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INEQUALITIES INVOLVING STOLARSKY AND GINI MEANS

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Abstract: This paper deals with inequalities for the Stolarsky and Gini means. Inequalities involving the means in question and their products are established. Some of these results provide refinements of known inequalities for the particular means of two variables. The Ky Fan type inequalities for the means discussed in this paper are also obtained.

1. Introduction and notation

This paper deals with the inequalities for two families of the two-parameter means of variables $x > 0$ and $y > 0$. In order to avoid trivialities we will always assume that $x \neq y$.

First class of means studied in this paper was introduced by K. Stolarsky [20]. For $a, b \in \mathbb{R}$ they are denoted by $D_{a,b}(\cdot,\cdot)$ and defined as

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$$
(1.1)\quad D_{a,b}(x,y)=\begin{cases}\left[\dfrac{b(x^a-y^a)}{a(x^b-y^b)}\right]^{1/(a-b)}, & ab(a-b)\neq 0\\ \exp\left(-\dfrac{1}{a}+\dfrac{x^a\ln x-y^a\ln y}{x^a-y^a}\right), & a=b\neq 0\\ \left[\dfrac{x^a-y^a}{a(\ln x-\ln y)}\right]^{1/a}, & a\neq 0,\ b=0\\ \sqrt{xy}, & a=b=0.\end{cases}
$$

Stolarsky means are sometimes called the difference means (see, e.g., [10], [8]).

The identric, logarithmic, and power means of order a ($a\neq 0$) will be denoted by I_a, L_a, and A_a, respectively. They are all contained in the family of means under discussion. We have $I_a=D_{a,a}$, $L_a=D_{a,0}$, and $A_a=D_{2a,a}$. When $a=1$ we will write I, L, and A instead of I_1, L_1, and A_1. There is a simple relationship between means of order a ($a\neq 1$) and those of order one. We have

$$
I_a(x,y)=[I(x^a,y^a)]^{1/a}
$$

with similar formulas for the remaining means mentioned above. Finally, the geometric mean of x and y is $G(x,y)=D_{0,0}(x,y)$.

Second family of bivariate means studied here was introduced by C. Gini [4]. Throughout the sequel they will be denoted by $S_{a,b}(\cdot,\cdot)$ and they are defined as follows

$$
S_{a,b}(x,y)=\begin{cases}\left[\dfrac{x^a+y^a}{x^b+y^b}\right]^{1/(a-b)}, & a\neq b\\ \exp\left(\dfrac{x^a\ln x+y^a\ln y}{x^a+y^a}\right), & a=b\neq 0\\ \sqrt{xy}, & a=b=0.\end{cases}
$$

Gini means are also called the sum means. It follows from (1.2) that $S_{0,-1}=H-$ the harmonic mean, $S_{0,0}=G$, and $S_{1,0}=A$. The following mean $J=S_{1,1}$ will play an important role in the discussion that follows.

Alzer and Ruscheweyh [1] have proven that the joint members in the families of the Stolarsky and Gini means are exactly the power means.

This paper is organized as follows. Basic properties of the means under discussion together with the comparison theorems are given in Section 2. They are included here for the sake of presentation. The main results of this paper are contained in Sections 3 in 4. Some of the results obtained in Section 3 provide generalizations of certain inequalities for the particular means. The Ky Fan type inequalities for the Stolarsky and Gini means are derived in Section 4.

2. Basic properties and the comparison theorems for $\mathbf{D}_{a,b}$ and $\mathbf{S}_{a,b}$

For the reader's convenience we give here a list of basic properties of the Stolarsky and Gini means. They follow directly from the defining formulas (1.1) and (1.2) and most of them can be found in [5] and [20]. Although they are formulated for the Stolarsky means they remain valid for the Gini means, too. In what follows we assume that $a, b, c \in \mathbb{R}$.

(P1) $D_{a,b}\ (\cdot,\cdot)$ is symmetric in its parameters, i.e., $D_{a,b}\ (\cdot,\cdot) = D_{b,a}\ (\cdot,\cdot)$.

(P2) $D_{\cdot,\cdot}(x,y)$ is symmetric in the variables x and y, i.e., $D_{\cdot,\cdot}(x,y) = D_{\cdot,\cdot}(y,x)$.

(P3) $D_{a,b}(x,y)$ is a homogeneous function of order one in its variables, i.e., $D_{a,b}(\lambda x, \lambda y) = \lambda D_{a,b}(x,y)$, $\lambda > 0$.

(P4) $D_{a,b}(x^c, y^c) = [D_{ac,bc}(x,y)]^c$.

(P5) $D_{a,b}(x,y)D_{-a,-b}(x,y) = xy$.

(P6) $D_{a,b}(x^c, y^c) = (xy)^c D_{a,b}(x^{-c}, y^{-c})$.

(P7) $D_{a,b}(x,y)S_{a,b}(x,y) = D_{a,b}(x^2,y^2) = D^2_{2a,2b}(x,y)$.

(P8) $D_{a,b}$ increases with increase in either a or b.

(P9) If $a > 0$ and $b > 0$, then $D_{a,b}$ is log-concave in both a and b. If $a < 0$ and $b < 0$, then $D_{a,b}$ is log-convex in both a and b.

Property (P8) for the Stolarsky means is established in [5] and [20]. F. Qi [13] has established (P8) for the Gini means. Logarithmic concavity (convexity) property for the Stolarsky means is established in [12].

(P10) If $a \neq b$, then

$$\ln D_{a,b} = \frac{1}{b-a}\int_a^b \ln I_t\, dt \quad \text{and} \quad \ln S_{a,b} = \frac{1}{b-a}\int_a^b \ln J_t\, dt.$$

First formula in (P10) is derived in [20] while the proof of the second one is an elementary exercise in calculus.

We shall prove now the property (P9) for the Gini means. The following result will be utilized.

Lemma 2.1 [14]. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function. If f is increasing (decreasing) and/or convex (concave), then the function*

$$g(a,b) = \frac{1}{b-a}\int_a^b f(t)dt$$

is increasing (decreasing) and/or convex (concave) function in both variables a and b.

Let $r = (x/y)^t$ and let $\mu(t) = \ln J_t$ ($t \in \mathbb{R}$). It follows from (1.2) that $t\mu(t) = t\ln x - (\ln r)/(r+1)$. Hence $t^2\mu'(t) = r[(\ln r)/(r+1)]^2 > > 0$ and $t^3\mu''(t) = r(1-r)[(\ln r)/(r+1)]^3 < 0$ for all $t \neq 0$. Thus the function $\mu(t)$ is strictly concave for $t > 0$ and strictly convex or $t < 0$. This in conjunction with the Lemma 2.1 and the second formula of (P10) gives the desired result.

We close this section with three comparison theorems for the means under discussion. The following functions will be used throughout the sequel. Let

$$k(x,y) = \begin{cases} \dfrac{|x|-|y|}{x-y}, & x \neq y \\ \operatorname{sign}(x), & x = y \end{cases}$$

and let

$$l(x,y) = \begin{cases} L(x,y), & x > 0,\ y > 0 \\ 0, & x \cdot y = 0, \end{cases}$$

where $L(x,y) = \frac{x-y}{\ln x - \ln y}$ ($x \neq y$), $L(x,x) = x$ is the logarithmic mean of x and y. Function $l(x,y)$ is defined for nonnegative values of x and y only.

The comparison theorem for the Stolarsky means reads as follows.

Theorem A ([10], [6]). *Let $a,b,c,d \in \mathbb{R}$. Then the comparison inequality*

$$D_{a,b}(x,y) \leq D_{c,d}(x,y)$$

holds true if and only if $a+b \leq c+d$ and

$$\begin{array}{ll} l(a,b) \leq l(c,d) & \text{if } 0 \leq \min(a,b,c,d), \\ k(a,b) \leq k(c,d) & \text{if } \min(a,b,c,d) < 0 < \max(a,b,c,d), \\ -l(-a,-b) \leq -l(-c,-d) & \text{if } \max(a,b,c,d) \leq 0. \end{array}$$

A comparison result for the Gini means is contained in the following.

Theorem B ([9]). *Let $a, b, c, d \in \mathbb{R}$. Then the comparison inequality*

$$S_{a,b}(x,y) \le S_{c,d}(x,y)$$

is valid if and only if $a + b \le c + d$ and

$$\begin{array}{ll} \min(a,b) \le \min(c,d) & \text{if } 0 \le \min(a,b,c,d), \\ k(a,b) \le k(c,d) & \text{if } \min(a,b,c,d) < 0 < \max(a,b,c,d), \\ \max(a,b) \le \max(c,d) & \text{if } \max(a,b,c,d) \le 0. \end{array}$$

A comparison result for the Stolarsky and Gini means is obtained in [8].

Theorem C. *Let $a, b \in \mathbb{R}$. If $a + b > 0$, then*

$$D_{a,b}(x,y) < S_{a,b}(x,y)$$

with the inequality reversed if $a + b < 0$. Moreover, $D_{a,b}(x,y) = S_{a,b}(x,y)$ if and only if $a + b = 0$.

A new proof of Th. C is included below.

Proof. There is nothing to prove when $a + b = 0$ because $D_{a,-a} = S_{a,-a} = G$. Define $r = (x/y)^t$ and $\phi(t) = \ln I_t - \ln J_t$. One can verify easily that

$$t\phi(t) = \frac{2r\ln r}{(r+1)(r-1)} - 1 = \frac{H(r,1)}{L(r,1)} - 1 < 0,$$

where that last inequality follows from the harmonic-logarithmic mean inequality. Also, $\phi(-t) = -\phi(t)$ for $t \in \mathbb{R}$. Hence $\phi(t) < 0$ if $t > 0$ and $\phi(t) > 0$ for $t < 0$. Let $a \ne b$. Making use of (P10) we obtain

$$\ln D_{a,b} - \ln S_{a,b} = \frac{1}{b-a}\int_a^b \phi(t)dt < 0,$$

where the last inequality holds true provided $a + b > 0$. The same argument can be employed to show that $D_{a,b} > S_{a,b}$ if $a + b < 0$. Assume now that $a = b \ne 0$. Sándor and Rasa [17] have proven that $D_{a,a} < S_{a,a}$ for $a > 0$ with the inequality reversed if $a < 0$. This completes the proof. ◊

3. Inequalities

Proofs of some results in this section utilize a refinement of the classical inequality which is due to Hermite and Hadamard.

Let $f : [a,b] \to \mathbb{R}$ be a convex function. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{1}{2}[f(a)+f(b)] \tag{3.1}$$

with the inequalities reversed if f is concave on $[a,b]$. Equalities hold in (3.1) if and only if f is a polynomial of degree one or less (see, e.g., [11]).

To obtain a refinement of (3.1) we introduce a uniform partition of $[a,b]$ with the breakpoints α_k, i.e., $a = \alpha_0 < \alpha_1 < \cdots < \alpha_n = b$ with $\alpha_k - \alpha_{k-1} = h > 0$. Also, let $\beta_1 < \beta_2 < \cdots < \beta_n$ be the midpoints of the consecutive subintervals. Thus

$$\alpha_k = \frac{(n-k)a+kb}{n} \quad (0 \le k \le n)$$

and

$$\beta_k = \frac{(2n-2k+1)a+(2k-1)b}{2n} \quad (1 \le k \le n).$$

Let n be a positive integer. We define

$$M_n = \frac{1}{n}\sum_{k=1}^{n} f(\beta_k)$$

and

$$T_n = \frac{1}{n}\left\{\frac{1}{2}[f(a)+f(b)] + \sum_{k=1}^{n-1} f(\alpha_k)\right\}.$$

Lemma 3.1. *Let f be a convex function on $[a,b]$. Then*

$$M_n \le \frac{1}{b-a}\int_a^b f(t)dt \le T_n. \tag{3.2}$$

Inequalities (3.2) *are reversed if f is concave on $[a,b]$.*

Proof. Applying (3.1) to each of the integrals

$$\frac{1}{h}\int_{\alpha_{k-1}}^{\alpha_k} f(t)dt$$

$(h = (b-a)/n)$ and next summing the resulting expression, for $k = 1,2,\ldots,n$, we obtain the assertion. ◊

It is easy to verify that if f is a convex function on $[a,b]$, then

$$M_n \ge f\left[\frac{1}{n}\sum_{k=1}^{n}\beta_k\right] = f\left(\frac{a+b}{2}\right)$$

and

$$T_n \le \frac{1}{2n}[f(a)+f(b)] + \frac{1}{n}\sum_{k=1}^{n-1}[(n-k)f(a)+kf(b)] = \frac{1}{2}[f(a)+f(b)].$$

Thus (3.2) gives the refinement of the Hermite-Hadamard inequality (3.1).

Inequality (3.2), when $n = 2$, appears in [3]. See also [2].

We shall use Lemma 3.1 in the proof of the following.

Theorem 3.2. *Let a and b $(a \ne b)$ be nonnegative numbers. Then*

$$\left(\sqrt{I_a I_b}\prod_{k=1}^{n-1} I_{\alpha_k}\right)^{1/n} \le D_{a,b} \le \left(\prod_{k=1}^{n} I_{\beta_k}\right)^{1/n}, \tag{3.3}$$

$$\frac{n}{\sum_{k=1}^{n} I_{\beta_k}} \le \frac{1}{D_{a,b}} \le \frac{1}{2n}\left(\frac{1}{I_a}+\frac{1}{I_b}+2\sum_{k=1}^{n-1}\frac{1}{I_{\alpha_k}}\right), \tag{3.4}$$

and

$$\left(\sqrt{I_{2a} I_{2b}}\prod_{k=1}^{n-1} I_{2\alpha_k}\right)^{2/n} \le D_{a,b}S_{a,b} \le \left(\prod_{k=1}^{n} I_{2\beta_k}\right)^{2/n}. \tag{3.5}$$

Inequalities (3.3) *and* (3.5) *are reversed if $a \le 0$ and $b \le 0$ $(a \ne b)$.*

Proof. Assume that $a \ge 0$ and $b \ge 0$, $a \ne b$. For the proof of (3.3) we use Lemma 3.1 with $f(t) = \ln I_t$ and property (P9) to obtain

$$\frac{1}{n}\left(\ln\sqrt{I_a I_b} + \sum_{k=1}^{n-1}\ln I_{\alpha_k}\right) \le \frac{1}{b-a}\int_a^b \ln I_t\,dt \le \frac{1}{n}\sum_{k=1}^{n}\ln I_{\beta_k}.$$

Application of (P10) to the middle term gives the assertion. In order to establish inequality (3.4) we use inequality (3.3) to obtain

$$\prod_{k=1}^{n}\left(\frac{1}{I_{\beta_k}}\right)^{1/n} \le \frac{1}{D_{a,b}} \le \left(\frac{1}{I_a I_b}\right)^{1/2n}\prod_{k=1}^{n-1}\left(\frac{1}{I_{\alpha_k}}\right)^{1/n}.$$

Application of the geometric mean-harmonic mean inequality together with the use of the arithmetic mean-geometric mean inequality completes the proof of (3.4). Inequality (3.5) follows from (3.3). Replacing a by $2a$ and b by $2b$ and next using the duplication formula $D_{a,b}S_{a,b} = D^2_{2a,2b}$ (see (P7)) we obtain the desired result. Case $a \le 0$ and $b \le 0$ is treated in an analogous manner, hence it is not included here. ◊

Inequalities (3.3) and (3.4) are valid for the Gini means with the identric means being replaced by J-means of the appropriate order.

Bounds on the product $D_{a,b}S_{a,b}$ are obtained below.

Theorem 3.3. *Let $a, b \in \mathbb{R}$. Assume that $a + b \geq 0$. Then*

$$D_{a,b}S_{a,b} \leq A_q^2 \tag{3.6}$$

if and only if $q = \max(r_1, r_2)$, where $r_1 = \frac{2}{3}(a+b)$ and

$$r_2 = \begin{cases} (\ln 4)l(a,b) & \text{if } a \geq 0 \text{ and } b \geq 0, \\ 0 & \text{if } a < 0 \text{ or } b < 0. \end{cases}$$

If $a + b \leq 0$, then the inequality (3.6) is reversed if and only if $q = \min(r_1, r_2)$, where r_1 is the same as above and

$$r_2 = \begin{cases} -(\ln 4)l(-a,-b) & \text{if } a \leq 0 \text{ and } b \leq 0, \\ 0 & \text{if } a > 0 \text{ or } b > 0. \end{cases}$$

Proof. We shall use again the duplication formula $\sqrt{D_{a,b}S_{a,b}} = D_{2a,2b}$. Assume that $a \geq 0$ and $b \geq 0$. Using Th. A we see that $D_{2a,2b} \leq D_{2q,q}$ if and only if $2(a+b) \leq 3q$ and $l(2a, 2b) \leq l(2q, q)$. Solving these inequalities for q we obtain $q \geq r_1$ and $q \geq r_2$. Assume now that $a \geq 0$, $b \leq 0$ with $a + b \geq 0$. Invoking Th. A again we obtain $q \geq r_1$ and $k(2a, 2b) \leq k(2q, q)$. The last inequality can be written as $(a+b)/(a-b) \leq 1$. Clearly it is satisfied for all values of a and b in the stated domain because $0 \leq a + b \leq a - b$. Case when $a \leq 0$, $b \geq 0$ with $a + b \geq 0$ is treated in the same way. We omit the proof of theorem when $a + b \leq 0$ because it goes along the lines introduced above. ◊

Numerous inequalities for the particular means are contained in those of Ths. 3.2 and 3.3.

Corollary 3.4. *We have*

$$A_{2/3} < \sqrt{I_{5/6}I_{7/6}} < I, \tag{3.7}$$

$$\sqrt{AL} < \sqrt{I_{1/2}I_{3/2}} < I, \tag{3.8}$$

$$\sqrt{AL} < A_{2/3}, \tag{3.9}$$

$$\sqrt{IJ} < A_{\ln 4}. \tag{3.10}$$

Proof. First inequalities in (3.7–3-8) follow from the second inequalities in (3.3) and (3.5) by letting $n = 2$ and putting $(a, b) = \left(\frac{4}{3}, \frac{2}{3}\right)$ and $(a, b) = (1, 0)$, respectively while the second inequalities are an obvious consequence of the logarithmic concavity of the identric mean.

Inequalities (3.9–3.10) follow from (3.6) by letting $(a, b) = (1, 0)$ and $(a, b) = (1, 1)$, respectively. ◊

Combining (3.7) and (3.9) we obtain $\sqrt{AL} < A_{2/3} < I$ (see [15]). The second inequality in the last result is also established in [21].

The following result

$$A^2_{4a/3} \le I_a J_a \le A^2_{(\ln 4)a}, \quad a \ge 0 \tag{3.11}$$

is also worth mentioning. Inequalities (3.11) are reversed if $a \le 0$. Let $a \ge 0$. Then the second inequality in (3.11) follows immediately from (3.6). For the proof of the first inequality in (3.11) we use (3.7) and the duplication formula (P7) to obtain

$$A^2_{4/3}(x, y) = A_{2/3}(x^2, y^2) < I(x^2, y^2) = I(x, y)J(x, y).$$

This completes the proof when $a = 1$. A standard argument is now used to complete the proof when $a \ge 0$. ◊

Our next result reads as follows.

Theorem 3.5. *Let $a \le 0$ and $b \le 0$. Then*

$$D_{a,b} \le L(I_a, I_b). \tag{3.12}$$

If $a \ge 0$ and $b \ge 0$, then

$$D_{a,b} \ge \frac{I_a I_b}{L(I_a, I_b)}. \tag{3.13}$$

Proof. There is nothing to prove when $a = b$. Assume that $a \le 0$, $b \le 0$, $a \ne b$. For the proof of (3.12) we use (P10), Jensen's inequality for integrals, logarithmic convexity of I_t and the formula

$$L(x, y) = \int_0^1 x^t y^{1-t} dt$$

(see [7]) to obtain

$$\ln D_{a,b} = \frac{1}{b-a} \int_a^b \ln I_t dt = \int_0^1 \ln I_{ta+(1-t)b} dt \le$$
$$\le \ln \left(\int_0^1 I_{ta+(1-t)b} dt \right) \le \ln \left(\int_0^1 I_a^t I_b^{1-t} dt \right) = \ln L(I_a, I_b).$$

Let now $a \ge 0$ and $b \ge 0$. For the proof of (3.13) we use (P5) and (3.12) to obtain

$$D_{a,b}(x,y) = \frac{xy}{D_{-a,-b}(x,y)} \geq \frac{xy}{L(I_{-a}, I_{-b})} = \frac{xy}{L\left(\frac{xy}{I_a}, \frac{xy}{I_b}\right)} =$$

$$= \frac{1}{L\left(\frac{1}{I_a}, \frac{1}{I_b}\right)} = \frac{I_a I_b}{L(I_a, I_b)}. \qquad \diamondsuit$$

Corollary 3.6. *The following inequality*

$$\frac{IG}{L(I,G)} < L \tag{3.14}$$

is valid.

Proof. In (3.13) put $(a,b) = (1,0)$. $\diamondsuit$

Inequalities similar to those in (3.12)–(3.13) hold true for the Gini means. We have

$$S_{a,b} \leq L(J_a, J_b) \quad (a \leq 0,\ b \leq 0)$$

and

$$S_{a,b} \geq \frac{J_a J_b}{L(J_a, J_b)} \quad (a \geq 0,\ b \geq 0).$$

Theorem 3.7. *Let $a, b, c \in \mathbb{R}$ $(c \neq 0)$. Then*

$$[D_{a,b}(x^c, y^c)]^{1/c} \geq D_{a,b}(x,y) \tag{3.15}$$

if and only if $(a+b)(c-1) \geq 0$. A similar result is valid for the Gini means.

Proof. We shall use (P4) in the form

$$[D_{a,b}(x^c, y^c)]^{1/c} = D_{ac,bc}(x,y). \tag{3.16}$$

It follows from Th. A that $D_{a,b} \leq D_{ac,bc}$ if and only if $a+b \leq c(a+b)$ and if one of the remaining three inequalities of the above mentioned theorem is valid. Assume that $a+b \geq 0$ and consider the case when $c > 1$. If $a \geq 0$ and $b \geq 0$, then $\min(a,b,ac,bc) \geq 0$ and $l(ac,bc) = cl(a,b) \geq l(a,b)$. Making use of (3.16) we obtain the desired inequality (3.15). Now let $a \geq 0$, $b \leq 0$ with $a+b \geq 0$. Then $\min(a,b,ac,bc) \leq 0 \leq$ $\leq \max(a,b,ac,bc)$ and $k(ac,bc) = k(a,b)$ which completes the proof of (3.15) in the case under discussion. Cases $0 < c < 1$ and $c < 0$ are treated in a similar way, hence they are not discussed here in detail. For the proof of the counterpart of (3.15) for the Gini means one uses the comparison inequality of Th. B. $\diamondsuit$

We close this section with the result which can be regarded as the Chebyshev type inequality for the Stolarsky and Gini means.

Theorem 3.8. *Let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ be positive vectors. Assume that $0 < p_1 < p_2$ and $0 < q_1 < q_2$ or that $0 < p_2 < p_1$ and $0 < q_2 < q_1$. Let $s = (s_1, s_2)$, where $s_1 = p_1q_1$ and $s_2 = p_2q_2$. If $a + b \geq 0$, then*

$$D_{a,b}(p)D_{a,b}(q) \leq D_{a,b}(s). \tag{3.17}$$

If $a + b \leq 0$, then the inequality (3.17) is reversed. A similar result is valid for the Gini means.

Proof. The following function

$$\psi(t) = \ln I_t(s) - \ln[I_t(p)I_t(q)] \quad (t \in \mathbb{R}) \tag{3.18}$$

plays an important role in the proof of (3.17). We shall prove that $\psi(-t) = -\psi(t)$ and also that $\psi(t) \geq 0$ for $t \geq 0$. We have

$$\begin{aligned} &\psi(t) + \psi(-t) = \\ &= \ln I_t(s) + \ln I_{-t}(s) - [\ln I_t(p) + \ln I_{-t}(p)] - [\ln I_t(q) + \ln I_{-t}(q)] = \\ &= 2[\ln G(s) - \ln G(p) - \ln G(q)] = 0. \end{aligned}$$

Here we have used the identity $\ln I_t + \ln I_{-t} = 2\ln G$ which is a special case of (P5) when $a = b = t$. Nonnegativity of the function $\psi(t)$ $(t \geq 0)$ can be established as follows. Let $0 \leq u \leq 1$ and let $v = (u, 1-u)$. The dot product of v and p, denoted by $v \cdot p$, is defined in the usual way $v \cdot p = up_1 + (1-u)p_2$. Using the integral representation for the identric mean of order one

$$\ln I(p) = \int_0^1 \ln(v \cdot p)du$$

we obtain

$$\ln[I(p)I(q)] = \int_0^1 \ln[(v \cdot p)(v \cdot q)]du.$$

Application of the Chebyshev inequality

$$(v \cdot p)(v \cdot q) \leq v \cdot s$$

gives

$$\ln[I(p)I(q)] \leq \int_0^1 \ln(v \cdot s)du = \ln I(s).$$

This implies the inequality $I_t(p)I_t(q) \leq I_t(s)$ $(t \geq 0)$ with the inequality reversed if $t \leq 0$. This completes the proof of (3.17) when $a = b = t$ and shows that $\psi(t) \geq 0$ for $t \geq 0$. Assume now that $a \neq b$. Let $a + b \geq 0$. Using (P10) together with the two properties of the function ψ we obtain

$$\ln[D_{a,b}(p)D_{a,b}(q)] = \frac{1}{b-a}\int_a^b \ln[I_t(p)I_t(q)]dt \le$$
$$\le \frac{1}{b-a}\int_a^b \ln I_t(s)dt = \ln D_{a,b}(s).$$

If $a+b \le 0$, then the last inequality is reversed. Proof of the corresponding inequality for the Gini means goes along the lines introduced above. We omit further details. $\diamond$

4. Ky Fan type inequalities

The goal of this section is to obtain the Ky Fan type inequalities for the means discussed in this paper.

To this end we will assume that $0 < x,\ y \le \frac{1}{2}$ with $x \ne y$. We define $x' = 1-x,\ y' = 1-y$ and write G' for the geometric mean of x' and y', i.e., $G' = G(x', y')$. The same convention will be used for the remaining means which appear in this section.

We need the following.

Lemma 4.1. *Let $a \ne 0$. Then*

$$\left|\frac{x^a - y^a}{x^a + y^a}\right| > \left|\frac{(1-x)^a - (1-y)^a}{(1-x)^a + (1-y)^a}\right|. \tag{4.1}$$

Proof. For the proof of (4.1) we define a function

$$\phi_a(t) = \frac{t^a - 1}{t^a + 1} \quad (t > 0).$$

Clearly ϕ_a is an odd function in a, i.e., $\phi_{-a} = -\phi_a$. In what follows we will assume that $a > 0$. Also, $\phi_a(t) > 0$ for $t > 1$ and $\phi_a(t) < 0$ for $0 < t < 1$. Since both sides of the inequality (4.1) are symmetric, we may assume, without a loss of generality, that $x > y > 0$. Let $z = x/y$ and $w = (1-x)/(1-y)$. It is easy to verify that $z > 1 > w > 0$ and $zw > 1$. In order to prove (4.1) it suffices to show that $|\phi_a(z)| > |\phi_a(w)|$ for $a > 0$. Using the inequalities which connect z and w we obtain

$$z^a - w^a > 0 \quad \text{and} \quad (zw)^a > 1.$$

Hence

$$z^a - w^a + (zw)^a - 1 > z^a - w^a - (zw)^a + 1$$

or what is the same that $(z^a - 1)(1 + w^a) > (z^a + 1)(1 - w^a)$. This implies that $\phi_a(z) > -\phi_a(w) > 0$. The proof is complete. $\diamond$

Proposition 4.2. *Let $a \geq 0$. Then*

$$\frac{G}{G'} \leq \frac{I_a}{I'_a} \leq \frac{A_a}{A'_a} \leq \frac{J_a}{J'_a}. \tag{4.2}$$

Inequalities (4.2) *are reversed if $a \leq 0$ and they become equalities if and only if $a = 0$.*

Proof. There is nothing to prove when $a = 0$. Assume that $a \neq 0$. We need the following series expansions

$$A = G \exp\left[\sum_{k=1}^{\infty} \frac{1}{2k}\left(\frac{x-y}{x+y}\right)^{2k}\right], \tag{4.3}$$

$$I = G \exp\left[\sum_{k=1}^{\infty} \frac{1}{2k+1}\left(\frac{x-y}{x+y}\right)^{2k}\right], \tag{4.4}$$

$$J = G \exp\left[\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x-y}{x+y}\right)^{2k}\right] \tag{4.5}$$

(see [18], [19], [16]). It follows from (4.3)–(4.5) that for any $a \neq 0$

$$A_a = G \exp\left[\frac{1}{a}\sum_{k=1}^{\infty} \frac{1}{2k}\left(\frac{x^a-y^a}{x^a+y^a}\right)^{2k}\right], \tag{4.6}$$

$$I_a = G \exp\left[\frac{1}{a}\sum_{k=1}^{\infty} \frac{1}{2k+1}\left(\frac{x^a-y^a}{x^a+y^a}\right)^{2k}\right], \tag{4.7}$$

$$J_a = G \exp\left[\frac{1}{a}\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x^a-y^a}{x^a+y^a}\right)^{2k}\right]. \tag{4.8}$$

Eliminating G between equations (4.6) and (4.7) and next between (4.6) and (4.8), we obtain

$$I_a = A_a \exp\left[-\frac{1}{a}\sum_{k=1}^{\infty} \frac{1}{2k(2k+1)}\left(\frac{x^a-y^a}{x^a+y^a}\right)^{2k}\right] \tag{4.9}$$

and

$$J_a = A_a \exp\left[\frac{1}{a}\sum_{k=1}^{\infty}\frac{1}{2k(2k-1)}\left(\frac{x^a-y^a}{x^a+y^a}\right)^{2k}\right]. \tag{4.10}$$

Assume that $a > 0$. For the proof of the first inequality in (4.2) we use (4.7) to obtain

$$\frac{I_a}{I'_a} = \frac{G}{G'}\exp\left[\frac{1}{a}\sum_{k=1}^{\infty}\frac{1}{2k+1}\left(u^{2k}-v^{2k}\right)\right], \tag{4.11}$$

where

$$u = \frac{x^a-y^a}{x^a+y^a} \quad \text{and} \quad v = \frac{(1-x)^a-(1-y)^a}{(1-x)^a+(1-y)^a}.$$

Making use of Lemma 4.1 we obtain $u^{2k} - v^{2k} > 0$ for $k = 1, 2, \dots$. This in conjunction with (4.11) gives the desired result. Second and third inequalities in (4.2) can be established in an analogous manner using (4.9) and (4.10), respectively. The case $a < 0$ is treated in the same way. ◊

The main result of this section reads as follows.

Theorem 4.3. *Let $a, b \in \mathbb{R}$. If $a + b \geq 0$, then*

$$\frac{G}{G'} \leq \frac{D_{a,b}}{D'_{a,b}} \leq \frac{S_{a,b}}{S'_{a,b}}. \tag{4.12}$$

Inequalities (4.12) are reversed if $a + b \leq 0$ and they become equalities if and only if $a + b = 0$.

Proof. There is nothing to prove when $a+b = 0$. Assume that $a+b > 0$. For the proof of the first inequality in (4.12) we use (P5) twice with $a = b = t$ to obtain

$$\ln\frac{I_t}{I'_t} + \ln\frac{I_{-t}}{I'_{-t}} = 2\ln\frac{G}{G'}. \tag{4.13}$$

Let us define $h(t) = \ln\frac{G}{G'} - \ln\frac{I_t}{I'_t}$. It follows from (4.13) that $h(t) = -h(-t)$. Also $h(t) \geq 0$ for $t \geq 0$ and $h(t) \leq 0$ for $t \leq 0$. This is an immediate consequence of the first inequality in (4.2). Making use of (P10) we obtain

$$0 \geq \frac{1}{b-a}\int_a^b h(t)dt = \ln\frac{G}{G'} - \ln\frac{D_{a,b}}{D'_{a,b}}.$$

For the proof of the second inequality in (4.12) we define now $h(t)$ as

$$h(t) = \ln \frac{I_t}{I'_t} - \ln \frac{J_t}{J'_t}.$$

It follows that $h(t) = (\ln I_t - \ln J_t) - (\ln I'_t - \ln J'_t)$. Since both terms on the right side are odd functions in t (see proof of Th. C) it follows that function $h(t)$ is also odd as a function of variable t. Using (4.2) we see that $h(t) \le 0$ for $t > 0$ with the inequality reversed if $t > 0$. This in conjunction with (P10) gives

$$0 \ge \frac{1}{b-a} \int_a^b h(t)dt = \ln \frac{D_{a,b}}{D'_{a,b}} - \ln \frac{S_{a,b}}{S'_{a,b}}.$$

This completes the proof in the case when $a+b \ge 0$. Case when $a+b \le 0$ is treated in an analogous manner. ◊

Corollary 4.4. *The following inequalities are valid*

$$\frac{H}{H'} \le \frac{G}{G'} \le \frac{L}{L'} \le \frac{A}{A'}.$$

Proof. To obtain the inequalities in question we use Th. 4.3 twice letting $(a,b) = (-1,0)$ and $(a,b) = (1,0)$. ◊

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