МАТЕМАТИЧКИ ВЕСНИК
64, 1 (2012), 25–31
March 2012

оригинални научни рад
research paper

GENERALIZATIONS OF PRIMAL IDEALS IN COMMUTATIVE RINGS

Ahmad Yousefian Darani

Abstract. Let R be a commutative ring with identity. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be a function where $\mathfrak{I}(R)$ denotes the set of all ideals of R. Let I be an ideal of R. An element $a \in R$ is called ϕ-prime to I if $ra \in I - \phi(I)$ (with $r \in R$) implies that $r \in I$. We denote by $S_\phi(I)$ the set of all elements of R that are not ϕ-prime to I. I is called a ϕ-primal ideal of R if the set $P := S_\phi(I) \cup \phi(I)$ forms an ideal of R. So if we take $\phi_\emptyset(Q) = \emptyset$ (resp., $\phi_0(Q) = 0$), a ϕ-primal ideal is primal (resp., weakly primal). In this paper we study the properties of several generalizations of primal ideals of R.

1. Introduction

Throughout, R will be a commutative ring with identity. (However, in most places the existence of an identity plays no role.) By a proper ideal I of R we mean an ideal I with $I \neq R$. Fuchs [5] introduced a new class of ideals of R: *primal ideals*. Later Ebrahimi Atani and the author gave a generalization of primal ideals: *weakly primal ideals*. Let I be an ideal of R. An element $a \in R$ is called prime (resp. weakly prime) to I if $ra \in I$ (resp. $0 \neq ra \in I$) (where $r \in R$) implies that $r \in I$. Denote by $S(I)$ (resp. $w(I)$) the set of elements of R that are not prime (resp. are not weakly prime) to I. A proper ideal I of R is said to be primal if $S(I)$ forms an ideal of R (so 0 is not necessarily primal); this ideal is always a prime ideal, called the adjoint prime ideal P of I. In this case we also say that I is a P-primal ideal of R [5]. Not that if $r \in R$ and $a \in S(I)$, then clearly $ra \in S(I)$. So what we require for I being primal is that if a and b are not prime to I, then their difference is also not prime to I. Also, a proper ideal I of R is called weakly primal if the set $P = w(I) \cup \{0\}$ forms an ideal; this ideal is always a weakly prime ideal [4, Proposition 4], where a proper ideal P of R is called weakly prime if whenever $a, b \in R$ with $0 \neq ab \in P$, then either $a \in P$ or $b \in P$ [2]. In this case we also say that I is a P-weakly primal ideal. If R is not an integral domain, then 0 is a 0-weakly primal ideal of R (by definition), so a weakly primal ideal need not be primal.

2010 AMS Subject Classification: 13A15, 13A10.
Keywords and phrases: Primal ideal; weakly primal ideal; ϕ-primal ideal.

Bhatwadekar and Sharma [3] recently defined a proper ideal I of an integral domain R to be almost prime if for $a, b \in R$ with $ab \in I - I^2$, then either $a \in I$ or $b \in I$. This definition can obviously be made for any commutative ring R. Thus a weakly prime ideal is almost prime and any proper idempotent ideal is almost prime. The concept of almost primal ideals in a commutative ring was introduced and studied in [6]. Let I be an ideal of R, and let $n \geq 2$ be an integer. An element $a \in R$ is called almost prime (resp. n-almost prime) to I if $ra \in I - I^2$ (resp. $ra \in I - I^n$) (with $r \in R$) implies that $r \in I$. Denote by $S_2(I)$ (resp. $S_n(I)$), the set of all elements of R that are not almost prime (resp. n-almost prime) to I. Then I is called almost primal (resp. n-almost primal) if the set $P = S_2(I) \cup I^2$ (resp. $P = S_n(I) \cup I^n$) forms an ideal of R. This ideal is an almost prime (resp. n-almost prime) ideal of R [6, Lemma 4], called the almost (resp. n-almost) prime adjoint ideal of I. In this case we also say that I is a P-almost (resp. P-n-almost) primal ideal.

In this paper we give some more generalizations of primal ideals and study the basic properties of these classes of ideals.

2. Results

Let R be a commutative ring, $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function and I an ideal of R. Since $I - \phi(I) = I - (I \cap \phi(I))$, there is no loss of generality in assuming that $I \subseteq \phi(I)$. We henceforth make this assumption throughout this paper.

Definition 2.1. Let R be a commutative ring and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function. Let I be an ideal of R. An element $a \in R$ is called ϕ-prime to I if $ra \in I - \phi(I)$ (with $r \in R$) implies that $r \in I$.

Remarks 2.2. Let R be a commutative ring, and I a proper ideal of R. Denote by $S_\phi(I)$ the set of all elements of R that are not ϕ-prime to I. Then

(1) Every element of $\phi(I)$ is ϕ-prime to I.

(2) If an element of R is prime to I, then it is ϕ-prime to I. So $S_\phi(I) \subseteq S(I)$.

(3) The converse of (2) is not necessarily true. For example assume that $\phi = \phi_0$, where $\phi_0(Q) = 0$ for every ideal Q of R. Let $R = \mathbb{Z}/24\mathbb{Z}$, and $I = 8\mathbb{Z}/24\mathbb{Z}$. Then, $\overline{6}$ is ϕ-prime to I, but it is not prime to I since $\overline{12}.\overline{6} = \overline{0} \in I$ with $\overline{12} \notin I$. Consequently $\overline{6}$ is ϕ-prime to I while it is not prime to I.

Definition 2.3. Let R be a commutative ring and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function. A proper ideal I of R is said to be a ϕ-primal ideal of R if $S_\phi(I) \cup \phi(I)$ forms an ideal of R.

Let R be a commutative ring and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function. We recall from [1] that a proper ideal P of R is called ϕ-prime if for every $x, y \in R$, $xy \in P - \phi(P)$ implies $x \in P$ or $y \in P$.

Proposition 2.4. *If I is a ϕ-primal ideal of R, then $P = S_\phi(I) \cup \phi(I)$ is a ϕ-prime ideal of R.*

Proof. Let $x, y \in R$ be such that $xy \in P - \phi(P)$ and $x \notin P$. Then $xy \in S_\phi(I)$ so xy is not ϕ-prime to I. Hence $rxy \in I - \phi(I)$ for some $r \in R - I$. There exists $r \in R - I$ with $rxy \in I - \phi(I)$. If $x \notin P$, then $ry \in I - \phi(I)$ implies that y is not ϕ-prime to I. So $y \in S_\phi(I) = P$. Since x is ϕ-prime to I, from $x(ry) = rxy \in I - \phi(I)$ we get $ry \in I - \phi(I)$. This implies that y is not ϕ-prime to I, that is $y \in S_\phi(I) \subseteq P$. So P is ϕ-prime. ■

NOTATION 2.5. Let I be a ϕ-primal ideal of R. By Lemma 2.4, $P = S_\phi(I) \cup \phi(I)$ is a ϕ-prime ideal of R. In this case P is called the ϕ-prime adjoint ideal (simply adjoint ideal) of I, and I is called a P-ϕ-primal ideal of R.

THEOREM 2.6. *Let R be a commutative ring with identity. Then every ϕ-prime ideal of R is ϕ-primal.*

Proof. Assume that P is a ϕ-prime ideal of R. It suffices to show that $P - \phi(P)$ consists exactly of elements of R that are not ϕ-prime to P. By Lemma 2.7, $P \subseteq S_\phi(P) \cup P$. So $P - \phi(P) \subseteq S_\phi(P)$. Now assume that $a \in S_\phi(P)$. Then $ab \in P - \phi(P)$ for some $b \in R - P$. Since P is ϕ-prime we have $a \in P - \phi(P)$. Consequently $P = S_\phi(P) \cup \phi(P)$. This implies that P is a P-ϕ-primal ideal of R. ■

LEMMA 2.7. *Let R be a commutative ring and I an ideal of R.*

(1) If I is proper in R, then $I \subseteq S_\phi(I) \cup \phi(I)$.

(2) If I is a P-ϕ-primal ideal of R, then $I \subseteq P$.

Proof. (1) For every $a \in I - \phi(I)$ we have $a.1_R = a \in I - \phi(I)$ with $1_R \in R - I$. This implies that a is not ϕ-prime to I, that is $a \in S_\phi(I)$.

(2) It follows from (1). ■

EXAMPLE 2.8. Lest R be a commutative ring. Define the following functions $\phi_\alpha : \Im(R) \to \Im(R) \cup \{\emptyset\}$ and the corresponding ϕ_α-primal ideals:

(1) $\phi_\emptyset$	$\phi(I) = \emptyset$	a ϕ-primal ideal is primal.
(2) ϕ_0	$\phi(I) = 0$	a ϕ-primal ideal is weakly primal.
(3) ϕ_2	$\phi(I) = I^2$	a ϕ-primal ideal is almost primal.
(4) $\phi_n (n \geq 2)$	$\phi(I) = I^n$	a ϕ-primal ideal is n-almost primal.
(5) ϕ_ω	$\phi(I) = \bigcap_{n=1}^{\infty} I^n$	a ϕ-primal ideal is ω-primal.
(6) ϕ_1	$\phi(I) = I$	a ϕ-primal ideal is any ideal.

The next result provides several characterizations of ϕ-primal ideals of a commutative ring R.

THEOREM 2.9. *Let R be a commutative ring and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function. Let I and P be proper ideals of R. The following are equivalent.*

(1) I is P-ϕ-primal.

(2) For every $x \notin P - \phi(I)$, $(I :_R x) = I \cup (\phi(I) :_R x)$; and for every $x \in P - \phi(I)$, $(I :_R x) \supsetneqq I \cup (\phi(I) :_R x)$.

(3) for every $x \notin P - \phi(I)$, $(I :_R x) = I$ or $(I :_R x) = (\phi(I) :_R x)$; and for every $x \in P - \phi(I)$, $(I :_R x) \supsetneqq I$ and $(I :_R x) \supsetneqq (\phi(I) :_R x)$.

Proof. $1 \Rightarrow 2$) Assume that I is P-ϕ-primal. Then $P - \phi(I)$ consists entirely of elements of R that are not ϕ-prime to I. Let $x \notin P - \phi(I)$. Then x is ϕ-prime to I. Clearly $I \cup (\phi(I) :_R x) \subseteq (I :_R x)$. For every $r \in (I :_R x)$, if $rx \in \phi(I)$, then $r \in (\phi(I) :_R x)$, and if $rx \notin \phi(I)$, then x ϕ-prime to I gives $r \in I$. Hence $r \in I \cup (\phi(I) :_R x)$, that is $(I :_R x) \subseteq I \cup (\phi(I) :_R x)$. Therefore $(I :_R x) = I \cup (\phi(I) :_R x)$.

Now assume that $x \in P - \phi(I)$. Then x is not ϕ-prime to I. So there exists $r \in R - I$ such that $rx \in I - \phi(I)$. Hence $r \in (I :_R x) - (I \cup (\phi(I) :_R x))$.

$2 \Rightarrow 3$) Let $x \notin P - \phi(I)$. Since $(I :_R x)$ is an ideal of R and $(I :_R x) = I \cup (\phi(I) :_R x)$, either $(\phi(I) :_R x) \subseteq I$ or $I \subseteq (\phi(I) :_R x)$. So either $(I :_R x) = I$ or $(I :_R x) = (\phi(I) :_R x)$. Moreover, for every $x \in P - \phi(I)$, $(I :_R x) \supsetneqq I \cup (\phi(I) :_R x)$. Hence $(I :_R x) \supsetneqq I$ and $(I :_R x) \supsetneqq (\phi(I) :_R x)$.

$3 \Rightarrow 1$) By (3), $P - \phi(I)$ consists exactly of all elements of R that are not ϕ-prime to I. Hence I is P-ϕ-primal. ∎

Example 2.10. In this example we show that the concepts "primal ideal" and "ϕ-primal ideal" are different. In fact we show that neither implies the other. Let R be a commutative ring and assume that $\phi = \phi_0$. Then

(1) Let us to denote the set of all zero-divisors of R by $Z(R)$. If R is not an integral domain such that $Z(R)$ is not an ideal of R (for example the ring $\mathbb{Z}/6\mathbb{Z}$), then the zero ideal of R is a ϕ-primal ideal which is not primal. Hence a ϕ-primal ideal need not be primal.

(2) Let $R = \mathbb{Z}/24\mathbb{Z}$, and consider the ideal $I = 8\mathbb{Z}/24\mathbb{Z}$ of R. It is not difficult to show that I is not a ϕ-primal ideal of R. Now set $P = 2\mathbb{Z}/24\mathbb{Z}$. Then every element of P is not prime to I. Assume that $\bar{a} \notin P$. If $\bar{a}.\bar{n} \in I$ for some $\bar{n} \in R$, then 8 divides n, that is $\bar{n} \in I$. Hence $\bar{a}$ is prime to I. We have shown that $S(I) = P$, that is I is P-primal. This example shows that a primal ideal need not be ϕ-primal.

According to Example 2.10, a ϕ-primal ideal need not necessarily be primal. In Theorems 2.11 and 2.12 we provide some conditions under which a ϕ-primal ideal is primal.

Theorem 2.11. *Let R be a commutative ring and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function. Suppose that I is a P-ϕ-primal ideal of R with $I^2 \nsubseteq \phi(I)$. If P is a prime ideal of R, then I is primal.*

Proof. Assume that $a \in P$. Then either $a \in \phi(I)$ or $a \in S_\phi(I)$. If the former case holds, then $a \in \phi(I) \subseteq I \subseteq S(I)$, and if the latter case holds, then $a \in S_\phi(I) \subseteq S(I)$ by Remark 2.2. So in any case a is not prime to I. Now assume that $b \in R$ is not prime to I. So $rb \in I$ for some $r \in R-I$. If $rb \notin \phi(I)$, then b is not ϕ-prime to I, so $b \in P$. Thus assume that $rb \in \phi(I)$. First suppose that $bI \nsubseteq \phi(I)$. Then, there exists $r_0 \in I$ such that $br_0 \notin \phi(I)$. Then $b(r+r_0) = br + br_0 \in I - \phi(I)$ with $r + r_0 \in R - I$, implies that b is not ϕ-prime to I, that is $b \in P$. Now we may assume that $bI \subseteq \phi(I)$. If $rI \nsubseteq \phi(I)$, then $rc \notin \phi(I)$ for some $c \in I$. In this case $b + c)r = br + cr \in I - \phi(I)$ with $r \in R - I$, that is $b \in P$. So we can assume that $rI \subseteq \phi(I)$. Since $I^2 \nsubseteq \phi(I)$, there are $a_0, b_0 \in I$ with $a_0b_0 \notin \phi(I)$. Then

$(b+a_0)(r+b_0) \in I - \phi(I)$ with $r+b_0 \in R-I$ implies that $b+a_0 \in P$. On the other hand $a_0 \in I \subseteq P$ by Lemma 2.7. So that $b \in P$. We have already shown that P is exactly the set of all elements of R that are not prime to I. Hence I is P-primal. ■

A commutative ring is called decomposable if there exist nontrivial commutative rings R_1 and R_2 such that $R \cong R_1 \times R_2$. A ring R that is not decomposable is called indecomposable. An ideal I of $R = R_1 \times R_2$ will have the form $I_1 \times I_2$ where I_1 and I_2 are ideals of R_1 and R_2, respectively. It is a well-known, and easily proved, result that I is prime if and only if $I = P_1 \times R_2$ or $I = R_1 \times P_2$ where P_i is a prime ideal of R_i. It has also proved in [4, Lemma 13] that if I_1 is a primal ideal of R_1 and I_1 is a primal ideal of R_2, then $I_1 \times R_2$ and $R_1 \times I_2$ are primal ideals of R. In the following Theorem we provide some conditions under which a ϕ-primal ideal of a decomposable ring is primal.

Theorem 2.12. *Let R_1 and R_2 be commutative rings and $R = R_1 \times R_2$. Let $\psi_i : \mathfrak{I}(R_i) \to \mathfrak{I}(R_i) \cup \{\emptyset\}$ $(i = 1, 2)$ be functions with $\psi_i(R_i) \neq R_i$, and set $\phi = \psi_1 \times \psi_2$. Assume that P is an ideal of R with $\phi(P) \neq P$. If I is a P-ϕ-primal ideal of R, then either $I = \phi(I)$ or I is primal.*

Proof. Suppose that I is a P-ϕ-primal ideal of R. We may assume that $I = I_1 \times I_2$ and $I \neq \phi(I)$. By Proposition 2.4, P is a ϕ-prime ideal of R. Therefore, by [1, Theorem 16], we have the following cases:

Case 1. $P = P_1 \times P_2$ where P_i is a proper ideal of R with $\psi_i(P_i) = P_i$. In this case we have $\phi(P) = \psi_1(P_1) \times \psi_2(P_2) = P_1 \times P_2 = P$ which is a contradiction.

Case 2. $P = P_1 \times R_2$ where P_1 is a ψ_1-prime ideal of R_1. Since $\psi_2(R_2) \neq R_2$, P_1 is a prime ideal of R_1 and hence P is a prime ideal of R. We show that $I_2 = R_2$. Since $I \neq \phi(I)$, there exists $(a, b) \in I - \phi(I)$. Then we have $(a, 1)(1, b) = (a, b) \in I - \phi(I)$. If $(a, 1) \notin I$, then $(1, b)$ is not ϕ-prime to I. Hence $(1, b) \in P = P_1 \times R_2$ and so $1 \in P_1$ a contradiction. Thus $(a, 1) \in I$ and so $1 \in I_2$, that is $I_2 = R_2$. Now we prove that I_1 is a P_1-primal ideal of R_1. Pick an element $a_1 \in P_1$. Then $(a_1, 0) \in P_1 \times R_2 = P = S_\phi(I) \cup \phi(I)$. If $(a_1, 0) \in \phi(I) = \psi_1(I_1) \times \psi_2(R_2)$, than $a_1 \in \psi_1(I_1) \subseteq I_1 \subseteq S(I_1)$. Hence a_1 is not prime to I_1. So assume that $(a_1, 0) \in S_\phi(I)$. In this case $(a_1, 0)(r_1, r_2) \in I - \phi(I)$ for some $(r_1, r_2) \in R - I$. So $a_1 r_1 \in I_1 - \psi_1(I_1)$ with $r_1 \in R_1 - I_1$ implies that a_1 is not ψ_1-prime to I_1. Hence a_1 is not prime to I_1 by Remark 2.2. Conversely, assume that b_1 is not prime to I_1. Then $b_1 c_1 \in I_1$ for some $c_1 \in R_1 - I_1$. In this case since $1 \in R_2 - \psi_2(R_2)$, we have $(b_1, 1)(c_1, 1) = (b_1 c_1, 1) \in I_1 \times R_2 - (I_1 \times \psi_2(R_2) \subseteq I - \phi(I)$ with $(c_1, 1) \in R - I$. Hence $(b_1, 1)$ is not ϕ-prime to I. Therefore $(b_1, 1) \in P = P_1 \times R_2$ and hence $b_1 \in P_1$. We have already shown that P_1 consists exactly of those elements of R_1 that are not prime to I_1. Hence I_1 is a P_1 primal ideal of R_1. Now I is a P-primal ideal of R by [4, Lemma 13].

Case 3. $P = R_1 \times P_2$ where P_2 is a ψ_2-prime ideal of R_2. A similar argument as in the Case 2 shows that I is P-primal. ■

Let J be an ideal of R and $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ a function. As in [1] we define $\phi_J : \mathcal{I}(R/J) \to \mathcal{I}(R/J) \cup \{\emptyset\}$ by $\phi_J(I/J) = (\phi(I) + J)/J$ for every ideal $I \in \mathfrak{I}(R)$ with $J \subseteq I$ (and $\phi_J(I/J) = \emptyset$ if $\phi(I) = \emptyset$).

THEOREM 2.13. *Let R be a commutative ring and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function. Let I and J be ideals of R with $J \subseteq \phi(I)$. Then I is a ϕ-primal ideal of R if and only if I/J is a ϕ_J-primal ideal of R/J.*

Proof. Assume that I is a P-ϕ-primal ideal of R. Suppose that $a + J$ is an element of R/J that is not ϕ_J-prime to I/J. There exists $b \in R - I$ with $(a + J)(b + J) \in I/J - \phi_J(I/J)$. In this case $ab \in I - \phi(I)$ with $b \in R - I$ implies that a is not ϕ-prime to I. Hence $a \in S_\phi(I) \subseteq P$, and so $a + J \in P/J$. Now assume that $c + J \in P/J$. Then $c \in P = S_\phi(I) \cup \phi(I)$. If $c \in \phi(I)$, then $c + J \in \phi_J(I/J)$. So assume that $c \in S_\phi(I)$, that is c is not ϕ-prime to I. Then $cd \in I - \phi(I)$ for some $d \in R - I$. Consequently, $(c + J)(d + J) \in I/J - (\phi(I)/J) = I/J - \phi_J(I/J)$ with $d + J \in R/J - I/J$. This implies that $c + J$ is not ϕ_J-prime to I/J; so $c + J \in S_{\phi_J}(I/J)$. We have already shown that $P/J = S_{\phi_J}(I/J) \cup \phi_J(I/J)$. Therefore I/J is ϕ_J-primal.

Conversely, suppose that I/J is ϕ_J-primal in R/J with the adjoint ideal P/J. For every $a \in P - \phi(I)$, we have $a + J \in P/J - \phi_J(I/J)$. So $a + J$ is not ϕ_J-prime to I/J. So $(a+J)(b+J) \in I/J - \phi_J(I/J)$ for some $b + J \in R/J - I/J$. In this case $b \in R - I$ and $ab \in I - \phi(I)$ implies that a is not ϕ-prime to I. Conversely, assume that $c \in R$ is not ϕ-prime to I. In this case $cd \in I - \phi(I)$ for some $d \in R - I$. Then $(c + J)(d + J) \in I/J - \phi_J(I/J)$ with $d + J \notin I/J$, that is $c + J$ is not ϕ_J-prime to I/J. Hence $c + J \in P/J - \phi_J(I/J)$, and hence $c \in P - \phi(I)$. It follows that $P = S_\phi(I) \cup \phi(I)$ which implies that I is P-ϕ-primal in R. ∎

Until further notice, let T be a multiplicatively closed subset of the commutative ring R and let $f : R \to R_T$ denote the natural ring homomorphism given by $r \mapsto r/1$. If J is an ideal of R_T, define $J \cap R = f^{-1}(J)$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be a function and define $\phi_T : \mathcal{I}(R_T) \to \mathcal{I}(R_T) \cup \{\emptyset\}$ by $\phi_T(J) = (\phi(J \cap R))_T$ (and $\phi_T(J) = \emptyset$ if $\phi(J \cap R) = \emptyset$) for every ideal J of R_T. Note that $\phi_T(J) \subseteq J$. In the remainder of this paper we study the relations between the set of ϕ-primal ideals of R and ϕ_T-primal ideals of R_T.

LEMMA 2.14. *Let R be a commutative ring and $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ a function. Let T be a multiplicatively closed subset of R and let I be a P-ϕ-primal ideal of R with $P \cap T = \emptyset$. Let $\lambda \in I_T - (\phi(I))_T$. Then every representation $\lambda = a/s$ of λ as a formal fraction (with $a \in R$ and $s \in T$) must have its numerator in I. Moreover if $(\phi(I))_T \neq I_T$, then $I = I_T \cap R$.*

Proof. Assume that $\lambda = a/s \in I_T - (\phi(I))_T$. Then $a/s = b/t$ for some $b \in I$ and $t \in T$. In this case $uta = usb \in I$ for some $u \in T$. If $uta \in \phi(I)$, then $a/s = (uta)/(uts) \in (\phi(I))_T$ a contradiction. So we have $uta \in I - \phi(I)$. If $a \notin I$, then ut is not ϕ-prime to I; so $ut \in P \cap T$ which contradicts the hypothesis. Therefore $a \in I$.

For the last part, it is clear that $I \subseteq I_T \cap R$. Now pick an element $a \in I_T \cap R$. Then $sa \in I$ for some $s \in T$. If $sa \notin \phi(I)$ and $a \notin I$, then s is not ϕ-prime to I, so $s \in P \cap T$ a contradiction. So a must be in I. If $sa \in \phi(I)$, then $a/1 = (sa)/s \in (\phi(I))_T$, and so $a \in (\phi(I))_T \cap R$. Therefore $I_T \cap R = I \cup ((\phi(I))_T \cap R)$. Hence either $I_T \cap R = I$ or $I_T \cap R = (\phi(I))_T \cap R$. But the latter case does not hold, for otherwise $I_T = (\phi(I))_T$ which is a contradiction. ■

Let R be a commutative ring and M an R-module. An element $a \in R$ is called a zero-divisor on M if $am = 0$ for some $rm = 0$. We denote by $Z_R(M)$ the set of all zero-divisors of R on M.

THEOREM 2.15. *Let R be a commutative ring and $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ a function. Suppose that T is a multiplicatively closed subset of R and I a P-ϕ-primal ideal of R with $P \cap T = \emptyset$, $T \cap Z_R(R/\phi(I)) = \emptyset$ and $(\phi(I))_T \subseteq \phi_T(I_T)$. Then I_T is a ϕ_T-primal ideal of R_T with the adjoint ideal P_T.*

Proof. Suppose that $a/s \in P_T - \phi_T(I_T)$. Since $(\phi(I))_T \subseteq \phi_T(I_T)$ we have $a \notin \phi(I)$. Hence, by Theorem 2.6 and 2.14, $a \in P - \phi(I)$. Thus a is not ϕ-prime to I; so $ab \in I - \phi(I)$ for some $b \in R - I$. If $(ab)/s \in \phi_T(I_T)$, then $(ab)/s = c/t$ for some $c \in \phi(I_T \cap R)$ and $t \in T$. One can shows that $c \in \phi(I)$ and so $utab = usc \in \phi(I)$ shows that $ut \in T \cap Z_R(R/\phi(I))$ a contradiction. So $(ab)/s \notin \phi_T(I_T)$. In this case, by Lemma 2.14, $b/1 \notin I_T$ and $(a/s)(b/1) = (ab)/s \in I_T - \phi_T(I_T)$ implies that a/s is not ϕ_T-prime to I_T. Conversely assume that $a/s \in R_T$ is not ϕ_T-prime to I_T. Then $a/s \notin \phi_T(I_T)$ and $(a/s)(b/t) \in I_T - \phi_T(I_T)$ for some $b/t \in R_T - I_T$. Since $(\phi(I))_T \subseteq \phi_T(I_T)$ we have $(ab)/(st) \in I_T - (\phi(I))_T$. Then, by Lemma 2.14, $ab \in I - \phi(I)$ and $b \in R - I$ implies that a is not ϕ-prime to I. So $a \in P$ and hence $a/s \in P_T - \phi_T(I_T)$. Consequently $P_T = S_{\phi_T}(I_T) \cup \phi_T(I_T)$ shows that I_T is a P_T-ϕ_T-primal ideal of R_T. ■

ACKNOWLEDGEMENT. The author thanks the referees for their valuable comments.

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(received 09.08.2010; in revised form 18.11.2010; available online 20.12.2010)

Department of Mathematics, University of Mohaghegh Ardabili, P. O. Box 179, Ardabil, Iran
E-mail: `yousefian@uma.ac.ir, youseffian@gmail.com`