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CONVERGENCE OF ALIKHANOV'S METHOD FOR FRACTIONAL DIFFUSION EQUATION WITH DRIFT AND FUNCTIONAL DELAY

Dedicated to the blessed memory of Professor N. V. Azbelev

Abstract. In this paper, a second-order method in time and space steps is constructed for a fractional diffusion equation in the presence of drift and functional delay. The basis of the algorithm is Alikhanov's method. To take into account the effect of functional delay, the interpolation and extrapolation constructions are used. The local error of the method is investigated. Using the discrete Gronwall inequality and some additional estimates, the convergence of the method is proved and the orders of convergence with respect to the partitioning steps in time and space are obtained. The results of numerical experiments on test examples are presented.

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Key words and phrases. Fractional diffusion equations, drift, functional delay, numerical methods, order of convergence.

რეზიუმე. ნაშრომში აგებულია მეორე რიგის მეთოდი დროისა და სივრცის ბიჯით ფრაქციული დიფუზიის განტოლებისთვის დრეიფისა და ფუნქციური დაგვიანების არსებობისას. ალგორითმის საფუძველია ალიხანოვის მეთოდი. ფუნქციური დაგვიანების ეფექტის გასათვალისწინებლად გამოიყენება ინტერპოლაციისა და ექსტრაპოლაციის კონსტრუქციები. გამოკვლეულია მეთოდის ლოკალური ცდომილება. გრონვოლის დისკრეტული უტოლობისა და ზოგიერთი დამატებითი შეფასების გამოყენებით დამტკიცებულია მეთოდის კრებადობა და დადგენილია კრებადობის რიგი დროში და სივრცეში დაყოფის ბიჯის მიხედვით. სატესტო მაგალითებზე წარმოდგენილია რიცხვითი ექსპერიმენტების შედეგები.

1 Introduction

Fractional differential equations, including those with delay, are used in the mathematical modeling of various objects (see, e.g., [2]). The presence of time-fractional derivatives in diffusion equations in combination with the nonlinear delay effect requires a complex technique of fractional discrete Gronwall inequalities to prove the convergence [3, 5, 6, 12]. The additional presence of a term with the first derivative (drift) somewhat simplifies the technique. So, in [7], a family of $h^2 + \Delta$-order methods based on the [4, p. 49] $L1$-algorithm is constructed for an equation with a drift and functional delay, for which their stability and convergence are studied. In particular, the unconditional stability of a purely implicit method is proved.

In [13], for a fractional (in time and space) equation with a drift and constant lumped delay, an analogue of Alikhanov's method [1] of order of Δ^2 in time step is developed and studied. In this paper, using interpolation procedures with the given properties, the results of [13] are generalized to the case of functional delay. To prove the convergence, the technique of discrete, but not fractional, Gronwall inequality is used [8, 11, 13]. To take into account the functional delay, the method of interpolation and extrapolation with specified properties [9] is used.

2 Problem definition

We consider an equation of the form

$$\frac{\partial u(x,t)}{\partial t} + \lambda \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + f\big(x, t, u(x,t), u_t(x,\cdot)\big), \tag{2.1}$$

where $0 \leqslant t \leqslant T$, $0 \leqslant x \leqslant X$ are independent variables, $u(x,t)$ is the desired function of solution, $u_t(x,\cdot) = \{u(x,t+s), \tau \leqslant s < 0\}$ is a history of the desired function at time t, τ is the value of delay, $\lambda \geqslant 0$. The Caputo fractional derivative of order α, $0 < \alpha < 1$, is determined by the formula

$$\frac{d^\alpha F(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{F'(\xi)}{(t-\xi)^\alpha}\, d\xi, \quad t > 0.$$

We set the boundary conditions

$$u(0,t) = u_0(t), \quad u(X,t) = u_1(t), \quad 0 \leqslant t \leqslant T, \tag{2.2}$$

and the initial conditions

$$u(x,t) = \varphi(x,t), \quad 0 \leqslant x \leqslant X, \quad -\tau \leqslant t \leqslant 0. \tag{2.3}$$

2.1 Assumptions

We assume that the solution $u(x,t)$ of problem (2.1)–(2.3) exists and is unique. In addition, in proving the convergence of numerical algorithms, we assume the necessary smoothness of the solution $u(x,t)$.

Denote by $Q = Q[-\tau, 0)$ the set of functions $v(s)$, piecewise continuous on $[-\tau, 0)$ with a finite number of break points of the first kind, at the break points continuous on the right. We define the norm of functions on Q by the relation

$$\|v(\cdot)\|_Q = \sup_{s\in[-\tau,0)} |v(s)|.$$

In addition, we assume that the functional $f(x,t,u,v(\cdot))$ is defined on $[0,X] \times [0,T] \times R \times Q$ and is Lipschitz by the last two arguments, i.e., there is a constant L_f such that for all $x \in [0,X]$, $t \in [0,T]$, $u^1 \in R$, $u^2 \in R$, $v^1(\cdot) \in Q$, $v^2(\cdot) \in Q$,

$$\big|f(x,t,u^1,v^1(\cdot)) - f(x,t,u^2,v^2(\cdot))\big| \leqslant L_f\big(|u^1 - u^2| + \|v^1(\cdot) - v^2(\cdot)\|_Q\big). \tag{2.4}$$

3 Discretization

We introduce the time step $\Delta = \frac{\tau}{M_0}$, where M_0 is a natural number and let $M = [\frac{T}{\Delta}]$. Enter the points $t_j = j\Delta$, $j = -M_0, \dots, M$. We divide the segment $[0, X]$ into parts with the step $h = X/N$, by entering the points $x_i = ih$, $i = 0, \dots, N$. The approximation of the function $u(x_i, t_j)$ at the nodes of the grid will be denoted by u_i^j.

3.1 Interpolation

For every fixed $i = 0, \dots, N$, we introduce a discrete prehistory to the moment t_m, $m = 0, \dots, M$: $\{u_i^j\}_m = \{u_i^j, m - M_0 \leqslant j \leqslant m\}$. The interpolation operator (with extrapolation by step) of a discrete prehistory is a mapping I: that associates a discrete prehistory $\{u_i^j\}_m$ with a function $u^m(t)_i$ defined on $[t_m - \tau, t_m + \Delta]$.

We say that the interpolation operator has the order of error p on the exact solution $u(x, t)$ if there are constants C_1 and C_2 such that for all i, m and $t \in [t_m - \tau, t_m + \Delta]$, the inequality

$$u^m(t)_i - u(x_i, t)| \leqslant C_1 \max_{m - M_0 \leqslant j \leqslant m} \|u^m(t_j)_i - u(x_i, t_j)\| + C_2 \Delta^p$$

is fulfilled.

In what follows, for the method under consideration, we will use a piecewise linear interpolation

$$u^m(t)_i = \frac{1}{\Delta}\left((t_j - t)u_i^{j-1} + (t - t_{j-1})u_i^j\right), \quad t_{j-1} \leqslant t \leqslant t_j, \quad j \leqslant m, \tag{3.1}$$

with extrapolation by step

$$u^m(t)_i = \frac{1}{\Delta}\left((t_m - t)u_i^{m-1} + (t - t_{m-1})u_i^m\right), \quad t_m \leqslant t \leqslant t_m + \Delta. \tag{3.2}$$

This interpolation operator is of second order if the exact solution $u(x, t)$ is twice continuously differentiable with respect to t [9].

Note also that the operator of a piecewise linear interpolation with extension extrapolation is Lipschitz with the Lipschitz constant $L_I = 2$ in the following sense: if $u^m(t)_i$ and $v^m(t)_i$ are the results of a piecewise linear interpolation with extrapolation by continuation of two histories, respectively, $\{u_i^j\}_m$ and $\{u_i^j\}_m$, then for every $t \in [t_m - \tau, t_{m+1}$,

$$|u^m(t)_i - v^m(t)_i| \leqslant L_I \max_{m - M_0 \leqslant j \leqslant m} \|u_i^j - v_i^j\| \tag{3.3}$$

is executed.

3.2 Approximation of the fractional derivative

Given a set of numbers $\{y^j\}_{j=0}^{m+1}$, we introduce a difference operator that approximates the Caputo derivative (Alikhanov's method [1]) at the point $t_{m+\sigma} = t_m + \sigma\Delta$, $\sigma = 1 - \frac{\alpha}{2}$,

$$D_\Delta^\alpha y^j|_{m+\sigma} = \frac{\Delta^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{m} c_{m-j}(y^{j+1} - y^j), \tag{3.4}$$

where

if $m = 0$, then $c_0 = a_0$;

if $m \geqslant 1$, then

$$c_j = \begin{cases} a_0 + b_1, & j = 0, \\ a_j + b_{j+1} + b_j, & 1 \leqslant j \leqslant m - 1, \\ a_m - b_m, & j = m, \end{cases}$$

in which
$$a_0 = \sigma^{1-\alpha}, \quad a_l = (l+\sigma)^{1-\alpha} - (l-1+\sigma)^{1-\alpha} \ \ (l \geqslant 1)$$
and
$$b_l = \frac{1}{2-\alpha}\left[(l+\sigma)^{2-\alpha} - (l-1+\sigma)^{2-\alpha}\right] - \frac{1}{2}\left[(l+\sigma)^{1-\alpha} - (l-1+\sigma)^{1-\alpha}\right].$$

This operator can be written as
$$D_\Delta^\alpha y^j\big|_{m+\sigma} = \frac{\Delta^{-\alpha}}{\Gamma(2-\alpha)}\Big(c_0 y^{m+1} - \sum_{j=1}^{m}(c_{m-j} - c_{m-j+1})y^j - c_m y^0\Big). \tag{3.5}$$

Note the properties of the coefficients [1]
$$c_0 \geqslant c_1 \geqslant \cdots \geqslant c_m \geqslant 0.$$

If the exact solution $u(x,t)$ to problem (2.1)–(2.3) is thrice continuously differentiable with respect to t, then [1]
$$\frac{\partial^\alpha u(x_i, t_{m+\sigma})}{\partial t^\alpha} = D_\Delta^\alpha u(x_i, t_j)\big|_{m+\sigma} + Q_i^m, \ \ |Q_i^m| \leqslant C_4 \Delta^{3-\alpha}. \tag{3.6}$$

3.3 Approximation of integer derivatives

We introduce a difference operator that approximates the second derivative with respect to x,
$$\delta_x^2 u_i^m = \frac{u_{i-1}^m - 2u_i^m + u_{i+1}^m}{h^2}. \tag{3.7}$$

If the exact solution $u(x,t)$ to problem (2.1)–(2.3) is four times continuously differentiable with respect to x, then
$$\frac{\partial^2 u}{\partial x^2}\, u(x_i, t_m) = \delta_x^2 u(x_i, t_m) + P_i^m, \ \ |P_i^m| \leqslant C_5 h^2. \tag{3.8}$$

To approximate the term with a drift, for any $m = 0, 1, \ldots, M-1$, we use the formula for numerical differentiation with respect to three nodes t_{m-1}, t_m, t_{m+1}, at the point $t_{m+\sigma}$ and get the operator
$$\delta_t u_i^m\big|_{m+\sigma} = \frac{(2\sigma+1)u_i^{m+1} - 4\sigma u_i^m + (2\sigma-1)u_i^{m-1}}{2\Delta}. \tag{3.9}$$

The values u_i^{-1} and u_i^0 will be taken from the functional initial conditions (2.3).

If the exact solution $u(x,t)$ of problem (2.1)–(2.3) is thrice continuously differentiable with respect to t, then [13]
$$\frac{\partial u(x_i, t_{m+\sigma})}{\partial t} = \delta_t u(x_i, t_m)\big|_{m+\sigma} + R_i^m, \ \ |R_i^m| \leqslant C_6 \Delta^3. \tag{3.10}$$

3.4 Difference scheme

Let us discretize (2.1) at the nodes $(x_i, t_{m+\sigma})$ by using approximations (3.4), (3.7), (3.9) and also the piecewise linear interpolation (3.1) with extrapolation (3.2) of the prehistory of the discrete model. We obtain a scheme
$$\delta_t u_i^m\big|_{m+\sigma} + \lambda D_\Delta^\alpha u_i^j\big|_{m+\sigma} = \sigma\delta_x^2 u_i^{m+1} + (1-\sigma)\delta_x^2 u_i^m + f\big(x_i, t_{m+\sigma}, u^m(t_{m+\sigma})_i, u^m_{t_{m+\sigma}}(\,\cdot\,)_i\big), \tag{3.11}$$
$$i = 1, \ldots, N-1, \ \ m = 0, \ldots, M-1.$$

The scheme is supplemented with the initial conditions
$$u_i^j = \varphi(x_i, t_j), \ \ j = -M_0, \ldots, 0, \ \ i = 1, \ldots, N-1,$$
and the boundary conditions
$$u_0^j = u_0(t_j), \ \ u_N^j = u_1(t_j), \ \ j = 0, \ldots, m,$$

where $u^m(t_{m+\sigma})_i$ is the result of interpolation at a point $t_{m+\sigma}$, $u^m_{t_{m+\sigma}}(\,\cdot\,)_i$ is the history of interpolation with extrapolation at this point. Let us transform schema (3.11). Using (3.4), (3.9) and (3.7), we get

$$\frac{(2\sigma+1)u_i^{m+1} - 4\sigma u_i^m + (2\sigma-1)u_i^{m-1}}{2\Delta} + \lambda \frac{\Delta^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{m} c^m_{m-j}(u_i^{j+1} - u_i^j)$$
$$= \sigma \frac{u_{i-1}^{m+1} - 2u_i^{m+1} + u_{i+1}^{m+1}}{h^2} + (1-\sigma)\frac{u_{i-1}^m - 2u_i^m + u_{i+1}^m}{h^2} + f\big(x_i, t_{m+\sigma}, u^m(t_{m+\sigma})_i, u^m_{t_{m+\sigma}}(\,\cdot\,)_i\big),$$

$$-\frac{2\sigma\Delta}{h^2}u_{i-1}^{m+1} + \Big[1 + 2\sigma + \frac{2\lambda c_0\Delta^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{4\sigma\Delta}{h^2}\Big]u_i^{m+1} - \frac{2\sigma\Delta}{h^2}u_{i-1}^{m+1}$$
$$= \frac{2(1-\sigma)\Delta}{h^2}u_{i-1}^m + \frac{2(1-\sigma)\Delta}{h^2}u_{i+1}^m + \Big(4\sigma - \frac{(1-\sigma)4\Delta}{h^2} + \frac{2\lambda c_0\Delta^{1-\alpha}}{\Gamma(2-\alpha)}\Big)u_i^m$$
$$+ 2\Delta f\big(x_i, t_{m+\sigma}, u^m(t_{m+\sigma})_i, u^m_{t_{m+\sigma}}(\,\cdot\,)_i\big)$$
$$-(2\sigma-1)u_i^{m-1} - \frac{2\lambda\Delta^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{m-1} c^m_{m-j}(u_i^{j+1} - u_i^j).$$

We denote

$$A = -\frac{2\sigma\Delta}{h^2}, \quad B = \Big(1 + 2\sigma + \frac{2\lambda c_0\Delta^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{4\sigma\Delta}{h^2}\Big),$$
$$F_i^m = \frac{2(1-\sigma)\Delta}{h^2}u_{i-1}^m + \frac{2(1-\sigma)\Delta}{h^2}u_{i+1}^m + \Big(4\sigma - \frac{(1-\sigma)4\Delta}{}h^2 + \frac{2\lambda c_0\Delta^{1-\alpha}}{\Gamma(2-\alpha)}\Big)u_i^m$$
$$-(2\sigma-1)u_i^{m-1} - \frac{2\lambda\Delta^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{m-1} c^m_{m-j}(u_i^{j+1} - u_i^j) + 2\Delta f\big(x_i, t_{m+\sigma}, u^m(t_{m+\sigma})_i, u^m_{t_{m+\sigma}}(\,\cdot\,)_i\big)$$

and get the system

$$Au_{i-1}^{m+1} + Bu_i^{m+1} + Au_{i-1}^{m+1} = F_i^m.$$

This system has diagonal dominance

$$D = |B| - 2|A| = 1 + 2\sigma + \frac{2\lambda c_0\Delta^{1-\alpha}}{\Gamma(2-\alpha)} \geqslant 1.$$

4 Residual analysis

For $i = 1, \dots, N-1$ and $m = 0, \dots, M-1$, the residual (without interpolation) of method (3.11) is called the grid function

$$\psi_i^m = \delta_t u(x_i, t_m)\big|_{m+\sigma} + \lambda D^\alpha_\Delta u(x_i, t_j)\big|_{m+\sigma}$$
$$- \sigma\delta_x^2 u(x_i, t_{m+1}) - (1-\sigma)\delta_x^2 u(x_i, t_m) - f\big(x_i, t_{m+\sigma}, u(x_i, t_{m+\sigma}), u_{t_{m+\sigma}}(x_i, \,\cdot\,)\big).$$

Lemma 4.1 (Residual order without interpolation)**.** *If the function $u(x,t)$ of the exact solution to problem* (2.1)–(2.3) *is four times continuously differentiable with respect to x and thrice continuously differentiable with respect to t, and also the second derivative with respect to x is twice continuously differentiable with respect to t, then for the residual (without interpolation) of method* (3.11),

$$|\psi_i^m| \leqslant C_7(h^2 + \Delta^2), \quad i = 1, \dots, N-1, \quad m = 1, \dots, M-1,$$

is executed.

Proof. Using (3.10), (3.6) and (3.8) we get

$$\psi_i^m = \frac{\partial u(x_i,t_{m+\sigma})}{\partial t} - R_i^m + \lambda\Big(\frac{\partial^\alpha u(x_i,t_{m+\sigma})}{\partial t^\alpha} - Q_i^m\Big) - \sigma\Big(\frac{\partial^2 u}{\partial x^2}\, u(x_i,t_{m+1}) - P_i^{m+1}\Big) - (1-\sigma)\Big(\frac{\partial^2 u}{\partial x^2}\, u(x_i,t_m) - P_i^m\Big) - f\big(x_i,t_{m+\sigma},u(x_i,t_{m+\sigma}),u_{t_{m+\sigma}}(x_i,\,\cdot\,)\big). \quad (4.1)$$

Since the second derivative of the exact solution with respect to x is twice continuously differentiable with respect to t, we have the representations

$$\frac{\partial^2 u}{\partial x^2}\, u(x_i,t_m) = \frac{\partial^2 u}{\partial x^2}\, u(x_i,t_{m+\sigma}) - \sigma\Delta\,\frac{\partial^3 u}{\partial x^2 \partial t}\, u(x_i,t_{m+\sigma}) + \check{P}_i^m,\ \ |\check{P}_i^m| \leqslant \check{C}_5\Delta^2,$$
$$\frac{\partial^2 u}{\partial x^2}\, u(x_i,t_{m+1}) = \frac{\partial^2 u}{\partial x^2}\, u(x_i,t_{m+\sigma}) + (1-\sigma)\Delta\,\frac{\partial^3 u}{\partial x^2 \partial t}\, u(x_i,t_{m+\sigma}) + \breve{P}_i^m,\ \ |\breve{P}_i^m| \leqslant \breve{C}_5\Delta^2,$$

hence it follows that

$$\sigma\,\partial^2 u \partial x^2\, u(x_i,t_{m+1}) + (1-\sigma)\,\frac{\partial^2 u}{\partial x^2}\, u(x_i,t_m) = \frac{\partial^2 u}{\partial x^2}\, u(x_i,t_{m+\sigma}) + \acute{P}_i^m,\ \ |\acute{P}_i^m| \leqslant \acute{C}_5\Delta^2.$$

Substituting this equality into (4.1) and using the fact that $u(x_i,t_{m+\sigma})$ is the exact solution of equation (2.1), as well as the estimate for P_i^m, Q_i^m, R_i^m and $\acute{P}_i^m$, we get

$$|\psi_i^m| \leqslant C_7(h^2+\Delta^2),\ \ C_7 = C_6 + \lambda C_4 T^{1-\alpha} + C_5 + \acute{C}_5. \qquad \square$$

The residual with a piecewise linear interpolation of method (3.11) is called the grid function

$$\widehat{\psi}_i^m = \delta_t u(x_i,t_m)\big|_{m+\sigma} + \lambda D_\Delta^\alpha u(x_i,t_j)\big|_{m+\sigma} - \sigma\delta_x^2 u(x_i,t_{m+1}) - (1-\sigma)\delta_x^2 u(x_i,t_m) - f\big(x_i,t_{m+\sigma},\widehat{u}^m(x_i,t_{m+\sigma}),\widehat{u}^m_{t_{m+\sigma}}(x_i,\,\cdot\,)\big), \quad (4.2)$$

where $\widehat{u}^m(x_i,t)$ for $t\in[\max\{0,t_m-\tau\},t_{m+1}]$ is the result of the piecewise linear interpolation (3.1) with extrapolation to continuation (3.2) of the discrete history of the exact solution, and when $-\tau \le t \le 0$ $\widehat{u}^m(x_i,t) = \varphi(x_i,t)$.

Lemma 4.2 (Residual order with a piecewise linear interpolation)**.** *Under the conditions of the previous lemma, for the residual with a piecewise constant interpolation of method* (3.11)*,*

$$|\widehat{\psi}_i^m| \leqslant C_8(h^2+\Delta^2),\ \ i=1,\dots,N-1,\ \ m=1,\dots,M-1,$$

is performed.

Proof. The residual with interpolation and the residual without interpolation are related by the relationship

$$\widehat{\psi}_i^m = \psi_i^m + f\big(x_i,t_{m+\sigma},u(x_i,t_{m+\sigma}),u_{t_{m+\sigma}}(x_i,\,\cdot\,)\big) - f\big(x_i,t_{m+\sigma},\widehat{u}^m(x_i,t_{m+\sigma}),\widehat{u}^m_{t_{m+\sigma}}(x_i,\,\cdot\,)\big).$$

Using the Lipschitz property (2.4) of f and the fact that a piecewise linear interpolation with extension extrapolation has second order, we get

$$\begin{aligned}|\widehat{\psi}_i^m| &\leqslant |\psi_i^m| + \Big|f\big(x_i,t_{m+\sigma},\widehat{u}^m(x_i,t_{m+\sigma}),\widehat{u}^m_{t_{m+\sigma}}(x_i,\,\cdot\,)\big) - f\big(x_i,t_{m+\sigma},u(x_i,t_{m+\sigma}),u_{t_{m+\sigma}}(x_i,\,\cdot\,)\big)\Big| \\ &\leqslant |\psi_i^m| + L_f\Big(\big|\widehat{u}^m(x_i,t_{m+\sigma}) - u(x_i,t_{m+\sigma})\big| + \big\|\widehat{u}^m_{t_{m+\sigma}}(x_i,\,\cdot\,) - u_{t_{m+\sigma}}(x_i,\,\cdot\,)\big\|_Q\Big) \\ &\leqslant |\psi_i^m| + 2L_f C_2\Delta^2.\end{aligned}$$

Using the assertion of the previous Lemma 4.1, we obtain the estimate

$$|\widehat{\psi}_i^m| \leqslant C_8(h^2+\Delta^2),\ \ C_8 = C_7 + 2L_f C_2. \qquad \square$$

5 Error analysis

We determine the error of method (3.11)

$$\varepsilon_i^j = u(x_i, t_j) - u_i^j, \ \ j = 0, \dots, M, \ \ i = 0, \dots, N.$$

Denote

$$\varepsilon_i^{j+\sigma} = \sigma \varepsilon_i^{j+1} + (1-\sigma)\varepsilon_i^j.$$

Under the absolute value of the accumulated error by the moment t_m during the time τ, we mean the value

$$\widetilde{\varepsilon}_i^m = \max_{m-M_0 \leqslant j \leqslant m} |\varepsilon_i^j|, \ \ m = 0, \dots, M, \ \ i = 0, \dots, N.$$

Under the absolute value of the accumulated error by the moment t_m for the entire time, we mean the value

$$\widehat{\varepsilon}_i^m = \max_{0 \leqslant j \leqslant m} |\varepsilon_i^j|, \ \ m = 0, \dots, M, \ \ i = 0, \dots, N.$$

A layered error vector ε^m, $m = 0, \dots, M$, is a vector with coordinates $(\varepsilon_1^m, \dots, \varepsilon_{N-1}^m)$. The layered vectors $\widetilde{\varepsilon}^m$ by the moment t_m at the time τ and $\widehat{\varepsilon}^m$ by t_m for the entire time are defined similarly.

For the vectors $u = (u_1, \dots, u_{N-1})$ and $v = (v_1, \dots, v_{N-1})$, we introduce the scalar product $(u,v) = h \sum\limits_{i=1}^{N-1} u_i v_i$ and energy norm $\|u\|^2 = (u,u)$.

Let the time t_m, $m = 0, \dots, M$, be fixed. Denote $E^m = \|\varepsilon^m\|$ and, respectively,

$$\widetilde{E}^m = \|\widetilde{\varepsilon}^m\| \ \text{ and } \ \widehat{E}^m = \|\widehat{\varepsilon}^m\|.$$

We note the estimates that follow from the definitions

$$E^m \leqslant \widetilde{E}^m \leqslant \widehat{E}^m, \quad \widehat{E}^m \leqslant \widehat{E}^{m+1}.$$

Lemma 5.1 ([1])**.** *Let a set of numbers y_i^m, $m = 0, 1, \dots, M-1$, be given. Then*

$$\big[\sigma y_i^{m+1} + (1-\sigma) y_i^m\big] D_\Delta^\alpha y_i^j \Big|_{m+\sigma} \geqslant \frac{1}{2} D_\Delta^\alpha (y_i^j)^2 \Big|_{m+\sigma}.$$

Lemma 5.2 ([13])**.** *Let there exist a set of $(N-1)$-dimensional vectors y^m, $m = 0, 1, \dots, M-1$. Then*

$$\big(\sigma y^{m+1} + (1-\sigma) y^m, \delta_t y^m|_{m+\sigma}\big) \geqslant \frac{1}{4\Delta}\,(G^{m+1} - G^m),$$
$$G^{m+1} = (2\sigma+1)\|y^{m+1}\|^2 - (2\sigma-1)\|y^m\|^2 + (2\sigma^2+\sigma-1)\|y^{m+1} - y^m\|^2.$$

Besides,

$$G^{m+1} \geqslant \frac{1}{\sigma}\,\|y^{m+1}\|^2.$$

The following statement gives the form of the sum over l from 1 to m of the operators $D_\Delta^\alpha y^j|_{l+\sigma}$. This representation significantly shortens the derivation of the main estimate in the Order of Convergence Theorem as compared to the proof of the similar Order of Convergence Theorem given in [13].

Lemma 5.3. *Given a set of numbers $\{y^j\}_{j=0}^{m+1}$, then for every $m \geqslant 1$ the following is performed:*

$$\sum_{l=1}^{m} D_\Delta^\alpha y^j\big|_{l+\sigma} = \frac{\Delta^{-\alpha}}{\Gamma(2-\alpha)} \left(\sum_{j=2}^{m+1} c_{m-j+1} y^j - (c_0 - c_m) y^1 - \Big(\sum_{j=1}^{m} c_j\Big) y^0 \right). \tag{5.1}$$

Proof. Let us verify the assertion of the lemma by induction on m.

From (3.5) with $m = 1$, we get

$$D_{\Delta}^{\alpha} y^j\big|_{1+\sigma} = \frac{\Delta^{-\alpha}}{\Gamma(2-\alpha)}\left(c_0 y^2 - (c_0 - c_1)y^1 - c_1 y^0\right),$$

and equality (5.1) holds for $m = 1$.

Suppose that equality (5.1) holds for $m-1$, i.e.,

$$\sum_{l=1}^{m-1} D_{\Delta}^{\alpha} y^j\big|_{l+\sigma} = \frac{\Delta^{-\alpha}}{\Gamma(2-\alpha)}\left(\sum_{j=2}^{m} c_{m-j} y^j - (c_0 - c_{m-1})y^1 - \Big(\sum_{j=1}^{m-1} c_j\Big) y^0\right).$$

Adding this equality to (3.5), we get that (5.1) holds for index m. □

Lemma 5.4 (Gronwall inequality, see, e.g., [11])**.** *Let a set of non-negative numbers* y^m*,* $m = 0, 1, \dots, M-1$*, be given and*

$$y^{m+1} \leqslant A + \Delta B \sum_{l=0}^{m} y^l.$$

Then

$$y^{m+1} \leqslant A \exp(Bm\Delta).$$

Theorem 5.1 (Order of Convergence Theorem)**.** *Suppose that the smoothness conditions of the solution formulated in Lemma 4.1 are satisfied, then for the accumulated error of method* (3.11)

$$\widehat{E}^m \leqslant C(h^2 + \Delta^2), \quad m = 1, \dots, M,$$

is performed.

Proof. From (3.11) and (4.2), for any $m = 0, \dots, M-1$, $i = 1, \dots, N-1$, we get the error equation

$$\delta_t \varepsilon_i^m\big|_{m+\sigma} + \lambda D_{\Delta}^{\alpha} \varepsilon_i^j\big|_{m+\sigma} = \sigma \delta_x^2 \varepsilon_i^{m+1} + (1-\sigma)\delta_x^2 \varepsilon_i^m + \widehat{\psi}_i^m$$
$$+ f\big(x_i, t_{m+\sigma}, \widehat{u}^m(x_i, t_{m+\sigma}), \widehat{u}^m_{t_{m+\sigma}}(x_i, \cdot\,)\big) - f\big(x_i, t_{m+\sigma}, u^m(t_{m+\sigma})_i, u^m_{t_{m+\sigma}}(\,\cdot\,)_i\big).$$

Multiply this equality by $h\varepsilon_i^{m+\sigma}$ and sum over i from 1 to $N-1$, we get

$$h\sum_{i=1}^{N-1} \varepsilon_i^{m+\sigma} \delta_t \varepsilon_i^m\big|_{m+\sigma} + \lambda h \sum_{i=1}^{N-1} \varepsilon_i^{m+\sigma} D_{\Delta}^{\alpha} \varepsilon_i^j\big|_{m+\sigma}$$
$$= \sigma h \sum_{i=1}^{N-1} \varepsilon_i^{m+\sigma} \delta_x^2 \varepsilon_i^{m+1} + (1-\sigma) h \sum_{i=1}^{N-1} \varepsilon_i^{m+\sigma} \delta_x^2 \varepsilon_i^m + h \sum_{i=1}^{N-1} \varepsilon_i^{m+\sigma} \widehat{\psi}_i^m$$
$$+ h \sum_{i=1}^{N-1} \varepsilon_i^{m+\sigma}\Big(f\big(x_i, t_{m+\sigma}, \widehat{u}^m(x_i, t_{m+\sigma}), \widehat{u}^m_{t_{m+\sigma}}(x_i, \cdot\,)\big) - f\big(x_i, t_{m+\sigma}, u^m(t_{m+\sigma})_i, u^m_{t_{m+\sigma}}(\,\cdot\,)_i\big)\Big) \quad (5.2)$$

or

$$LP = \big(\varepsilon^{m+\sigma}, \delta_t \varepsilon^m|_{m+\sigma}\big) + \lambda\big(\varepsilon^{m+\sigma}, D_{\Delta}^{\alpha} \varepsilon^j|_{m+\sigma}\big)$$
$$= (\varepsilon^{m+\sigma}, \delta_x^2 \varepsilon^{m+\sigma}) + (\varepsilon^{m+\sigma}, \widehat{\psi}^m) + (\varepsilon^{m+\sigma}, f^m - \widehat{f}^m) = RP. \quad (5.3)$$

Let us estimate from above each term on the right-hand side of (5.3).

By definition (5.3), the operator $-\delta_x^2$ is positive definite and self-adjoint; moreover, the conditions [10, p. 315]

$$\kappa\|y\|^2 \leqslant (-\delta_x^2 y, y) \leqslant \frac{4}{h}\,\|y\|^2, \quad \kappa = \frac{9}{X}\,,$$

hold, whence

$$(\varepsilon^{m+\sigma}, \delta_x^2 \varepsilon^{m+\sigma}) \leqslant -\kappa \|\varepsilon^{m+\sigma}\|^2, \ \ \kappa = \frac{9}{X}.$$

By definition of the scalar product, we obtain

$$(\varepsilon^{m+\sigma}, \widehat{\psi}^m) = \Big(\sqrt{\kappa}\varepsilon^{m+\sigma}, \frac{\widehat{\psi}^m}{\sqrt{\kappa}}\Big) \leqslant \frac{\kappa}{2}\, \|\varepsilon^{m+\sigma}\|^2 + \frac{1}{2\kappa}\, \|\widehat{\psi}^m\|^2.$$

In a similar way, we obtain

$$(\varepsilon^{m+\sigma}, f^m - \widehat{f}^m) \leqslant \frac{\kappa}{2}\, \|\varepsilon^{m+\sigma}\|^2 + \frac{1}{2\kappa}\, \|f^m - \widehat{f}^m\|^2.$$

Thus, the entire right-hand side of relation (5.3) is estimated from above by the quantity

$$RP \leqslant \frac{1}{2\kappa}\, \|\widehat{\psi}^m\|^2 + \frac{1}{2\kappa}\, \|f^m - \widehat{f}^m\|^2. \tag{5.4}$$

By virtue of Lemma 4.2, we have the estimate

$$\|\widehat{\psi}^m\|^2 = h\sum_{i=1}^{N-1} |\widehat{\psi}_i^m|^2 \leqslant X(C_8)^2(h^2+\Delta^2)^2. \tag{5.5}$$

Since the function f and the interpolation operator are Lipschets (2.4), (3.3), we have

$$\begin{aligned}
&\Big| f\big(x_i, t_{m+\sigma}, \widehat{u}^m(x_i, t_{m+\sigma}), \widehat{u}^m_{t_{m+\sigma}}(x_i, \cdot\,)\big) - f\big(x_i, t_{m+\sigma}, u^m(t_{m+\sigma})_i, u^m_{t_{m+\sigma}}(\,\cdot\,)_i\big)\Big| \\
&\qquad \leqslant L_f\Big(|\widehat{u}^m(x_i, t_{m+\sigma}) - u^m(t_{m+\sigma})_i| + \|\widehat{u}^m_{t_{m+\sigma}}(x_i, \cdot\,) - u^m_{t_{m+\sigma}}(\,\cdot\,)_i\|_Q\Big) \\
&\qquad\qquad \leqslant 2L_f L_I \max_{m-M_0\leqslant j\leqslant m} \|u_i^j - u(x_i, t_j)\| = 2L_f L_I \widetilde{\varepsilon}_i^m,
\end{aligned}$$

hence

$$\begin{aligned}
&\|f^m - \widehat{f}^m\|^2 \\
&\quad = h\sum_{i=1}^{N-1} \Big| f\big(x_i, t_{m+\sigma}, \widehat{u}^m(x_i, t_{m+\sigma}), \widehat{u}^m_{t_{m+\sigma}}(x_i, \cdot\,)\big) - f\big(x_i, t_{m+\sigma}, u^m(t_{m+\sigma})_i, u^m_{t_{m+\sigma}}(\,\cdot\,)_i\big)\Big|^2 \\
&\qquad \leqslant (2L_f L_I)^2 (\widetilde{E}^m)^2 \leqslant (2L_f L_I)^2 (\widehat{E}^m)^2.
\end{aligned} \tag{5.6}$$

Thus, from (5.4)–(5.6) we get

$$RP \leqslant \frac{1}{2\kappa}\, X(C_8)^2(h^2+\Delta^2)^2 + \frac{1}{2\kappa}\, (2L_f L_I)^2 (\widehat{E}^m)^2. \tag{5.7}$$

Let us estimate from below each term on the left-hand side of the relation (5.3).
Lemma 5.2 implies that

$$\begin{aligned}
(\varepsilon^{m+\sigma}, \delta_t \varepsilon^m|_{m+\sigma}) &\geqslant \frac{1}{4\Delta}(G^{m+1} - G^m), \\
G^{m+1} = (2\sigma+1)\|\varepsilon^{m+1}\|^2 &- (2\sigma-1)\|\varepsilon^m\|^2 + (2\sigma^2+\sigma-1)\|\varepsilon^{m+1} - \varepsilon^m\|^2.
\end{aligned} \tag{5.8}$$

Besides,

$$G^{m+1} \geqslant \frac{1}{\sigma}\, \|\varepsilon^{m+1}\|^2. \tag{5.9}$$

Lemma 5.1 implies

$$\lambda\big(\varepsilon^{m+\sigma}, D_\Delta^\alpha \varepsilon^j|_{m+\sigma}\big) = \lambda h\sum_{i=1}^{N-1} \varepsilon_i^{m+\sigma} D_\Delta^\alpha \varepsilon_i^j\big|_{m+\sigma} \geqslant \frac{\lambda}{2}\, h\sum_{i=1}^{N-1} D_\Delta^\alpha (\varepsilon_i^j)^2\big|_{m+\sigma} = \frac{\lambda}{2}\, D_\Delta^\alpha \|\varepsilon^j\|^2\big|_{m+\sigma},$$

thus

$$LP \geqslant \frac{1}{4\Delta}\,(G^{m+1} - G^m) + \frac{\lambda}{2}\, D^{\alpha}_{\Delta}\|\varepsilon^j\|^2\big|_{m+\sigma}. \tag{5.10}$$

Substituting into (5.3) the estimates (5.7) and (5.10), we obtain

$$\frac{1}{4\Delta}\,(G^{m+1} - G^m) + \frac{\lambda}{2}\, D^{\alpha}_{\Delta}(E^j)^2\big|_{m+\sigma} \leqslant \frac{1}{2\kappa}\, X(C_8)^2(h^2+\Delta^2)^2 + \frac{1}{2\kappa}\,(2L_f L_I)^2(\widehat{E}^m)^2. \tag{5.11}$$

First, consider this inequality for $m = 0$. Given (5.9) and also that $E^0 = \widetilde{E}^0 = \widehat{E}^0 = 0$,

$$D^{\alpha}_{\Delta}(E^j)^2\big|_{\sigma} = \frac{\Delta^{-\alpha}}{\Gamma(2-\alpha)}\, a_0(E^1)^2,$$

we obtain

$$\frac{1}{4\sigma\Delta}\,(E^1)^2 + \frac{\lambda\Delta^{-\alpha}a_0}{2\Gamma(2-\alpha)}\,(E^1)^2 \leqslant \frac{1}{2\kappa}\, X(C_8)^2(h^2+\Delta^2)^2.$$

Since the second term on the left-hand side is non-negative, we obtain the estimate

$$(E^1)^2 \leqslant C_9\Delta(h^2+\Delta^2)^2, \quad C_9 = \frac{2\sigma}{\kappa}\, X(C_8)^2. \tag{5.12}$$

From (5.8) with $m = 0$, we get $G^1 = (2\sigma^2+3\sigma)(E^1)^2$, hence it follows that

$$G^1 \leqslant C_{10}\Delta(h^2+\Delta^2)^2, \quad C_{10} = \frac{C_9}{2\sigma^2+3\sigma}\,. \tag{5.13}$$

Now we replace the index m in inequality (5.11) by l and sum it over all l from 1 to m, then we get

$$\frac{1}{4\Delta}\,(G^{m+1} - G^1) + \frac{\lambda}{2}\sum_{l=1}^{m} D^{\alpha}_{\Delta}(E^j)^2\big|_{l+\sigma} \leqslant \frac{m}{2\kappa}\, X(C_8)^2(h^2+\Delta^2)^2 + \frac{1}{2\kappa}\,(2L_f L_I)^2\sum_{l=1}^{m}(\widehat{E}^l)^2. \tag{5.14}$$

Lemma 5.3, taking into account $E^0 = 0$, implies

$$\begin{aligned}\frac{\lambda}{2}\sum_{l=1}^{m} D^{\alpha}_{\Delta}(E^j)^2\big|_{l+\sigma} &= \frac{\lambda\Delta^{-\alpha}}{2\Gamma(2-\alpha)}\Big(c_0(E^{m+1})^2 + \sum_{j=2}^{m} c_{m-j+1}(E^j)^2 - (c_0-c_m)(E^1)^2\Big)\\ &\geqslant \frac{\lambda\Delta^{-\alpha}}{2\Gamma(2-\alpha)}\big(-(c_0-c_m)(E^1)^2\big),\end{aligned}$$

thus inequality (5.14) implies

$$\begin{aligned}\frac{1}{4\Delta}\,(G^{m+1} - G^1) &- \frac{\lambda\Delta^{-\alpha}}{2\Gamma(2-\alpha)}\big((c_0-c_m)(E^1)^2\big)\\ &\leqslant \frac{m}{2\kappa}\, X(C_8)^2(h^2+\Delta^2)^2 + \frac{1}{2\kappa}\,(2L_f L_I)^2\sum_{l=1}^{m}(\widehat{E}^l)^2,\end{aligned}$$

or given (5.9) implies

$$\begin{aligned}(E^{m+1})^2 \leqslant G^1 &+ \frac{4\sigma\lambda\Delta^{1-\alpha}}{2\Gamma(2-\alpha)}\big((c_0-c_m)(E^1)^2\big)\\ &+ \frac{2\sigma\Delta m}{\kappa}\, X(C_8)^2(h^2+\Delta^2)^2 + \frac{2\sigma\Delta}{\kappa}\,(2L_f L_I)^2\sum_{l=1}^{m}(\widehat{E}^l)^2.\end{aligned}$$

Using (5.12) and (5.13), we get

$$(E^{m+1})^2 \leqslant C_{10}\Delta(h^2+\Delta^2)^2 + \frac{4\sigma\lambda\Delta^{1-\alpha}}{2\Gamma(2-\alpha)}(c_0-c_m)C_9\Delta(h^2+\Delta^2)^2 + \frac{2\sigma\Delta m}{\kappa}X(C_8)^2(h^2+\Delta^2)^2 + \frac{2\sigma\Delta}{\kappa}(2L_fL_I)^2\sum_{l=1}^{m}(\widehat{E}^l)^2$$

or

$$(E^{m+1})^2 \leqslant C_{11}(h^2+\Delta^2)^2 + C_{12}\Delta\sum_{l=1}^{m}(\widehat{E}^l)^2, \tag{5.15}$$

$$C_{11} = C_{10}T + \frac{4\sigma\lambda T^{2-\alpha}(c_0-c_m)C_9}{2\Gamma(2-\alpha)} + \frac{2\sigma T}{\kappa}X(C_8)^2, \quad C_{12} = \frac{2\sigma}{\kappa}(2L_fL_I)^2.$$

Consider a non-decreasing sequence $\widehat{E}^0 = 0, \widehat{E}^1, \dots, \widehat{E}^m, \dots$.

Let k be the first number when the condition $\widehat{E}^{k+1} > \widehat{E}^k$ is satisfied (such a number can be found, since otherwise any error is zero and the assertion of the theorem trivially holds), then $\widehat{E}^{k+1} = E^{k+1}$ and (5.15) implies

$$(\widehat{E}^{k+1})^2 \leqslant C_{11}(h^2+\Delta^2)^2 + C_{12}\Delta\sum_{l=1}^{k}(\widehat{E}^l)^2.$$

By definition, the same relation holds for all indices less than k.

Take any index $m > k$. Two situations are possible.

A). $\widehat{E}^{m+1} > \widehat{E}^m$, then $\widehat{E}^{m+1} = E^{m+1}$ and

$$(\widehat{E}^{m+1})^2 \leqslant C_{11}(h^2+\Delta^2)^2 + C_{12}\Delta\sum_{l=1}^{m}(\widehat{E}^l)^2. \tag{5.16}$$

B). $\widehat{E}^{m+1} = \widehat{E}^m$, then there is an index $s < m$ such that $\widehat{E}^{m+1} = \widehat{E}^m = \dots = \widehat{E}^{s+1} > \widehat{E}^s$. Since $\widehat{E}^{s+1} = E^{s+1}$, from (5.15) it follows that

$$\widehat{E}^{m+1} = \widehat{E}^{s+1} \leqslant C_{11}(h^2+\Delta^2)^2 + C_{12}\Delta\sum_{l=1}^{s}(\widehat{E}^l)^2 \leqslant C_{11}(h^2+\Delta^2)^2 + C_{12}\Delta\sum_{l=1}^{m}(\widehat{E}^l)^2.$$

Thus, relation (5.16) holds for all indices $m = 0, \dots, M-1$.

Applying the Gronwall inequality (Lemma 5.4), we obtain

$$(\widehat{E}^{m+1})^2 \leqslant C_{11}(h^2+\Delta^2)^2\exp(C_{12}m\Delta) \leqslant C_{13}(h^2+\Delta^2)^2, \;\; C_{13} = C_{11}\exp(C_{12}T).$$

Extracting the square root, we obtain the assertion of the theorem for $C = \sqrt{C_{13}}$. □

6 Numerical experiments

We consider the standard rates

$$\rho^x_{\Delta,h} = \log_2\left(\frac{\varepsilon_{\Delta,2h}}{\varepsilon_{\Delta,h}}\right), \quad \rho^t_{\Delta,h} = \log_2\left(\frac{\varepsilon_{2\Delta,h}}{\varepsilon_{\Delta,h}}\right).$$

Example 1. Consider the problem with a constant delay

$$\frac{\partial u(x,t)}{\partial t} + 0.5\cdot\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(x,t) - u^3(x,t) - u(x,t-\tau) + \left(3t^2 - t^3 + (t-\tau)^3 + \frac{3^{2-\alpha}}{\Gamma(4-\alpha)}\right)x^2(1-x)^2 + t^9x^6(1-x)^6 \tag{6.1}$$

for $\alpha = 0.5$, $\tau = 0.1$ with the boundary conditions

$$u(0,t) = 0, \ \ u(1,t) = t^3, \ \ 0 \leqslant t \leqslant 1$$

and the initial conditions

$$u(x,t) = u(x,t) = t^3 x^2 (1-x^2), \ \ 0 \leqslant x \leqslant 1, \ \ -\tau \leqslant t \leqslant 0.$$

The exact solution of the equation is $u(x,t) = t^3 x^2(1-x^2)$. Table 1 shows the numerical study of convergence for problem (6.1) in both space and time. This table shows a comparison of the results of calculations by the method from [7] and method (3.11).

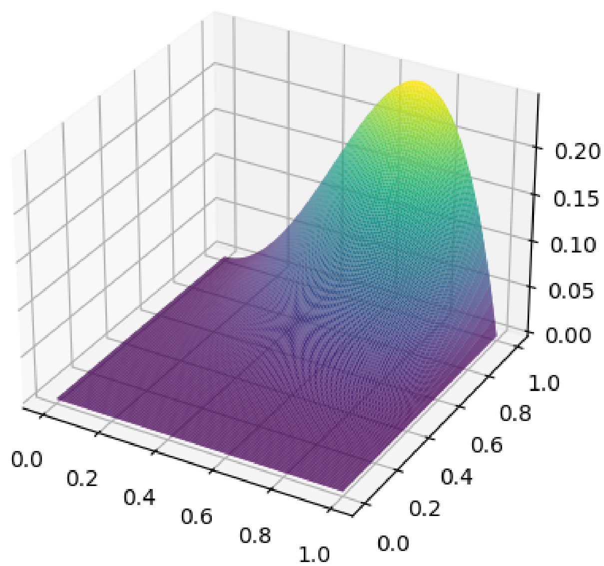

Figure 1. Approximate solution of equation (6.1) by method (3.11) for $h = 0.01$, $\Delta = 0.005$.

Table 1. Absolute errors and standard convergence rates when approximating the solution u of (6.1)

h	Δ	method [7] $\varepsilon_{\Delta,h}$	method [7] $\rho^t_{\Delta,h}$	method (3.11) $\varepsilon_{\Delta,h}$	method (3.11) $\rho^t_{\Delta,h}$
0.5	0.125×2^{-1}	3.1351×10^{-2}	-	6.4571×10^{-3}	-
	0.125×2^{-2}	1.9841×10^{-2}	0.6601	1.1141×10^{-3}	2.5350
	0.125×2^{-3}	9.1058×10^{-3}	1.1236	2.6678×10^{-4}	2.0621
0.25	0.125×2^{-1}	1.1854×10^{-2}	-	1.5184×10^{-3}	-
	0.125×2^{-2}	7.3457×10^{-3}	0.6904	3.4573×10^{-4}	2.1348
	0.125×2^{-3}	3.0051×10^{-3}	1.2894	1.0032×10^{-4}	1.7850
0.125	0.125×2^{-1}	2.3789×10^{-3}	-	1.6731×10^{-3}	-
	0.125×2^{-2}	6.9892×10^{-4}	1.7670	4.1732×10^{-4}	2.0032
	0.125×2^{-3}	4.2192×10^{-4}	0.7281	1.2360×10^{-4}	1.7554

Example 2. Consider the problem with a variable delay

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(x,t) - u(x, t-2t^2) + (-t^2+2t) + \frac{2e^x t^{2-\alpha}}{\Gamma(3-\alpha)} x^2(1-x)^2 + t^9 x^6 (1-x)^6 \quad (6.2)$$

for $\alpha = 0.5$ with the boundary conditions

$$u(0,t) = t^2, \;\; u(1,t) = et^2, \;\; 0 \leqslant t \leqslant 1$$

and the initial conditions

$$u(x,t) = u(x,t) = t^2 e^x, \;\; 0 \leqslant x \leqslant 1, \;\; -\tau \leqslant t \leqslant 0.$$

The exact solution of the equation is $u(x,t) = t^2 e^x$. The exact solution of the equation is $u(x,t) = t^3 x^2 (1 - x^2)$. Unlike the previous example, this problem cannot be numerically implemented without interpolation procedures. This is the main difference between method (3.11) and the algorithm of [13]. Table 2 shows the numerical study of convergence for problem (6.2) in both space and time.

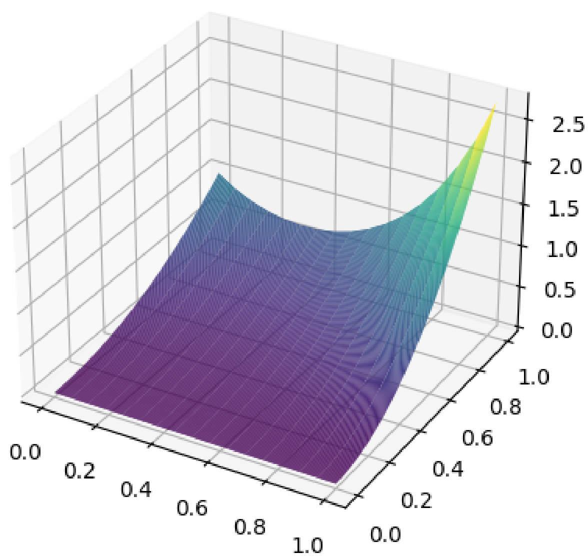

Figure 2. Approximate solution of equation (6.2) by method (3.11) for $h = 0.0625$, $\Delta = 0.005$.

Table 2. Absolute errors and standard convergence rates when approximating the solution u of (6.2)

h	Δ	method [7] $\varepsilon_{\Delta,h}$	method [7] $\rho^t_{\Delta,h}$	method (3.11) $\varepsilon_{\Delta,h}$	method (3.11) $\rho^t_{\Delta,h}$
0.5	0.125×2^{-1}	2.2699×10^{-1}	-	5.8928×10^{-2}	-
	0.125×2^{-2}	1.1841×10^{-1}	0.9388	1.8094×10^{-2}	1.7034
	0.125×2^{-3}	4.1058×10^{-2}	1.5280	5.0610×10^{-3}	1.8380
0.25	0.125×2^{-1}	7.0917×10^{-2}	-	1.5892×10^{-2}	-
	0.125×2^{-2}	4.5189×10^{-2}	0.6501	4.6024×10^{-3}	1.7878
	0.125×2^{-3}	3.0045×10^{-2}	0.5888	1.5975×10^{-3}	1.5265
0.125	0.125×2^{-1}	3.4889×10^{-2}	-	5.3466×10^{-3}	-
	0.125×2^{-2}	1.9892×10^{-2}	0.8105	1.8449×10^{-3}	1.5350
	0.125×2^{-3}	1.0304×10^{-2}	0.9489	4.1760×10^{-4}	3.6784

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