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CAPUTO–HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH INTEGRAL BOUNDARY CONDITIONS VIA FIXED POINT THEORY

Abstract. In this work, we study the uniqueness and existence results for higher order fractional differential inclusions and equations involving the Caputo–Hadamard fractional derivative subject to integral boundary conditions (IBCs). Our results are obtained via the fixed point theorems (FPTs) for multi- and single-valued analysis. An example demonstrating the effectiveness of the theoretical findings is presented.

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1 Introduction

In the latest years, fractional differential equations (FDEs) theory has received very broad regard in the fields of pure and applied mathematics (see [18, 27, 32, 37]). FDE's and Fractional differential inclusions (FDIs) emerge naturally in diverse scopes of science, with many applications (see, e.g., [13, 14, 16, 17, 21, 24, 34, 39, 42]).

Throughout the years, research in this field to discuss the qualitative properties of solutions of FDEs and FDIs such as the existence, uniqueness, stability, controllability, and optimizations, etc., has been of great interest to researchers and scholars (see [2–15, 18–20, 23, 28, 35, 36, 40, 41]). Some researches are devoted to discussing more analytical properties of solutions of this type of equations and inclusions, and others are already oriented toward numerical applications and solutions. Other articles on existence, uniqueness, and stability of FDEs involving various types of fractional derivatives (FDs), can be found in [1, 8–10, 29–31].

Recently in [3], the authors discussed the existence results of FDIs subject to nonlocal boundary conditions of the form

$$\begin{cases} {}^{C}D^{r_1}\varkappa(\tau) \in Q(\tau, \varkappa(\tau)), \quad \tau \in [0,1], \; m-1 < r_1 \leq m, \; m \geq 2, \; m \in \mathbb{N}, \\ \varkappa(0) = \varkappa'(0) = \varkappa''(0) = \cdots = \varkappa^{(m-2)}(0) = 0, \quad \varkappa(1) = \beta\varkappa(\eta), \end{cases}$$

where ${}^{C}D^{r_1}$ is the Caputo fractional derivative, $\eta \in (0,1)$, $\beta\eta^{m-1} \neq 1$, $\beta \in \mathbb{R}$ and $Q : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map.

In [11], Arioua et al. studied the existence of solutions of the following problem of FDEs:

$$\begin{cases} {}^{C}_{H}\mathfrak{D}^{r_1}_{1} x(\tau) = q(\tau, \varkappa(\tau)), \quad \tau \in (1,e), \; 2 < r_1 \leq 3, \\ \varkappa(1) = \varkappa'(1) = 0, \; {}^{C}_{H}\mathfrak{D}^{r_1-1}_{1}\varkappa(e) = {}^{C}_{H}\mathfrak{D}^{r_1-2}_{1}\varkappa(e) = 0, \end{cases}$$

where ${}^{C}_{H}\mathfrak{D}^{r_1}_{1}$ is the FD in Caputo–Hadamard sense of order r_1 and $q : [1,e] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

In [19], Duraisamy et al. used some fixed point theorems to discuss the existence of solutions of higher order FDEs given by

$$\begin{cases} {}^{C}D^{r_1}\varkappa(\tau) = q(\tau, \varkappa(\tau)), \quad \tau \in [0,1], \\ \varkappa(0) = \varkappa'(0) = \varkappa''(0) = \cdots = \varkappa^{(m-2)}(0) = 0, \\ \varkappa(1) = \displaystyle\sum_{i=1}^{m} \gamma_i \big[I^{\beta_i}\varkappa(\eta_i) - I^{\beta_i}\varkappa(\zeta_i)\big]. \end{cases}$$

Inspired and motivated by the aforementioned works, we prove the existence and uniqueness of solutions for higher order FDEs and FDIs with IBCs

$$\begin{cases} {}^{C}_{H}\mathfrak{D}^{r_1}_{1}\varkappa(\tau) = q(\tau, \varkappa(\tau)), \quad \tau \in (1,\tau_1), \\ \varkappa(1) = \varkappa'(1) = \varkappa''(1) = \cdots = \varkappa^{(m-2)}(1) = 0, \\ \varkappa(\tau_1) = \lambda \displaystyle\int_1^{\tau_1} \varkappa(\zeta)\,\frac{d\zeta}{\zeta} + d, \end{cases} \tag{1.1}$$

where $m-1 < r_1 \leq m$, $m \geq 2$, $m \in \mathbb{N}$, $\lambda, d \in \mathbb{R}$ and $q : [1,\tau_1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and

$$\begin{cases} {}^{C}_{H}\mathfrak{D}^{r_1}_{1}\varkappa(\tau) \in Q(\tau, \varkappa(\tau)), \quad 1 < \tau < \tau_1, \\ \varkappa(1) = \varkappa'(1) = \varkappa''(1) = \cdots = \varkappa^{(m-2)}(1) = 0, \\ \varkappa(\tau_1) = \lambda \displaystyle\int_1^{\tau_1} \varkappa(\zeta)\,\frac{d\zeta}{\zeta} + d, \quad \lambda, d \in \mathbb{R}, \end{cases} \tag{1.2}$$

where Q is a multi-valued map from $[1,\tau_1] \times \mathbb{R}$ to the family of $\mathcal{P}(\mathbb{R})$ (all non-empty subsets of $\mathbb{R}$), and Q is convex-valued.

This paper is organized as follows. In Section 2, we give some fundamentals ideas of fractional calculus (FC), set-valued analysis and FP techniques. In Section 3, we demonstrate the existence and uniqueness outcomes for (1.1) by using the FPTs of Banach and Schauder. In Section 4, we study the existence results for inclusion problem (1.2) relying on FPT of Leray–Schauder. In Section 5, an example is given.

2 Preliminaries

In this part, we give some fundamental ideas of FC, set-valued analysis and FPTs that prerequisite in our analysis.

2.1 Fractional calculus

Let $J_1 = [1, \tau_1]$. By $\mathcal{C} = C(J_1, \mathbb{R})$ we denote the Banach space of all continuous functions $\varkappa : J_1 \to \mathbb{R}$ with the norm

$$\|\varkappa\| = \sup \{ |\varkappa(\tau)| : \tau \in J_1 \}.$$

Let $L^1(J_1, \mathbb{R})$ be the Banach space of Lebesgue integrable functions $\varkappa : J_1 \to \mathbb{R}$ with the norm

$$\|\varkappa\|_{L^1} = \int_{J_1} |\varkappa(\tau)|\, d\tau.$$

$AC(J_1, \mathbb{R})$ is the space of absolutely continuous functions and

$$AC^m(J_1) = \Big\{ \varkappa : J_1 \to \mathbb{R} : \ \varkappa, \varkappa', \varkappa'', \dots, \varkappa^{m-1} \in \mathcal{C} \text{ and } \varkappa^{m-1} \in AC(J_1, \mathbb{R}) \Big\}.$$

Definition 2.1 ([27])**.** The Hadamard fractional integral of order $r_1 > 0$ for a function $x \in L^1(J_1)$ is defined as

$${}^{H}\mathfrak{I}_1^{r_1} x(\tau) = \frac{1}{\Gamma(r_1)} \int_1^{\tau} \left(\log \frac{\tau}{\zeta} \right)^{r_1 - 1} x(\zeta) \, \frac{d\zeta}{\zeta}, \quad r_1 > 0.$$

Set $\delta = (\tau \frac{d}{d\tau})$, $n = [r_1] + 1$, where r_1 denotes the integer part of r_1. Define the space

$$AC_\delta^n(J_1) = \{ \varkappa : J_1 \to \mathbb{R} : \ \delta^{n-1} \varkappa(\tau) \in AC(J_1, \mathbb{R}) \}.$$

Definition 2.2 ([27])**.** The Hadamard FD of order $r_1 > 0$ for a function $\varkappa \in AC_\delta^n(J_1)$ is defined as

$${}^{H}\mathfrak{D}_1^{r_1} \varkappa(\tau) = \delta^n ({}^{H}\mathfrak{I}^{n-r_1} \varkappa)(\tau) = \frac{1}{\Gamma(n - r_1)} \left(\tau \frac{d}{d\tau} \right)^n \int_1^{\tau} \left(\log \frac{\tau}{\zeta} \right)^{n - r_1 - 1} \varkappa(\zeta) \, \frac{d\zeta}{\zeta}.$$

Definition 2.3 ([26])**.** The Caputo–Hadamard FD of order $r_1 > 0$ for a function $\varkappa \in AC_\delta^n(J_1)$ is defined as

$${}_{H}^{C}\mathfrak{D}_1^{r_1} \varkappa(\tau) = ({}^{H}\mathfrak{I}_1^{n-r_1} \delta^n \varkappa)(\tau) = \frac{1}{\Gamma(n - r_1)} \int_1^{\tau} \left(\log \frac{\tau}{\zeta} \right)^{n - r_1 - 1} \delta^n \varkappa(\zeta) \, \frac{d\zeta}{\zeta}.$$

Lemma 2.1 ([26])**.** *Let $r_1 > 0$ and $m = [r_1] + 1$. If $\varkappa \in AC_\delta^n(J_1)$, then the Caputo–Hadamard FDE*

$${}_{H}^{C}\mathfrak{D}_1^{r_1} \varkappa(\tau) = 0,$$

has a solution

$$\varkappa(\tau) = \sum_{k=0}^{m-1} c_k (\log \tau)^k,$$

and the following formula holds:

$$ {}^{H}\mathfrak{I}_1^{r_1}({}_{H}^{C}\mathfrak{D}_1^{r_1}\varkappa(\tau)) = \varkappa(\tau) + \sum_{k=0}^{m-1} c_k(\log\tau)^k, $$

where $c_k \in \mathbb{R}$, $k = 1, 2, \dots, m-1$.

To study the nonlinear problem (1.1), we need the following lemma.

Lemma 2.2. *Let*

$$ \Delta = m(\log\tau_1)^{m-1} - \lambda(\log\tau_1)^m \neq 0. $$

For any $\omega \in \mathcal{C}$, *the solution of the boundary value problem*

$$ \begin{cases} {}_{H}^{C}\mathfrak{D}_1^{r_1}\varkappa(\tau) = \omega(\tau), \quad \tau \in (1,\tau_1), \quad m-1 < r_1 \le m, \quad m \ge 2, \quad m \in \mathbb{N}, \\ \varkappa(1) = \varkappa'(1) = \varkappa''(1) = \cdots = \varkappa^{(m-2)}(1) = 0, \\ \varkappa(\tau_1) = \lambda \displaystyle\int_1^{\tau_1} \varkappa(\zeta)\,\frac{d\zeta}{\zeta} + d, \end{cases} \tag{2.1} $$

is obtained as

$$ \begin{aligned} \varkappa(\tau) = {} & \frac{1}{\Gamma(r_1)}\int_1^{\tau}\Big(\log\frac{\tau}{\zeta}\Big)^{r_1-1}\omega(\zeta)\,\frac{d\zeta}{\zeta} + \frac{m(\log\tau)^{m-1}}{\Delta}\Bigg(\frac{1}{\Gamma(r_1)}\int_1^{\tau_1}\Big(\log\frac{\tau_1}{\zeta}\Big)^{r_1-1}\omega(\zeta)\,\frac{d\zeta}{\zeta} \\ & - \lambda\int_1^{\tau_1}\Big(\frac{1}{\Gamma(r_1)}\int_1^{\zeta}\Big(\log\frac{\zeta}{\sigma}\Big)^{r_1-1}\omega(\sigma)\,\frac{d\sigma}{\sigma}\Big)\frac{d\zeta}{\zeta} - d\Bigg). \end{aligned} \tag{2.2} $$

Proof. Using ${}^{H}\mathfrak{I}_1^{r_1}$ to (2.1), and by Lemma 2.1, we have

$$ \varkappa(\tau) = {}^{H}\mathfrak{I}_1^{r_1}\omega(\tau) - c_0 - c_1\log\tau - c_2(\log\tau)^2 - \cdots - c_{m-1}(\log\tau)^{m-1}, \tag{2.3} $$

then

$$ \begin{aligned} \varkappa'(\tau) = {} & \frac{1}{\Gamma(r_1-1)\tau}\int_1^{\tau}\Big(\log\frac{\tau}{\zeta}\Big)^{r_1-2}\omega(\zeta)\,\frac{d\zeta}{\zeta} \\ & - \frac{c_1}{\tau} - c_2\,\frac{2\log\tau}{\tau} - \cdots - c_{m-1}\frac{(m-1)(\log\tau)^{m-2}}{\tau}, \\ \varkappa''(\tau) = {} & \frac{-1}{\Gamma(r_1-1)\tau^2}\int_1^{\tau}\Big(\log\frac{\tau}{\zeta}\Big)^{r_1-2}\omega(\zeta)\,\frac{d\zeta}{\zeta} + \frac{1}{\Gamma(r_1-2)\tau}\int_1^{\tau}\Big(\log\frac{\tau}{\zeta}\Big)^{r_1-3}\omega(\zeta)\,\frac{d\zeta}{\zeta} \\ & - c_1\Big(\frac{-1}{\tau^2}\Big) - 2c_2\Big(\frac{1}{\tau^2} - \frac{\log\tau}{\tau^2}\Big) - \cdots \\ & - (m-1)c_{m-1}\Big(\frac{(\log\tau)^{m-2}}{\tau^2} - \frac{(m-2)(\log\tau)^{m-3}}{\tau^2}\Big), \dots. \end{aligned} $$

Applying the boundary conditions, we have

$$ c_0 = c_1 = c_2 = \cdots = c_{m-2} = 0. \tag{2.4} $$

Substituting (2.4) into (2.3), we get

$$ \varkappa(\tau) = {}^{H}\mathfrak{I}_1^{r_1}\omega(\tau) - c_{m-1}(\log\tau)^{m-1}. \tag{2.5} $$

From the integral condition of (2.1), we have

$$\frac{1}{\Gamma(r_1)}\int_1^{\tau_1}\left(\log\frac{\tau_1}{\zeta}\right)^{r_1-1}\omega(\zeta)\,\frac{d\zeta}{\zeta}-c_{m-1}(\log\tau_1)^{m-1}$$
$$=\lambda\int_1^{\tau_1}\left(\frac{1}{\Gamma(r_1)}\int_1^{\zeta}\left(\log\frac{\zeta}{\sigma}\right)^{r_1-1}\omega(\sigma)\,\frac{d\sigma}{\sigma}\right)\frac{d\zeta}{\zeta}-\frac{\lambda c_{m-1}(\log\tau_1)^m}{m}+d,$$

so,

$$c_{m-1}=\frac{m}{\Delta}\left(\frac{1}{\Gamma(r_1)}\int_1^{\tau_1}\left(\log\frac{\tau_1}{\zeta}\right)^{r_1-1}\omega(\zeta)\,\frac{d\zeta}{\zeta}-\lambda\int_1^{\tau_1}\left(\frac{1}{\Gamma(r_1)}\int_1^{\zeta}\left(\log\frac{\zeta}{\sigma}\right)^{r_1-1}\omega(\sigma)\,\frac{d\sigma}{\sigma}\right)\frac{d\zeta}{\zeta}-d\right). \tag{2.6}$$

Substituting (2.6) into (2.5), we obtain (2.2). Since each step is reversible, the converse follows easily. □

2.2 Multi-valued analysis

Let $(E,\|\cdot\|)$ be a Banach space and $\xi: E\to\mathcal{P}(E)$ be a set-valued map, then

(a) ξ is closed (convex) valued if $\xi(\varkappa)$ is closed (convex) for any $\varkappa\in\mathcal{P}(E)$,

(b) ξ is bounded if $\xi(\mathcal{B})=\bigcup_{\varkappa\in\mathcal{B}}\xi(\varkappa)$ is bounded for all bounded sets $\mathcal{B}$ of E, i.e.,

$$\sup_{\varkappa\in\mathcal{B}}\left\{\sup\left\{|\mathfrak{y}|:\ \mathfrak{y}\in\xi(\varkappa)\right\}\right\}<\infty,$$

(c) ξ is measurable if $\forall\,\mathfrak{y}\in\mathbb{R}$, the function

$$\tau\to d(\mathfrak{y},\xi(\varkappa))=\inf\left\{|\mathfrak{y}-\eta|:\ \eta\in\xi(\varkappa)\right\},$$

is measurable.

For standard definitions, e.g., completely continuity, upper semi-continuity, we refer to [7]. Moreover, a collection of selections of Q at the point $\varkappa\in\mathcal{C}$ is defined by

$$S_{Q,\varkappa}=\left\{\upsilon\in L^1(J_1,\mathbb{R}):\ \upsilon(\tau)\in Q(\tau,\varkappa)\text{ for a.e. }\tau\in J_1\right\}.$$

Next, we denote

$$\mathcal{P}_k(E)=\left\{Y\in\mathcal{P}(E):Y\text{ is nonempty and has property }k\right\},$$

where $\mathcal{P}_{cp}$, $\mathcal{P}_b$, $\mathcal{P}_{cl}$, $\mathcal{P}_c$ denote the classes of all compact, bounded, closed and convex subsets of E, respectively. Also, $\mathcal{P}_{cp,c}$ denotes the class of all compact and convex subsets of E.

Definition 2.4. A set-valued map Q from $[a,b]\times\mathbb{R}$ to $\mathcal{P}(\mathbb{R})$ is called Carathéodory whenever the map $\tau\to Q(\tau,\varkappa)$ is measurable for any $\varkappa\in\mathbb{R}$, and the map $\varkappa\to Q(\tau,\varkappa)$ is upper semi-continuous (u.s.c.) for (a.e.) all $\tau\in J_1$.

Moreover, a set-valued map Q is called L^1-Carathéodory if $\forall\,\rho>0$ there exists $\varphi_\rho\in L^1(J_1,\mathbb{R}^+)$ such that

$$\|Q(\tau,\varkappa)\|=\sup\left\{|\upsilon|:\ \upsilon\in Q(\tau,\varkappa)\right\}\le\varphi_\rho(\tau),$$

for all $\|\varkappa\|\le\rho$ and for a.e. $\tau\in J_1$.

We need the following lemmas, which play an important role in the achievement of the desired outcomes in this research.

Lemma 2.3 ([17, Proposition 1.2]). *Let* $\mathrm{Gr}(\xi) = \{(\varkappa, \mathfrak{y}) \in E \times Y,\ \rho \in \xi(\varkappa)\}$ *be a graph of* ξ. *Reciprocally, if* $\xi : E \to \mathcal{P}_{cl}(E)$ *is u.s.c., then* $\mathrm{Gr}(\xi)$ *is a closed subset of* $E \times Y$. *If* ξ *is completely continuous and has a closed graph, then it is u.s.c.*

Lemma 2.4 ([33]). *Let* E *be a separable Banach space,* $Q : J_1 \times E \to \mathcal{P}_{cp,c}(E)$ *be an* L^1*-Carathéodory set-valued map, and* $\Theta : L^1(J_1, E) \to C(J_1, E)$ *be a linear continuous mapping. Then the operator*

$$\Theta \circ S_Q : C(J_1, E) \to \mathcal{P}_{cp,c}(C(J_1, E)), \quad \varkappa \to (\Theta \circ S_Q)(\varkappa) = \Theta(S_{Q,\varkappa}),$$

is a closed graph operator in $C(J_1, E) \times C(J_1, E)$.

For more details on multi-valued maps, we refer to [17, 21, 25].

2.3 Fixed point theorems

In this portion, we recall some fixed point theorems.

Theorem 2.1 (Banach theorem [38]). *Let* Ω *be a non-empty closed convex subset of a Banach space* $(E, \|\cdot\|)$. *Suppose that* $\Phi : \Omega \to \Omega$ *is a contraction mapping. Then* Φ *admits a unique fixed point.*

Theorem 2.2 (Schauder theorem [38]). *Let* Ω *be a non-empty closed bounded convex subset of a Banach space* E. *Suppose that* $\Phi : \Omega \to \Omega$ *is completely continuous. Then* Φ *has a fixed point in* Ω.

Lemma 2.5 (Nonlinear alternative of Kakutani maps [22]). *Let* E *be a Banach space,* $C \subset E$ *be a closed convex subset, and let* $\mathcal{D} \subset C$ *be an open set such that* $0 \in \mathcal{D}$. *If* $\Psi : \overline{\mathcal{D}} \to \mathcal{P}_{cp,c}(C)$ *is an u.s.c compact map. Then either*

(a) Ψ *has a fixed point in* $\overline{\mathcal{D}}$, *or*

(b) *there exist* $u \in \partial\mathcal{D}$ *and* $\mu_0 \in (0, 1)$ *with* $u \in \mu_0 \Psi(u)$.

3 Existence and uniqueness results for single-valued problems

Initially, to transform problem (1.1) into a FP problem, we consider the operator $\Phi : \mathcal{C} \to \mathcal{C}$ as

$$(\Phi\varkappa)(\tau) = \frac{1}{\Gamma(r_1)} \int_1^{\tau} \Big(\log\frac{\tau}{\zeta}\Big)^{r_1-1} q(\zeta, \varkappa(\zeta))\,\frac{d\zeta}{\zeta} + \frac{m(\log\tau)^{m-1}}{\Delta} \Big(\frac{1}{\Gamma(r_1)} \int_1^{\tau_1} \Big(\log\frac{\tau_1}{\zeta}\Big)^{r_1-1} q(\zeta, \varkappa(\zeta))\,\frac{d\zeta}{\zeta}$$
$$- \lambda \int_1^{\tau_1} \Big(\frac{1}{\Gamma(r_1)} \int_1^{\zeta} \Big(\log\frac{\zeta}{\sigma}\Big)^{r_1-1} q(\sigma, \varkappa(\sigma))\,\frac{d\sigma}{\sigma}\Big)\frac{d\zeta}{\zeta} - d\Big),$$

with $\Delta \neq 0$. Clearly, the solution of (1.1) is an FP of the operator Φ.

Our first result deals with the existence of a unique solution of (1.1) relying on the Banach theorem.

Theorem 3.1. *Suppose that*

(As1) *there is a constant* $L_q > 0$ *such that*

$$|q(\tau, \varkappa) - q(\tau, \mathfrak{y})| \leq L_q |\varkappa - \mathfrak{y}|, \ \ \forall\, (\tau, \varkappa, \mathfrak{y}) \in J_1 \times \mathbb{R} \times \mathbb{R}.$$

(As2)

$$\beta = \frac{(\log\tau_1)^{r_1} L_q}{\Gamma(r_1+1)} + \frac{m(\log\tau_1)^{m-1}}{|\Delta|} \Big(\frac{(\log\tau_1)^{r_1} L_q}{\Gamma(r_1+1)} + \frac{|\lambda|(\log\tau_1)^{r_1+1} L_q}{\Gamma(r_1+2)}\Big) < 1.$$

Then (1.1) *has a unique solution on* J_1.

Proof. For $\varkappa, \mathfrak{y} \in \mathcal{C}$, using the hypothesis (As1), we get

$$\begin{aligned}\big|(\Phi\varkappa)(\tau) - (\Phi\mathfrak{y})(\tau)\big| &= \frac{1}{\Gamma(r_1)} \int_1^\tau \Big(\log\frac{\tau}{\zeta}\Big)^{r_1-1} \big|q(\zeta,\varkappa(\zeta)) - q(\zeta,\mathfrak{y}(\zeta))\big| \frac{d\zeta}{\zeta} \\ &\quad + \frac{m(\log\tau)^{m-1}}{|\Delta|}\Bigg(\frac{1}{\Gamma(r_1)}\int_1^{\tau_1}\Big(\log\frac{\tau_1}{\zeta}\Big)^{r_1-1}\big|q(\zeta,\varkappa(\zeta)) - q(\zeta,\mathfrak{y}(\zeta))\big|\frac{d\zeta}{\zeta} \\ &\quad + |\lambda|\int_1^{\tau_1}\Big(\frac{1}{\Gamma(r_1)}\int_1^{\zeta}\Big(\log\frac{\zeta}{\sigma}\Big)^{r_1-1}|q(\zeta,\varkappa(\zeta)) - q(\zeta,\mathfrak{y}(\zeta))|\frac{d\sigma}{\sigma}\Big)\frac{d\zeta}{\zeta}\Bigg) \\ &\leq \bigg(\frac{(\log\tau_1)^{r_1}L_q}{\Gamma(r_1+1)} + \frac{m(\log\tau_1)^{m-1}L_q}{|\Delta|}\Big(\frac{(\log\tau_1)^{r_1}}{\Gamma(r_1+1)} + \frac{|\lambda|(\log\tau_1)^{r_1+1}}{\Gamma(r_1+2)}\Big)\bigg)\|\varkappa - \mathfrak{y}\|.\end{aligned}$$

Thus

$$\|\Phi\varkappa - \Phi\mathfrak{y}\| \leq \beta\|\varkappa - \mathfrak{y}\|.$$

From (As2), Φ is a contraction mapping. As an outcome of the Banach theorem, Φ has a unique FP which corresponds to a unique solution of (1.1) on J_1. □

We next prove an existence result for (1.1) by means of the Schauder theorem.

Theorem 3.2. *Suppose that*

(As3) *there exists a function $\varsigma \in L^1(J_1, \mathbb{R}^+)$ such that*

$$|q(\tau,\varkappa)| \leq \varsigma(\tau), \quad \forall\,(\tau,\varkappa) \in J_1 \times \mathbb{R}.$$

Then (1.1) *has a solution on J_1.*

Proof. Consider the non-empty bounded closed convex subset

$$\Omega = \big\{\varkappa \in \mathcal{C} :\ \|\varkappa\| \leq M_0\big\},$$

where M_0 is chosen such that

$$M_0 \geq \frac{(\log\tau_1)^{r_1}\varsigma*}{\Gamma(r_1+1)} + \frac{m(\log\tau_1)^{m-1}}{|\Delta|}\Big(\frac{(\log\tau_1)^{r_1}\varsigma*}{\Gamma(r_1+1)} + \frac{|\lambda|(\log\tau_1)^{r_1+1}\varsigma*}{\Gamma(r_1+2)} + |d|\Big),$$

with $\varsigma* = \sup\{\varsigma(\tau) :\ \tau \in J_1\}$. It is known that the continuity of the function q implies that the operator Φ is continuous. It remains to demonstrate that the operator Φ is compact and will be presented as follows.

Step 1. We show that $\Phi(\Omega) \subset \Omega$.

For $\varkappa \in \Omega$, we have

$$\begin{aligned}|(\Phi\varkappa)(\tau)| &\leq \frac{1}{\Gamma(r_1)}\int_1^\tau\Big(\log\frac{\tau}{\zeta}\Big)^{r_1-1}\big|q(\zeta,\varkappa(\zeta))\big|\frac{d\zeta}{\zeta} \\ &\quad + \frac{m(\log\tau)^{m-1}}{|\Delta|}\Bigg(\frac{1}{\Gamma(r_1)}\int_1^{\tau_1}\Big(\log\frac{\tau_1}{\zeta}\Big)^{r_1-1}\big|q(\zeta,\varkappa(\zeta))\big|\frac{d\zeta}{\zeta} \\ &\quad + |\lambda|\int_1^{\tau_1}\Big(\frac{1}{\Gamma(r_1)}\int_1^{\zeta}\Big(\log\frac{\zeta}{\sigma}\Big)^{r_1-1}\big|q(\sigma,\varkappa(\sigma))\big|\frac{d\sigma}{\sigma}\Big)\frac{d\zeta}{\zeta} + |d|\Bigg) \\ &\leq \frac{(\log\tau_1)^{r_1}\varsigma*}{\Gamma(r_1+1)} + \frac{m(\log\tau_1)^{m-1}}{|\Delta|}\Big(\frac{(\log\tau_1)^{r_1}\varsigma*}{\Gamma(r_1+1)} + \frac{|\lambda|(\log\tau_1)^{r_1+1}\varsigma*}{\Gamma(r_1+2)} + |d|\Big),\end{aligned}$$

and, consequently, $\|\Phi\varkappa\| \le M_0$. Hence $\Phi(\Omega) \subset \Omega$, and the set $\Phi(\Omega)$ is uniformly bounded.

Step 2. Φ sends bounded sets of $\mathcal{C}$ into equicontinuous sets.

For $t_1, t_2 \in J_1$, $t_1 < t_2$ and for $\varkappa \in \Omega$, we have

$$\begin{aligned}\big|(\Phi\varkappa)(t_2) - (\Phi\varkappa)(t_1)\big| \le{}& \frac{\varsigma*}{\Gamma(r_1+1)}\big((\log t_2)^{r_1} - (\log t_1)^{r_1}\big)\\ &+ \frac{m((\log t_2)^{m-1} - (\log t_1)^{m-1})}{|\Delta|}\left(\frac{(\log \tau_1)^{r_1}\varsigma*}{\Gamma(r_1+1)} + \frac{|\lambda|(\log\tau_1)^{r_1+1}\varsigma*}{\Gamma(r_1+2)} + |d|\right).\end{aligned}$$

As $t_1 \to t_2$, we obtain

$$\big|(\Phi\varkappa)(t_2) - (\Phi\varkappa)(t_1)\big| \to 0.$$

Hence $\Phi(B_{r_0})$ is equicontinuous. By means of the Arzelaá–Ascoli theorem, we infer that Φ is compact. So, by the Schauder theorem, we demonstrate that Φ has a FP $\varkappa \in \Omega$ which is a solution of (1.1) on J_1. □

4 Existence results for multi-valued problems

Definition 4.1. $\varkappa \in AC^m(J_1, \mathbb{R})$ is considered as a solution of (1.2) if there is an integrable function $\upsilon \in L^1(J_1, \mathbb{R})$ with $\upsilon(\tau) \in Q(\tau, \varkappa(\tau))$ for all $\tau \in J_1$ satisfying the boundary conditions

$$\varkappa(1) = \varkappa'(1) = \varkappa''(1) = \cdots = \varkappa^{(m-2)}(1) = 0, \quad \varkappa(\tau_1) = \lambda \int_1^{\tau_1} \varkappa(\zeta)\,\frac{d\zeta}{\zeta} + d,$$

and

$$\begin{aligned}\varkappa(\tau) = \frac{1}{\Gamma(r_1)}\int_1^{\tau}\Big(\log\frac{\tau}{\zeta}\Big)^{r_1-1}\upsilon(\zeta)\,\frac{d\zeta}{\zeta} + \frac{m(\log\tau)^{m-1}}{\Delta}\Bigg(&\frac{1}{\Gamma(r_1)}\int_1^{\tau_1}\Big(\log\frac{\tau_1}{\zeta}\Big)^{r_1-1}\upsilon(\zeta)\,\frac{d\zeta}{\zeta}\\ &- \lambda\int_1^{\tau_1}\Big(\frac{1}{\Gamma(r_1)}\int_1^{\zeta}\Big(\log\frac{\zeta}{\sigma}\Big)^{r_1-1}\upsilon(\sigma)\,\frac{d\sigma}{\sigma}\Big)\frac{d\zeta}{\zeta} - d\Bigg).\end{aligned}$$

In what follows, to demonstrate our main results, we will apply the Leray–Schauder nonlinear alternative theorem.

Theorem 4.1. *Set*

$$\Lambda = \frac{(\log\tau_1)^{r_1}}{\Gamma(r_1+1)} + \frac{m(\log\tau_1)^{m-1}}{|\Delta|}\Big(\frac{(\log\tau_1)^{r_1}}{\Gamma(r_1+1)} + \frac{|\lambda|(\log\tau_1)^{r_1+1}}{\Gamma(r_1+2)}\Big),$$

and assume that

(Hy1) $Q : J_1 \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is an* L^1*-Carathéodory multi-valued map,*

(Hy2) *there is a positive nondecreasing continuous function* G *on* $\mathbb{R}^+$ *and* $\psi \in C(J_1, \mathbb{R}^+)$ *such that*

$$\|Q(\tau,\varkappa)\|_{\mathcal{P}} = \sup\big\{|\mathfrak{y}| :\ \mathfrak{y} \in Q(\tau,\varkappa)\big\} \le \psi(\tau)G(|\varkappa|),$$

for each $(\tau, \varkappa) \in J_1 \times \mathbb{R}$,

(Hy3) *there is a positive constant* M_0 *such that*

$$\Lambda\|\psi\|G(M_0) + \frac{m(\log\tau_1)^{m-1}|d|}{|\Delta|} < M_0.$$

Then (1.2) *has a solution on* J_1.

Proof. Initially, to transform (1.2) into FP problem, we define the operator $\Psi : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ as

$$\Psi(\varkappa) = \left\{ \begin{array}{l} \phi \in \mathcal{C}, \\ \phi(\tau) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(r_1)} \displaystyle\int_1^{\tau} \left(\log \frac{\tau}{\zeta} \right)^{r_1 - 1} \upsilon(\zeta) \, \frac{d\zeta}{\zeta} \\ \quad + \dfrac{m(\log \tau)^{m-1}}{\Delta} \left(\dfrac{1}{\Gamma(r_1)} \displaystyle\int_1^{\tau_1} \left(\log \frac{\tau_1}{\zeta} \right)^{r_1 - 1} \upsilon(\zeta) \, \frac{d\zeta}{\zeta} \right. \\ \qquad \left. - \lambda \displaystyle\int_1^{\tau_1} \left(\frac{1}{\Gamma(r_1)} \int_1^{\zeta} \left(\log \frac{\zeta}{\sigma} \right)^{r_1 - 1} \upsilon(\sigma) \, \frac{d\sigma}{\sigma} \right) \frac{d\zeta}{\zeta} - d \right), \end{array} \right. \end{array} \right\}$$

for $\upsilon \in S_{Q,\varkappa}$. Clearly, the solution of (1.2) is a FP of the operator Φ. The proof steps will be presented as follows.

Step 1. The set-valued map $\Psi(\varkappa)$ is convex for any $\varkappa \in \mathcal{C}$.

Let $\phi_1, \phi_2 \in \Psi(\varkappa)$. Then there exist $\upsilon_1, \upsilon_2 \in S_{Q,\varkappa}$ such that

$$\begin{aligned} \phi_i(\tau) = {} & \frac{1}{\Gamma(r_1)} \int_1^{\tau} \left(\log \frac{\tau}{\zeta} \right)^{r_1 - 1} \upsilon_i(\zeta) \, \frac{d\zeta}{\zeta} \\ & + \frac{m(\log \tau)^{m-1}}{\Delta} \left(\frac{1}{\Gamma(r_1)} \int_1^{\tau_1} \left(\log \frac{\tau_1}{\zeta} \right)^{r_1 - 1} \upsilon_i(\zeta) \, \frac{d\zeta}{\zeta} \right. \\ & \left. - \lambda \int_1^{\tau_1} \left(\frac{1}{\Gamma(r_1)} \int_1^{\zeta} \left(\log \frac{\zeta}{\sigma} \right)^{r_1 - 1} \upsilon_i(\sigma) \, \frac{d\sigma}{\sigma} \right) \frac{d\zeta}{\zeta} - d \right), \quad i = 1, 2, \ \ \forall \tau \in J_1. \end{aligned}$$

Let $\theta \in [0, 1]$. Then for any $\tau \in J_1$,

$$\begin{aligned} \big[\theta\phi_1 + (1 - \theta)\phi_2\big](\tau) = {} & \frac{1}{\Gamma(r_1)} \int_1^{\tau} \left(\log \frac{\tau}{\zeta} \right)^{r_1 - 1} \big[\theta\upsilon_1(\zeta) + (1 - \theta)\upsilon_2(\zeta)\big] \, \frac{d\zeta}{\zeta} \\ & + \frac{m(\log \tau)^{m-1}}{\Delta} \left(\frac{1}{\Gamma(r_1)} \int_1^{\tau_1} \left(\log \frac{\tau_1}{\zeta} \right)^{r_1 - 1} \big[\theta\upsilon_1(\zeta) + (1 - \theta)\upsilon_2(\zeta)\big] \, \frac{d\zeta}{\zeta} \right. \\ & \left. - \lambda \int_1^{\tau_1} \left(\frac{1}{\Gamma(r_1)} \int_1^{\zeta} \left(\log \frac{\zeta}{\sigma} \right)^{r_1 - 1} \big[\theta\upsilon_1(\sigma) + (1 - \theta)\upsilon_2(\sigma)\big] \, \frac{d\sigma}{\sigma} \right) \frac{d\zeta}{\zeta} - d \right). \end{aligned}$$

Since Q has convex values, $S_{Q,\varkappa}$ is convex and $\theta\upsilon_1(\zeta) + (1-\theta)\upsilon_2(\zeta) \in S_{Q,\varkappa}$. So, $\theta\phi_1 + (1-\theta)\phi_2 \in \Psi(\varkappa)$.

Step 2. Ψ is bounded on the bounded sets of $\mathcal{C}$.

For a constant $r_0 > 0$, let $B_{r_0} = \{\varkappa \in \mathcal{C} : \ \|\varkappa\| \leq r_0\}$ be a bounded set in $\mathcal{C}$. Then for each $\phi \in \Psi(\varkappa)$, $\varkappa \in B_{r_0}$, there exists $\upsilon \in S_{Q,\varkappa}$ such that

$$\begin{aligned} \phi(\tau) = {} & \frac{1}{\Gamma(r_1)} \int_1^{\tau} \left(\log \frac{\tau}{\zeta} \right)^{r_1 - 1} \upsilon(\zeta) \, \frac{d\zeta}{\zeta} \\ & + \frac{m(\log \tau)^{m-1}}{\Delta} \left(\frac{1}{\Gamma(r_1)} \int_1^{\tau_1} \left(\log \frac{\tau_1}{\zeta} \right)^{r_1 - 1} \upsilon(\zeta) \, \frac{d\zeta}{\zeta} \right. \\ & \left. - \lambda \int_1^{\tau_1} \left(\frac{1}{\Gamma(r_1)} \int_1^{\zeta} \left(\log \frac{\zeta}{\sigma} \right)^{r_1 - 1} \upsilon(\sigma) \, \frac{d\sigma}{\sigma} \right) \frac{d\zeta}{\zeta} - d \right). \end{aligned}$$

Under the assumption (Hy2) and for any $\tau \in J_1$, we attain

$$\begin{aligned}
|\phi(\tau)| \leq {} & \frac{1}{\Gamma(r_1)} \int_1^{\tau} \Big(\log\frac{\tau}{\zeta}\Big)^{r_1-1} |\upsilon(\zeta)| \,\frac{d\zeta}{\zeta} \\
& + \frac{m(\log\tau)^{m-1}}{|\Delta|} \Bigg(\frac{1}{\Gamma(r_1)} \int_1^{\tau_1} \Big(\log\frac{\tau_1}{\zeta}\Big)^{r_1-1} |\upsilon(\zeta)| \,\frac{d\zeta}{\zeta} \\
& + |\lambda| \int_1^{\tau_1} \Big(\frac{1}{\Gamma(r_1)} \int_1^{\zeta} \Big(\log\frac{\zeta}{\sigma}\Big)^{r_1-1} |\upsilon(\sigma)| \,\frac{d\sigma}{\sigma} \Big) \frac{d\zeta}{\zeta} + |d| \Bigg) \\
\leq {} & \|\psi\| G(r_0) \Bigg(\frac{(\log\tau_1)^{r_1}}{\Gamma(r_1+1)} + \frac{m(\log\tau_1)^{m-1}}{|\Delta|} \Big(\frac{(\log\tau_1)^{r_1}}{\Gamma(r_1+1)} + \frac{|\lambda|(\log\tau_1)^{r_1+1}}{\Gamma(r_1+2)} \Big) \Bigg) + \frac{m(\log\tau_1)^{m-1}|d|}{|\Delta|},
\end{aligned}$$

thus

$$\|\phi\| \leq \Lambda \|\psi\| G(r_0) + \frac{m(log\tau_1)^{m-1}|d|}{|\Delta|}.$$

Step 3. Ψ sends bounded sets of $\mathcal{C}$ into equicontinuous sets.

Let $\varkappa \in B_{r_0}$ and $\phi \in \Psi(\varkappa)$. Then there is a function $\upsilon \in S_{Q,\varkappa}$ such that

$$\begin{aligned}
\phi(\tau) = {} & \frac{1}{\Gamma(r_1)} \int_1^{\tau} \Big(\log\frac{\tau}{\zeta}\Big)^{r_1-1} \upsilon(\zeta) \,\frac{d\zeta}{\zeta} \\
& + \frac{m(\log\tau)^{m-1}}{\Delta} \Bigg(\frac{1}{\Gamma(r_1)} \int_1^{\tau_1} \Big(\log\frac{\tau_1}{\zeta}\Big)^{r_1-1} \upsilon(\zeta) \,\frac{d\zeta}{\zeta} \\
& - \lambda \int_1^{\tau_1} \Big(\frac{1}{\Gamma(r_1)} \int_1^{\zeta} \Big(\log\frac{\zeta}{\sigma}\Big)^{r_1-1} \upsilon(\sigma) \,\frac{d\sigma}{\sigma} \Big) \frac{d\zeta}{\zeta} - d \Bigg).
\end{aligned}$$

Let $t_1, t_2 \in J_1$, $t_1 < t_2$. Then

$$\begin{aligned}
\big|(\Phi\varkappa)(t_2) - (\Phi\varkappa)(t_1)\big| \leq {} & \frac{\|\psi\| G(r_0)}{\Gamma(r_1+1)} \big((\log t_2)^{r_1} - (\log t_1)^{r_1} \big) \\
& + \frac{m((\log t_2)^{m-1} - (\log t_1)^{m-1})}{|\Delta|} \Big(\frac{(\log\tau_1)^{r_1} \|\psi\| G(r_0)}{\Gamma(r_1+1)} + \frac{|\lambda|(\log\tau_1)^{r_1+1} \|\psi\| G(r_0)}{\Gamma(r_1+2)} + |d| \Big).
\end{aligned}$$

As $t_1 \to t_2$, we obtain $|\phi(t_2) - \phi(t_1)| \to 0$. Hence $\Psi(B_{r_0})$ is equicontinuous. From the above-mentioned Steps 2 and 3 along with the Arzelá–Ascoli theorem, we deduce that Ψ is completely continuous.

Step 4. We prove that the graph of Ψ is closed.

Let $\varkappa_m \to \varkappa_*$, $\phi_m \in \Psi(\varkappa_m)$ and let ϕ_m tend to ϕ_*. We show that $\phi_* \in \Psi(\varkappa_*)$. Since $\phi_m \in \Psi(\varkappa_m)$, there exists $\upsilon_m \in S_{Q,\varkappa_m}$ such that

$$\begin{aligned}
\phi_m(\tau) = {} & \frac{1}{\Gamma(r_1)} \int_1^{\tau} \Big(\log\frac{\tau}{\zeta}\Big)^{r_1-1} \upsilon_m(\zeta) \,\frac{d\zeta}{\zeta} \\
& + \frac{m(\log\tau)^{m-1}}{\Delta} \Bigg(\frac{1}{\Gamma(r_1)} \int_1^{\tau_1} \Big(\log\frac{\tau_1}{\zeta}\Big)^{r_1-1} \upsilon_m(\zeta) \,\frac{d\zeta}{\zeta} \\
& - \lambda \int_1^{\tau_1} \Big(\frac{1}{\Gamma(r_1)} \int_1^{\zeta} \Big(\log\frac{\zeta}{\sigma}\Big)^{r_1-1} \upsilon_m(\sigma) \,\frac{d\sigma}{\sigma} \Big) \frac{d\zeta}{\zeta} - d \Bigg), \quad \forall \tau \in J_1.
\end{aligned}$$

Therefore, we have to prove that there is $\upsilon_* \in S_{Q,\varkappa_*}$ such that

$$\begin{aligned}\phi_*(\tau) = & \frac{1}{\Gamma(r_1)} \int_1^\tau \left(\log \frac{\tau}{\zeta}\right)^{r_1-1} \upsilon_*(\zeta) \frac{d\zeta}{\zeta} \\ & + \frac{m(\log \tau)^{m-1}}{\Delta} \left(\frac{1}{\Gamma(r_1)} \int_1^{\tau_1} \left(\log \frac{\tau_1}{\zeta}\right) \upsilon_*(\zeta) \frac{d\zeta}{\zeta} \right. \\ & \left. - \lambda \int_1^{\tau_1} \left(\frac{1}{\Gamma(r_1)} \int_1^{\zeta} \left(\log \frac{\zeta}{\sigma}\right)^{r_1-1} \upsilon_*(\sigma) \frac{d\sigma}{\sigma} \right) \frac{d\zeta}{\zeta} - d \right), \quad \forall \tau \in J_1.\end{aligned}$$

Define the continuous linear operator $\Theta : L^1(J_1, E) \to C(J_1, E)$ as follows:

$$\begin{aligned}\upsilon \to \Theta(\upsilon)(\tau) = & \frac{1}{\Gamma(r_1)} \int_1^\tau \left(\log \frac{\tau}{\zeta}\right)^{r_1-1} \upsilon(\zeta) \frac{d\zeta}{\zeta} \\ & + \frac{m(\log \tau)^{m-1}}{\Delta} \left(\frac{1}{\Gamma(r_1)} \int_1^{\tau_1} \left(\log \frac{\tau_1}{\zeta}\right)^{r_1-1} \upsilon(\zeta) \frac{d\zeta}{\zeta} \right. \\ & \left. - \lambda \int_1^{\tau_1} \left(\frac{1}{\Gamma(r_1)} \int_1^{\zeta} \left(\log \frac{\zeta}{\sigma}\right)^{r_1-1} \upsilon(\sigma) \frac{d\sigma}{\sigma} \right) \frac{d\zeta}{\zeta} - d \right).\end{aligned}$$

Notice that

$$\begin{aligned}\|\phi_m - \phi_*\| = & \left\| \frac{1}{\Gamma(r_1)} \int_1^\tau \left(\log \frac{\tau}{\zeta}\right)^{r_1-1} (\upsilon_m(\zeta) - \upsilon_*(\zeta)) \frac{d\zeta}{\zeta} \right. \\ & + \frac{m(\log \tau)^{m-1}}{\Delta} \left(\frac{1}{\Gamma(r_1)} \int_1^{\tau_1} \left(\log \frac{\tau_1}{\zeta}\right) (\upsilon_m(\zeta) - \upsilon_*(\zeta)) \frac{d\zeta}{\zeta} \right. \\ & \left. \left. - \lambda \int_1^{\tau_1} \left(\frac{1}{\Gamma(r_1)} \int_1^{\zeta} \left(\log \frac{\zeta}{\sigma}\right)^{r_1-1} (\upsilon_m(\sigma) - \upsilon_*(\sigma)) \frac{d\sigma}{\sigma} G \right) \frac{d\zeta}{\zeta} \right) \right\| \to 0,\end{aligned}$$

when $m \to \infty$. So, in view of Lemma 2.4, $\Theta \circ S_{Q,\varkappa}$ is a closed graph operator. Moreover, we have

$$\phi_m \in \Theta(S_{Q,\varkappa_m}).$$

Since $\varkappa_m \to \varkappa_*$, Lemma 2.4 gives

$$\begin{aligned}\phi_*(\tau) = & \frac{1}{\Gamma(r_1)} \int_1^\tau \left(\log \frac{\tau}{\zeta}\right)^{r_1-1} \upsilon_*(\zeta) \frac{d\zeta}{\zeta} \\ & + \frac{m(\log \tau)^{m-1}}{\Delta} \left(\frac{1}{\Gamma(r_1)} \int_1^{\tau_1} \left(\log \frac{\tau_1}{\zeta}\right) \upsilon_*(\zeta) \frac{d\zeta}{\zeta} \right. \\ & \left. - \lambda \int_1^{\tau_1} \left(\frac{1}{\Gamma(r_1)} \int_1^{\zeta} \left(\log \frac{\zeta}{\sigma}\right)^{r_1-1} \upsilon_*(\sigma) \frac{d\sigma}{\sigma} \right) \frac{d\zeta}{\zeta} - d \right),\end{aligned}$$

for some $\upsilon_* \in S_{Q,\varkappa_*}$.

Step 5. We show that there exists an open set $\mathcal{D} \subseteq \mathcal{C}$ with $\varkappa \notin \mu_0 \Psi(\varkappa)$ for each $0 < \mu_0 < 1$ and $\forall \varkappa \in \partial \mathcal{D}$.

Let $\mu_0 \in (0,1)$ and $\varkappa \in \mu_0 \Psi(\varkappa)$. Then there exists $\upsilon \in S_{Q,\varkappa}$ such that

$$\begin{aligned}\varkappa(\tau) = \frac{\mu}{\Gamma(r_1)} \int_1^\tau \Big(\log\frac{\tau}{\zeta}\Big)^{r_1-1} \upsilon(\zeta)\,\frac{d\zeta}{\zeta} \\ + \frac{\mu m(\log\tau)^{m-1}}{\Delta}\bigg(\frac{1}{\Gamma(r_1)}\int_1^{\tau_1}\Big(\log\frac{\tau_1}{\zeta}\Big)^{r_1-1}\upsilon(\zeta)\frac{d\zeta}{\zeta} \\ - \lambda\int_1^{\tau_1}\bigg(\frac{1}{\Gamma(r_1)}\int_1^{\zeta}\Big(\log\frac{\zeta}{\sigma}\Big)^{r_1-1}\upsilon(\sigma)\,\frac{d\sigma}{\sigma}\bigg)\,\frac{d\zeta}{\zeta} - d\bigg).\end{aligned}$$

Thus we have

$$|\varkappa(\tau)| \le \Lambda\|\psi\|G(\|\varkappa\|) + \frac{m(\log\tau_1)^{m-1}|d|}{|\Delta|},$$

consequently, we obtain

$$\|\varkappa\| \le \Lambda\|\psi\|G(\|\varkappa\|) + \frac{m(\log\tau_1)^{m-1}|d|}{|\Delta|}.$$

Under the hypothesis (Hy3), there is a positive constant M_0 such that $\|\varkappa\| \ne M_0$. We build the set $\mathcal{D}$ as follows:

$$\mathcal{D} = \{\varkappa \in \mathcal{C} : \|\varkappa\| < M_0\}.$$

From Steps 1–4, the operator $\Psi : \overline{\mathcal{D}} \to \mathcal{P}(\mathcal{C})$ is u.s.c. and completely continuous. From the choice of $\mathcal{D}$, there is no $\varkappa \in \partial\mathcal{D}$ such that $\varkappa \in \mu_0\Psi(\varkappa)$ for some $\mu_0 \in (0,1)$. So, by the Leray–Schauder theorem for set-valued maps, we infer that (1.2) has a solution $\varkappa \in \overline{\mathcal{D}}$. □

5 Example

In this section, to validate the existence results, we consider the FDE or FDI of the form.

$$\begin{cases} {}^{C}_{H}\mathfrak{D}_1^{\frac{5}{2}}\varkappa(\tau) = q(\tau,\varkappa(\tau) \ \text{ or } \ \in Q(\tau,\varkappa(\tau)),\ \tau\in[1,e], \\ \varkappa(1) = \varkappa'(1) = 0, \quad \varkappa(e) = \dfrac{1}{2}\displaystyle\int_1^e \varkappa(\zeta)\,\frac{d\zeta}{\zeta} + 3. \end{cases} \tag{5.1}$$

Here, $r_1 = \frac{5}{2}$, $\lambda = \frac{1}{2}$, $d = 3$, $\tau_1 = e$, $m = 3$. With these data we find that $\Delta = 2.5 \ne 0$.

5.1 Single-valued case

Let

$$q(\tau,\varkappa) = \frac{\sin(\tau)}{\exp(\tau-1)+1}\Big(\frac{|\varkappa|}{|\varkappa|+1}\Big).$$

For $\varkappa, \mathfrak{y} \in \mathbb{R}$, we have

$$\big|q(\tau,\varkappa) - q(\tau,\mathfrak{y})\big| \le \frac{1}{2}\,|\varkappa - \mathfrak{y}|.$$

Thus the assumption (As1) is satisfied with $L_q = \frac{1}{2}$ and

$$\beta = \frac{(\log\tau_1)^{r_1}L_q}{\Gamma(r_1+1)} + \frac{m(\log\tau_1)^{m-1}}{|\Delta|}\Big(\frac{(\log\tau_1)^{r_1}L_q}{\Gamma(r_1+1)} + \frac{|\lambda|(\log\tau_1)^{r_1+1}L_q}{\Gamma(r_1+2)}\Big) \simeq 0.36 < 1.$$

Therefore all the hypotheses of Theorem 3.1 are valid. Then there exists a unique solution of (5.1) on $[1,e]$. With the same function q, we see that

$$|q(\tau,\varkappa)| \le \frac{1}{\exp(\tau-1)+1} = \varsigma(\tau).$$

Then, in the light of Theorem 3.2, we infer that problem (5.1) has at least one solution on $[1,e]$.

5.2 Multi-valued case

Consider the set-valued map $Q : [1, e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ defined by

$$\varkappa \to Q(\tau, \varkappa) = \left[\frac{1}{(\exp(t^2) + 4)} \frac{x^2}{(x^2 + 1)}, \frac{1}{2\sqrt{\log t + 1)}} \frac{|x|}{|x| + 1} \right]. \tag{5.2}$$

Obviously, the set-valued map Q satisfies hypothesis (Hy1) and

$$\|Q(\tau, \varkappa)\|_{\mathcal{P}} = \sup \left\{ |\mathfrak{y}| : \mathfrak{y} \in Q(\tau, \varkappa) \right\} \leq \frac{1}{2\sqrt{\log t + 1}} = \psi(\tau) G(|\varkappa|),$$

where $\|\psi\| = \frac{1}{2}$ and $G(\|\varkappa\|) = 1$. Thus, the assumption (Hy2) is fulfilled, and by (Hy3), we get $M_0 > 4.85$. Therefore, all the hypotheses of Theorem 4.1 are valid. Then there exists at least one solution of (5.1) on $[1, e]$ with Q given by (5.2).

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