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COVERING OF CURVES, GONALITY, AND SCROLLAR INVARIANTS

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Abstract. Let $f\colon X \to Y$ be a degree k covering of smooth and connected projective curves with $p_a(Y) > 0$. Here we continue the study of the Brill-Noether theory of divisors on X.

1. Introduction

Let X (resp. Y) be a smooth and connected curve of genus g (resp. genus q) and $f\colon X \to Y$ a degree k covering, $k \geqslant 2$. Thus $g \geqslant kq - k + 1$ (Riemann-Hurwitz). Let $u\colon X \to \mathbf{P}^1$ be a degree z morphism computing the gonality $\operatorname{gon}(X)$ of X. We always have $z \leqslant k \cdot \operatorname{gon}(Y)$ and if $z = k \cdot \operatorname{gon}(Y)$, then at least one degree z pencil $X \to \mathbf{P}^1$ factors through f. By Brill-Noether theory we have $\operatorname{gon}(X) \leqslant \lfloor (g+3)/2 \rfloor$ and $\operatorname{gon}(Y) \leqslant \lfloor (q+3)/2 \rfloor$. Hence $z \leqslant \min\{\lfloor (g+3)/2 \rfloor, k \cdot \lfloor (q+3)/2 \rfloor\}$. If $z \leqslant (g - kq)/(k-1)$, then u factors through f by Castelnuovo-Severi inequality [5]. In the first part we will consider several examples in which u does not factor through f and study their scrollar invariants in the sense of [3]. To state our first result we need the following notation/observation.

Remark 1. Let $f\colon X \to Y$ be a finite morphism between smooth and connected projective curves and $D = \sum n_i P_i$ any divisor on X. Set $f_!(D) := \sum n_i f(P_i)$. A key property of rational equivalence says that if D and D' are linearly equivalent divisors on X, then $f_!(D)$ and $f_!(D')$ are linearly equivalent divisors on Y; here the smoothness of Y is essential, because it implies that rational equivalence and linear equivalence are the same on Y. Hence for any $d \in \mathbf{Z}$ the map $f_!$ induces a map $f_! : \operatorname{Pic}^d(X) \to \operatorname{Pic}^d(Y)$ such that $h^0(Y, f_!(L)) \geqslant h^0(X, L)$ for all $L \in \operatorname{Pic}^d(X)$. Furthermore, if L is base point free, then $f_!(L)$ is base point free.

A modification of the proof of [2], Th. 1, (see section 2) will give the following result.

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Theorem 1. *Fix integers* d, q, k, g *such that* $q > 0$, $k \geqslant 2$, $g \geqslant kq - k + 1$ *and* $kd - d - k + kq + 1 - (\lfloor d/2 \rfloor + 1 - q) \cdot (\lfloor k/2 \rfloor + 1) \leqslant g \leqslant kd - d - k + kq + 1$. *Set* $a := kd - d - k + kq + 1 - g$ *and let* $\delta(a)$ *be the maximal integer* t *such that* $(k-1-t)(d+q-1) \geqslant a$. *Let* Y *be a smooth and connected curve of genus* q *such that there is a base point free* $M \in \mathrm{Pic}^d(Y)$. *Then there exist a smooth and connected genus* g *curve* X, *a degree* k *covering* $f\colon X \to Y$ *and a base point free* $L \in \mathrm{Pic}^d(X)$ *such that* $f_!(L) \cong M$ *and there is no base point free* $R \in \mathrm{Pic}(Y)$ *such that* $f^*(R) = L$ *and* $h^0(Y, R) = h^0(X, L)$. *Furthermore,* $e_{d-1}(L) \geqslant \max\{e_{d-1}(M), \delta(a) + 1\}$. *If* $(d + k - 1)(t + 1 - k) > (t + 1)q$, *then* $e_{d-1}(L) \leqslant k - 2$.

Obviously, in the statement of Theorem 1 we have $\delta(a) \leqslant k - 1$. Notice that $\delta(a) = k - 1$ if $a = 0$ and that $\delta(a) = k - 2$ if $a \leqslant d + q - 1$.

Remark 2. Take the notation of the statement of Theorem 1. Obviously, we have $\mathrm{gon}(X) \leqslant d$ and $\mathrm{gon}(Y) \leqslant d$. Hence, if $\mathrm{gon}(Y) > d/k$, then the gonality of X is computed by a pencil not coming from Y, while if $\mathrm{gon}(Y) = d/k$, then there is at least one pencil on X computing $\mathrm{gon}(X)$, but not coming from Y. If either $1 \leqslant q \leqslant 2$ or $q \geqslant 3$ and Y has general moduli, then $\mathrm{gon}(Y) = \lfloor (q + 2)/2 \rfloor$. Hence if $\lfloor (g+3)/2 \rfloor < k \cdot \lfloor (q+3)/2 \rfloor\}$ (resp. $\lfloor (g+3)/2 \rfloor = k \cdot \lfloor (q+3)/2 \rfloor\}$, then we may apply the first part of this remark. Hence either $1 \leqslant q \leqslant 2$ or $q \geqslant 3$ and Y has general moduli, then there is a small, but non empty, interval of integers g for which both Theorem 1 and the first part of this remark may be applied.

In section 3 we will continue [1] and study the rank $k - 1$ vector bundle $E_f := f_*(\mathcal{O}_X)/\mathcal{O}_Y$.

We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$.

2. Proof of Theorem 1

Remark 3. Let Y be a smooth and connected projective curve. Set $q := p_a(Y)$ and $S := Y \times \mathbf{P}^1$. Hence $h^1(S, \mathcal{O}_S) = q$. Let $\pi_1\colon S \to Y$ and $\pi_2\colon S \to \mathbf{P}^1$ denote the two projections. For any $R \in \mathrm{Pic}(S)$ there are unique $M \in \mathrm{Pic}(Y)$ and $k \in \mathbf{Z}$ such that $R \cong \pi_1^*(M) \otimes \pi_2^*(\mathcal{O}_{\mathbf{P}^1}(k))$. Set $\mathcal{O}_S(M, k) := \pi_1^*(M) \otimes \pi_2^*(\mathcal{O}_{\mathbf{P}^1}(k))$. If $k < 0$, then $h^0(S, \mathcal{O}_S(M, k)) = 0$ and $h^1(S, \mathcal{O}_S(M, k)) = (-k - 1) \cdot h^0(Y, M)$ (Künneth formula). If $k \geqslant 0$, then $h^0(S, \mathcal{O}_S(M, k)) = (k + 1) \cdot h^0(Y, M)$ and $h^1(S, \mathcal{O}_S(M, k)) = (k + 1) \cdot h^1(Y, M)$ (Künneth formula). Furthermore, if M is spanned and $k \geqslant 0$, then $\mathcal{O}_S(M, k)$ is spanned, while if M is (birationally) very ample and $k > 0$, then $\mathcal{O}_S(M, k)$ is (birationally) very ample. Fix integers $k \geqslant 2$ and $d > 0$ and $M \in \mathrm{Pic}^d(Y)$ such that $|M|$ has no base point. Let $C \subset S$ be an integral curve in the linear system $|\mathcal{O}_S(M, k)|$ and $\nu\colon X \to C$ the normalization map. Set $A(C) := \mathrm{Sing}(C)$ and let $B(C) \subset S$ the conductor of ν. We recall that $B(C)_{red} = A(C)$ and that $B(C) = A(C)$ if each singular point of C is either an ordinary double point or an ordinary cusp. For any $A \in \mathrm{Pic}(Y)$ and any integer x set $\mathcal{O}_C(A, x) := \mathcal{O}_S(A, x)|C$ and $\mathcal{O}_X(A, x) := \nu^*(\mathcal{O}_C(A, x))$. We will also write $\mathcal{O}_C(0, x)$ (resp. $\mathcal{O}_X(0, x)$, resp. $\mathcal{O}_S(0, x)$) instead of $\mathcal{O}_C(\mathcal{O}_Y, x)$ (resp. $\mathcal{O}_X(\mathcal{O}_Y, x)$, resp. $\mathcal{O}_S(\mathcal{O}_Y, x)$). Notice that $\mathcal{O}_C(A, x)$ is a line bundle of degree $k \cdot \deg(A) + x \cdot$

$\deg(M)$. The morphism $\pi_1 \circ \nu \colon X \to Y$ is a degree k covering between smooth and projective curves. Since $\omega_S \cong \mathcal{O}_S(\omega_Y, -2)$, then $\omega_C \cong \mathcal{O}_C(M \otimes \omega_Y, k-2)$ (adjunction formula). Thus $p_a(C) = kd - d - k + kq + 1$.

Remark 4. Use the set-up of Remark 3. Notice that $h^0(S, \omega_S(\mathcal{O}_Y, -t)) = 0$ for all $t \geqslant 0$ and $h^1(S, \omega_S(\mathcal{O}_Y, -t)) = h^1(S, \mathcal{O}_S(\mathcal{O}_Y, t) = q(t+1)$ for all $t \geqslant -1$. Consider the exact sequence

$$0 \to \mathcal{O}_S(-M, t-k) \to \mathcal{O}_S(0,t) \to \mathcal{O}_C(0,t) \to 0 \tag{1}$$

Since $\mathcal{O}_S(M, k-t)$ is ample for all $t < k$, we have $h^1(S, \mathcal{O}_S(-M, -k+t) = 0$ for all $t < k$ (Kodaira's vanishing). Obviously, $h^0(S, \mathcal{O}_S(-M, -k+t) = 0$ for all $t < k$. Hence from (1) we get $h^0(C, \mathcal{O}_C(0,t) = t+1$ for $-1 \leqslant t < k$. Now assume $t \geqslant k$. Since $h^1(S, \mathcal{O}_S(0,t)) = (t+1)q$ and $h^1(S, \mathcal{O}_S(-M, t-k) = h^1(S, \mathcal{O}_S(M + \omega_Y, k-t-2)) = (d+q-1)(t+1-k)$ (Riemann-Roch and Serre duality), the long cohomology exact sequence of the exact sequence (1) gives $h^0(C, \mathcal{O}_C(0,t)) \geqslant t+1+(d+q-1)(t+1-k)-(t+1)q$. Hence if $(d+k-1)(t+1-k) > (t+1)q$, then $e_{d-1}(\mathcal{O}_C(0,1)) = k$ and hence $e_{d-1}(L) \leqslant k-2$, where $L := \nu^*(\mathcal{O}_C(1,0)) \in \mathrm{Pic}^d(X)$.

Remark 5. Take the set-up of Remarks 1 and 3. Notice that $f_!(L) \cong M$. Let $e_i(M)$, $1 \leqslant i \leqslant d-1$ (resp. $e_i(L)$, $1 \leqslant i \leqslant d-1$) denote the scrollar invariants of M (resp. L) [3]. Thus $e_1(L) \geqslant \cdots \geqslant e_{d-1}(L) \geqslant 0$, $e_1(M) \geqslant \cdots \geqslant e_{d-1}(M) \geqslant 0$, $e_1(L) + \cdots + e_{d-1}(L) = g-d+1$, and $e_1(M) + \cdots + e_{d-1}(M) = q+d-1$. We only need that the integer $m := e_{d-1} + 2$ is characterized by the property $h^0(X, R^{\otimes(m-1)}) = m$ and $h^0(X, R^{\otimes m}) \geqslant m+2$. Fix any integer $t \geqslant 1$ such that $h^0(Y, M^{\otimes t}) = t+1$. Hence $h^0(X, L^{\otimes t}) \leqslant h^0(Y, f_!(L^{\otimes t})) = h^0(Y, M^{\otimes t}) = t+1$. Hence $h^0(X, L^{\otimes t}) = t+1$. Thus $e_{d-1}(L) \geqslant e_{d-1}(M)$.

Remark 6. Take the set-up of Remarks 3 and 5, but assume that C is nodal (and hence $B(C) = A(C)$) and that its singular points are general in S. Set $A := \mathrm{Sing}(C)$ and $a := \sharp(A)$. Thus $a = p_a(C) - p_a(X) = kd - d - k + kq + 1 - g$. Let $\delta(a)$ be the maximal integer t such that $h^0(S, \mathcal{O}_S(M + \omega_Y, k-2-t)) \geqslant a$, i.e. $(k-1-t)(d+q-1) \geqslant a$. Obviously, $\delta(a)$ only depends from Y, M, k, a. By the generality of A we have $h^0(S, \mathcal{I}_A(M + \omega_Y, k-2-t)) = h^0(S, \mathcal{O}_S(M + \omega_Y, k-2-t)) - a$ for all $t \leqslant \delta(a)$. Thus by adjunction theory and Riemann-Roch we have $h^0(X, L^{\otimes t}) = h^0(C, \mathcal{O}_C(0,t))$ for all $t \leqslant \delta(a)$. By Remark 4 we get $e_{d-1}(L) \geqslant \min\{\delta(a) - 1, k-2\}$.

Proof of Theorem 1. Set $a := kd - d - k + 1 - g$. Let $A \subset S := Y \times \mathbf{P}^1$ be a general subset with $\sharp(A) = a$. The proof of [2], Th. 1, gives the existence of an integral nodal curve $C \in |\mathcal{O}_S(M,k)|$ such that $A = \mathrm{Sing}(C)$. Let $\nu \colon X \to C$ be the normalization map. Apply Remarks 3, 5 and 6. ∎

3. The vector bundle $E_f := f_*(\mathcal{O}_X)/\mathcal{O}_Y$

In this section we fix the following set-up. Fix positive integers k, d. Let Y be a smooth and connected projective curve, C an integral projective curve and

$u\colon C \to Y$ a degree k morphism. Let $\nu : X \to C$ be the normalization map. Set $f\colon u \circ \nu$, $q\colon = p_a(Y)$, $\gamma := p_a(C)$, and $g := p_a(X)$. Since Y is smooth, C and X are locally Cohen-Macaulay and u, f are finite, u and f are flat ([4], Prop. III.9.7). Furthermore, every torsion free finite rank sheaf on a one-dimensional regular local ring is free. Thus $u_*(\mathcal{O}_C)$ and $f_*(\mathcal{O}_X)$ are locally free. Since $\mathrm{char}(\mathbf{K}) = 0$, the trace map shows that $\mathcal{O}_Y$ is in a natural way a direct factor of $u_*(\mathcal{O}_C)$ and $f_*(\mathcal{O}_X)$. Hence there are rank k locally free sheaves E_u and E_f on Y such that $u_*(\mathcal{O}_C) \cong \mathcal{O}_Y \oplus E_u$ and $f_*(\mathcal{O}_X) \cong \mathcal{O}_Y \oplus E_f$. Following [1] we will say that E_u (resp. E_f) is the bundle associated to u (resp. f). Since X and C are integral, we have $h^0(X, \mathcal{O}_X) = h^0(C, \mathcal{O}_C) = 1$. Since f and u are finite, we have $R^1 f_*(\mathcal{O}_X) = R^1 u_*(\mathcal{O}_C) = 0$. Thus the Leray spectral sequences of f and u give $\chi(E_u) = \chi(u_*(\mathcal{O}_C)) - \chi(\mathcal{O}_Y) = q - \gamma$ and $\chi(E_f) = \chi(f_*(\mathcal{O}_X)) - \chi(\mathcal{O}_Y) = q - g$. Thus by Riemann-Roch we have $\deg(E_u) = \chi(E_u) + (k-1)(q-1) = kq - k - \gamma + 1$ and $\deg(E_f) = \chi(E_f) + (k-1)(q-1) = kq - k - g + 1$.

Castelnuovo-Severi inequality gives a strong restriction on the cohomological properties of the associated bundle E_f when f does not factor non-trivially through another smooth curve, i.e. when there are no (Y', f_1, f_2) such that Y' is a smooth curve, $f_1\colon X \to Y'$, $f_2\colon Y' \to Y$, $f = f_2 \circ f_1$ and $1 < \deg(f_1) < k$. Notice that this is always the case if k is prime.

THEOREM 2. *Let $f\colon X \to Y$ be a degree $k \geqslant 2$ covering between smooth projective curves which does not factor non-trivially through another smooth curve. Set $g := p_a(X)$ and $q := p_a(Y)$. Then there exists no effective divisor D on Y such that* $\deg(D) \leqslant (g - kq)/k(k-1)$, $h^0(Y, \mathcal{O}_Y(D)) = 1$, *and* $h^0(Y, E_f(D)) > 0$.

Proof. Set $L := f^*(\mathcal{O}_Y(D))$. By the projection formula we have $h^0(X, L) = h^0(Y, \mathcal{O}_Y(D)) + h^0(Y, E(D)) \geqslant 2$. Let B the base point of $|L|$. Hence $\deg(L(-B)) \leqslant \deg(L) < (g - kq)/(k-1)$. Hence the morphism induced by $|L(-B)|$ factors through f. Hence $L(-B) \cong f^*(\mathcal{O}_Y(D'))$ for some $D' \subseteq D$ such that the linear system $|D'|$ induces a non-constant morphism. Since $h^0(Y, \mathcal{O}_Y(D)) = 1$, this is absurd. ■

Motivated by Theorem 2 we now introduce the following invariants of a covering $f\colon X \to Y$ of smooth curve. Fix an integer $z \geqslant 1$. Let $\epsilon(f, z)$ denote the minimal integer $t \geqslant 0$ such that $h^0(Y, E_f(D)) \geqslant z$ for some effective degree t divisor D on Y. Set $\epsilon(f) := \epsilon(f, 1)$. Let $\eta(f, z)$ denote the minimal integer t such that that $h^0(Y, E_f \otimes R) \geqslant z$ for some $R \in \mathrm{Pic}^t(Y)$. Set $\eta(f) := \eta(f, 1)$. Hence $\eta(f, z) \leqslant \epsilon(f, z)$. The connectedness of X is equivalent to the inequality $\epsilon(f) > 0$. In many cases of coverings consideeed in [1] it is quite easy to compute these invariants and the divisors D (resp. line bundles R) which are "extremal", i.e. which compute $\epsilon(f, z)$ (resp. $\eta(f, z)$). For instance, in the case considered in [1], Th. 1.4, we have $\eta(f) = \eta(f, k-1) = \epsilon(f) = \epsilon(f, k-1) = b(\gamma - k + 1)/(k-1)$.

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