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GENERALIZED a-WEYL'S THEOREM FOR DIRECT SUMS

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Abstract. If T and S are Hilbert space operators obeying generalized a-Weyl's theorem, then it does not necessarily imply that the direct sum $T \oplus S$ obeys generalized a-Weyl's theorem. In this paper we explore certain conditions on T and S so that the direct sum $T \oplus S$ obeys generalized a-Weyl's theorem.

1. Introduction

Let H be an infinite dimensional separable Hilbert space. Let $B(H)$ be the algebra of all operators on H (bounded linear transformations of H into itself). For an operator $T \in B(H)$, let $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote the spectrum, point spectrum and approximate point spectrum of T, respectively. Let $\alpha(T)$ and $\beta(T)$ denote the dimension of the kernel $\ker T$ and the codimension of the range $R(T)$, respectively. An operator $T \in B(H)$ is called an upper semi-Fredholm if $\alpha(T) < \infty$ and $T(H)$ is closed, while $T \in B(H)$ is called a lower semi-Fredholm if $\beta(T) < \infty$. However, T is called a semi-Fredholm operator if T is either an upper or a lower semi-Fredholm and T is said to be a Fredholm operator if it is both an upper and a lower semi-Fredholm. If $T \in B(H)$ is semi-Fredholm, then the index of T is defined by

$$\operatorname{ind}(T) = \alpha(T) - \beta(T).$$

The ascent of T is defined by the smallest non-negative integer $p := p(T)$ such that $N(T^p) = N(T^{p+1})$. If such an integer does not exist we put $p(T) = \infty$. Analogously, the descent of T, is defined by the smallest nonnegative integer $q := q(T)$ such that $R(T^q) = R(T^{q+1})$ and if such an integer does not exist we put $q(T) = \infty$.

An operator $T \in B(H)$ is called a Weyl operator if it is a Fredholm operator of index 0, while $T \in B(H)$ is called a Browder if it is a Fredholm operator of finite ascent and descent. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_W(T)$

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of T are defined by

$$\sigma_e(T) = \{\lambda \in \mathbf{C} : T - \lambda I \ \text{ is not Fredholm}\},$$
$$\sigma_W(T) = \{\lambda \in \mathbf{C} : T - \lambda I \ \text{ is not Weyl}\}.$$

For $T \in B(H)$, define the set $LD(H)$ by

$$LD(H) = \{T \in B(H) : p(T) < \infty \ \text{ and } \ R(T^{p+1}) \text{ is closed}\}.$$

An operator $T \in B(H)$ is said to be left Drazin invertible if $T \in LD(H)$. We say that $\lambda \in \sigma_a(T)$ is a left pole of T, if $T - \lambda I \in LD(H)$. We denote by $\pi^a(T)$ the set of all left poles of T.

We say that Weyl's theorem holds for T, if

$$\sigma(T) \setminus \sigma_W(T) = E_0(T),$$

where $E_0(T)$ is the set of all isolated point of $\sigma(T)$ which are eigenvalues of finite multiplicity.

For a bounded linear operator T and a nonnegative integer n we define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into itself (in particular $T_0 = T$). If for some integer n, the range space $R(T^n)$ is closed and T_n is an upper (resp., a lower) semi-Fredholm operator, then T is called an upper (resp., a lower) semi B-Fredholm operator. In this situation, T_m is a semi-Fredholm operator and $\text{ind}(T_m) = \text{ind}(T_n)$ for each $m \geq n$ [3, Proposition 2.1]. It permits us to define the index of a semi B-Fredholm operator T as the index of the semi-Fredholm operator T_n where n is any integer such that $R(T^n)$ is closed and T_n is a semi-Fredholm operator. Moreover if T_n is a Fredholm operator, then T is called a B-Fredholm operator. A semi B-Fredholm operator is an upper or a lower semi B-Fredholm operator.

An operator $T \in B(H)$ is called a B-Weyl operator if it is a B-Fredholm operator of index 0. The B-Fredholm spectrum $\sigma_{BF}(T)$ and the B-Weyl spectrum $\sigma_{BW}(T)$ of T are defined as

$$\sigma_{BF}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \ \text{ is not a } B\text{-Fredholm operator}\},$$
$$\sigma_{BW}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \ \text{ is not a } B\text{-Weyl operator}\}.$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T),$$

where $E(T)$ is the set of isolated eigenvalues of T ([2], Definition 2.13), and that generalized Browder's theorem holds for T if $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$, where $\pi(T)$ is the set of all poles of T.

Let $SBF(H)$ be the class of all semi B-Fredholm operators on H, $USBF(H)$ be the class of all upper semi B-Fredholm operators on H and $USBF^-(H)$ be the class of all $T \in USBF(H)$ such that $\text{ind}(T) \leq 0$. Also let

$$\sigma_{usbf^-}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not in } USBF^-(\text{H})\}.$$

We say T obeys generalized a-Weyl's theorem if

$$\sigma_a(T) \setminus \sigma_{usbf^-}(T) = E^a(T),$$

where $E^a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ ([2], Definition 2.13). We know that [2]

generalized a-Weyl's theorem $\Rightarrow$ generalized Weyl's theorem $\Rightarrow$ Weyl's theorem.

We say T obeys generalized a-Browder's theorem if

$$\sigma_{usbf^-}(T) = \sigma_a(T) \setminus \pi^a(T).$$

An operator is called T polaroid if all isolated points of the spectrum of T are poles of the resolvent of T and is called isoloid if each $\lambda \in \sigma_{iso}(T)$ is an eigenvalue of T, where $\sigma_{iso}(T)$ is the set of isolated points of $\sigma(T)$. An operator T is called a-isoloid if every $\lambda \in \sigma_a^{iso}(T)$ is an eigenvalue of T, where $\sigma_a^{iso}(T)$ is the set of isolated points of $\sigma_a(T)$. Every a-isoloid operator is isoloid but the converse is generally not true.

2. Generalized a-Weyl's theorem for direct sums

Let H and K be nonzero complex Hilbert spaces. Although $T \in B(H)$ and $S \in B(K)$ satisfy generalized a-Weyl's theorem, we do not guarantee that their orthogonal direct sum $T \oplus S$ satisfies generalized a-Weyl's theorem.

Example 2.1. Let us define S for each $x \in (x_i) \in l^1$ by

$$S(x_1, x_2, x_3, \dots, x_k \dots) = (0, \alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_{k-1} x_{k-1}, \dots)$$

where (α_i) is a sequence of complex numbers such that $0 < |\alpha_i| \leq 1$ and $\sum_{i=1}^{\infty} |\alpha_i| < \infty$. $\sigma(S) = \sigma_a(S) = \{0\}$. It can be proved that $\overline{R(S^n)} \neq R(S^n)$ for any $n = 1, 2, \dots$ Thus, $\sigma_{usbf^-}(S) = \{0\}$. Since $E^a(S) = \phi$, it follows that S satisfies generalized a-Weyl's theorem. Define T on $X = l^1 \oplus l^1$ by $T = S \oplus 0$. Now $N(T) = \{0\} \oplus l^1, \sigma(T) = \sigma_a(T) = \{0\}$, $E^a(T) = \{0\}$. As $R(T^n) = R(S^n) \oplus \{0\}$, $R(T^n)$ is not closed for any $n \in \mathbf{N}$. So $T \notin USBF^-$ and $\sigma_{usbf^-}(T) = \{0\}$. Thus, $\sigma_a(T) \setminus \sigma_{usbf^-}(T) \neq E^a(T)$. Hence T does not satisfy generalized a-Weyl's theorem.

In this section we discuss certain conditions on T and S to ensure that generalized a-Weyl's theorem holds for $T \oplus S$. W.Y. Lee [6] proved that if $T \in B(H)$ and $S \in B(K)$ are isoloid and satisfy Weyl's theorem such that $\sigma_W(T \oplus S) = \sigma_W(T) \cup \sigma_W(S)$ then Weyl's theorem holds for $T \oplus S$. We now prove the result for generalized a-Weyl's theorem:

Theorem 2.2. *Suppose that generalized* a-*Weyl's theorem holds for* $T \in B(H)$ *and* $S \in B(K)$*. If* T *and* S *are* a-*isoloid and* $\sigma_{usbf^-}(T \oplus S) = \sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)$*, then generalized* a-*Weyl's theorem holds for* $T \oplus S$*.*

Proof. We know $\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$ for any pair of operators. If T and S are a-isoloid, then

$$E^a(T \oplus S) = [E^a(T) \cap \rho_a(S)] \cup [\rho_a(T) \cap E^a(S)] \cup [E^a(T) \cap E^a(S)]$$

where $\rho_a(.) = \mathbf{C} \setminus \sigma_a(.)$.

If generalized a-Weyl's theorem holds for T and S, then

$$[\sigma_a(T)\cup\sigma_a(S)]\setminus[\sigma_{usbf^-}(T)\cup\sigma_{usbf^-}(S)] \\ =[E^a(T)\cap\rho_a(S)]\cup[\rho_a(T)\cap E^a(S)]\cup[E^a(T)\cap E^a(S)].$$

Thus, $E^a(T\oplus S)=[\sigma_a(T)\cup\sigma_a(S)]\setminus[\sigma_{usbf^-}(T)\cup\sigma_{usbf^-}(S)]$.

If $\sigma_{usbf^-}(T\oplus S)=\sigma_{usbf^-}(T)\cup\sigma_{usbf^-}(S)$, then $E^a(T\oplus S)=\sigma_a(T\oplus S)\setminus\sigma_{usbf^-}(T\oplus S)$. Hence generalized a-Weyl's theorem holds for $T\oplus S$. ■

THEOREM 2.3. *Suppose $T\in B(H)$ has no isolated point in its approximate spectrum and $S\in B(K)$ satisfies generalized* a-*Weyl's theorem. If $\sigma_{usbf^-}(T\oplus S)=\sigma_a(T)\cup\sigma_{usbf^-}(S)$, then generalized* a-*Weyl's theorem holds for $T\oplus S$.*

Proof. As $\sigma_a(T\oplus S)=\sigma_a(T)\cup\sigma_a(S)$ for any pair of operators, we have

$$\begin{aligned}\sigma_a(T\oplus S)\setminus\sigma_{usbf^-}(T\oplus S)&=[\sigma_a(T)\cup\sigma_a(S)]\setminus[\sigma_a(T)\cup\sigma_{usbf^-}(S)]\\&=\sigma_a(S)\setminus[\sigma_a(T)\cup\sigma_{usbf^-}(S)]\\&=[\sigma_a(S)\setminus\sigma_{usbf^-}(S)]\setminus\sigma_a(T)\\&=E^a(S)\cap\rho_a(T)\end{aligned}$$

where $\rho_a(T)=\mathbf{C}\setminus\sigma_a(T)$.

Let $\sigma_a^{iso}(T)$ be the set of isolated points of $\sigma_a(T)$ and $\sigma_a^{iso}(T\oplus S)$ be the set of isolated points of $\sigma_a(T\oplus S)=\sigma_a(T)\cup\sigma_a(S)$. If $\sigma_a^{iso}(T)=\phi$ it implies that $\sigma_a(T)=\sigma_a^{acc}(T)$, where $\sigma_a^{acc}(T)=\sigma_a(T)\setminus\sigma_a^{iso}(T)$ is the set of all accumulation points of $\sigma_a(T)$. Thus we have

$$\begin{aligned}\sigma_a^{iso}(T\oplus S)&=[\sigma_a^{iso}(T)\cup\sigma_a^{iso}(S)]\setminus[(\sigma_a^{iso}(T)\cap\sigma_a^{acc}(S))\cup(\sigma_a^{acc}(T)\cap\sigma_a^{iso}(S))]\\&=(\sigma_a^{iso}(T)\setminus\sigma_a^{acc}(S))\cup(\sigma_a^{iso}(S)\setminus\sigma_a^{acc}(T))\\&=\sigma_a^{iso}(S)\setminus\sigma_a(T)\\&=\sigma_a^{iso}(S)\cap\rho_a(T).\end{aligned}$$

We have that $\sigma_p(T\oplus S)=\sigma_p(T)\cup\sigma_p(S)$ for every pair of operators, therefore

$$\begin{aligned}E^a(T\oplus S)&=\sigma_a^{iso}(T\oplus S)\cap\sigma_p(T\oplus S)\\&=\sigma_a^{iso}(S)\cap\rho_a(T)\cap\sigma_p(S)\\&=E^a(S)\cap\rho_a(T).\end{aligned}$$

Thus, $\sigma_a(T\oplus S)\setminus\sigma_{usbf^-}(T\oplus S)=E^a(T\oplus S)$. Hence $T\oplus S$ satisfies generalized a-Weyl's theorem. ■

Let $\sigma_1(T)$ denote the compliment of $\sigma_{usbf^-}(T)$ in $\sigma_a(T)$ i.e. $\sigma_1(T)=\sigma_a(T)\setminus\sigma_{usbf^-}(T)$. A straight forward application of Theorem 2.3 leads to the following corollaries.

COROLLARY 2.4. *Suppose $T\in B(H)$ is such that $\sigma_a^{iso}(T)=\phi$ and $S\in B(K)$ satisfies generalized* a-*Weyl's theorem with $\sigma_a^{iso}(S)\cap\sigma_p(S)=\phi$ and $\sigma_1(T\oplus S)=\phi$, then $T\oplus S$ satisfies generalized* a-*Weyl's theorem.*

Proof. Since S satisfies generalized a-Weyl's theorem, therefore given condition $\sigma_a^{iso}(S) \cap \sigma_p(S) = \phi$ implies that $\sigma_a(S) = \sigma_{usbf^-}(S)$. Now $\sigma_1(T \oplus S) = \phi$ gives that $\sigma_{usbf^-}(T \oplus S) = \sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_{usbf^-}(S)$. Thus from Theorem 2.3, we have that $T \oplus S$ satisfies generalized a-Weyl's theorem. ■

Corollary 2.5. *Suppose $T \in B(H)$ is such that $\sigma_1(T) \cup \sigma_a^{iso}(T) = \phi$ and $S \in B(K)$ satisfies generalized* a-*Weyl's theorem. If $\sigma_{usbf^-}(T \oplus S) = \sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)$, then generalized* a-*Weyl's theorem holds for $T \oplus S$.*

Definition 2.6. An operator $T \in B(H)$ is called left-polaroid if every isolated point of the spectrum of $\sigma_a(T)$ is left pole of T.

Theorem 2.7. *Suppose generalized* a-*Browder's theorem holds for $T \in B(H)$ and $S \in B(K)$. Suppose T and S are left-polaroid and $\sigma_{usbf^-}(T \oplus S) = \sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)$, then generalized* a-*Browder's theorem holds for $T \oplus S$.*

Proof. If T and S are left -polaroid , then

$$\pi^a(T \oplus S) = [\pi^a(T) \cap \rho_a(S)] \cup [\rho_a(T) \cap \pi^a(S)] \cup [\pi^a(T) \cap \pi^a(S)]$$

where $\rho_a(.) = \mathbf{C} \setminus \sigma_a(.)$.

Since generalized a-Browder's theorem holds for T and S, we have

$$[\sigma_a(T) \cup \sigma_a(S)] \setminus [\sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)] \\ = [\pi^a(T) \cap \rho_a(S)] \cup [\rho_a(T) \cap \pi^a(S)] \cup [\pi^a(T) \cap \pi^a(S)].$$

Thus, $\pi^a(T \oplus S) = [\sigma_a(T) \cup \sigma_a(S)] \setminus [\sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)]$.

If $\sigma_{usbf^-}(T \oplus S) = \sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)$, then $\pi^a(T \oplus S) = \sigma_a(T \oplus S) \setminus \sigma_{usbf^-}(T \oplus S)$. Hence, generalized a-Browder's theorem holds for $T \oplus S$. ■

3. Generalized Weyl's theorem for direct sums

We know that generalized a-Weyl's theorem ⇒ generalized Weyl's theorem [2]. Thus we have the following similar results of generalized Weyl's theorem for direct sum of operators:

Theorem 3.1. *Suppose that generalized Weyl's theorem holds for $T \in B(H)$ and $S \in B(K)$. If T and S are isoloid and $\sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S)$, then generalized Weyl's theorem holds for $T \oplus S$.*

A straight forward application of Theorem 3.1 leads to the following corollary.

Corollary 3.2. *Suppose $T \in B(H)$ is an isoloid operator that satisfies generalized Weyl's theorem, then $T \oplus S$ satisfies generalized Weyl's theorem whenever $S \in B(K)$ is a normal operator.*

Proof. It is shown in [4] that if K is a Hilbert space and an operator $S \in B(K)$ satisfies $\sigma_{BF}(S) = \sigma_{BW}(S)$, then $\sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S)$ for every

Hilbert space H and $T \in B(H)$. As $S \in B(K)$ is a normal operator we have $\sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S)$. Since every normal operator is isoloid and satisfies generalized Weyl's theorem [1], therefore S is isoloid and satisfies generalized Weyl's theorem. Hence the required result follows from the Theorem 3.1. ■

THEOREM 3.3. *Suppose $T \in B(H)$ has no isolated point in its spectrum and $S \in B(K)$ satisfies generalized Weyl's theorem. Suppose $\sigma_{BW}(T \oplus S) = \sigma(T) \cup \sigma_{BW}(S)$, then generalized Weyl's theorem holds for $T \oplus S$.*

We denote by $\sigma_0(T)$ the complement of $\sigma_{BW}(T)$ in $\sigma(T)$. We have the following consequences of the above result.

COROLLARY 3.4. *Suppose $T \in B(H)$ is such that $\sigma_{iso}(T) = \phi$ and $S \in B(K)$ satisfies generalized Weyl's theorem with $\sigma_{iso}(S) \cap \sigma_p(S) = \phi$ and $\sigma_0(T \oplus S) = \phi$, then $T \oplus S$ satisfies generalized Weyl's theorem.*

COROLLARY 3.5. *Suppose $T \in B(H)$ is such that $\sigma_0(T) \cup \sigma_{iso}(T) = \phi$ and $S \in B(K)$ satisfies generalized Weyl's theorem. If $\sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S)$, then generalized Weyl's theorem holds for $T \oplus S$.*

THEOREM 3.6. *Suppose generalized Browder's theorem holds for $T \in B(H)$ and $S \in B(K)$. Suppose T and S are polaroid and $\sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S)$, then generalized Browder's theorem holds for $T \oplus S$.*

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