (7.12), (7.14) *hold. Moreover, if there exist a* Δ*-derivative function* $\delta(t) > 0$ *and* $s_0 \ge 0$, $t_0 \le 2s_0 < t$, *respectively, satisfying the condition*

$$\limsup_{t\to\infty} \int_{2s_0}^{t} \left(\delta(s)\mathcal{G}(s) - \frac{\left(\delta^{\Delta}(s) - \delta(s)\mathcal{G}(s)(a(s) + p(s)\frac{s}{2})\right)^2}{4\delta(s)}\right)\Delta s = \infty, \tag{8.17}$$

then equation (7.9) *oscillates on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume to the contrary that (7.9) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (7.9). Then, without loss of generality, we assume that $y(t)$ is an eventually positive solution of (7.9) i.e., there exists $t_0 \le s_0$ such that $y(t) > 0$, $\forall\, t \in [s_0,\infty)_{\mathbb{T}}$. A similar argument holds also for the case if $y(t)$ is eventually negative. Now, from equations (8.2) and (8.12), we obtain

$$w^{\Delta}(t) \le \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\,w^{\sigma}(t) - \frac{\delta(t)}{(\delta^{\sigma}(t))^2}\,(w^{\sigma}(t))^2 + \delta(t)\,\frac{y^{\Delta\Delta}(t)}{y^{\sigma}(t)}\,. \tag{8.18}$$

From (8.14), we have

$$\delta(t)\,\frac{y^{\Delta\Delta}}{y^{\sigma}(t)} \le -\mathcal{G}(t)\delta(t) - a(t)\mathcal{G}(t)w(t) - p(t)\,\frac{t}{2}\,\mathcal{G}(t)w(t). \tag{8.19}$$

Substituting (8.19) into (8.18), we arrive at

$$w^{\Delta}(t) \le -\delta(t)\mathcal{G}(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\,w^{\sigma}(t) - \frac{\delta(t)}{(\delta^{\sigma}(t))^2}\,(w^{\sigma}(t))^2 - \mathcal{G}(t)\Big(a(t) + p(t)\,\frac{t}{2}\Big)w(t). \tag{8.20}$$

From (8.2), (7.16) and $y^{\Delta\Delta}(t) < 0$, we obtain the relation

$$w(t) \ge \frac{\delta(t)}{\delta^{\sigma}(t)}\,w^{\sigma}(t). \tag{8.21}$$

Substituting (8.21) into (8.20), we get

$$w^{\Delta}(t) \le -\delta(t)\mathcal{G}(t) + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\,w^{\sigma}(t) - \frac{\delta(t)}{(\delta^{\sigma}(t))^2}\,(w^{\sigma}(t))^2 - \frac{\delta(t)}{\delta^{\sigma}(t)}\,\mathcal{G}(t)\Big(a(t) + p(t)\,\frac{t}{2}\Big)w^{\sigma}(t),$$

which is equivalent to

$$w^{\Delta}(t) \le -\delta(t)\mathcal{G}(t) + \frac{\delta^{\Delta}(t) - \delta(t)\mathcal{G}(t)(a(t) + p(t)\frac{t}{2})}{\delta^{\sigma}(t)}\,w^{\sigma}(t) - \frac{\delta(t)}{(\delta^{\sigma}(t))^2}\,(w^{\sigma}(t))^2. \tag{8.22}$$

By following the similar steps of equations (8.7) and (8.8), equation (8.22) becomes

$$w^{\Delta}(t) \leq -\delta(t)\mathcal{G}(t) + \frac{\left(\delta^{\Delta}(t) - \delta(t)\mathcal{G}(t)(a(t) + p(t)\frac{s}{2})\right)^2}{4\delta(t)}. \tag{8.23}$$

Integrating equation (8.23) from $2s_0$ to t, we have

$$\int_{2s_0}^{t} \left[\delta(s)\mathcal{G}(s) - \frac{\left(\delta^{\Delta}(s) - \delta(s)\mathcal{G}(s)(a(s) + p(s)\frac{s}{2})\right)^2}{4\delta(s)}\right] \Delta s \leq w(2s_0) < \infty.$$

For sufficiently large t, we derive a contradiction to (8.17), as the left-hand side is finite, which completes the proof of our theorem. □

In view of the above theorem, we immediately obtain the following

Corollary 8.3. *Assume that* (O_1), (O_2) *and relations* (7.12), (7.14) *hold. Moreover, if there exist* Δ*-derivative function* $\delta(t) > 0$ *and* $s_0 \geq 0$, $t_0 \leq 2s_0 < t$, *respectively, satisfying the conditions*

$$\limsup_{t\to\infty} \int_{2s_0}^{t} \delta(s)\mathcal{G}(s)\, \Delta s = \infty$$

and

$$\limsup_{t\to\infty} \int_{2s_0}^{t} \frac{\left(\delta^{\Delta}(s) - \delta(s)\mathcal{G}(s)(a(s) + p(s)\frac{s}{2})\right)^2}{4\delta(s)} \Delta s < \infty,$$

then equation (7.9) *oscillates on* $[t_0, \infty)_{\mathbb{T}}$.

In order to present our next theorems, we first introduce Saker's result [59] as follows:

$$((t-s)^{\mathcal{N}})^{\Delta_s} \leq -\mathcal{N}(t-\sigma(s))^{\mathcal{N}-1} \leq 0 \;\text{ for }\; \mathcal{N} > 1 \;\text{ and }\; \sigma(s) \leq t. \tag{8.24}$$

By using an integral averaging technique of Kamenev-type, we present some new oscillation criteria of (7.9).

Theorem 8.4. *Assume that* $(O_1), (O_2)$ *and relations* (7.12), (7.14) *hold. If there exists a function* $\delta(t) > 0$ *and there exists* $\mathcal{N} > 1$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}} \int_{2s_0}^{t} (t-s)^{\mathcal{N}} \mathcal{F}(s) \Delta s = \infty, \tag{8.25}$$

where $s_0 \geq 0$, $t_0 \leq 2s_0 < t$,

$$\mathcal{F}(t) := \left(\delta^{\sigma}(t)\mathcal{G}(t) - \frac{\left(\delta^{\Delta}(t) - \mathcal{G}(t)\delta^{\sigma}(t)(a(t) + p(t)\frac{s}{2})\right)^2}{4\delta^{\sigma}(t)\mathcal{G}(t)}\right),$$

then equation (7.9) *oscillates on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume to the contrary that (7.9) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (7.9). Then, without loss of generality, we assume that $y(t)$ is an eventually positive function, i.e., there exists t_0 such that $y(t) > 0$, $\forall\, t \in [t_0, \infty)_{\mathbb{T}}$. A similar argument holds also for the case if $y(t)$ is eventually negative. From equation (8.8), we have

$$\mathcal{F}(t) \leq -w^{\Delta}(t) \;\text{ for }\; 2s_0 \leq t.$$

Multiplying the above relation by $(t-s)^{\mathcal{N}}$ and then integrating from $2s_0$ to t, we obtain

$$\int_{2s_0}^{t} (t-s)^{\mathcal{N}} \mathcal{F}(s)\,\Delta s \le -\int_{2s_0}^{t} (t-s)^{\mathcal{N}} w^{\Delta}(s)\,\Delta s. \tag{8.26}$$

Comparing the right-hand side of (8.26) with equation (1.2), we have

$$-\int_{2s_0}^{t} (t-s)^{\mathcal{N}} w^{\Delta}(s)\,\Delta s = (t-2s_0)^{\mathcal{N}} w(2s_0) + \int_{2s_0}^{t} ((t-s)^{\mathcal{N}})^{\Delta_s} w^{\sigma}(s)\,\Delta s. \tag{8.27}$$

From equations (8.24), (8.26) and (8.27), we arrive at

$$\int_{2s_0}^{t} (t-s)^{\mathcal{N}} \mathcal{F}(s)\,\Delta s \le (t-2s_0)^{\mathcal{N}} w(2s_0). \tag{8.28}$$

Thus

$$\frac{1}{t^{\mathcal{N}}}\int_{2s_0}^{t} (t-s)^{\mathcal{N}} \mathcal{F}(s)\,\Delta s \le \frac{(t-2s_0)^{\mathcal{N}}}{t^{\mathcal{N}}}\, w(2s_0) \;\text{ for }\; 2s_0 \le t. \tag{8.29}$$

Taking lim sup as $t \to \infty$ on both side of equation (8.29), we have

$$\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}}\int_{2s_0}^{t} (t-s)^{\mathcal{N}} \mathcal{F}(s)\,\Delta s < \infty. \tag{8.30}$$

Thus, we derive a contradiction to (8.25), which completes the proof of our theorem. □

Corollary 8.4. *Assume that* $(O_1), (O_2)$ *and relations* (7.12), (7.14) *hold,* $s_0 \ge 0$, $t_0 \le 2s_0 < t$. *If there exist* $\delta(t) > 0$ *and* $\mathcal{N} > 1$ *such that the following conditions hold*

$$\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}}\int_{2s_0}^{t} (t-s)^{\mathcal{N}} \delta^{\sigma}(t)\mathcal{G}(t)\,\Delta s = \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}}\int_{2s_0}^{t} (t-s)^{\mathcal{N}} \frac{\left(\delta^{\Delta}(t) - \mathcal{G}(t)\delta^{\sigma}(t)(a(t) + p(t)\frac{s}{2})\right)^2}{4\delta^{\sigma}(t)\mathcal{G}(t)}\,\Delta s < \infty,$$

then equation (7.9) *oscillates on* $[t_0, \infty)_{\mathbb{T}}$.

Theorem 8.5. *Assume that* $(O_1), (O_2)$ *and relations* (7.12), (7.14) *hold. If there exists a function* $\delta(t) > 0$ *such that* $\mu(t)\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)} < 1$, *and for* $\mathcal{N} > 1$, $s_0 \ge 0$, $t_0 \le 2s_0 < t$, *the following condition holds*

$$\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}}\int_{2s_0}^{t} \frac{(t-s)^{\mathcal{N}}}{1 - \frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\,\mu(s)} \left[\mathcal{G}(s)\delta(s) - \frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\right]\Delta s = \infty, \tag{8.31}$$

where $A(t)$ *is given by* (8.11), *then equation* (7.9) *oscillates on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume to the contrary that (7.9) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (7.9). Then, without loss of generality, we assume that $y(t)$ is an eventually positive function, i.e., there exists t_0 such that $y(t) > 0$, $\forall\, t \in [t_0\ \infty)_{\mathbb{T}}$. A similar argument holds for the case if $y(t)$ is eventually negative. From equation (8.16), we have

$$\frac{1}{1-\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}\,\mu(t)}\left[\mathcal{G}(t)\delta(t)-\frac{\delta(t)A^2(t)}{4\mathcal{G}(t)}\right] \le -w^{\Delta}(t) \ \text{ for } \ 2s_0 \le t.$$

Multiplying the above relation by $(t-s)^{\mathcal{N}}$ and then integrating from $2s_0$ to t, we obtain

$$\int_{2s_0}^{t} \frac{(t-s)^{\mathcal{N}}}{1-\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\,\mu(s)}\left[\mathcal{G}(s)\delta(s)-\frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\right]\Delta s \le -\int_{2s_0}^{t}(t-s)^{\mathcal{N}}w^{\Delta}(s)\,\Delta s. \tag{8.32}$$

By following the similar steps of equations (8.26)-(8.28), equation (8.32) becomes

$$\int_{2s_0}^{t} \frac{(t-s)^{\mathcal{N}}}{1-\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\,\mu(s)}\left[\mathcal{G}(s)\delta(s)-\frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\right]\Delta s \le (t-2s_0)^{\mathcal{N}}w(2s_0).$$

Thus, we have

$$\frac{1}{t^{\mathcal{N}}}\int_{2s_0}^{t} \frac{(t-s)^{\mathcal{N}}}{1-\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\,\mu(s)}\left[\mathcal{G}(s)\delta(s)-\frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\right]\Delta s \le \frac{(t-2s_0)^{\mathcal{N}}}{t^{\mathcal{N}}}\,w(2s_0). \tag{8.33}$$

Taking $\limsup$ as $t \to \infty$ in equation (8.33), we obtain

$$\limsup_{t\to\infty}\frac{1}{t^{\mathcal{N}}}\int_{2s_0}^{t} \frac{(t-s)^{\mathcal{N}}}{1-\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}\,\mu(s)}\left[\mathcal{G}(s)\delta(s)-\frac{\delta(s)A^2(s)}{4\mathcal{G}(s)}\right]\Delta s < \infty. \tag{8.34}$$

Thus, we derive a contradiction to (8.31), which completes the proof of our theorem. □

Theorem 8.6. *Assume that* $(O_1), (O_2)$ *and relations* (7.12), (7.14) *hold. If there exist* $\delta(t) > 0$ *and* $\mathcal{N} > 1$, $s_0 \ge 0$, $t_0 \le 2s_0 < t$ *such that*

$$\limsup_{t\to\infty}\frac{1}{t^{\mathcal{N}}}\int_{2s_0}^{t}(t-s)^{\mathcal{N}}\left(\delta(s)\mathcal{G}(s)-\frac{\left(\delta^{\Delta}(s)-\delta(s)\mathcal{G}(s)(a(s)+p(s)\frac{s}{2})\right)^2}{4\delta(s)}\right)\Delta s = \infty, \tag{8.35}$$

then equation (7.9) *oscillates on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume to the contrary that (7.9) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (7.9). Then, without loss of generality, we assume that $y(t)$ is an eventually positive function, i.e., there exists t_0 such that $y(t) > 0$, $\forall\, t \in [t_0, \infty)_{\mathbb{T}}$. From equation (8.23) and following the similar steps of equations (8.26)–(8.30), we easily obtain the relation

$$\frac{1}{t^{\mathcal{N}}}\int_{2s_0}^{t}(t-s)^{\mathcal{N}}\left(\delta(s)\mathcal{G}(s)-\frac{\left(\delta^{\Delta}(s)-\delta(s)\mathcal{G}(s)(a(s)+p(s)\frac{s}{2})\right)^2}{4\delta(s)}\right)\Delta s \le \left(1-\frac{2s_0}{t}\right)^{\mathcal{N}}w(2s_0) < \infty.$$

For all sufficiently large t, we derive a contradiction to (8.35). □

Corollary 8.5. *Assume that* $(O_1), (O_2)$ *and relations* (7.12), (7.14) *hold. If there exists a function* $\delta(t) > 0$ *and* $\mathcal{N} > 1$ $s_0 \ge 0$, $t_0 \le 2s_0 < t$ *such that the following conditions hold*

$$\limsup_{t\to\infty}\frac{1}{t^{\mathcal{N}}}\int_{2s_0}^{t}(t-s)^{\mathcal{N}}\delta(s)\mathcal{G}(s)\,\Delta s = \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\mathcal{N}}} \int\limits_{2s_0}^{t} (t-s)^{\mathcal{N}} \, \frac{\left(\delta^{\Delta}(s) - \delta(s)\mathcal{G}(s)(a(s)+p(s)\frac{s}{2})\right)^2}{4\delta(s)} \, \Delta s < \infty,$$

then equation (7.9) *oscillates on* $[t_0,\infty)_{\mathbb{T}}$.

Our next aim is to establish the Philos-type oscillation criteria for (7.9). We define some elementary assumptions as follows:

For any number $\eta \in \mathbb{R}$, we define positive and negative parts, η_+ and η_-, respectively, of η by

$$\eta_+ := \max\{0,\eta\} \text{ and } \eta_- := \max\{0,\eta\}.$$

Assume that the rd-continuous functions $H, h : \mathbb{D} \to \mathbb{R}$, where $\mathbb{D} = \{(t,s) : \ t_0 \le s_0 \le t\}$, is such that

$$H(t,t) \ge 0, \ \ t_0 \le t \ \text{ and } \ H(t,s) > 0 \ \text{ and } \ H^{\Delta_s}(t,s) < 0, \ \ t_0 \le s < t \tag{8.36}$$

and $H^{\Delta_s}(t,s)$ (Δ-derivative w.r.t second variable) is rd-continuous.

Theorem 8.7. *Assume that* (O_1), (O_2) *relations* (7.12), (7.14) *and equation* (8.36) *hold. Moreover, if there exist a* Δ*-derivative function* $\delta(t) > 0$ *and* $s_0 \ge 0$, $t_0 \le 2s_0 < t$, *respectively, satisfying the conditions*

$$H^{\Delta_s}(\sigma(t),s) + \frac{H^{\sigma}(\sigma(t),s)}{\delta(t)} \left(\delta^{\Delta}(t) - \mathcal{G}(t)\delta^{\sigma}(t)\Big(a(t)+p(t)\,\frac{t}{2}\Big)\right) = -\frac{h(t,s)}{\delta(t)}\sqrt{H^{\sigma}(\sigma(t),s)} \tag{8.37}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),2s_0)} \int\limits_{2s_0}^{\sigma(t)} \left(H^{\sigma}(\sigma(t),s)\delta^{\sigma}(s)\mathcal{G}(s) - \frac{(h_-(t,s))^2}{4\delta(s)\mathcal{G}(s)}\right) \Delta s = \infty, \tag{8.38}$$

then equation (7.9) *oscillates on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume to the contrary that (7.9) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (7.9). Then, without loss of generality, we assume that $y(t)$ is an eventually positive function, i.e., there exists t_0 such that $y(t) > 0$, $\forall\, t \in [t_0,\infty)_{\mathbb{T}}$. A similar argument holds for the case if $y(t)$ is eventually negative. We have defined a Riccati transformation function in (8.2). Now, from (8.7), we have

$$\begin{aligned} w^{\Delta}(t) \le & -\delta^{\sigma}(t)\mathcal{G}(t) + \frac{1}{\delta(t)}\left(\delta^{\Delta}(t) - \mathcal{G}(t)\delta^{\sigma}(t)\Big(a(t)+p(t)\,\frac{t}{2}\Big)\right) w(t) \\ & - \frac{\delta^{\sigma}(t)}{\delta^2(t)}\,\mathcal{G}(t)w^2(t) \ \text{ for } \ 2s_0 \le t. \end{aligned} \tag{8.39}$$

Multiplying equation (8.39) by $H^{\sigma}(\sigma(t),s)$, i.e., $H(\sigma(t),\sigma(s))$, and then integrating from $2s_0$ to $\sigma(t)$, we obtain

$$\begin{aligned} \int\limits_{2s_0}^{\sigma(t)} H^{\sigma}(\sigma(t),s)\delta^{\sigma}(s)\mathcal{G}(s)\,\Delta s \le & -\int\limits_{2s_0}^{\sigma(t)} H^{\sigma}(\sigma(t),s)w^{\Delta}(s)\,\Delta s \\ + \int\limits_{2s_0}^{\sigma(t)} H^{\sigma}(\sigma(t),s)\Bigg(\frac{1}{\delta(s)}\left(\delta^{\Delta}(s) - \mathcal{G}(s)\delta^{\sigma}(s)\Big(a(s)+p(s)\frac{s}{2}\Big)\right) & w(s) - \frac{\delta^{\sigma}(s)}{\delta^2(s)}\,\mathcal{G}(s)w^2(s)\Bigg)\,\Delta s. \end{aligned} \tag{8.40}$$

From (1.1), we obtain the right-hand side of (8.40) as follows:

$$\begin{aligned}
&\le H(\sigma(t),2s_0)w(2s_0)\\
&\quad+\int_{2s_0}^{\sigma(t)}\Big[H^{\Delta_s}(\sigma(t),s)+\frac{H^\sigma(\sigma(t),s)}{\delta(s)}\Big(\delta^\Delta(s)-\mathcal{G}(s)\delta^\sigma(s)\Big(a(s)+p(s)\frac{s}{2}\Big)\Big)\Big]w(t)\,\Delta s\\
&\quad-\int_{2s_0}^{\sigma(t)}H^\sigma(\sigma(t),s)\,\frac{\delta^\sigma(s)}{\delta^2(s)}\,\mathcal{G}(s)w^2(s)\,\Delta s. \qquad (8.41)
\end{aligned}$$

Substituting (8.37) into (8.41), we arrive at

$$\begin{aligned}
\int_{2s_0}^{\sigma(t)}H^\sigma(\sigma(t),s)\delta^\sigma(s)\mathcal{G}(s)\,\Delta s &\le H(\sigma(t),2s_0)w(2s_0)\\
&\quad+\int_{2s_0}^{\sigma(t)}\Big(\frac{h_-(t,s)\sqrt{H^\sigma(\sigma(t),s)}}{\delta(s)}\,w(s)-H^\sigma(\sigma(t),s)\,\frac{\delta^\sigma(s)}{\delta^2(s)}\,\mathcal{G}(s)w^2(s)\Big)\,\Delta s. \qquad (8.42)
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\int_{2s_0}^{\sigma(s)}H^\sigma(\sigma(t),s)\delta^\sigma(s)\mathcal{G}(s)\,\Delta s &\le H(\sigma(t),2s_0)w(2s_0)\\
&\quad+\int_{2s_0}^{\sigma(t)}\frac{(h_-(t,s))^2}{4\delta^\sigma(s)\mathcal{G}(s)}\,\Delta s-\int_{2s_0}^{\sigma(t)}\Big(\frac{\sqrt{H^\sigma(\sigma(t),s)\delta^\sigma(s)\mathcal{G}(s)}}{\delta(s)}\,w(s)-\frac{h_-(t,s)}{2\sqrt{\delta^\sigma(s)\mathcal{G}(s)}}\Big)^2\,\Delta s. \qquad (8.43)
\end{aligned}$$

Hence, we have

$$\int_{2s_0}^{\sigma(t)}H^\sigma(\sigma(t),s)\delta^\sigma(s)\mathcal{G}(s)\,\Delta s\le H(\sigma(t),2s_0)w(2s_0)+\int_{2s_0}^{\sigma(t)}\frac{(h_-(t,s))^2}{4\delta^\sigma(s)\mathcal{G}(s)}\,\Delta s. \qquad (8.44)$$

Dividing (8.44) by $H(\sigma(t),2s_0)$, we obtain

$$\frac{1}{H(\sigma(t),2s_0)}\int_{2s_0}^{\sigma(t)}\Big(H^\sigma(\sigma(t),s)\delta^\sigma(s)\mathcal{G}(s)-\frac{(h_-(t,s))^2}{4\delta(s)\mathcal{G}(s)}\Big)\,\Delta s\le w(2s_0)<\infty$$

for sufficiently large t. Thus, we derive a contradiction to (8.38). □

Corollary 8.6. *Assume that* (O_1), (O_2), *relations* (7.12), (7.14) *and the equation* (8.36) *hold. Moreover, if there exist a Δ-derivative function $\delta(t)>0$ and $s_0\ge 0$, $t_0\le 2s_0<t$, respectively, satisfying the conditions*

$$H^{\Delta_s}(\sigma(t),s)+\frac{H^\sigma(\sigma(t),s)}{\delta(t)}\Big(\delta^\Delta(t)-\mathcal{G}(t)\delta^\sigma(t)\Big(a(t)+p(t)\,\frac{t}{2}\Big)\Big)=-\frac{h(t,s)}{\delta(t)}\,\sqrt{H^\sigma(\sigma(t),s)}\,,$$

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),2s_0)}\int_{2s_0}^{\sigma(t)}H^\sigma(\sigma(t),s)\delta^\sigma(s)\mathcal{G}(s)\,\Delta s=\infty$$

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),2s_0)}\int_{2s_0}^{\sigma(t)}\frac{(h_-(t,s))^2}{4\delta(s)\mathcal{G}(s)}\,\Delta s<\infty,$$

then equation (7.9) *oscillates on* $[t_0,\infty)_{\mathbb{T}}$.

Theorem 8.8. *Assume that* (O_1), (O_2), *relations* (7.12), (7.14) *and equation* (8.36) *hold. Moreover, if there exist a* Δ*-derivative function* $\delta(t) > 0$ *and* $s_0 \geq 0$ *such that* $\mu(t)\frac{\delta^\Delta(t)}{\delta^\sigma(t)} < 1$ *and* $t_0 \leq 2s_0 < t$, *respectively, satisfying the conditions*

$$H^{\Delta_s}(\sigma(t), s) + \frac{H^\sigma(\sigma(t), s)}{1 - \frac{\delta^\Delta(t)}{\delta^\sigma(t)}\,\mu(t)}\,A(t) = -\frac{h(t,s)}{1 - \frac{\delta^\Delta(t)}{\delta^\sigma(t)}\,\mu(t)}\,\sqrt{H^\sigma(\sigma(t), s)}\,, \tag{8.45}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t), 2s_0)} \int_{2s_0}^{\sigma(t)} \Big(H^\sigma(\sigma(t), s)\mathcal{G}(s) - \frac{(h_-(t,s))^2}{4\mathcal{G}(s)}\Big)\,\frac{\delta(s)}{1 - \frac{\delta^\Delta(s)}{\delta^\sigma(s)}\,\mu(s)}\,\Delta s = \infty, \tag{8.46}$$

where $A(t)$ *is given by* (8.11), *then equation* (7.9) *oscillates on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume to the contrary that (7.9) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (7.9). Then, without loss of generality, we assume that $y(t)$ is an eventually positive function, i.e., there exists t_0 such that $y(t) > 0$, $\forall\, t \in [t_0\ \infty)_{\mathbb{T}}$. A similar argument holds for the case if $y(t)$ is eventually negative. We have defined a Riccati transformation function in (8.2). Now, from (8.15), we have

$$\Big(1 - \frac{\delta^\Delta(t)}{\delta^\sigma(t)}\,\mu(t)\Big)w^\Delta(t) \leq -\mathcal{G}(t)\delta(t) + A(t)w(t) - \frac{\mathcal{G}(t)}{\delta(t)}\,w^2(t),$$

which can be written as

$$\frac{\mathcal{G}(t)\delta(t)}{1 - \frac{\delta^\Delta(t)}{\delta^\sigma(t)}\,\mu(t)} \leq -w^\Delta(t) + \frac{A(t)}{1 - \frac{\delta^\Delta(t)}{\delta^\sigma(t)}\,\mu(t)}\,w(t) - \frac{\mathcal{G}(t)}{\delta(t)(1 - \frac{\delta^\Delta(t)}{\delta^\sigma(t)}\,\mu(t))}\,w^2(t). \tag{8.47}$$

Multiplying equation (8.47) by $H^\sigma(\sigma(t), s)$ and then integrating from $2s_0$ to $\sigma(t)$, we have

$$\begin{aligned}\int_{2s_0}^{\sigma(t)} \frac{H^\sigma(\sigma(t), s)\mathcal{G}(s)\delta(s)}{1 - \frac{\delta^\Delta(s)}{\delta^\sigma(s)}\,\mu(s)}\,\Delta s \leq &-\int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t), s)w^\Delta(s)\,\Delta s \\ &+ \int_{2s_0}^{\sigma(t)} \frac{H^\sigma(\sigma(t), s)A(s)}{1 - \frac{\delta^\Delta(s)}{\delta^\sigma(s)}\,\mu(s)}\,w(s)\,\Delta s - \int_{2s_0}^{\sigma(t)} \frac{H^\sigma(\sigma(t), s)\mathcal{G}(s)}{\delta(s)(1 - \frac{\delta^\Delta(s)}{\delta^\sigma(s)}\,\mu(s))}\,w^2(s)\,\Delta s.\end{aligned} \tag{8.48}$$

From (8.45), (8.48) and following the similar steps of equations (8.41)–(8.44), we obtain

$$\int_{2s_0}^{\sigma(t)} \frac{H^\sigma(\sigma(t), s)\mathcal{G}(s)\delta(s)}{1 - \frac{\delta^\Delta(s)}{\delta^\sigma(s)}\,\mu(s)}\,\Delta s \leq H(\sigma(t), 2s_0)w(2s_0) + \int_{2s_0}^{\sigma(t)} \frac{\delta(s)(h_-(t,s))^2}{4(1 - \frac{\delta^\Delta(s)}{\delta^\sigma(s)}\,\mu(s))\mathcal{G}(s)}\,\Delta s. \tag{8.49}$$

Dividing equation (8.49) by $H(\sigma(t), 2s_0)$, we obtain

$$\frac{1}{H(\sigma(t), 2s_0)} \int_{2s_0}^{\sigma(t)} \Big(H^\sigma(\sigma(t), s)\mathcal{G}(s) - \frac{(h_-(t,s))^2}{4\mathcal{G}(s)}\Big)\,\frac{\delta(s)}{1 - \frac{\delta^\Delta(s)}{\delta^\sigma(s)}\,\mu(s)}\,\Delta s < w(2s_0) < \infty.$$

For sufficiently large t, we derive a contradiction to (8.46), which completes the proof of our theorem. $\square$

Theorem 8.9. *Assume that* (O_1), (O_2), *relations* (7.12), (7.14) *and equation* (8.36) *hold. Moreover, if there exist a* Δ*-derivative function* $\delta(t) > 0$ *and* $s_0 \geq 0$, $t_0 \leq 2s_0 < t$, *respectively, satisfying the conditions*

$$H^{\Delta_s}(t, s) + \frac{H(t,s)}{\delta^\sigma(t)}\,\Big(\delta^\Delta(t) - \mathcal{G}(t)\delta(t)\Big(a(t) + p(t)\,\frac{t}{2}\Big)\Big) = -\frac{h(t,s)}{\delta^\sigma(t)}\,\sqrt{H(t,s)}\,, \tag{8.50}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,2s_0)}\int_{2s_0}^{t}\Big(H(t,s)\delta(s)\mathcal{G}(s)-\frac{(h_-(t,s))^2}{4\delta(s)}\Big)\,\Delta s=\infty,$$

then equation (7.9) *oscillates on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume to the contrary that (7.9) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (7.9). Then, without loss of generality, we assume that $y(t)$ is an eventually positive function, i.e., there exists t_0 such that $y(t)>0$, $\forall\, t\in[t_0\ \infty)_{\mathbb{T}}$. We have defined the Riccati transformation function in (8.2). Now, multiplying equation (8.23) by $H(t,s)$ and integrating from $2s_0$ to t, we have the relation

$$\begin{aligned}\int_{2s_0}^{t}H(t,s)\delta(t)\mathcal{G}(s)\,\Delta s&\le-\int_{2s_0}^{t}H(t,s)w^{\Delta}(s)\,\Delta s\\&+\int_{2s_0}^{t}H(t,s)\,\frac{\delta^{\Delta}(s)-\delta(s)\mathcal{G}(s)(a(s)+p(s)\frac{s}{2})}{\delta^{\sigma}(s)}\,w^{\sigma}(s)\,\Delta s-\int_{2s_0}^{t}\frac{H(t,s)\delta(s)}{(\delta^{\sigma}(s))^2}\,(w^{\sigma}(s))^2\,\Delta s.\end{aligned}$$

From (1.2) and (8.36), we obtain

$$\begin{aligned}&\int_{2s_0}^{t}H(t,s)\delta(s)\mathcal{G}(s)\,\Delta s\\&\le H(t,2s_0)w(2s_0)+\int_{2s_0}^{t}\Big(H^{\Delta}(t,s)+H(t,s)\,\frac{\delta^{\Delta}(s)-\delta(s)\mathcal{G}(s)(a(s)+p(s)\frac{s}{2})}{\delta^{\sigma}(s)}\Big)w^{\sigma}(s)\,\Delta s\\&\qquad-\int_{2s_0}^{t}\frac{H(t,s)\delta(s)}{(\delta^{\sigma}(s))^2}\,(w^{\sigma}(s))^2\,\Delta s.\end{aligned}\tag{8.51}$$

From equations (8.50), (8.51) and by following the similar steps of equations (8.26)–(8.30), we obtain a new relation

$$\frac{1}{H(t,2s_0)}\int_{2s_0}^{t}\Big(H(t,s)\delta(s)\mathcal{G}(s)-\frac{(h_-(t,s))^2}{4\delta(s)}\Big)\,\Delta s\le w(2s_0)<\infty$$

for sufficiently large t. Thus, we derive a contradiction to (8.9), which completes the proof of our theorem. ☐

Corollary 8.7. *Assume that* (O_1), (O_2), *relations* (7.12), (7.14) *and equation* (8.36) *hold. Moreover, if there exist a* Δ*-derivative function* $\delta(t)>0$ *and* $s_0\ge0$, $t_0\le 2s_0<t$, *respectively, satisfying the conditions*

$$H^{\Delta_s}(t,s)+\frac{H(t,s)}{\delta^{\sigma}(t)}\Big(\delta^{\Delta}(t)-\mathcal{G}(t)\delta(t)\Big(a(t)+p(t)\,\frac{t}{2}\Big)\Big)=-\frac{h(t,s)}{\delta^{\sigma}(t)}\sqrt{H(t,s)}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,2s_0)}\int_{2s_0}^{t}H(t,s)\delta(s)\mathcal{G}(s)\,\Delta s=\infty,\quad \limsup_{t\to\infty}\frac{1}{H(t,2s_0)}\int_{2s_0}^{t}\frac{(h_-(t,s))^2}{4\delta(s)}\,\Delta s<\infty,$$

then equation (7.9) *oscillates on* $[t_0,\infty)_{\mathbb{T}}$.

In [49], Negi et al. considered the following singular initial-value problem for the second-order dynamic equation on time scales with the initial point $a \in \mathbb{T} \subseteq \mathbb{R}$, and $t^* \le a$ for all $t^* \in \mathbb{T}$:

$$\left.\begin{aligned} (r^\beta(t)Y(t))^\Delta - F\big(t, y^\sigma(t), y^\Delta(t)\big) + \int_{t^*}^{t} \mathcal{O}(s)\mathcal{D}(s, y(s), y^\sigma(s))\,\Delta s = G(t, y(t), y^\Delta(t)),\\ y(a) = 0, \quad y^\Delta(a) = b \ \text{ for } b \in \mathbb{R}, \end{aligned}\right\} \tag{8.52}$$

where $\beta \neq 0$,

$$Y(t) = y^\Delta(t) + \operatorname{sgn}(y(t))K\left(t, y(t), y^\sigma(t), y^\Delta(t), \int_a^t p_3^*(t,s)|y^\Delta(s)|^\gamma\,\Delta s\right) - f(t)$$

and $\mathbb{T}$ is an unbounded time scale, i.e., $\sup \mathbb{T} = \infty$. The functions F, G, K, $\mathcal{D}$, r, p_3^*, g and f are rd-continuous on their respective domains. Throughout this paper, we denote the time scale interval $[t_1, \infty)_\mathbb{T} = [t_1, \infty) \cap \mathbb{T}$ for $t_1 \in \mathbb{T}$, and $\mathbb{C}_{rd}^{\Delta^n}(\mathbb{T}, \mathbb{R})$ denotes the set of all n^{th}-delta derivable rd-continuous functions, for $n \in \mathbb{N}$. By a solution of equation (8.52), we mean a nontrivial real-valued function $y(t) \in \mathbb{C}_{rd}^{\Delta^2}([t_y, \infty)_\mathbb{T}, \mathbb{R})$ and $r^\beta(t)Y(t) \in \mathbb{C}_{rd}^{\Delta}([t_y, \infty)_\mathbb{T}, \mathbb{R})$, $t_y \le t$.

In order to establish the oscillation criteria, we have not used the Riccati technique and provided a new way to establish the oscillation criteria. Moreover, some superior estimates are given in our main theorems by employing the generalized Opial's type inequality on time scales. Some results are also presented without using the inequality.

For the oscillatory results, we need the following assumptions which have a vital role in our analytic findings:

[A1]: (a) $\mathcal{O} : \mathbb{T} \to [0, \infty)$ is a rd-continuous function such that $\mathcal{O} > 0$ for $t_0 < t$; otherwise zero.

(b) $p_3^* : [t_0, \infty)_\mathbb{T} \times [t_0, \infty)_\mathbb{T} \to (0, \infty)_\mathbb{R}$, $r, q, p_3', p_i : [t_0, \infty)_\mathbb{T} \to (0, \infty)_\mathbb{R}$ for $i = 1, 2, 3, 4$, and $f : [t_0, \infty)_\mathbb{T} \to \mathbb{R}$ are rd-continuous functions such that $p_3^*(y, x) \ge p_3(y)p_3'(x)$ for $x, y \in [t_0, \infty)_\mathbb{T}$, $x \le y$, $p_1(t) \le p_2(t)$, and $p_3^{*\Delta_y}(y, x)$, $f^\Delta(t)$ and $r^\Delta(t)$ exist for each $y, x, t \in [t_0, \infty)_\mathbb{T}$.

[A2]: $F, G : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ and $\mathcal{D} : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ are rd-continuous functions such that $uG(t, u, v) > 0$, $u^*F(t, u^*, v) > 0$ and $u\mathcal{D}(t, u, u^*) > 0$, which satisfy

$$|F(t, u^*, v)| \le p_1(t)|u^*|^{\alpha^*}|v|^\alpha \ \text{ and } \ |G(t, u, v)| \le p_2(t)|u|^{\alpha^*}|v|^\alpha \ \text{ for } \ 0 \neq u, u^*v \in \mathbb{R},$$

where $\alpha > 0$, and α^* is a positive odd integer.

[A3]: $\displaystyle\int_{t_0}^{\infty} \frac{1}{p_4^{1/\alpha}(s)}\,\Delta s < \infty$ and $\displaystyle\int_{t_0}^{\infty} \frac{c_1}{r^\beta(s)}\,\Delta s < \infty$ for any real constant c_1.

[A4]: $K : [t_0, \infty)_\mathbb{T} \times \mathbb{R}^4 \to [0, \infty)_\mathbb{R}$ is delta-derivable on $\mathbb{T}$ and there is a delta-derivable function $g : [t_0, \infty)_\mathbb{T} \to \mathbb{R}$ such that

$$g(t) + K(t, u, u^*, v, X) \ge q(t)X \ \text{ for any } \ \gamma, u, u^* \neq 0, \ \ v, w, X \in \mathbb{R}.$$

9 Oscillatory results

The purpose of this section is to give new sufficient conditions of the oscillation for equation (8.52) on time scales $\mathbb{T}$, except for the time scale $\mathbb{T} = \{q^n : n \in \mathbb{N}\} \cup \{0\}$, $0 < q < 1$.

Theorem 9.1. *Suppose the conditions* [A1]–[A4] *and the relation $L(t_0, t, \alpha) \le q(t)p_3(t)r^\beta(t)$ hold for $t_0 \le t$ with $\gamma > 1 + \alpha$ and $\alpha^* = 1$. Furthermore, assume that*

$$\liminf_{t\to\infty} \int_{t_0}^{t} \left[f(X) + g(X) + q(X)p_3(X) \int_{t_0}^{X} \mathcal{M}(s)\,\Delta s \right] \Delta X = -\infty$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[f(X)-g(X)-q(X)p_3(X)\int_{t_0}^{X}\mathcal{M}(s)\,\Delta s\right]\Delta X=+\infty,$$

where $L(t_0,t,\alpha)$ *in defined by* (1.5) *and*

$$\mathcal{M}(t)=(1+\alpha)^{\frac{1+\alpha}{\gamma-1-\alpha}}(\gamma-1-\alpha)\gamma^{\gamma/1+\alpha-\gamma}(p_3'(t))^{(1+\alpha)/1+\alpha-\gamma}(p_4(t))^{\gamma/\gamma-1-\alpha},$$

then all nontrivial solutions of equation (8.52) *are oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume the contrary, then there exists a solution $y(t)$ of (8.52) which may be assumed to be nonnegative on $[t_0,\infty)_{\mathbb{T}}$ such that $y(t_0)=0$ and $y(t)>0$ for $t_0<t$. Similar proof can be done in case $y(t)<0$ for $t_0<t$. From equation (8.52), using the assumptions [A1] and [A2], we obtain

$$\begin{aligned}(r^{\beta}(t)Y(t))^{\Delta}&=G(t,y(t),y^{\Delta}(t))+F(t,y^{\sigma}(t),y^{\Delta}(t))-\int_{t^*}^{t}\mathcal{O}(s)\mathcal{D}(s,y(s),y^{\sigma}(t))\,\Delta s\\&\le p_1(t)y(t)|y^{\Delta}(t)|^{\alpha}+p_2(t)y^{\sigma}(t)|y^{\Delta}(t)|^{\alpha}\\&\qquad-\int_{t^*}^{t_0}\mathcal{O}(s)\mathcal{D}(s,y(s),y^{\sigma}(t))\,\Delta s-\int_{t_0}^{t}\mathcal{O}(s)\mathcal{D}(s,y(s),y^{\sigma}(t))\,\Delta s\\&\le p_2(t)|y^{\Delta}(t)|^{\alpha}|y(t)+y^{\sigma}(t)|.\end{aligned}\tag{9.1}$$

Integrating equation (9.1) from t_0 to t and using the assumption [A4], we have

$$\begin{aligned}&y^{\Delta}(t)+p_3(t)q(t)\int_{t_0}^{t}p_3'(s)|y^{\Delta}(t)|^{\gamma}\,\Delta s-f(t)-g(t)\\&\qquad\le\frac{c_1}{r^{\beta}(t)}+\frac{1}{r^{\beta}(t)}\int_{t_0}^{t}p_2(s)|y^{\Delta}(s)|^{\alpha}|y(s)+y^{\sigma}(s)|\,\Delta s.\end{aligned}\tag{9.2}$$

From equations (1.4), (9.2) and (1.5), we obtain

$$\begin{aligned}y^{\Delta}(t)&\le f(t)+g(t)+\frac{c_1}{r^{\beta}(t)}+\frac{L(t_0,t,\alpha)}{r^{\beta}(t)}\int_{t_0}^{t}p_4(s)|y^{\Delta}(s)|^{1+\alpha}\,\Delta s-q(t)p_3(t)\int_{t_0}^{t}p_3'(s)|y^{\Delta}(s)|^{\gamma}\,\Delta s\\&\le f(t)+g(t)+\frac{c_1}{r^{\beta}(t)}+q(t)p_3(t)\left[\int_{t_0}^{t}\left(p_4(s)|y^{\Delta}(s)|^{1+\alpha}-p_3'(s)|y^{\Delta}(s)|^{\gamma}\right)\Delta s\right],\end{aligned}\tag{9.3}$$

since $\gamma>1+\alpha$, substituting $A=|y^{\Delta}(t)|^{1+\alpha}$, $B=(\frac{1+\alpha}{\gamma})\frac{p_4(s)}{p_3'(s)}$, $\xi=\frac{\gamma}{1+\alpha}$ and $\eta=\frac{\gamma}{\gamma-1-\alpha}$ in part (1) of Lemma 1.1, we obtain

$$\begin{aligned}y^{\Delta}(t)&\le f(t)+g(t)+\frac{c_1}{r^{\beta}(t)}+q(t)p_3(t)\int_{t_0}^{t}\left[\frac{\gamma}{1+\alpha}p_3'(s)\Big(AB-\frac{1}{\xi}A^{\xi}\Big)\right]\Delta s\\&\le f(t)+g(t)+\frac{c_1}{r^{\beta}(t)}+q(t)p_3(t)\int_{t_0}^{t}\mathcal{M}(s)\,\Delta s.\end{aligned}\tag{9.4}$$

Integration of equation (9.4) from t_0 to t gives

$$y(t)\le\int_{t_0}^{t}\left[f(X)+g(X)+q(X)p_3(X)\int_{t_0}^{X}\mathcal{M}(s)\,\Delta s\right]\Delta X+\int_{t_0}^{t}\frac{c_1}{r^{\beta}(s)}\,\Delta s,$$

and taking lim inf on both sides as $t \to \infty$, we get a contradiction to the fact that $y(t)$ is eventually nonnegative. Hence the proof is done. $\square$

From Theorem 9.1, we may also obtain some results concerning the oscillation behavior of equation (8.52).

Corollary 9.1. *Suppose the conditions* [A1]–[A4] *and the relation* $L(t_0, t, \alpha) \leq q(t)p_3(t)r^{\beta}(t)$ *hold for* $t_0 \leq t$ *with* $\gamma > 1 + \alpha$ *and* $\alpha^* = 1$. *Furthermore, assume that*

$$\int_{t_0}^{\infty} q(X)p_3(X) \int_{t_0}^{X} \mathcal{M}(s)\, \Delta s\, \Delta X < \infty,$$

$$\liminf_{t\to\infty} \int_{t_0}^{t} \big[f(X) + g(X)\big]\, \Delta X = -\infty, \quad \limsup_{t\to\infty} \int_{t_0}^{t} \big[f(X) - g(X)\big]\, \Delta X = +\infty,$$

where $\mathcal{M}(t)$ *is from Theorem 9.1 and* $L(t_0, t, \alpha)$ *is defined by* (1.5). *Then all nontrivial solutions of equation* (8.52) *are oscillatory on* $[t_0.\infty)_{\mathbb{T}}$.

Similarly, the next result immediately follows from the above Theorem 9.1.

Corollary 9.2. *Suppose the conditions* [A1]–[A4] *and the relation* $L(t_0, t, \alpha) \leq q(t)p_3(t)r^{\beta}(t)$ *hold for* $t_0 \leq t$ *with* $\gamma > 1 + \alpha$ *and* $\alpha^* = 1$. *Furthermore, assume that*

$$\int_{t_0}^{\infty} q(X)p_3(X) \int_{t_0}^{X} \mathcal{M}(s)\, \Delta s\, \Delta X < \infty,$$

and for any real constant c_1,

$$\liminf_{t\to\infty} \int_{t_0}^{t} \Big[f(X) + g(X) + \frac{c_1}{r^{\beta}(s)}\Big]\, \Delta X = -\infty, \quad \limsup_{t\to\infty} \int_{t_0}^{t} \Big[f(X) - g(X) + \frac{c_1}{r^{\beta}(s)}\Big]\, \Delta X = +\infty,$$

where $\mathcal{M}(t)$ *is from Theorem 9.1 and* $L(t_0, t, \alpha)$ *is defined by* (1.5). *Then all nontrivial solutions of equation* (8.52) *are oscillatory on* $[t_0.\infty)_{\mathbb{T}}$.

We also obtain some more results concerning the oscillation behavior of equation (8.52).

Theorem 9.2. *Suppose the conditions* [A1]–[A4] *and the relation* $L(t_0, t, \alpha) \leq q(t)p_3(t)r^{\beta}(t)$ *hold for* $t_0 \leq t$ *with any* ζ *such that* $1 + \alpha < \zeta < \gamma$. *Furthermore, assume that there exists an rd-continuous function* $\eta : \mathbb{T} \to (0, \infty)$ *and*

$$\liminf_{t\to\infty} \int_{t_0}^{t} \Big[f(X) + g(X) + q(X)p_3(X) \int_{t_0}^{X} \mathcal{M}_1(s) + \mathcal{M}_2(s)\, \Delta s\Big]\, \Delta X = -\infty$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \Big[f(X) + g(X) + q(X)p_3(X) \int_{t_0}^{X} \mathcal{M}_1(s) + \mathcal{M}_2(s)\, \Delta s\Big]\, \Delta X = +\infty,$$

where $L(t_0, t, \alpha)$ *is defined by* (1.5) *and*

$$\mathcal{M}_1(t) = (1+\alpha)^{\frac{1+\alpha}{\zeta-1-\alpha}}(\zeta - 1 - \alpha)\zeta^{\zeta/1+\alpha-\zeta}(\eta(t))^{(1+\alpha)/1+\alpha-\zeta}(p_4(t))^{\zeta/\zeta-1-\alpha}$$

and

$$\mathcal{M}_2(t) = \zeta^{\frac{\zeta}{\gamma-\zeta}}(\gamma - \zeta)\gamma^{\gamma/\zeta-\gamma}(p_3'(t))^{\zeta/\zeta-\gamma}(\eta(t))^{\gamma/\gamma-\zeta}.$$

Then all nontrivial solutions of equation (8.52) *are oscillatory on* $[t_0.\infty)_{\mathbb{T}}$.

Proof. Suppose the contrary, then there exists a solution $y(t)$ of (8.52) which may be assumed to be nonnegative on $[t_0,\infty)_{\mathbb{T}}$ such that $y(t_0)=0$ and $y(t)>0$ for $t_0<t$. Similar proof can be done in case $y(t)<0$ for $t_0<t$. In Theorem 9.1, from equation (9.3), we obtain

$$y^{\Delta}(t)\le f(t)+g(t)+\frac{c_1}{r^{\beta}(t)}+q(t)p_3(t)\Big[\int_{t_0}^{t}\Big(p_4(s)|y^{\Delta}(s)|^{1+\alpha}-\eta(s)|y^{\Delta}(s)|^{\zeta}\Big)\,\Delta s\Big]$$
$$+q(t)p_3(t)\Big[\int_{t_0}^{t}\Big(\eta(s)|y^{\Delta}(s)|^{\zeta}-p_3'(s)|y^{\Delta}(s)|^{\gamma}\Big)\,\Delta s\Big].$$

Now, we observe equations (9.3) to (8.6) in Theorem 9.1, thus we obtain the inequality

$$y(t)\le\int_{t_0}^{t}\Big[f(X)+g(X)+q(X)p_3(X)\int_{t_0}^{X}\mathcal{M}_1(s)+\mathcal{M}_2(s)\,\Delta s\Big]\,\Delta X+\int_{t_0}^{t}\frac{c_1}{r^{\beta}(s)}\,\Delta s.$$

Taking lim inf on both sides as $t\to\infty$, we obtain a contradiction to the fact that $y(t)$ is eventually nonnegative. Hence the proof is done. $\square$

Introduce the following condition:

[A5]: $K:[t_0,\infty)_{\mathbb{T}}\times\mathbb{R}^4\to[0,\infty)_{\mathbb{R}}$, $S:[t_0,\infty)_{\mathbb{T}}\times[t_0,\infty)_{\mathbb{T}}\to[0,\infty)_{\mathbb{R}}$ are delta-derivable on $\mathbb{T}$ such that $S(t,s)\ge S_1(t)S_2(s)$ for $s\le t$ and $1\le q(t)p_3(t)r^{\beta}(t)\le S_1(t)r^{\beta}(t)$. Also, there is a delta-derivable function $g:[t_0,\infty)_{\mathbb{T}}\to\mathbb{R}$ such that

$$g(t)+K(t,u,u^*,v,X)\ge q(t)X+\int_{t_0}^{t}S(t,s)\big(|u|^{\alpha^*}+|u^*|^{\alpha^*}\big)^{\delta}\,\Delta s$$

for any $\delta,\gamma,u,u^*\ne0$, $X,v\in\mathbb{R}$.

If we consider [A5] in place of [A4] in the above theorem, we do not need the generalized Opial's type inequality for the sufficient conditions of the oscillation of equation (8.52) for all odd integer α^*.

Theorem 9.3. *Suppose the conditions* [A1]–[A3] *hold for* $t_0\le t$ *for all* α^*, $\gamma>2\alpha>0$ *and* $2<\delta$. *In addition to* [A5], *assume that the conditions*

$$\liminf_{t\to\infty}\int_{t_0}^{t}f(s)+g(s)+S_1(s)\Big[\int_{t_0}^{s}\mathcal{W}_2(\tau)+\mathcal{W}_1(\tau)\,\Delta\tau\Big]\,\Delta s=-\infty$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}f(s)-g(s)-S_1(s)\Big[\int_{t_0}^{s}\mathcal{W}_2(\tau)+\mathcal{W}_1(\tau)\,\Delta\tau\Big]\,\Delta s=+\infty$$

hold, where

$$\mathcal{W}_1(s)=(2\alpha)^{\frac{2\alpha}{\gamma-2\alpha}}(\gamma-2\alpha)\gamma^{\gamma/2\alpha-\gamma}(p_3'(s))^{2\alpha/2\alpha-\gamma}(p_2(s))^{\gamma/\gamma-2\alpha}$$

and

$$\mathcal{W}_2(s)=(2\alpha)^{\frac{2}{\delta-2}}(\delta-2)\delta^{\delta/2-\delta}(S_2(s))^{2/2-\delta}(p_2(s))^{\frac{\delta}{\delta-2}},$$

then all nontrivial solutions of equation (8.52) *are oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose the contrary, then there exists a solution $y(t)$ of (8.52) which may be assumed to be nonnegative on $[t_0,\infty)_{\mathbb{T}}$ such that $y(t_0)=0$ and $y(t)>0$ for $t_0<t$. Similar proof can be done in case $y(t)<0$ for $t_0<t$. From equation (9.1), we have

$$(r^{\beta}(t)Y(t))^{\Delta}\le p_2(t)|y^{\Delta}(t)|^{\alpha}\big(|y(t)|^{\alpha^*}+|y^{\sigma}(t)|^{\alpha^*}\big). \tag{9.5}$$

Integrating equation (9.5) from t_0 to t, we have

$$Y(t) \leq \frac{c_1}{r^\beta(t)} + \frac{c_1}{r^\beta(t)} \int_{t_0}^{t} p_2(s)|y^\Delta(s)|^\alpha \big(|y(s)|^{\alpha^*} + |y^\sigma(s)|^{\alpha^*}\big),$$

and since $1 \leq q(t)p_3(t)r^\beta(t)$, we obtain

$$\leq \frac{c_1}{r^\beta(t)} + q(t)p_3(t) \int_{t_0}^{t} p_2(s)|y^\Delta(s)|^{2\alpha}\,\Delta s + q(t)p_3(t) \int_{t_0}^{t} p_2(s)\big(|y(s)|^{\alpha^*} + |y^\sigma(s)|^{\alpha^*}\big)^2\,\Delta s.$$

Therefore, from the condition [**A5**], using the relation $1 \leq q(t)p_3(t)r^\beta(t) \leq S_1(t)r^\beta(t)$, we obtain

$$\begin{aligned} y^\Delta(t) &\leq f(t) + g(t) + \frac{c_1}{r^\beta(t)} \\ &\quad + S_1(t)\Big[\int_{t_0}^{t} p_2(s)\big(|y(s)|^{\alpha^*} + |y^\sigma(s)|^{\alpha^*}\big)^2 - S_2(s)\big(|y(s)|^{\alpha^*} + |y^\sigma(s)|^{\alpha^*}\big)^\delta\Big] \\ &\quad + S_1(t)\Big[\int_{t_0}^{t} p_2(s)|y^\Delta(s)|^{2\alpha} - p_3'(s)|y^\Delta(s)|^\gamma\Big]. \end{aligned}$$

For $2\alpha < \gamma$, $2 < \delta$ and observing equations (8.4), (8.5), we get

$$y^\Delta(t) \leq f(t) + g(t) + \frac{c_1}{r^\beta(t)} + S_1(t)\Big[\int_{t_0}^{t} \mathcal{W}_2(s) + \mathcal{W}_1(s)\,\Delta s\Big]. \tag{9.6}$$

Integrating equation (9.6) from t_0 to t, we have

$$y(t) \leq \int_{t_0}^{t} f(s) + g(s) + S_1(s)\Big[\int_{t_0}^{s} \mathcal{W}_2(\tau) + \mathcal{W}_1(\tau)\,\Delta\tau\Big]\,\Delta s + \int_{t_0}^{t} \frac{c_1}{r^\beta(s)}\,\Delta s,$$

and taking lim inf on both sides as $t \to \infty$, we get a contradiction to the fact that $y(t)$ is eventually nonnegative. Hence the proof is done. $\square$

Remark 9.1. In view of the above theorem, we may also obtain some results concerning the oscillation behavior of equation (8.52).

The technique we used above can also be applied to study the higher order dynamic equations on time scales in the following form:

$$\begin{aligned} &\Big[r^\beta(t)\Big(y^{\Delta^{n-1}}(t) + \operatorname{sgn}(y(t))\Big[K\big(t, y(t), y^\Delta(t), \ldots, y^{\Delta^{n-1}}(t)\big) - g(t)\Big] - f(t)\Big)\Big]^\Delta \\ &\quad - F\big(t, y^\sigma(t), y^\Delta(t), \ldots, y^{\Delta^{n-1}}(t)\big) = G\big(t, y(t), y^\Delta(t), \ldots, y^{\Delta^{n-1}}(t)\big) - \int_{t^*}^{t} \mathcal{O}(s)\mathcal{D}(s, y(s))\,\Delta s \end{aligned} \tag{9.7}$$

and

$$y(t_0) = 0,\ y^\Delta(t_0) = a_1,\ y^{\Delta^2}(t_0) = a_2, \ldots, y^{\Delta^{n-1}}(t_0) = a_{n-1} \ \text{ for each } \ a_i \in \mathbb{R}.$$

If we introduce one more condition [A6] in the above Theorem 9.1, i.e.,

[A6]: $\int\limits_{t_0}^{\infty}\int\limits_{t_0}^{\tau_{n-1}}\cdots\int\limits_{t_0}^{\tau_3}\int\limits_{t_0}^{\tau_2}\frac{1}{p_4^{1/\alpha}(\tau_1)}\,\Delta\tau_1\Delta\tau_2\cdots\Delta\tau_{n-1}<\infty$ and $\int\limits_{t_0}^{\infty}\int\limits_{t_0}^{\tau_{n-1}}\cdots\int\limits_{t_0}^{\tau_3}\int\limits_{t_0}^{\tau_2}\frac{c_1}{r^{\beta}(\tau_1)}\,\Delta\tau_1\Delta\tau_2\cdots\Delta\tau_{n-1}<\infty$ for any constant c_1,

the similar oscillation condition for equation (9.7) can be derived easily. The proofs are omitted, since they are quite similar to the the proof of Theorem 9.1.

Theorem 9.4. *Suppose the conditions* [A1]–[A4] *and the relation* $L(t_0,t,\alpha)\le q(t)p_3(t)r^{\beta}(t)$ *hold for* $t_0\le t$ *with for* $\gamma>1+\alpha$ *and* $\alpha^*=1$*. Furthermore, in addition to the condition* [A6]*, we assume that*

$$\liminf_{t\to\infty}\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau_{n-1}}\cdots\int\limits_{t_0}^{\tau_3}\int\limits_{t_0}^{\tau_2}\big[f(\tau_1)+g(\tau_1)+W_1(\tau_1)\big]\,\Delta\tau_1\Delta\tau_2\cdots\Delta\tau_{n-1}=-\infty$$

and

$$\limsup_{t\to\infty}\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau_{n-1}}\cdots\int\limits_{t_0}^{\tau_3}\int\limits_{t_0}^{\tau_2}\big[f(\tau_1)-g(\tau_1)-W_1(\tau_1)\big]\,\Delta\tau_1\Delta\tau_2\cdots\Delta\tau_{n-1}=+\infty,$$

where $W_1(\tau_1)=q(\tau_1)p_3(\tau_1)\int\limits_{t_0}^{\tau_1}\mathcal{M}(s)\,\Delta s$*, and* $\mathcal{M}(t)$ *is as in Theorem 9.1 and* $L(t_0,t,\alpha)$ *is defined by* (1.5)*, then all nontrivial solutions of equation* (9.7) *are oscillatory on* $[t_0,\infty)_{\mathbb{T}}$*.*

Proof. Comparing the solution of equation (8.52) and using the same argument as in Theorem 9.1, we can prove the oscillatory Theorem for (9.7). □

In view of the above theorems, some corollaries can also be carried out.

Corollary 9.3. *Suppose the conditions* [A1]–[A4] *and the relation* $L(t_0,t,\alpha)\le q(t)p_3(t)r^{\beta}(t)$*,* $\beta>0$ *hold for* $t_0\le t$ *with for* $\gamma>1+\alpha$ *and* $\alpha^*=1$*. Furthermore, in addition to the condition* [A6]*, we assume that*

$$\int\limits_{t_0}^{\infty}W_1(\tau_1)\,\Delta\tau=\int\limits_{t_0}^{\infty}q(\tau_1)p_3(\tau_1)\int\limits_{t_0}^{\tau_1}\mathcal{M}(s)\ \Delta s\,\Delta\tau_1<\infty,$$

$$\liminf_{t\to\infty}\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau_{n-1}}\cdots\int\limits_{t_0}^{\tau_3}\int\limits_{t_0}^{\tau_2}\big[f(\tau_1)+g(\tau_1)\big]\,\Delta\tau_1\Delta\tau_2\cdots\Delta\tau_{n-1}=-\infty$$

and

$$\limsup_{t\to\infty}\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau_{n-1}}\cdots\int\limits_{t_0}^{\tau_3}\int\limits_{t_0}^{\tau_2}\big[f(\tau_1)-g(\tau_1)\big]\,\Delta\tau_1\Delta\tau_2\cdots\Delta\tau_{n-1}=+\infty.$$

where $\mathcal{M}(t)$ *is as in Theorem 9.1 and* $L(t_0,t,\alpha)$ *is defined by* (1.5)*, then all nontrivial solutions of equation* (9.7) *are oscillatory on* $[t_0.\infty)_{\mathbb{T}}$*.*

Corollary 9.4. *Suppose the conditions* [A1]–[A4] *and the relation* $L(t_0,t,\alpha)\le q(t)p_3(t)r^{\beta}(t)$*,* $\beta>0$ *hold for* $t_0\le t$ *with for* $\gamma>1+\alpha$ *and* $\alpha^*=1$*. Furthermore, in addition to the condition* [A6]*, we assume that*

$$\int\limits_{t_0}^{\infty}W_1(\tau_1)\,\Delta\tau=\int\limits_{t_0}^{\infty}q(\tau_1)p_3(\tau_1)\int\limits_{t_0}^{\tau_1}\mathcal{M}(s)\ \Delta s\,\Delta\tau_1<\infty,$$

and for any real constant c_1*,*

$$\liminf_{t\to\infty}\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau_{n-1}}\cdots\int\limits_{t_0}^{\tau_3}\int\limits_{t_0}^{\tau_2}\Big[f(\tau_1)+g(\tau_1)+\frac{c_1}{r^{\beta}(s)}\Big]\,\Delta\tau_1\Delta\tau_2\cdots\Delta\tau_{n-1}=-\infty$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\int_{t_0}^{\tau_{n-1}}\cdots\int_{t_0}^{\tau_3}\int_{t_0}^{\tau_2}\left[f(\tau_1)-g(\tau_1)+\frac{c_1}{r^{\beta}(s)}\right]\Delta\tau_1\Delta\tau_2\cdots\Delta\tau_{n-1}=+\infty,$$

where $\mathcal{M}(t)$ *is as in Theorem 9.1 and* $L(t_0,t,\alpha)$ *is defined by* (1.5)*, then all nontrivial solutions of equation* (9.7)*) are oscillatory on* $[t_0,\infty)_{\mathbb{T}}$*.*

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