

## INTEGRAL INEQUALITIES FOR MAXIMAL SPACE-LIKE SUBMANIFOLDS IN THE INDEFINITE SPACE FORM

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**Abstract.** In this note, we give two intrinsic integral inequalities for compact maximal space-like submanifolds in the indefinite space form and a sufficient and necessary condition for such submanifolds to be totally geodesic.

### 1. Introduction

Let  $M_p^{n+p}(c)$  be an  $(n+p)$ -dimensional connected semi-Riemannian manifold of constant curvature  $c$  whose index is  $p$ . It is called an indefinite space form of index  $p$  and simply a space form when  $p = 0$ . If  $c > 0$ , we call it a de Sitter space of index  $p$ . Let  $M^n$  be an  $n$ -dimensional Riemannian manifold immersed in  $M_p^{n+p}(c)$ . As the semi-Riemannian metric of  $M_p^{n+p}(c)$  induces the Riemannian metric of  $M^n$ ,  $M^n$  is called a space-like submanifold. A space-like submanifold with vanishing mean curvature is called a maximal space-like submanifold. Kobayashi [5] gave the Weierstrass-Enneper representation formulas for maximal space-like surfaces in 3-dimensional Minkowski space and exhibited various examples. In particular, he determined the maximal space-like surfaces which are rotation surfaces or ruled surfaces. Montiel [6] gave an integral inequality for compact space-like hypersurfaces in a de Sitter space and by use of this integral inequality, he studied the constant mean curvature space-like hypersurfaces. Also, Akutagawa [1] and Ramanathan [8] investigated space-like hypersurfaces in a de Sitter space and proved independently that a complete space-like hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature  $H$  satisfies  $H^2 \leq c$  when  $n = 2$  and  $n^2 H^2 < 4(n-1)c$  when  $n \geq 3$ . Later, Cheng [3] generalized this result to general submanifolds in a de Sitter space.

In this paper, we study compact maximal space-like submanifolds in the indefinite space form with flat normal bundle and obtain two intrinsic integral inequalities for such submanifolds. We also give a sufficient and necessary condition for such submanifolds to be totally geodesic. We will prove the following

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**THEOREM 1.** *Let  $M^n$  be an  $n$ -dimensional compact maximal space-like submanifold in  $M_p^{n+p}(c)$  with flat normal bundle. Then*

$$\int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + \sum R_{mj}^2 - ncR \right\} * 1 \leq 0.$$

**THEOREM 2.** *Let  $M^n$  be an  $n$ -dimensional compact maximal space-like submanifold in  $M_p^{n+p}(c)$  with flat normal bundle. Then*

$$\int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + \frac{1}{n} S^2 + (n-2)cS - n(n-1)c^2 \right\} * 1 \leq 0.$$

**THEOREM 3.** *Let  $M^n$  be an  $n$ -dimensional compact maximal space-like submanifold in  $M_p^{n+p}(c)$  with flat normal bundle. Then  $M^n$  is totally geodesic if and only if*

$$\int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + (n-2)cS - n(n-1)c^2 \right\} * 1 = 0.$$

In the above theorems,  $\sum R_{mijk}^2$  is the square length of the Riemannian curvature tensor,  $\sum R_{mj}^2$  the square length of the Ricci curvature tensor,  $S$  the square length of the second fundamental form,  $R$  the scalar curvature. All these are intrinsic properties of  $M^n$ .

## 2. Preliminaries

Let  $M_p^{n+p}(c)$  be an  $(n+p)$ -dimensional semi-Riemannian manifold of constant curvature  $c$  whose index is  $p$ . Let  $M^n$  be an  $n$ -dimensional Riemannian manifold immersed in  $M_p^{n+p}(c)$ . We choose a local field of semi-Riemannian orthonormal frames  $e_1, \dots, e_{n+p}$  in  $M_p^{n+p}(c)$  such that at each point of  $M^n$ ,  $e_1, \dots, e_n$  span the tangent space of  $M^n$  and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n+p; \quad 1 \leq i, j, k, \dots \leq n; \quad n+1 \leq \alpha, \beta, \gamma \leq n+p.$$

Let  $\omega_1, \dots, \omega_{n+p}$  be its dual frame field so that the semi-Riemannian metric of  $M_p^{n+p}(c)$  is given by  $d\bar{s}^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \epsilon_A \omega_A^2$ , where  $\epsilon_i = 1$  and  $\epsilon_\alpha = -1$ . Then the structure equations of  $M_p^{n+p}(c)$  are given by

$$\begin{aligned} d\omega_A &= \sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} &= \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \\ K_{ABCD} &= c \epsilon_A \epsilon_B (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}). \end{aligned}$$

Restricting these forms to  $M^n$ , we obtain  $\omega_\alpha = 0$ ,  $n+1 \leq \alpha \leq n+p$ , and the Riemannian metric of  $M^n$  is written as  $ds^2 = \sum_i \omega_i^2$ . From Cartan's lemma we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

From these formulas, we obtain the structure equations of  $M^n$ :

$$\begin{aligned} d\omega_i &= \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} K_{ijkl} \omega_k \wedge \omega_l, \\ R_{ijkl} &= c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_{\alpha} (h_{ik}^{\alpha} h_{jl}^{\alpha} - h_{il}^{\alpha} h_{jk}^{\alpha}), \end{aligned} \quad (1)$$

where  $R_{ijkl}$  are the components of the curvature tensor of  $M^n$ .

For details on indefinite Riemannian manifolds we refer to O'Neill [7].

We call

$$h = \sum_{\alpha} h_{\alpha} e_{\alpha} = \sum_{i,j,\alpha} h_{ij}^{\alpha} \omega_i \otimes \omega_j \otimes e_{\alpha}$$

the second fundamental form of  $M^n$  and the square length of the second fundamental form is defined by

$$S = \sum_{\alpha} \text{tr}(h_{\alpha})^2 = \sum_{\alpha,i,j} (h_{ij}^{\alpha})^2.$$

The mean curvature vector  $N$  of  $M^n$  is defined by

$$N = \frac{1}{n} \sum_{\alpha} \text{tr}(h_{\alpha}) e_{\alpha} = \frac{1}{n} \sum_{\alpha} (\sum_i h_{ii}^{\alpha}) e_{\alpha},$$

and it is well known that  $N$  is independent of the choice of unit normal vectors  $e_{n+1}, \dots, e_{n+p}$  to  $M^n$ . The length of the mean curvature vector is called the mean curvature of  $M^n$ , denoted by  $H$ .

If  $M^n$  is maximal, then

$$\sum_i h_{ii}^{\alpha} = 0, \quad \alpha = n+1, \dots, n+p. \quad (2)$$

Define the first and the second covariant derivatives of  $\{h_{ij}^{\alpha}\}$ , say  $\{h_{ij,k}^{\alpha}\}$  and  $\{h_{ij,kl}^{\alpha}\}$ , by

$$\begin{aligned} \sum_k h_{ij,k}^{\alpha} \omega_k &= dh_{ij}^{\alpha} + \sum_k h_{kj}^{\alpha} \omega_{ki} + \sum_k h_{ik}^{\alpha} \omega_{kj} + \sum_{\beta} h_{ij}^{\beta} \omega_{\beta\alpha}, \\ \sum_l h_{ij,kl}^{\alpha} \omega_l &= dh_{ij,k}^{\alpha} + \sum_m h_{mj,k}^{\alpha} \omega_{mi} + \sum_m h_{im,k}^{\alpha} \omega_{mj} + \sum_m h_{ij,m}^{\alpha} \omega_{mk} + \sum_{\beta} h_{ij,k}^{\beta} \omega_{\beta\alpha}. \end{aligned}$$

Then we have

$$h_{ijk}^{\alpha} = h_{ikj}^{\alpha}, \quad (3)$$

$$h_{ij,kl}^{\alpha} - h_{ij,lk}^{\alpha} = \sum_m h_{im}^{\alpha} R_{mjkl} + \sum_m h_{jm}^{\alpha} R_{mikl} + \sum_{\beta} h_{ij}^{\beta} R_{\alpha\beta kl}, \quad (4)$$

where  $R_{\alpha\beta kl}$  are the components of the normal curvature tensor of  $M^n$ , that is

$$R_{\alpha\beta kl} = \sum_i (h_{ik}^{\alpha} h_{il}^{\beta} - h_{il}^{\alpha} h_{ik}^{\beta}).$$

If  $R_{\alpha\beta kl} = 0$  at a point  $x$  of  $M^n$  we say that the normal connection of  $M^n$  is flat at  $x$ , and it is well known [2] that  $R_{\alpha\beta kl} = 0$  at  $x$  if and only if  $h_\alpha$  are simultaneously diagonalizable at  $x$ .

The Laplacian  $\Delta h_{ij}^\alpha$  of the fundamental form  $h_{ij}^\alpha$  is defined to be  $\sum_k h_{ijkk}^\alpha$ , and hence, if  $M^n$  has flat normal bundle, from (3) and (4) we have

$$\begin{aligned}\Delta h_{ij}^\alpha &= \sum_k (h_{ijkk}^\alpha - h_{ikjk}^\alpha) + \sum_k (h_{ikjk}^\alpha - h_{ikkj}^\alpha) + \sum_k (h_{ikkj}^\alpha - h_{kkij}^\alpha) \\ &= \sum_{m,k} h_{im}^\alpha R_{mkjk} + \sum_{m,k} h_{mk}^\alpha R_{mijk}\end{aligned}\quad (5)$$

### 3. Proofs of the Theorems

*Proof of Theorem 1.* From (1), (2) and (5), we have

$$\begin{aligned}\sum h_{ij}^\alpha \Delta h_{ij}^\alpha &= \sum h_{ij}^\alpha h_{mk}^\alpha R_{mijk} + \sum h_{ij}^\alpha h_{im}^\alpha R_{mkjk} \\ &= \frac{1}{2} \sum (h_{ij}^\alpha h_{mk}^\alpha - h_{mj}^\alpha h_{ik}^\alpha) R_{mijk} + \sum (h_{ij}^\alpha h_{im}^\alpha - h_{ii}^\alpha h_{jm}^\alpha) R_{mj} \\ &= \frac{1}{2} \sum [c(\delta_{ij}\delta_{mk} - \delta_{mj}\delta_{ik}) - R_{imjk}] R_{mijk} \\ &\quad + \sum [c(\delta_{ij}\delta_{im} - \delta_{ii}\delta_{jm}) + R_{ijim}] R_{mj} \\ &= \frac{1}{2} \sum R_{mijk}^2 + \sum R_{mj}^2 - ncR.\end{aligned}$$

Since  $\int_{M^n} \{\sum h_{ij}^\alpha \Delta h_{ij}^\alpha\} * 1 \leq 0$ , we have

$$\int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + \sum R_{mj}^2 - ncR \right\} * 1 \leq 0. \quad (6)$$

Theorem 1 is proved. ■

In order to prove Theorem 2, we need the following algebraic lemma

LEMMA. *Let  $a_1, \dots, a_n$  be real numbers. Then*

$$\sum (a_i)^2 \geq \frac{1}{n} (\sum a_i)^2, \quad (7)$$

and the equality holds if and only if  $a_1 = \dots = a_n$ .

In fact,

$$n \sum (a_i)^2 - (\sum a_i)^2 = \sum (a_i - a_j)^2, \quad (8)$$

and the Lemma follows immediately from (8).

*Proof of Theorem 2.* From (1), we have

$$R_{mj} = (n-1)c\delta_{mj} + \sum h_{mi}^\alpha h_{ij}^\alpha, \quad \text{and} \quad R = n(n-1)c + S.$$

Since  $M^n$  has flat normal bundle, we can diagonalize the second fundamental form simultaneously so that  $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$ , and then from (7), we have

$$\begin{aligned} R_{mj} &= (n-1)c\delta_{mj} + \sum(\lambda_j^\alpha)^2 \delta_{mj}, \\ \sum R_{mj}^2 &= n(n-1)^2 c^2 + 2(n-1)cS + \sum(\lambda_j^\alpha)^4 \\ &\geq n(n-1)^2 c^2 + 2(n-1)cS + \frac{1}{n} \{\sum(\lambda_j^\alpha)^2\}^2 \\ &= n(n-1)^2 c^2 + 2(n-1)cS + \frac{1}{n} S^2. \end{aligned}$$

Therefore, from (6) we have

$$\int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + \frac{1}{n} S^2 + (n-2)cS - n(n-1)c^2 \right\} * 1 \leq 0. \quad (9)$$

Theorem 2 is proved. ■

*Proof of Theorem 3.* From (9) we have

$$\int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + (n-2)cS - n(n-1)c^2 \right\} * 1 \leq 0. \quad (10)$$

If  $M^n$  is totally geodesic, i.e.,  $S = 0$ ,  $h_{ij}^\alpha = 0$ , then from (1) we have

$$R_{mijk} = c(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}), \quad \sum R_{mijk}^2 = 2n(n-1)c^2,$$

and in this case, (10) becomes an equality.

Conversely, if (10) becomes an equality, then  $S = 0$ , and  $M^n$  is totally geodesic. ■

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