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SOME PROPERTIES OF NOOR INTEGRAL OPERATOR OF $(n+p-1)$-th ORDER

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Abstract. Let $A(p)$ be the class of functions $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ $(p \in N = \{1, 2, \dots\})$ which are analytic in the unit disc $U = \{z : |z| < 1\}$. The object of the present paper is to give some properties of Noor integral operator $I_{n+p-1}f(z)$ of $(n+p-1)$-th order, where $I_{n+p-1}f(z) = \left[\frac{z^p}{(1-z)^{n+p}}\right]^{(-1)} * f(z)$ $(n > -p,\ f(z) \in A(p))$ and $*$ denotes convolution (Hadamard product).

1. Introduction

Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in N = \{1, 2, \dots\}) \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. For functions $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k, \tag{1.2}$$

we define the Hadamard product (convolution) $f_1 * f_2(z)$ of functions $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k. \tag{1.3}$$

The integral operator $I_{n+p-1} : A(p) \to A(p)$ is defined as follows, see [2].

For any integer n greater than $-p$, let $f_{n+p-1}(z) = \frac{z^p}{(1-z)^{n+p}}$ and let $f^{(-1)}_{n+p-1}(z)$ be defined such that

$$f_{n+p-1}(z) * f^{(-1)}_{n+p-1}(z) = \frac{z^p}{(1-z)^{1+p}}. \tag{1.4}$$

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Then

$$I_{n+p-1}f(z) = f_{n+p-1}^{(-1)}(z) * f(z) = \left[\frac{z^p}{(1-z)^{n+p}}\right]^{(-1)} * f(z). \tag{1.5}$$

From (1.4) and (1.5) and a well known identity for the Ruscheweyh derivative [1, 10], it follows that

$$z(I_{n+p}f(z))' = (n+p)I_{n+p-1}f(z) - nI_{n+p}f(z). \tag{1.6}$$

For $p = 1$, the identity (1.6) is given by Noor and Noor [8]. If $f(z)$ is given by (1.1), then from (1.4) and (1.5), we deduce that

$$I_{n+p-1}f(z) = [z^p\ {}_2F_1(1, 1+p, n+p; z)] * f(z) \quad (n > -p), \tag{1.7}$$

where ${}_2F_1$ is the hypergeometric function. We also note that $I_{p-1}f(z) = \frac{zf'(z)}{p}$ and $I_pf(z) = f(z)$. Moreover, the operator $I_{n+p-1}f(z)$ defined by (1.5) is called the Noor integral operator of $(n + p - 1)$-th order of $f(z)$ [2]. For $p = 1$, the operator I_nf was introduced by Noor [5] and Noor and Noor [8]. Several classes of analytic functions, defined by using the operator $I_{n+p-1}f$, have been studied by many authors [6, 7, 9].

We define a function $G_{n,p}(\alpha, \beta; z)$ by

$$G_{n,p}(\alpha, \beta; z) = \alpha I_{n+p}f(z) + \beta I_{n+p-1}f(z), \tag{1.8}$$

for $f(z) \in A(p), n > -p$, $p \in N$ and α and β are complex numbers. For $\alpha = 1 - \beta$ $(\beta \in C)$ we define a function $G_{n,p}(\beta; z)$ by

$$G_{n,p}(\beta; z) = (1-\beta)I_{n+p}f(z) + \beta I_{n+p-1}f(z), \tag{1.9}$$

for $f(z) \in A(p)$, $n > -p$, $p \in N$ and $\beta \in C$. Also for $n = 0$, we obtain

$$G_{0,p}(\alpha, \beta; z) = \alpha f(z) + \beta\frac{zf'(z)}{p}, \quad f(z) \in A(p), \;\; p \in N \text{ and } \beta \in C. \tag{1.10}$$

2. Some properties of $G_{n,p}(\alpha, \beta; z)$

In order to prove our main results, we recall here the following lemma.

LEMMA 1. [3,4] *Let $\varphi(u, v)$ be a complex-valued function, $\varphi : D \to C$, $D \subset C \times C$ (C is the complex plane) and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that $\varphi(u, v)$ satisfies the following conditions:*

(i) $\varphi(u, v)$ is continuous in D;

(ii) $(1, 0) \in D$ and $\Re\{\varphi(1, 0)\} > 0$;

(iii) $\Re\{\varphi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.

Let $q(z) = 1 + q_1z + q_2z^2 + \cdots$ be regular in the unit disc U such that $(q(z), zq'(z)) \in D$ for all $z \in U$. If $\Re\{\varphi(q(z), zq'(z))\} > 0$ $(z \in U)$, then $\Re\{q(z)\} > 0 (z \in U)$.

Applying the above lemma, we derive the following

THEOREM 1. *Let a function* $G_{n,p}\alpha,\beta;z)$ *be defined by* (1.8) *for* $\alpha\in C$, $\beta\in C$ $(\Re(\beta)\geq 0)$, $\alpha+\beta\in R$, $n>-p$, $p\in N$ *and* $f(z)\in A(p)$. *If*

$$\Re\left\{\frac{G_{n,p}(\alpha,\beta;z)}{z^p}\right\}>\gamma\quad(z\in U)\tag{2.1}$$

for some γ $(\gamma<\alpha+\beta)$, *then*

$$\Re\left\{\frac{I_{n+p}f(z)}{z^p}\right\}>\frac{2(n+p)\gamma+\Re(\beta)}{2(n+p)(\alpha+\beta)+\Re(\beta)}\quad(z\in U).\tag{2.2}$$

Proof. Define the function $q(z)$ by

$$\frac{I_{n+p}f(z)}{z^p}=\delta+(1-\delta)q(z),\tag{2.3}$$

with

$$\delta=\frac{2(n+p)\gamma+\Re(\beta)}{2(n+p)(\alpha+\beta)+\Re(\beta)}<1.\tag{2.4}$$

Then $q(z)=1+q_1z+q_2z^2+\dots$ is regular in U, and

$$\begin{aligned}\frac{G_{n,p}(\alpha,\beta;z)}{z^p}&=\alpha\frac{I_{n+p}f(z)}{z^p}+\beta\frac{I_{n+p-1}f(z)}{z^p}\\&=(\alpha+\beta)\delta+(\alpha+\beta)(1-\delta)q(z)+\beta\frac{(1-\delta)}{(n+p)}zq'(z).\end{aligned}\tag{2.5}$$

Therefore, we have

$$\begin{aligned}&\Re\left\{\frac{G_{n,p}(\alpha,\beta;z)}{z^p}-\gamma\right\}\\&\quad=\Re\left\{(\alpha+\beta)\delta-\gamma+(\alpha+\beta)(1-\delta)q(z)+\beta\frac{(1-\delta)}{(n+p)}zq'(z)\right\}.\end{aligned}\tag{2.6}$$

If we define the function $\varphi(u,v)$ by

$$\varphi(u,v)=(\alpha+\beta)\delta-\gamma+(\alpha+\beta)(1-\delta)u+\beta\frac{(1-\delta)}{(n+p)}v,\tag{2.7}$$

with $u=u_1+iu_2$ and $v=v_1+iv_2$, then

(i) $\varphi(u,v)$ is continuous in $D=C\times C=C^2$;

(ii) $(1,0)\in D$ and $\Re\{\varphi(1,0)\}=(\alpha+\beta)-\gamma>0$;

(iii) for all $(iu_2,v_1)\in D$ such that $v_1\leq-\frac{1}{2}(1+u_2^2)$,

$$\begin{aligned}\Re\{\varphi(iu_2,v_1)\}&\leq(\alpha+\beta)\delta-\gamma-\frac{(1-\delta)}{(n+p)}\Re(\beta)-\frac{(1-\delta)}{2(n+p)}\Re(\beta)u_2^2\\&=-\frac{(1-\delta)}{2(n+p)}\Re(\beta)u_2^2\leq 0.\end{aligned}$$

Therefore, the function $\varphi(u,v)$ satisfies the conditions in Lemma 1. This implies that $\Re\{q(z)\} > 0$ $(z \in U)$, that is,

$$\Re\left\{\frac{I_{n+p}f(z)}{z^p}\right\} > \delta = \frac{2(n+p)\gamma + \Re(\beta)}{2(n+p)(\alpha+\beta)+\Re(\beta)} \quad (z \in U). \tag{2.8}$$

This completes the proof of Theorem 1. ■

Putting $\alpha = 1-\beta$ in Theorem 1, we obtain

Corollary 1. *Let a function $G_{n,p}(\beta; z)$ be defined by* (1.9) *for $\beta \in C$ ($\Re(\beta) \geq 0$), $n > -p$, $p \in N$ and $f(z) \in A(p)$. If*

$$\Re\left\{\frac{G_{n,p}(\beta;z)}{z^p}\right\} > \gamma \quad (z \in U),$$

for some γ ($\gamma < 1$), then

$$\Re\left\{\frac{I_{n+p}f(z)}{z^p}\right\} > \frac{2(n+p)\gamma + \Re(\beta)}{2(n+p)+\Re(\beta)} \quad (z \in U).$$

Putting $n = 0$ in Corollary 1, we have

Corollary 2. *If $f(z) \in A(p)$, $\beta \in C$ ($\Re(\beta) \geq 0$), $p \in N$, and*

$$\Re\left\{\frac{G_{0,p}(\beta;z)}{z^p}\right\} > \gamma \quad (z \in U),$$

for some γ ($\gamma < 1$), then

$$\Re\left\{\frac{f(z)}{z^p}\right\} > \frac{2p\gamma + \Re(\beta)}{2p+\Re(\beta)} \quad (z \in U).$$

Taking $\alpha = \overline{\beta}$ in Theorem 1, we have

Corollary 3. *If $f(z) \in A(p)$, $\beta \in C$ ($\Re(\beta) > 0$), $n > -p$, $p \in N$, and*

$$\Re\left\{\frac{G_{n,p}(\overline{\beta},\beta;z)}{z^p}\right\} > \gamma \quad (z \in U),$$

for some γ ($\gamma < 2\Re(\beta)$), then

$$\Re\left\{\frac{I_{n+p}f(z)}{z^p}\right\} > \frac{2(n+p)\gamma + \Re(\beta)}{[4(n+p)+1]\Re(\beta)} \quad (z \in U).$$

Further, if

$$\Re\left\{\frac{G_{n,p}(\overline{\beta},\beta;z)}{z^p}\right\} > \frac{3}{2}\Re(\beta) \quad (z \in U),$$

then

$$\Re\left\{\frac{I_{n+p}f(z)}{z^p}\right\} > \frac{3(n+p)+1}{4(n+p)+1} \quad (z \in U).$$

Putting $n=0$ in Corollary 3, we obtain

Corollary 4. *If* $f(z)\in A(p)$, $\beta\in C$ $(\Re(\beta)>0)$, $p\in N$, *and*

$$\Re\left\{\frac{G_{0,p}(\overline{\beta},\beta;z)}{z^p}\right\}>\gamma\quad(z\in U),$$

for some γ $(\gamma<2\Re(\beta))$, *then*

$$\Re\left\{\frac{f(z)}{z^p}\right\}>\frac{2p\gamma+\Re(\beta)}{(4p+1)\Re(\beta)}\quad(z\in U).$$

Further, if

$$\Re\left\{\frac{G_{0,z}(\overline{\beta},\beta;z)}{z^p}\right\}>\frac{3}{2}\Re(\beta)\quad(z\in U),$$

then

$$\Re\left\{\frac{f(z)}{z^p}\right\}>\frac{3p+1}{4p+1}\quad(z\in U).$$

Next, we prove

Theorem 2. *Let a function* $G_{n,p}(\alpha,\beta;z)$ *be defined by* (1.8) *for* $\alpha\in C$, $\beta\in C$ $(\Re(\beta)\geq 0)$, $\alpha+\beta\in R$, $n>-p$, $p\in N$ *and* $f(z)\in A(p)$. *If*

$$\Re\left\{\frac{G_{n,p}(\alpha,\beta;z)}{z^p}\right\}<\gamma\quad(z\in U),\tag{2.9}$$

for some γ $(\gamma>\alpha+\beta)$, *then*

$$\Re\left\{\frac{I_{n+p}f(z)}{z^p}\right\}<\frac{2(n+p)\gamma+\Re(\beta)}{2(n+p)(\alpha+\beta)+\Re(\beta)}\quad(z\in U).\tag{2.10}$$

Proof. Define the function $q(z)$ by

$$\frac{I_{n+p}f(z)}{z^p}=\delta+(1-\delta)q(z),\tag{2.11}$$

with

$$\delta=\frac{2(n+p)\gamma+\Re(\beta)}{2(n+p)(\alpha+\beta)+\Re(\beta)}>1.\tag{2.12}$$

Then we observe that $q(z)=1+q_1z+q_2z^2+\cdots$ is regular in the unit disc U, and that

$$\Re\left\{\gamma-\frac{G_{n,p}(\alpha,\beta;z)}{z^p}\right\}$$
$$=\Re\{\gamma-(\alpha+\beta)\delta-(\alpha+\beta)(1-\delta)q(z)-\beta\frac{(1-\delta)}{(n+p)}zq'(z)\}>0.\tag{2.13}$$

Taking $q(z) = u = u_1 + iu_2$ and $zq'(z) = v = v_1 + iv_2$, we define the function $\varphi(u, v)$ by

$$\varphi(u, v) = \gamma - (\alpha + \beta)\delta - (\alpha + \beta)(1 - \delta)u - \beta\frac{(1 - \delta)}{(n + p)}v. \tag{2.14}$$

Then it follows from (2.14) that:

(i) $\varphi(u, v)$ is continuous in $D = C \times C = C^2$;

(ii) $(1, 0) \in D$ and $\Re\{\varphi(1, 0)\} = \gamma - (\alpha + \beta) > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1}{2}(1 + u_2^2)$,

$$\begin{aligned}\Re\{\varphi(iu_2, v_1)\} &= \gamma - (\alpha + \beta)\delta - \frac{(1 - \delta)}{(n + p)}v_1\Re(\beta) \\ &\leq \gamma - (\alpha + \beta)\delta + \frac{(1 - \delta)}{2(n + p)}\Re(\beta)(1 + u_2^2) \\ &= -\frac{(1 - \delta)}{2(n + p)}\Re(\beta)u_2^2 \leq 0.\end{aligned}$$

Consequently, applying Lemma 1, we have that $\Re\{q(z)\} > 0$ $(z \in U)$, which implies that

$$\Re\left\{\frac{I_{n+p}f(z)}{z^p}\right\} < \delta = \frac{2(n + p)\gamma + \Re(\beta)}{2(n + p)(\alpha + \beta) + \Re(\beta)} \quad (z \in U). \tag{2.15}$$

This completes the proof of Theorem 2. ■

Putting $\alpha = 1 - \beta$ in Theorem 2, we obtain

Corollary 5. *Let a function $G_{n,p}(\beta; z)$ be defined by (1.9) for $\beta \in C$ $(\Re(\beta) \geq 0)$, $n > -p$, $p \in N$ and $f(z) \in A(p)$. If*

$$\Re\left\{\frac{G_{n,p}(\beta; z)}{z^p}\right\} < \gamma \quad (z \in U),$$

for some γ $(\gamma < 1)$, then

$$\Re\left\{\frac{I_{n+p}f(z)}{z^p}\right\} < \frac{2(n + p)\gamma + \Re(\beta)}{2(n + p) + \Re(\beta)} \quad (z \in U).$$

Putting $n = 0$ in Corollary 5, we have

Corollary 6. *If $f(z) \in A(p)$, $\beta \in C$ $(\Re(\beta) \geq 0)$, $p \in N$, and*

$$\Re\left\{\frac{G_{0,p}(\beta; z)}{z^p}\right\} < \gamma \quad (z \in U),$$

for some γ $(\gamma < 1)$, then

$$\Re\left\{\frac{f(z)}{z^p}\right\} < \frac{2p\gamma + \Re(\beta)}{2p + \Re(\beta)} \quad (z \in U).$$

Taking $\alpha = \overline{\beta}$ in Theorem 2, we have

Corollary 7. *If $f(z) \in A(p)$, $\beta \in C$ $(\Re(\beta) > 0)$, $n > -p$, $p \in N$ and*

$$\Re\left\{\frac{G_{n,p}(\overline{\beta},\beta;z)}{z^p}\right\} < \gamma \quad (z \in U),$$

for some γ $(\gamma < 2\Re(\beta))$, then

$$\Re\left\{\frac{I_{n+p}f(z)}{z^p}\right\} < \frac{2(n+p)\gamma + \Re(\beta)}{[4(n+p)+1]\Re(\beta)} \quad (z \in U).$$

Further, if

$$\Re\left\{\frac{G_{n,p}(\overline{\beta},\beta;z)}{z^p}\right\} < \frac{3}{2}\Re(\beta) \quad (z \in U),$$

then

$$\Re\left\{\frac{I_{n+p}f(z)}{z^p}\right\} < \frac{3(n+p)+1}{4(n+p)+1} \quad (z \in U).$$

Putting $n = 0$ in Corollary 7, we obtain

Corollary 8. *If $f(z) \in A(p)$, $\beta \in C$ $(\Re(\beta) > 0)$, $p \in N$, and*

$$\Re\left\{\frac{G_{0,z}(\overline{\beta},\beta;z)}{z^p}\right\} < \gamma \quad (z \in U),$$

for some γ $(\gamma < 2\Re(\beta))$, then

$$\Re\left\{\frac{f(z)}{z^p}\right\} < \frac{2p\gamma + \Re(\beta)}{(4p+1)\Re(\beta)} \quad (z \in U).$$

Further, if

$$\Re\left\{\frac{G_{0,z}(\overline{\beta},\beta;z)}{z^p}\right\} < \frac{3}{2}\Re(\beta) \quad (z \in U),$$

then

$$\Re\left\{\frac{f(z)}{z^p}\right\} < \frac{3p+1}{4p+1}(z \in U).$$

Using the same technique as in the proof of Theorem 1 and Theorem 2 (or putting $\frac{zf'(z)}{p}$ instead of $f(z)$ in Theorem 1 and Theorem 2, respectively), we have the following results.

Theorem 3. *Let a function $G_{n,p}(\alpha,\beta;z)$ be defined by* (1.8) *for $\alpha \in C$, $\beta \in C$ $(\Re(\beta) \geq 0)$, $\alpha + \beta \in R$, $n > -p$, $p \in N$ and $f(z) \in A(p)$. If*

$$\Re\left\{\frac{G'_{n,p}(\alpha,\beta;z)}{pz^{p-1}}\right\} > \gamma \quad (z \in U), \tag{2.16}$$

for some γ $(\gamma < \alpha + \beta)$, *then*

$$\Re\left\{\frac{(I_{n+p}f(z))'}{pz^{p-1}}\right\} > \frac{2(n+p)\gamma + \Re(\beta)}{2(n+p)(\alpha+\beta) + \Re(\beta)} \quad (z \in U). \tag{2.17}$$

Putting $\alpha = 1 - \beta$ in Theorem 3, we have

Corollary 9. *Let a function* $G_{n,p}(\beta; z)$ *be defined by* (1.9) *for* $\beta \in C$ $(\Re(\beta) \geq 0)$, $n > -p$, $p \in N$ *and* $f(z) \in A(p)$. *If*

$$\Re\left\{\frac{G'_{n,p}(\beta; z)}{pz^{p-1}}\right\} > \gamma \quad (z \in U),$$

for some γ $(\gamma < 1)$, *then*

$$\Re\left\{\frac{(I_{n+p}f(z))'}{pz^{p-1}}\right\} > \frac{2(n+p)\gamma + \Re(\beta)}{2(n+p) + \Re(\beta)} \quad (z \in U).$$

Putting $n = 0$ in Corollary 9, we have

Corollary 10. *If* $f(z) \in A(p)$, $\beta \in C$ $(\Re(\beta) \geq 0)$, $p \in N$, *and*

$$\Re\left\{\frac{G'_{0,p}(\beta; z)}{pz^{p-1}}\right\} > \gamma \quad (z \in U),$$

for some γ $(\gamma < 1)$, *then*

$$\Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} > \frac{2p\gamma + \Re(\beta)}{2p + \Re(\beta)} \quad (z \in U).$$

Taking $\alpha = \overline{\beta}$ in Theorem 3, we have

Corollary 11. *If* $f(z) \in A(p)$, $\beta \in C$ $(\Re(\beta) > 0)$, $n > -p$, $p \in N$ *and*

$$\Re\left\{\frac{G'_{n,p}(\overline{\beta}, \beta; z)}{pz^{p-1}}\right\} > \gamma \quad (z \in U),$$

for some γ $(\gamma < 2\Re(\beta))$, *then*

$$\Re\left\{\frac{(I_{n+p}f(z))'}{pz^{p-1}}\right\} > \frac{2(n+p)\gamma + \Re(\beta)}{[4(n+p)+1]\Re(\beta)} \quad (z \in U).$$

Further, if

$$\Re\left\{\frac{G'_{n,p}(\overline{\beta}, \beta; z)}{pz^{p-1}}\right\} > \frac{3}{2}\Re(\beta) \quad (z \in U),$$

then

$$\Re\left\{\frac{(I_{n+p}f(z))'}{pz^{p-1}}\right\} > \frac{3(n+p)+1}{4(n+p)+1} \quad (z \in U).$$

Putting $n=0$ in Corollary 11, we have

Corollary 12. *If* $f(z) \in A(p)$, $\beta \in C$ $(\Re(\beta) > 0)$, $p \in N$ *and*

$$\Re\left\{\frac{G'_{0,p}(\overline{\beta},\beta;z)}{pz^{p-1}}\right\} > \gamma \quad (z \in U),$$

for some γ $(\gamma < 2\Re(\beta))$, *then*

$$\Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} > \frac{2p\gamma + \Re(\beta)}{(4p+1)\Re(\beta)} \quad (z \in U).$$

Further, if

$$\Re\left\{\frac{G'_{0,z}(\overline{\beta},\beta;z)}{pz^{p-1}}\right\} > \frac{3}{2}\Re(\beta) \quad (z \in U),$$

then

$$\Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} > \frac{3p+1}{4p+1} \quad (z \in U).$$

Theorem 4. *Let a function* $G_{n,p}(\alpha,\beta;z)$ *be defined by* (1.8) *for* $\alpha \in C$, $\beta \in C$ $(\Re(\beta) \geq 0)$, $\alpha+\beta \in R$, $n > -p$, $p \in N$ *and* $f(z) \in A(p)$. *If*

$$\Re\left\{\frac{G'_{n,p}(\alpha,\beta;z)}{pz^{p-1}}\right\} < \gamma \quad (z \in U), \tag{2.18}$$

for some γ $(\gamma > \alpha+\beta)$, *then*

$$\Re\left\{\frac{(I_{n+p}f(z))'}{pz^{p-1}}\right\} < \frac{2(n+p)\gamma + \Re(\beta)}{2(n+p)(\alpha+\beta) + \Re(\beta)} \quad (z \in U). \tag{2.19}$$

Putting $\alpha = 1-\beta$ in Theorem 4, we have

Corollary 13. *Let a function* $G_{n,p}(\beta;z)$ *be defined by* (1.9) *for* $\beta \in C$ $(\Re(\beta) \geq 0)$, $n > -p$, $p \in N$ *and* $f(z) \in A(p)$. *If*

$$\Re\left\{\frac{G'_{n,p}(\beta;z)}{pz^{p-1}}\right\} < \gamma \quad (z \in U)$$

for some γ $(\gamma < 1)$, *then*

$$\Re\left\{\frac{(I_{n+p}f(z))'}{pz^{p-1}}\right\} < \frac{2(n+p)\gamma + \Re(\beta)}{2(n+p) + \Re(\beta)} \quad (z \in U).$$

Putting $n=0$ in Corollary 13, we have

Corollary 14. *If* $f(z) \in A(p)$, $\beta \in C$ $(\Re(\beta) \geq 0)$, $p \in N$ *and*

$$\Re\left\{\frac{G'_{0,p}(\beta; z)}{pz^{p-1}}\right\} < \gamma \quad (z \in U),$$

for some γ $(\gamma < 1)$, *then*

$$\Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} < \frac{2p\gamma + \Re(\beta)}{2p + \Re(\beta)} \quad (z \in U).$$

Taking $\alpha = \overline{\beta}$ in Theorem 4, we have

Corollary 15. *If* $f(z) \in A(p)$, $\beta \in C$ $(\Re(\beta) > 0)$, $n > -p$, $p \in N$ *and*

$$\Re\left\{\frac{G'_{n,p}(\overline{\beta}, \beta; z)}{pz^{p-1}}\right\} < \gamma \quad (z \in U),$$

for some γ $(\gamma < 2\Re(\beta))$, *then*

$$\Re\left\{\frac{(I_{n+p}f(z))'}{pz^{p-1}}\right\} < \frac{2(n+p)\gamma + \Re(\beta)}{[4(n+p)+1]\Re(\beta)} \quad (z \in U).$$

Further, if

$$\Re\left\{\frac{G'_{n,p}(\overline{\beta}, \beta; z)}{pz^{p-1}}\right\} < \frac{3}{2}\Re(\beta) \quad (z \in U),$$

then

$$\Re\left\{\frac{(I_{n+p}f(z))'}{pz^{p-1}}\right\} < \frac{3(n+p)+1}{4(n+p)+1} \quad (z \in U).$$

Putting $n = 0$ in Corollary 15, we have

Corollary 16. *If* $f(z) \in A(p)$, $\beta \in C$ $(\Re(\beta) > 0)$, $p \in N$, *and*

$$\Re\left\{\frac{G'_{0,p}(\overline{\beta}, \beta; z)}{pz^{p-1}}\right\} < \gamma \quad (z \in U),$$

for some γ $(\gamma < 2\Re(\beta))$, *then*

$$\Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} < \frac{2p\gamma + \Re(\beta)}{(4p+1)\Re(\beta)} \quad (z \in U).$$

Further, if

$$\Re\left\{\frac{G'_{0,z}(\overline{\beta}, \beta; z)}{pz^{p-1}}\right\} < \frac{3}{2}\Re(\beta) \quad (z \in U),$$

then

$$\Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} < \frac{3p+1}{4p+1} \quad (z \in U).$$

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