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SINGULAR ANISOTROPIC ELLIPTIC PROBLEMS WITH VARIABLE EXPONENTS

Abstract. In this paper, we prove the existence and regularity results of positive solutions for anisotropic elliptic problems with variable exponents and a singular nonlinearity having also a variable exponent. The functional setting involves anisotropic Sobolev spaces with variable exponents.

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1 Introduction

Our aim is to prove the existence of at least one positive solution u to the singular anisotropic equation

$$\begin{cases} -\sum_{i=1}^{N} D_i\big(|D_i u|^{p_i(x)-2} D_i u\big) = \dfrac{f}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, and f is assumed to be a nonnegative function belonging to $L^m(\Omega)$, $m \geq 1$. We assume that the variable exponent $\gamma(\,\cdot\,) : \overline{\Omega} \to (0, +\infty)$ is a smooth continuous function, and $p_i(\,\cdot\,) : \overline{\Omega} \to (1, +\infty)$, $i = 1, \dots, N$, are continuous functions such that

$$p_1(x) \leq p_2(x) \leq \cdots \leq p_N(x), \quad \forall\, x \in \overline{\Omega}, \tag{1.2}$$

and

$$\overline{p}(x) < N, \tag{1.3}$$

where

$$\frac{1}{\overline{p}(x)} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i(x)}\,, \quad \forall\, x \in \overline{\Omega}.$$

Anisotropic operators with variable exponents are involved in various branches of applied sciences. In some cases, they provide realistic models for studying natural phenomena in electro-rheological fluids (see the references in [1–3,14,18]). Other important application is related to the image processing [7]. The corresponding results in the isotropic case are developed in [4–6, 13].

In [5], Boccardo and Orsina studied problem (1.1) in the isotropic constant case with a positive constant γ and f in a certain Lebesgue space. They proved some existence and regularity results. In [6], Carmona and Martínez studied the same case for the singular nonlinearity with a variable exponent. Additionally, singular nonlinear elliptic equations in $\mathbb{R}^{\mathbb{N}}$ were studied in [4]. Then, the existence results for quasilinear nonlocal elliptic problems with variable singular exponent were proved in [13].

In this paper, we prove the existence and regularity results of positive solutions for anisotropic problems with variable exponents and a singular nonlinearity having also a variable exponent, where it was addressed to the treatment of cases $m = \frac{\overline{p}_N^-}{\overline{p}_N^- - 1}$ in Theorem 3.1, and $m = \frac{N(\alpha - 1 + \overline{p}^-)}{N(\overline{p}^- - 1) + \alpha \overline{p}^-}$ in Theorem 3.2, where $\overline{p}^- = \min_{x \in \overline{\Omega}} \overline{p}(x)$, $\overline{p}_N^- = \min_{x \in \overline{\Omega}} \overline{p}_N(x)$, $\alpha > \max\{1, \gamma^+\}$, and $\gamma^+ = \max_{x \in \overline{\Omega}} \gamma(x)$. This is explained under certain conditions in each of the two Theorems 3.1 and 3.2.

The proof requires a priori estimates of the sequence of suitable approximate solutions (u_n), which, in turn, proves its existence, and then, by passing to the limit, the functional setting involves Lebesgue and Sobolev spaces with variable exponent $L^{p(\,\cdot\,)}$, $W^{1,\overrightarrow{p}(\,\cdot\,)}$, $W^{1,\overrightarrow{p}(\,\cdot\,)}_{\mathrm{loc}}$, and $\mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}$.

We prove the strong convergence. Equipped with this convergence, we pass to the limit in the weak formulation.

2 Preliminaries

In this section, we recall some facts on anisotropic spaces with variable exponents and give some of their properties. For further details on the Lebesgue–Sobolev spaces with variable exponents, we refer to [10, 11, 16] and the references therein. Here, we set

$$C_+(\overline{\Omega}) = \big\{p \in C(\overline{\Omega}) :\ p(x) > 1 \text{ for any } x \text{ in } \overline{\Omega}\big\}.$$

For any $p \in C_+(\overline{\Omega})$, we denote

$$p^+ = \max_{x \in \overline{\Omega}} p(x) \ \text{ and } \ p^- = \min_{x \in \overline{\Omega}} p(x).$$

We define the Lebesgue space with a variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u:\Omega\to\mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u)=\int_\Omega |u|^{p(x)}\,dx$$

is finite. The expression

$$\|u\|_{p(\cdot)}:=\|u\|_{L^{p(\cdot)}(\Omega)}=\inf\Big\{\lambda>0:\ \rho_{p(\cdot)}\Big(\frac{u}{\lambda}\Big)\le 1\Big\}$$

defines a norm on $L^{p(\cdot)}(\Omega)$ called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega),\|u\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1<p^-\le p^+<+\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex and hence, reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$. For all $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$, the Hölder type inequality

$$\Big|\int_\Omega uv\,dx\Big|\le\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}\le 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}$$

holds. We also define the Banach space

$$W^{1,p(\cdot)}(\Omega)=\big\{u\in L^{p(\cdot)}(\Omega):\ |\nabla u|\in L^{p(\cdot)}(\Omega)\big\},$$

which is equipped with the following norm:

$$\|u\|_{1,p(\cdot)}=\|u\|_{W^{1,p(\cdot)}(\Omega)}=\|u\|_{p(\cdot)}+\|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega),\|u\|_{1,p(\cdot)})$ is a Banach space. Next, we also define $W_0^{1,p(\cdot)}(\Omega)$, the Sobolev space with zero boundary values, by

$$W_0^{1,p(\cdot)}(\Omega)=\big\{u\in W^{1,p(\cdot)}(\Omega):\ u=0 \text{ on } \partial\Omega\big\}$$

endowed with the norm $\|\cdot\|_{1,p(\cdot)}$. The space $W_0^{1,p(\cdot)}(\Omega)$ is separable and reflexive, provided $1<p^-\le p^+<+\infty$. For $u\in W_0^{1,p(\cdot)}(\Omega)$ with $p\in C_+(\overline{\Omega})$, the Poincaré inequality

$$\|u\|_{p(\cdot)}\le C\|\nabla u\|_{p(\cdot)} \tag{2.1}$$

holds for some $C>0$ depending on Ω and $p(\cdot)$. Therefore, $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(\cdot)}(u)$ of the space $L^{p(\cdot)}(\Omega)$. We have the following results.

Proposition 2.1 ([10]). *If $u_n,u\in L^{p(\cdot)}(\Omega)$ and $p^+<+\infty$, then the following properties hold:*

- $\|u\|_{p(\cdot)}<1$ (*resp.* $=1,>1$) $\Longleftrightarrow \rho_{p(\cdot)}(u)<1$ (*resp.* $=1,>1$),
- $\min\big(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}},\rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\big)\le\|u\|_{p(\cdot)}\le\max\big(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}},\rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\big)$,
- $\min\big(\|u\|_{p(\cdot)}^{p^-},\|u\|_{p(\cdot)}^{p^+}\big)\le\rho_{p(\cdot)}(u)\le\max\big(\|u\|_{p(\cdot)}^{p^-},\|u\|_{p(\cdot)}^{p^+}\big)$,
- $\|u\|_{p(\cdot)}\le\rho_{p(\cdot)}(u)+1$,
- $\|u_n-u\|_{p(\cdot)}\to 0\Longleftrightarrow\rho_{p(\cdot)}(u_n-u)\to 0$.

Remark 2.1. Note that the inequality

$$\int_\Omega |f|^{p(x)}\,dx\le C\int_\Omega |Df|^{p(x)}\,dx,$$

in general, does not hold (see [12]). But by Proposition 2.1 and (2.1) we have

$$\int_\Omega |f|^{p(x)}\,dx\le C\max\big\{\|Df\|_{p(\cdot)}^{p^+},\|Df\|_{p(\cdot)}^{p^-}\big\}. \tag{2.2}$$

In this paper, we will also need the space $W^{1,p(\,\cdot\,)}_{\rm loc}(\Omega)$, which is defined as follows:

$$W^{1,p(\,\cdot\,)}_{\rm loc}(\Omega) = \big\{u \in W^{1,p(\,\cdot\,)}(U) \text{ for all open } U \Subset \Omega\big\}.$$

We equip $W^{1,p(\,\cdot\,)}_{\rm loc}(\Omega)$ with the initial topology induced by the embeddings

$$W^{1,p(\,\cdot\,)}_{\rm loc}(\Omega) \hookrightarrow W^{1,p(\,\cdot\,)}(U) \ \text{ for all open } \ U \Subset \Omega.$$

Now, we present the anisotropic Sobolev space with a variable exponent which is used to study problem (1.1). First of all, let $p_i(\,\cdot\,) : \overline{\Omega} \to [1,+\infty)$, $i = 1,\dots,N$, be continuous functions, we set $\overrightarrow{p}(\,\cdot\,) = (p_1(\,\cdot\,),\dots,p_N(\,\cdot\,))$ and $p_+(x) = \max\limits_{1\le i\le N} p_i(x)$ for all $x \in \overline{\Omega}$. The anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ is defined as

$$W^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega) = \big\{u \in L^{p_+(\,\cdot\,)}(\Omega) :\ D_i u \in L^{p_i(\,\cdot\,)}(\Omega),\ i = 1,\dots,N\big\},$$

which is a Banach space with respect to the norm

$$\|u\|_{W^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)} = \|u\|_{p_+(\,\cdot\,)} + \sum_{i=1}^{N} \|D_i u\|_{p_i(\,\cdot\,)}.$$

We denote by $W^{1,\overrightarrow{p}(\,\cdot\,)}_0(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$, and we define

$$\mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega) = W^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega) \cap W^{1,1}_0(\Omega)$$

and

$$W^{1,\overrightarrow{p}(\,\cdot\,)}_{\rm loc}(\Omega) = \bigcap_{i=1}^{N} W^{1,p_i(\,\cdot\,)}_{\rm loc}(\Omega).$$

If Ω is a bounded open set with Lipschitz boundary $\partial\Omega$, then

$$\mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega) = \Big\{u \in W^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega) :\ u|_{\partial\Omega} = 0\Big\}.$$

It is well-known that in the constant exponent case, that is, when $\overrightarrow{p}(\,\cdot\,) = \overrightarrow{p} \in [1,+\infty)^N$, we have $W^{1,\overrightarrow{p}}_0(\Omega) = \mathring{W}^{1,\overrightarrow{p}}(\Omega)$. However, in the variable exponent case, in general, $W^{1,\overrightarrow{p}(\,\cdot\,)}_0(\Omega) \subset \mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ and smooth functions, in general, are not dense in $\mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$, but if for each $i = 1,\dots,N$, p_i is log-Hölder continuous, that is, there exists a positive constant L such that

$$|p_i(x) - p_i(y)| \le \frac{L}{-\ln|x-y|}\,, \quad \forall\, x,y \in \Omega,\ \ |x-y| \le \frac{1}{2}\,,$$

then $C_0^\infty(\Omega)$ is dense in $\mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ Thus $W^{1,\overrightarrow{p}(\,\cdot\,)}_0(\Omega) = \mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$. For more details on the study of anisotropic variable exponent Sobolev spaces $W^{1,\overrightarrow{p}(\,\cdot\,)}_0(\Omega)$ and $\mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$, we refer to the work audited in [11, 15, 16].

For all $x \in \overline{\Omega}$ we set

$$\overline{p}(x) = \frac{N}{\sum\limits_{i=1}^{N} \frac{1}{p_i(x)}}\,, \quad p_+(x) = \max_{1\le i\le N} p_i(x),$$

$$p_+^+ = \max_{x\in\overline{\Omega}} p_+(x), \quad p_-(x) = \min_{1\le i\le N} p_i(x), \quad p_-^- = \min_{x\in\overline{\Omega}} p_-(x),$$

and define

$$\overline{p}^{\,\star}(x) = \begin{cases} \dfrac{N\overline{p}(x)}{N-\overline{p}(x)} & \text{for } \ \overline{p}(x) < N, \\ +\infty & \text{for } \ \overline{p}(x) \ge N. \end{cases}$$

We have the following embedding results.

Lemma 2.1 ([11]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\,\cdot\,) \in (C_+(\overline{\Omega}))^N$. If $q(\,\cdot\,) \in C_+(\overline{\Omega})$ and for all $x \in \overline{\Omega}$, $q(x) < \max(p_+(x), \overline{p}^\star(x))$, then the embedding*

$$\mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega) \hookrightarrow L^{q(\,\cdot\,)}(\Omega)$$

is compact.

Lemma 2.2 ([11]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\,\cdot\,) \in (C_+(\overline{\Omega}))^N$. Suppose that*

$$p_+(x) < \overline{p}^\star(x), \quad \forall\, x \in \overline{\Omega}. \tag{2.3}$$

Then the following Poincaré-type inequality holds:

$$\|u\|_{L^{p_+(\,\cdot\,)}(\Omega)} \le C \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\,\cdot\,)}(\Omega)}, \quad \forall\, u \in \mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega), \tag{2.4}$$

where C is a positive constant independent of u. Thus $\sum\limits_{i=1}^{N} \|D_i u\|_{L^{p_i(\,\cdot\,)}(\Omega)}$ is an equivalent norm in $\mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$.

The following embedding results for the anisotropic constant exponent Sobolev space are well-known (see [17,19,20]).

Lemma 2.3. *Let $\alpha \ge 1$, $i = 1, \dots, N$, we pose $\vec{\alpha} = (\alpha_1, \dots, \alpha_N)$. Suppose $u \in W_0^{1,\vec{\alpha}}(\Omega)$ and set*

$$\frac{1}{\overline{\alpha}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\alpha_i}, \qquad r = \begin{cases} \overline{\alpha}^\star = \dfrac{N\overline{\alpha}}{N - \overline{\alpha}} & \text{if } \overline{\alpha} < N, \\ \text{any number in } [1, +\infty) & \text{if } \overline{\alpha} \ge N. \end{cases}$$

Then there exists a constant C depending on $N, \alpha_1, \dots, \alpha_N$ if $\overline{\alpha} < N$, and also on r and $|\Omega|$ if $\overline{\alpha} \ge N$, such that

$$\|u\|_{L^r(\Omega)} \le C \prod_{i=1}^{N} \|D_i u\|_{L^{\alpha}(\Omega)}^{1/N}. \tag{2.5}$$

The next Lemma is Lemma 4.1 given in [8].

Lemma 2.4. *For all u in $W^{1,p_i}(\Omega) \cap L^\infty(\Omega)$, where $\overline{p} < N$, we have*

$$\left(\int_\Omega |u|^a \right)^{\frac{N}{\overline{p}} - 1} \le \prod_{i=1}^{N} \left(|D_i u|^{p_i} |u|^{b_i p_i} \right)^{\frac{1}{p_i}} \tag{2.6}$$

for any choice of a and b_i, where

$$\frac{1}{a} = \frac{c_i(N-1) - 1 + \frac{1}{p_i}}{1 + b_i} \quad \text{with} \quad \sum_{i=1}^{N} c_i = 1.$$

3 Main results

Definition 3.1. We say that $u \in \mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ is a positive solution of problem (1.1) if $u > 0$ almost everywhere in Ω,

$$\frac{f(x)}{u^{\gamma(x)}} \in L^1_{\mathrm{loc}}(\Omega)$$

and

$$\int_\Omega \sum_{i=1}^{N} |D_i u|^{p_i(x)-2} D_i u D_i \varphi \, dx = \int_\Omega \frac{f\varphi}{u^{\gamma(x)}} \, dx \tag{3.1}$$

for every $\varphi \in C_0^1(\Omega)$.

Our main results are the following statements.

Theorem 3.1. *Let* $f \in L^m(\Omega)$*, where* $m = \frac{p_N^-}{p_N^- - 1}$ *, let* $p_i(\,\cdot\,) : \overline{\Omega} \to (1, +\infty)$ *be continuous functions such that* (1.2)*,* (1.3)*, and* (2.3) *hold, and let* $\gamma(\,\cdot\,) : \overline{\Omega} \to (0, +\infty)$ *be a smooth continuous function. Then problem* (1.1) *has a positive solution* u *in* $\mathring{W}^{1,\overrightarrow{p}(x)}(\Omega)$ *.*

Theorem 3.2. *For some* $\alpha > \max\{1, \gamma^+\}$*, let* $f \in L^m(\Omega)$*, where* $m = \frac{N(\alpha-1+\overline{p}^-)}{N(\overline{p}^- - 1) + \alpha\overline{p}^-}$*, let* $p_i(\,\cdot\,) : \overline{\Omega} \to (1, +\infty)$ *be continuous functions such that* (1.2)*,* (1.3)*, and* (2.3) *hold, and let* $\gamma(\,\cdot\,) : \overline{\Omega} \to (0, +\infty)$ *be a smooth continuous function. Then problem* (1.1) *has a positive solution* u *in* $L^{r(x)}(\Omega)$*, where* $r(x) = N\,\frac{\overline{p}^*(x)}{\overline{p}(x)}\,(\alpha - 1 + \overline{p}(x))$*, and this solution belongs to* $W^{1,\overrightarrow{p}(x)}_{loc}(\Omega)$ *.*

Remark 3.1. From (1.2) we have $p_+(x) = p_N(x)$ for all $x \in \overline{\Omega}$, while $r^- = N\,\frac{(\overline{p}^-)^*}{\overline{p}^-}\,(\alpha - 1 + \overline{p}^-)$, where $\overline{p}^-$ is the harmonic mean of $\{p_i^-,\ i = 1, \dots, N\}$.

4 Existence

We use the following problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D_i\big(|D_i u_n|^{p_i(x)-2} D_i u_n\big) = \frac{f_n}{(u_n + \frac{1}{n})^{\gamma(x)}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $f_n = T_n(f)$, $n \geq 1$.

We are going to prove the existence of a positive solution u_n to problem (4.1).

Lemma 4.1. *Let* $f \in L^m(\Omega)$ *and let* $\gamma(\,\cdot\,) : \overline{\Omega} \to (0, +\infty)$*,* $p_i(\,\cdot\,) : \overline{\Omega} \to (1, +\infty)$*,* $i = 1, \dots, N$*, be continuous functions. Assume that* (2.3) *holds. Then there exists at least one nonnegative solution* $u_n \in \mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ *to problem* (4.1) *in the sense that*

$$\sum_{i=1}^{N} \int_{\Omega} |D_i u_n|^{p_i(x)-2} D_i u_n D_i \varphi \, dx = \int_{\Omega} \frac{f_n \varphi}{(u_n + \frac{1}{n})^{\gamma(x)}} \, dx \tag{4.2}$$

for every $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega) \cap L^\infty(\Omega)$*.*

Proof. Consider the following problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D_i\big(|D_i u_n|^{p_i(x)-2} D_i u_n\big) = \frac{f_n}{(|u_n| + \frac{1}{n})^{\gamma(x)}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Fix $n \in \mathbb{N}^*$ and consider for $X = L^{P_N(\,\cdot\,)}(\Omega)$ the operator

$$\psi : X \times [0, 1] \to X, \quad (v_n, \sigma) \mapsto u_n = \psi(v_n, \sigma),$$

where u_n is the only solution of the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D_i\big(|D_i u_n|^{p_i(x)-2} D_i u_n\big) = \sigma \frac{f_n}{(|v_n| + \frac{1}{n})^{\gamma(x)}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

It is clear that problem (4.4) has a unique solution whenever the right-hand side belongs to $L^{q'(\,\cdot\,)}(\Omega)$, where $q(\,\cdot\,)$ is defined as in Lemma 2.1, i.e., $q(\,\cdot\,) < \overline{p}^*(\,\cdot\,)$ in $\overline{\Omega}$ (see [8, 9, 11]).

- It is clear that $\psi(v_n,0)=0$ for all $v_n\in X$, since the only weak solution to the problem

$$\begin{cases} -\sum_{i=1}^{N} D_i\big(|D_iu_n|^{p_i(x)-2}D_iu_n\big)=0 & \text{in } \Omega,\\ u_n=0 & \text{on } \partial\Omega \end{cases} \tag{4.5}$$

is $u_n=0\in X$.

- *Let us estimate the elements of X such that $v_n=\psi(v_n,\sigma)$.*

For all $\varphi\in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we have

$$\sum_{i=1}^{N}\int_\Omega |D_iv_n|^{p_i(x)-2}D_iv_nD_i\varphi\,dx=\sigma\int_\Omega \frac{f_n\varphi}{(|v_n|+\frac{1}{n})^{\gamma(x)}}\,dx. \tag{4.6}$$

Choosing $\varphi=v_n$ in (4.6), we get

$$\int_\Omega\sum_{i=1}^{N}|D_iv_n|^{p_i(x)}\,dx\le n^{1+\gamma^+}\int_\Omega |v_n|\,dx. \tag{4.7}$$

Recall Young's inequality: for any $\varepsilon>0$, and $a,b\ge 0$,

$$ab\le \varepsilon a^p+c(\varepsilon)b^{p'}, \tag{4.8}$$

where $c(\varepsilon)=\frac{1}{(\varepsilon p)^{\frac{p'}{p}}}\cdot\frac{1}{p'}$ and $\frac{1}{p}+\frac{1}{p'}=1$.

It follows from (4.8) that for any $\varepsilon>0$, there exists a constant $C(\varepsilon)$ depending on ε such that

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega |D_iv_n|^{p_i(x)}\,dx &\le n^{1+\gamma^+}\Big(\varepsilon\int_\Omega |v_n|^{p_-^-}\,dx+C(\varepsilon)\Big)\\ &\le n^{1+\gamma^+}\Big(\varepsilon c_1\int_\Omega |D_iv_n|^{p_-^-}\,dx+C(\varepsilon)\Big)\le n^{1+\gamma^+}\Big(\varepsilon c_2\Big(1+\int_\Omega |D_iv_n|^{p_i(x)}\,dx\Big)+C(\varepsilon)\Big)\\ &\le n^{1+\gamma^+}\Big(\varepsilon c_2\Big(1+\sum_{i=1}^{N}\int_\Omega |D_iv_n|^{p_i(x)}\,dx\Big)+C(\varepsilon)\Big),\end{aligned}$$

where c_1, c_2 are positive constants.

Now, we choose $\varepsilon=1/(2n^{1+\gamma^+}c_2)$, then

$$\sum_{i=1}^{N}\int_\Omega |D_iv_n|^{p_i(x)}\,dx\le c(n). \tag{4.9}$$

On the other hand, we have

$$\sum_{i=1}^{N}\int_\Omega |D_iv_n|^{p_i(x)}\,dx\ge\sum_{i=1}^{N}\min\big\{\|D_iv_n\|_{p_i(x)}^{p_i^-},\|D_iv_n\|_{p_i(x)}^{p_i^+}\big\},$$

where $\|\cdot\|_{p_i(\cdot)}=\|\cdot\|_{L^{p_i(\cdot)}(\Omega)}$.

We define

$$\beta_i=\begin{cases}p_+^+ & \text{if } \|D_iv_n\|_{p_i(\cdot)}<1,\\ p_-^- & \text{if } \|D_iv_n\|_{p_i(\cdot)}\ge 1.\end{cases}$$

We obtain

$$\sum_{i=1}^{N} \min\left\{\|D_i v_n\|_{p_i(\cdot)}^{p_i^-}, \|D_i v_n\|_{p_i(\cdot)}^{p_i^+}\right\}$$

$$\geq \sum_{i=1}^{N} \|D_i v_n\|_{p_i(\cdot)}^{\beta_i} \geq \sum_{i=1}^{N} \|D_i v_n\|_{p_i(\cdot)}^{p_-^-} - \sum_{\{i,\beta_i=p_+^+\}} \left(\|D_i v_n\|_{p_i(\cdot)}^{p_-^-} - \|D_i v_n\|p_{+\,p_i(\cdot)}^{+}\right)$$

$$\geq \sum_{i=1}^{N} \|D_i v_n\|_{p_i(\cdot)}^{p_-^-} - \sum_{\{i,\beta_i=p_+^+\}} \|D_i v_n\|_{p_i(\cdot)}^{p_-^-} \geq \Big(\frac{1}{N}\sum_{i=1}^{N} \|D_i v_n\|_{p_i(\cdot)}\Big)^{p_-^-} - N.$$

From (2.4) we get

$$\sum_{i=1}^{N} \int_{\Omega} |D_i v_n|^{p_i(x)}\,dx \geq \Big(\frac{1}{N}\,\|v_n\|_X\Big)^{p_-^-} - N. \tag{4.10}$$

From (4.9) and (4.10) we conclude that

$$\|v_n\|_X \leq C(n). \tag{4.11}$$

Then it follows from Leray–Schauder theorem that the operator $\psi_1 : X \to X$ defined by $\forall\, x \in X$: $\psi_1(x) = \psi(x,1)$ has a fixed point.

So, by the Sobolev compact embedding in Lemma 2.1, we conclude that the approximation problem (4.4) has the solution in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ for every fixed $n \in \mathbb{N}^*$.

Since $\frac{f_n}{(|u_n|+\frac{1}{n})^{\gamma(x)}} \geq 0$, the maximum principle (see [9]) implies that $u_n \geq 0$, thus u_n solves (4.3). $\square$

4.1 A priori estimates

In this section, we state and prove a uniform estimate for the approximate solutions u_n of problem (4.2).

Following the same proof steps as in [5,6], we obtain the following two lemmas below.

Lemma 4.2. *The sequence (u_n) is increasing with respect to n.*

Proof. From (4.2) we have

$$\sum_{i=1}^{N} \int_{\Omega} |D_i u_n|^{p_i(x)-2} D_i u_n D_i \varphi\,dx = \int_{\Omega} \frac{f_n \varphi}{(u_n + \frac{1}{n})^{\gamma(x)}}\,dx,$$

$$\sum_{i=1}^{N} \int_{\Omega} |D_i u_{n+1}|^{p_i(x)-2} D_i u_{n+1} D_i \varphi\,dx = \int_{\Omega} \frac{f_{n+1} \varphi}{(u_{n+1} + \frac{1}{n+1})^{\gamma(x)}}\,dx,$$

then

$$\sum_{i=1}^{N} \int_{\Omega} \left(|D_i u_n|^{p_i(x)-2} D_i u_n - |D_i u_{n+1}|^{p_i(x)-2} D_i u_{n+1}\right) D_i \varphi\,dx$$

$$= \int_{\Omega} \Big(\frac{f_n}{(u_n + \frac{1}{n})^{\gamma(x)}} - \frac{f_{n+1}}{(u_{n+1} + \frac{1}{n+1})^{\gamma(x)}}\Big)\varphi\,dx. \tag{4.12}$$

Taking $\varphi = (u_n - u_{n+1})^+$ as a test function in (4.12) and observing that if $0 \leq f_n \leq f_{n+1}$, then

$$\frac{f_n}{(u_n + \frac{1}{n})^{\gamma(x)}} - \frac{f_{n+1}}{(u_{n+1} + \frac{1}{n+1})^{\gamma(x)}} \leq f_{n+1}\Big(\frac{1}{(u_n + \frac{1}{n+1})^{\gamma(x)}} - \frac{1}{(u_{n+1} + \frac{1}{n+1})^{\gamma(x)}}\Big),$$

we find that the right-hand side gives

$$\sum_{i=1}^{N}\int_{\Omega}\big(|D_iu_n|^{p_i(x)-2}D_iu_n-|D_iu_{n+1}|^{p_i(x)-2}D_iu_{n+1}\big)D_i(u_n-u_{n+1})^+\,dx$$
$$\leq\int_{\Omega}f_{n+1}\Big(\frac{1}{(u_n+\frac{1}{n+1})^{\gamma(x)}}-\frac{1}{(u_{n+1}+\frac{1}{n+1})^{\gamma(x)}}\Big)(u_n-u_{n+1})^+\,dx.$$

Since $f_{n+1}\geq 0$, $(u_n-u_{n+1})^+\geq 0$ and

$$\frac{1}{(u_n+\frac{1}{n+1})^{\gamma(x)}}-\frac{1}{(u_{n+1}+\frac{1}{n+1})^{\gamma(x)}}\leq 0\ \text{ in } \{x\in\Omega:\ u_n(x)\geq u_{n+1}(x)\},$$

we get

$$\sum_{i=1}^{N}\int_{\Omega}\big(|D_iu_n|^{p_i(x)-2}D_iu_n-|D_iu_{n+1}|^{p_i(x)-2}D_iu_{n+1}\big)D_i(u_n-u_{n+1})^+\,dx\leq 0. \tag{4.13}$$

We recall the following well-known inequalities that hold for any two real vectors ξ,η and a real $p>1$:

$$\big(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\big)(\xi-\eta)\geq\begin{cases}2^{2-p}|\xi-\eta|^p & \text{if } p\geq 2,\\ (p-1)\,\dfrac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}} & \text{if } 1<p<2.\end{cases} \tag{4.14}$$

For all $i=1,\dots,N$, we put

$$\Omega_i^1=\{x\in\Omega,\ p_i(x)\geq 2\}\ \text{ and }\ \Omega_i^2=\{x\in\Omega,\ 1<p_i(x)<2\},$$

then, by virtue of (4.14), we have

$$\int_{\Omega}\big(|D_iu_n|^{p_i(x)-2}D_iu_n-|D_iu_{n+1}|^{p_i(x)-2}D_iu_{n+1}\big)D_i(u_n-u_{n+1})\,dx$$
$$\geq 2^{2-p_i^+}\int_{\Omega_i^1}|D_i(u_n-u_{n+1})|^{p_i(x)}. \tag{4.15}$$

On the other hand, by the Hölder inequality, (4.14) and Proposition 2.1, we have

$$\int_{\Omega_i^2}|D_i(u_n-u_{n+1})|^{p_i(x)}\,dx$$
$$\leq\int_{\Omega_i^2}\frac{|D_i(u_n-u_{n+1})|^{p_i(x)}}{(|D_iu_n|+|D_iu_{n+1}|)^{\frac{p_i(x)(2-p_i(x))}{2}}}\,\big(|D_iu_n|+|D_iu_{n+1}|\big)^{\frac{p_i(x)(2-p_i(x))}{2}}\,dx$$
$$\leq 2\bigg\|\frac{|D_i(u_n-u_{n+1})|^{p_i(x)}}{(|D_iu_n|+|D_iu_{n+1}|)^{\frac{p_i(x)(2-p_i(x))}{2}}}\bigg\|_{L^{\frac{2}{p_i(\cdot)}}(\Omega_i^2)}\Big\|\big(|D_iu_n|+|D_iu_{n+1}|\big)^{\frac{p_i(x)(2-p_i(x))}{2}}\Big\|_{L^{\frac{2}{2-p_i(\cdot)}}(\Omega_i^2)}$$
$$\leq 2\max\left\{\Big(\int_{\Omega_i^2}\frac{|D_i(u_n-u_{n+1})|^2}{(|D_iu_n|+|D_iu_{n+1}|)^{2-p_i(x)}}\,dx\Big)^{\frac{p_i^-}{2}},\Big(\int_{\Omega_i^2}\frac{|D_i(u_n-u_{n+1})|^2}{(|D_iu_n|+|D_iu_{n+1}|)^{2-p_i(x)}}\,dx\Big)^{\frac{p_i^+}{2}}\right\}$$
$$\times\max\left\{\Big(\int_{\Omega}\big(|D_iu_n|+|D_iu_{n+1}|\big)^{p_i(x)}\,dx\Big)^{\frac{2-p_i^+}{2}},\Big(\int_{\Omega}\big(|D_iu_n|+|D_iu_{n+1}|\big)^{p_i(x)}\,dx\Big)^{\frac{2-p_i^-}{2}}\right\}$$
$$\leq 2c\max\left\{\Big(\int_{\Omega}\Big(|D_iu_n|^{p_i(x)-2}D_iu_n-|D_iu_{n+1}|^{p_i(x)-2}D_iu_{n+1}\Big)D_i(u_n-u_{n+1})\,dx\Big)^{\frac{p_i^-}{2}},\right.$$

$$\Big(\int\limits_{\Omega}\Big(|D_iu_n|^{p_i(x)-2}D_iu_n-|D_iu_{n+1}|^{p_i(x)-2}D_iu_{n+1}\Big)D_i(u_n-u_{n+1})\,dx\Big)^{\frac{p_i^+}{2}}\Big\}$$
$$\times\Big(\big(\rho_{p_i}\big(|D_iu_n|+|D_iu_{n+1}|\big)\big)^{\frac{2-p_-^-}{2}}+1\Big). \quad (4.16)$$

Since $u_n,u_{n+1}\in\mathring{W}^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega;\mathbb{R}^m)$, from (4.15), (4.16) and (4.13) we obtain

$$\int\limits_{\Omega}|D_i(u_n-u_{n+1})^+|^{p_i(x)}\,dx\le 0\quad\text{for all }\ i=1,\dots,N.$$

Hence

$$u_n\le u_{n+1}. \quad (4.17)$$

The lemma is proved. □

Lemma 4.3. *For all $n\in\mathbb{N}^*$, $u_n\in L^\infty(\Omega)$ and for all $\omega\Subset\Omega$, there exists $C_\omega>0$ (independent of n) such that*

$$u_n\ge C_\omega>0. \quad (4.18)$$

Proof. As the right-hand side of (4.1) belongs to $L^\infty(\Omega)$, therefore the $L^\infty(\Omega)$ estimate of $(u_n)_n$ is a direct consequence of Stampachia's result [19].

Now, since we have

$$-\sum_{i=1}^{N}D_i\big(|D_iu_1|^{p_i(x)-2}D_iu_1\big)=\frac{f_1}{(u_1+1)^{\gamma(x)}}\ge\frac{f_1}{(\|u_1\|_\infty+1)^{\gamma(x)}}\ge 0,$$

the strong maximum principle and (4.17) give (4.18). □

Lemma 4.4. *Let m, $\gamma(\,\cdot\,)$, and $p_i(\,\cdot\,)$ be restricted as in Theorem 3.1. Then (u_n) is bounded in $\mathring{W}^{1,\overrightarrow{p}(x)}(\Omega)$.*

Proof. Choosing $\varphi=u_n$ in (4.2) and letting $\Omega_\delta=\{x\in\Omega,\ \operatorname{dist}(x,\partial\Omega)<\delta\}$, we have

$$\begin{aligned}\sum_{i=1}^{N}\int\limits_{\Omega}|D_iu_n|^{p_i(x)}\,dx&=\int\limits_{\Omega}\frac{f_nu_n}{(u_n+\frac1n)^{\gamma(x)}}\,dx\\
&\le\int\limits_{\overline{\Omega_\delta}}\frac{f_nu_n}{(u_n+\frac1n)^{\gamma(x)}}\,dx+\int\limits_{\Omega\setminus\overline{\Omega_\delta}}\frac{f_nu_n}{(u_n+\frac1n)^{\gamma(x)}}\,dx\le\int\limits_{\overline{\Omega_\delta}}fu_n^{1-\gamma(x)}\,dx+\int\limits_{\Omega\setminus\overline{\Omega_\delta}}\frac{fu_n}{(u_n+\frac1n)^{\gamma(x)}}\,dx\\
&\le\int\limits_{\overline{\Omega_\delta}\cap\{u_n\le1\}}f_nu_n^{1-\gamma(x)}\,dx+\int\limits_{\overline{\Omega_\delta}\cap\{u_n>1\}}\frac{f_nu_n}{(u_n+\frac1n)^{\gamma(x)}}\,dx+\big(1+C_{\Omega\setminus\overline{\Omega_\delta}}^{-\gamma^+}\big)\int\limits_{\Omega\setminus\overline{\Omega_\delta}}f_nu_n\,dx\\
&\le\int\limits_{\Omega}f\,dx+\big(2+C_{\Omega\setminus\overline{\Omega_\delta}}^{-\gamma^+}\big)\int\limits_{\Omega}fu_n\,dx.\end{aligned}$$

Using the Hölder inequality and (2.4), we obtain

$$\begin{aligned}\int\limits_{\Omega}\sum_{i=1}^{N}|D_iu_n|^{p_i(x)}\,dx&\le c\|f\|_{L^m(\Omega)}+C\big(2+C_{\Omega\setminus\overline{\Omega_\delta}}^{-\gamma^+}\big)\|f\|_{L^m(\Omega)}\|u_n\|_{L^{p_N^-}(\Omega)}\\
&\le c\|f\|_{L^m(\Omega)}+C\big(2+C_{\Omega\setminus\overline{\Omega_\delta}}^{-\gamma^+}\big)\|f\|_{L^m(\Omega)}\Big(\sum_{i=1}^{N}\|D_iu_n\|_{L^{p_i(\cdot)}(\Omega)}\Big). \quad (4.19)\end{aligned}$$

On the other hand, we have

$$\sum_{i=1}^{N}\int_{\Omega}|D_iu_n|^{p_i(x)}\,dx \geq \Big(\frac{1}{N}\sum_{i=1}^{N}\|D_iu_n\|_{p_i(\,\cdot\,)}\Big)^{p_-^-} - N. \tag{4.20}$$

From (4.19) and (4.20) we get the desired result. □

Lemma 4.5. *Let m, α, $\gamma(\,\cdot\,)$, and $p_i(\,\cdot\,)$ be restricted as in Theorem 3.2. Then (u_n) is bounded in $L^{r^-}(\Omega)$ with*

$$r(x) = N\,\frac{\overline{p}^{\,*}(x)}{\overline{p}(x)}\,(\alpha - 1 + \overline{p}(x)).$$

Proof. Chousing $\varphi = u_n^{\alpha}$ in (4.2) and using the Hölder inequality, we have

$$\begin{aligned}
\alpha\int_{\Omega}|D_iu_n|^{p_i(x)}u_n^{\alpha-1}\,dx &\leq \int_{\Omega}\frac{f_nu_n^{\alpha}}{(u_n+\frac{1}{n})^{\gamma(x)}}\,dx \leq \int_{\overline{\Omega_\delta}} fu_n^{\alpha-\gamma(x)}\,dx + \big(1+C_{\Omega\setminus\overline{\Omega_\delta}}^{-\gamma^+}\big)\int_{\Omega\setminus\overline{\Omega_\delta}} fu_n^{\alpha}\,dx \\
&\leq \int_{\Omega} f\,dx + \big(2+C_{\Omega\setminus\overline{\Omega_\delta}}^{-\gamma^+}\big)\int_{\Omega} fu_n^{\alpha}\,dx \leq c\|f\|_{L^m(\Omega)} + \big(2+C_{\Omega\setminus\overline{\Omega_\delta}}^{-\gamma^+}\big)\|f\|_{L^m(\Omega)}\|u_n^{\alpha}\|_{L^{m'}(\Omega)} \\
&\leq c\|f\|_{L^m(\Omega)} + \big(2+C_{\Omega\setminus\overline{\Omega_\delta}}^{-\gamma^+}\big)\|f\|_{L^m(\Omega)}(\|u_n\|_{L^{r^-}(\Omega)})^{\frac{r^-}{\beta}},
\end{aligned}$$

where $\beta = \frac{r^-}{\alpha}$. From the fact that

$$|\Omega| + \int_{\Omega}|D_iu_n|^{p_i(x)}u_n^{\alpha-1}\,dx \geq \int_{\Omega}|D_iu_n|^{p_i^-}u_n^{\alpha-1}\,dx$$

we get

$$\int_{\Omega}|D_iu_n|^{p_i^-}u_n^{\alpha-1}\,dx \leq c_1 + c_2(\|u_n\|_{L^{r^-}(\Omega)})^{\frac{r^-}{\beta}}. \tag{4.21}$$

From (4.21) we obtain

$$\prod_{i=1}^{N}\Big(\int_{\Omega}|D_iu_n|^{p_i^-}u_n^{\alpha-1}\,dx\Big)^{\frac{1}{p_i^-}} \leq \Big(c_1 + c_2(\|u_n\|_{L^{r^-}(\Omega)})^{\frac{r^-}{\beta}}\Big)^{\frac{N}{\overline{P}^-}}. \tag{4.22}$$

From (4.22), after applying Lemma 2.4 with respect to

$$b_i = \frac{r^- - 1}{p_i^-}, \quad a = r^-, \quad c_i = \frac{1}{N-1}\Big(\frac{1+b_i}{a} + 1 - \frac{1}{p_i^-}\Big),$$

we get

$$(\|u_n\|_{L^{r^-}(\Omega)}^{r^-})^{\frac{N}{\overline{P}^-}-1} \leq \big(c_1 + c_2(\|u_n\|_{L^{r^-}(\Omega)}^{r^-})^{\frac{1}{\beta}}\big)^{\frac{N}{\overline{P}^-}}.$$

Therefore, we obtain

$$(\|u_n\|_{L^{r^-}(\Omega)}^{r^-})^{1-\frac{\overline{P}^-}{N}} \leq c_1 + c_2(\|u_n\|_{L^{r^-}(\Omega)}^{r^-})^{\frac{1}{\beta}}. \tag{4.23}$$

Since $\|u_n\|_{L^{r^-}(\Omega)}^{r^-} \leq 1$, we have

$$\|u_n\|_{L^{r^-}(\Omega)} \leq 1. \tag{4.24}$$

Since $\|u_n\|_{L^{r^-}(\Omega)}^{r^-} > 1$, from (4.23) we get

$$(\|u_n\|_{L^{r^-}(\Omega)}^{r^-})^{1-\frac{\overline{P}^-}{N}} \leq (c_1 + c_2)(\|u_n\|_{L^{r^-}(\Omega)}^{r^-})^{\frac{1}{\beta}}.$$

Since $\frac{1}{\beta} < 1 - \frac{\overline{P}^-}{N}$, we have

$$\|u_n\|_{L^{r^-}(\Omega)} \leq C. \tag{4.25}$$

Then (4.24) and (4.25) imply that (u_n) is bounded in $L^{r^-}(\Omega)$ with $r(x) = N\,\frac{\overline{p}^{\,*}(x)}{\overline{p}(x)}\,(\alpha - 1 + \overline{p}(x))$. □

4.2 Proof of Theorems 3.1 and 3.2

4.2.1 Proof of Theorem 3.1

By Lemma 4.4, (u_n) is bounded in $\mathring{W}^{1,\overrightarrow{p}(x)}(\Omega)$. Consequently, we can extract a subsequence (denoted again by (u_n)) such that

$$u_n \rightharpoonup u \text{ weakly in } \mathring{W}^{1,\overrightarrow{p}(x)}(\Omega).$$

From here and Lemma 2.1 we obtain

$$u_n \to u \text{ strongly in } L^{q(x)}(\Omega),$$

where $q(\cdot) < \overline{p}^*(\cdot)$ in $\overline{\Omega}$. Thus

$$u_n \to u \text{ a.e. in } \Omega.$$

So, for all $\varphi \in C_0^1(\Omega)$, we get

$$\sum_{i=1}^{N} \int_\Omega |D_i u_n|^{p_i(x)-2} D_i u_n D_i \varphi \, dx \longrightarrow \sum_{i=1}^{N} \int_\Omega |D_i u|^{p_i(x)-2} D_i u D_i \varphi \, dx \quad \text{as } n \to +\infty. \tag{4.26}$$

For all $\varphi \in C_0^1(\Omega)$, $\varphi \neq 0$, and on the set where $u_n \geq C_{\Omega\setminus\overline{\Omega_\delta}}$, $\Omega \setminus \overline{\Omega_\delta}$ being the support of φ, we have

$$0 \leq \left| \frac{f_n \varphi}{(u_n + \frac{1}{n})^{\gamma(x)}} \right| \leq \big(1 + C_{\Omega\setminus\overline{\Omega_\delta}}^{-\gamma^+}\big) \|\varphi\|_{L^\infty(\Omega)} f.$$

Then the dominated Lebesgue's theorem permits us to conclude that

$$\int_\Omega \frac{f_n \varphi}{(u_n + \frac{1}{n})^{\gamma(x)}} \, dx \longrightarrow \int_\Omega \frac{f\varphi}{u^{\gamma(x)}} \, dx \quad \text{as } n \to +\infty. \tag{4.27}$$

4.2.2 Proof of Theorem 3.2

By Lemma 4.5 and the continuous embedding $L^{r(x)}(\Omega) \hookrightarrow L^{r^-}(\Omega)$ we find that (u_n) is bounded in $L^{r(x)}(\Omega)$. Then, by the monotone convergence theorem, we have

$$u_n \to u \text{ strongly in } L^{r(x)}(\Omega).$$

Now, we can pass to the limit in the weak formulation (4.2) prove (4.26) and (4.27) in a similar way.

On the other hand, we find that

$$\sum_{i=1}^{N} \int_\Omega |D_i u_n|^{p_i(x)} u_n^{\alpha-1} \, dx \leq C.$$

By the strong maximum principle for every compact $K \Subset \Omega$ we have

$$C_K^{\alpha-1} \sum_{i=1}^{N} \int_\Omega |D_i u_n| \, dx \leq C.$$

Thus we obtain

$$u_n \rightharpoonup u \text{ weakly in } W_{\text{loc}}^{1,\overrightarrow{p}(x)}(\Omega).$$

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References

[1] R. P. Agarwal, O. Bazighifan and M. A. Ragusa, Nonlinear neutral delay differential equations of fourth-order: oscillation of solutions. *Entropy* **23** (2021), no. 2, Paper No. 129, 10 pp.

[2] R. P. Agarwal, S. Gala and M. A. Ragusa, A regularity criterion in weak spaces to Boussinesq equations. *Mathematics* **8** (2020), no. 6, 920.

[3] A. M. Alghamdi, S. Gala, C. Qian and M. A. Ragusa, The anisotropic integrability logarithmic regularity criterion for the 3D MHD equations. *Electron. Res. Arch.* **28** (2020), no. 1, 183–193.

[4] C. O. Alves, J. V. Goncalves and L. A. Maia, Singular nonlinear elliptic equations in $\mathbf{R}^N$. *Abstr. Appl. Anal.* **3** (1998), no. 3-4, 411–423.

[5] L. Boccardo and L. Orsina, Semilinear elliptic equations with singular nonlinearities. *Calc. Var. Partial Differential Equations* **37** (2010), no. 3-4, 363–380.

[6] J. Carmona and P. J. Martínez-Aparicio, A singular semilinear elliptic equation with a variable exponent. *Adv. Nonlinear Stud.* **16** (2016), no. 3, 491–498.

[7] Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.* **66** (2006), no. 4, 1383–1406.

[8] A. Di Castro, Existence and regularity results for anisotropic elliptic problems. *Adv. Nonlinear Stud.* **9** (2009), no. 2, 367–393.

[9] A. Di Castro, Elliptic problems for some anisotropic operators. *Ph.D. Thesis, University of Rome "Sapienza"*, a. y. 2008/2009.

[10] L. Diening, T. Harjulehto, P. Hästö and M. Růžička, *Lebesque and Sobolev Spaces with Variable Exponents.* Lecture Notes in Mathematics, Vol. 2017, Springer-Verlag, Berlin, 2011.

[11] X. Fan, Anisotropic variable exponent Sobolev spaces and $\overrightarrow{p(x)}$-Laplacian equations. *Complex Var. Elliptic Equ.* **56** (2011), no. 7-9, 623–642.

[12] X. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *J. Math. Anal. Appl.* **263** (2001), no. 2, 424–446.

[13] P. Garain and T. Mukherjee, Quasilinear nonlocal elliptic problems with variable singular exponent. *Commun. Pure Appl. Anal.* **19** (2020), no. 11, 5059–5075.

[14] M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **462** (2006), no. 2073, 2625–2641.

[15] N. Mokhtar and M. B. Benboubker, Distributional solutions of anisotropic nonlinear elliptic systems with variable exponents: existence and regularity. *Adv. Oper. Theory* **7** (2022), no. 2, Paper No. 17.

[16] N. Mokhtar and F. Mokhtari, Anisotropic nonlinear elliptic systems with variable exponents and degenerate coercivity. *Appl. Anal.* **100** (2021), no. 11, 2347–2367.

[17] J. Rákosník, Some remarks to anisotropic Sobolev spaces. II. *Beiträge Anal.*, No. 15 (1980), 127–140 (1981).

[18] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory.* Lecture Notes in Mathematics, 1748. Springer-Verlag, Berlin, 2000.

[19] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. (French) *Ann. Inst. Fourier (Grenoble)* **15** (1965), fasc. 1, 189–258.

[20] M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi. (Italian) *Ricerche Mat.* **18** (1969), 3–24.

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