МАТЕМАТИЧКИ ВЕСНИК
64, 1 (2012), 53–59
March 2012

оригинални научни рад
research paper

ON THE BEST CONSTANTS IN SOME INEQUALITIES FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

Milutin R. Dostanić

Abstract. In this paper we prove that the constants in some inequalities for entire functions of exponential type (trigonometric polynomials) are best possible. Also, we prove an inequality of Bernstein type for entire functions having some additional properties.

1. Introduction

If f is an entire function of exponential type σ such that

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty \qquad (p > 0),$$

then the inequality.

$$\int_{\mathbb{R}} |f'(x)|^p \, dx \leq \sigma^p \int_{\mathbb{R}} |f|^p \, dx \tag{1}$$

holds (Bernstein's inequality). Similar inequality holds for trigonometric polynomials, i.e.

$$\int_{-\pi}^{\pi} |T_n'(x)|^p \, dx \leq n^p \int_{-\pi}^{\pi} |T_n(x)|^p \, dx \tag{2}$$

where T_n is an arbitrary trigonometric polynomial of degree n.

The cases $p \geq 1$ of inequalities (1) and (2) are widely known (see for example [1, 3, 6] but in the case $0 < p < 1$ they are proved (in more general form) in [7] and [2].

If the entire function f (especially trigonometric polynomial T_n) has some additional properties, then the constant σ^p (n^p) in (1) ((2)) can be improved (decreased).

In this paper we prove that the constants in some inequalities for entire functions of exponential type (trigonometric polynomials) are best possible. Also, we prove an inequality of Bernstein type for entire functions having some additional properties.

2010 AMS Subject Classification: Primary 41A17, 30D15; Secondary 26D05.
Keywords and phrases: Bernstein inequality, entire functions of exponential type.
Partially supported by MNZZS Grant No.ON174017

2. Results

THEOREM 1. *Let f be an entire function of exponential type σ such that $\int_{\mathbb{R}} |f(x)|^p\,dx < \infty$ $(p > 0)$, and $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. Then we have*

$$\int_{\mathbb{R}} |i\sigma f(x) + f'(x)|^p\,dx \le c_p \sigma^p \int_{\mathbb{R}} |f(x)|^p\,dx \tag{3}$$

where

$$c_p = \frac{\sqrt{\pi}\,\Gamma(1+\frac{p}{2})}{\Gamma(\frac{1+p}{2})}$$

is the best possible constant.

REMARK 1. The inequalities analogous to (3) hold for trigonometric polynomials: If T_n is an arbitrary trigonometric polynomial of the degree n then inequality

$$\int_{-\pi}^{\pi} |inT_n(x) + T_n'(x)|^p\,dx \le c_p n^p \int_{-\pi}^{\pi} |T_n(x)|^p\,dx \tag{4}$$

holds (see [8, Corollary 14.6.7, p. 555]). Equality holds in (4) for $T_n(x) = a\cos(nx+\alpha)$, $a, \alpha \in \mathbb{R}$. If trigonometric polynomial T_n has at least one coefficient which is not real, then (4) might not be true. For example, if $p = 4$, $T_n = i + \omega \cos nx$ $(\omega \in \mathbb{R})$ and ω is large enough, then (4) is not true.

REMARK 2. In [5], an inequality correlated with inequality (3) was proved without proving that the constant in it is best possible.

The inequality (3) was proved in [7] but it was not proved that the constant c_p is the best possible.

In order to prove Theorem 1, we need the following lemma.

LEMMA 1. *If $p > 0$, $\nu > 1$, then*

$$\lim_{\lambda\to+\infty} \int_{\mathbb{R}} \left|\frac{\sin x}{x}\right|^\nu |\cos\lambda x|^p\,dx = \frac{1}{c_p} \int_{\mathbb{R}} \left|\frac{\sin x}{x}\right|^\nu dx$$

holds.

Proof. In [4] it was proved that for all $\nu > 1$ and $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ the equality

$$\lim_{\lambda\to+\infty} \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \frac{\sin\frac{n\pi+x}{\lambda}}{\frac{n\pi+x}{\lambda}} \right|^\nu = \frac{1}{\pi} \int_{\mathbb{R}} |\frac{\sin t}{t}|^\nu\,dt \tag{5}$$

holds and that there exists a constant $K < \infty$ such that

$$\frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \frac{\sin\frac{n\pi+x}{\lambda}}{\frac{n\pi+x}{\lambda}} \right|^\nu \le K \tag{6}$$

for all $\lambda \ge 1$ and $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Since

$$\lim_{\lambda\to+\infty} \int_0^{+\infty} \left|\frac{\sin x}{x}\right|^\nu |\cos\lambda x|^p\,dx = \lim_{\lambda\to+\infty} \sum_{n=1}^{\infty} \int_{\frac{(2n-1)\pi}{2\lambda}}^{\frac{(2n+1)\pi}{2\lambda}} \left|\frac{\sin x}{x}\right|^\nu |\cos\lambda x|^p\,dx,$$

after change of variable and application of Lebesgue dominated convergence theorem (it can be applied according to (6)) we obtain

$$\lim_{\lambda\to+\infty}\int_0^{+\infty}\left|\frac{\sin x}{x}\right|^{\nu}|\cos\lambda x|^p\,dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|\cos x|^p\Big(\lim_{\lambda\to+\infty}\frac{1}{\lambda}\sum_{n=1}^{\infty}\left|\frac{\sin\frac{n\pi+x}{\lambda}}{\frac{n\pi+x}{\lambda}}\right|^{\nu}\Big)\,dx. \tag{7}$$

From (5) and (7) the statement of the lemma follows. ■

2.1. Proof of Theorem 1.

In [7] the inequality

$$\int_{\mathbb{R}}|\sin\alpha\, f'(t)-\sigma\cos\alpha f(t)|^p\,dt\le\sigma^p\int_{\mathbb{R}}|f(t)|^p\,dt \tag{8}$$

was proved, where $\alpha\in\mathbb{R}$, $p>0$, f is an entire function of the exponential type σ, such that $\int_{\mathbb{R}}|f(t)|^p\,dt<\infty$. Having in mind that

$$\int_{-\pi}^{\pi}|a\cos\varphi+b\sin\varphi|^p d\varphi=\frac{2\sqrt{\pi}\,\Gamma(1+\frac{p}{2})}{\Gamma(\frac{1+p}{2})}(a^2+b^2)^{\frac{p}{2}}\qquad(a,b\in\mathbb{R},\ p>0),$$

by integrating (8) with respect to α over $(-\pi,\pi)$ and applying Fubini Theorem, we obtain (3).

We will now prove that the constant c_p in (3) is best possible.

Let B_p be the smallest constant such that

$$\int_{\mathbb{R}}|i\sigma f(t)+f'(t)|^p\,dt\le B_p\int_{\mathbb{R}}|f(t)|^p\,dt$$

holds for all entire functions of exponential type σ satisfying the following conditions:

(i) $f(x)\in\mathbb{R}$ for all $x\in\mathbb{R}$,

(ii) $\int_{\mathbb{R}}|f(t)|^p\,dt<\infty$.

Obviously, $B_p\le c_p\sigma^p$. We will prove the reverse inequality.

Let $0<\varepsilon<\frac{\sigma}{2}$ and $r=1+[\frac{1}{p}]$. Consider the function

$$f_\varepsilon(z)=\Big(\frac{\sin\frac{\varepsilon z}{r}}{z}\Big)^r\cos(\sigma-\varepsilon)z.$$

We have

$$f'_\varepsilon(x)+i\sigma f_\varepsilon(x)=g_{1,\varepsilon}(x)+g_{2,\varepsilon}(x)+g_{3,\varepsilon}(x) \tag{9}$$

where

$$g_{1,\varepsilon}(x)=\Big(\frac{\sin\frac{\varepsilon x}{r}}{x}\Big)^{r-1}\frac{r(\cos\frac{\varepsilon x}{r}\cdot\frac{\varepsilon x}{r}-\sin\frac{\varepsilon x}{r})}{x^2}\cos(\sigma-\varepsilon)x,$$

$$g_{2,\varepsilon}(x)=\varepsilon\sin(\sigma-\varepsilon)x\cdot\Big(\frac{\sin\frac{\varepsilon x}{r}}{x}\Big)^r,$$

$$g_{3,\varepsilon}(x)=i\sigma\Big(\frac{\sin\frac{\varepsilon x}{r}}{x}\Big)^r e^{i(\sigma-\varepsilon)x}.$$

According to Lemma 1 ($\nu = pr > 1$) we have

$$\|f_\varepsilon\|_p^p \sim \left(\frac{\varepsilon}{r}\right)^{pr-1} \frac{1}{c_p} \int_{\mathbb{R}} \left|\frac{\sin x}{x}\right|^{pr} dx, \qquad \varepsilon \to 0+. \tag{10}$$

Since

$$\|g_{1,\varepsilon}\|_p^p = \sigma^p \left(\frac{\varepsilon}{r}\right)^{pr-1} \int_{\mathbb{R}} |\frac{\sin x}{x}|^{pr} dx,$$

it follows from (10) that

$$\lim_{\varepsilon\to 0+} \frac{\|g_{1,\varepsilon}\|_p^p}{\|f_\varepsilon\|_p^p} = \sigma^p c_p. \tag{11}$$

Having in mind that

$$\begin{aligned}\|g_{3,\varepsilon}\|_p^p &= r^p \left(\frac{\varepsilon}{r}\right)^{pr+p-1} \int_{\mathbb{R}} \left|\frac{\sin x}{x}\right|^{p(r-1)} \left|\frac{x\cos x - \sin x}{x^2}\right|^p |\cos\lambda x|^p\, dx \le \\ &\le r^p \left(\frac{\varepsilon}{r}\right)^{pr+p-1} \int_{\mathbb{R}} \left|\frac{\sin x}{x}\right|^{p(r-1)} \left|\frac{x\cos x - x}{x^2}\right|^p |\cos\lambda x|^p\, dx,\end{aligned}$$

($\lambda = \frac{\sigma-\varepsilon}{\varepsilon} r$), it follows from (10) that

$$\lim_{\varepsilon\to 0+} \frac{\|g_{3,\varepsilon}\|_p^p}{\|f_\varepsilon\|_p^p} = 0. \tag{12}$$

Since

$$\begin{aligned}\|g_{2,\varepsilon}\|_p^p &= \varepsilon^p \left(\frac{\varepsilon}{r}\right)^{pr-1} \int_{\mathbb{R}} \left|\frac{\sin x}{x}\right|^{pr} |\sin\lambda x|^p\, dx \le \\ &\le \varepsilon^p \left(\frac{\varepsilon}{r}\right)^{pr-1} \int_{\mathbb{R}} \left|\frac{\sin x}{x}\right|^{pr} dx, \qquad \left(\lambda = \frac{\sigma-\varepsilon}{\varepsilon} r\right),\end{aligned}$$

it follows from (10) that

$$\lim_{\varepsilon\to 0+} \frac{\|g_{2,\varepsilon}\|_p^p}{\|f_\varepsilon\|_p^p} = 0. \tag{13}$$

From (9) it follows that

$$\|f_\varepsilon' + i\sigma f_\varepsilon\|_p = \|g_{1,\varepsilon}\|_p - \|g_{2,\varepsilon}\|_p - \|g_{3,\varepsilon}\|_p. \tag{14}$$

Since

$$B_p \ge \frac{\|f_\varepsilon' + i\sigma f_\varepsilon\|_p}{\|f_\varepsilon\|_p},$$

from (11)–(14) it follows (when $\varepsilon \to 0$) $B_p^{\frac{1}{p}} \ge \sigma c_p^{\frac{1}{p}}$, i.e. $B_p \ge c_p \sigma^p$. ∎

Remark 3. The inequality (4) can be proved analogously, starting from the inequality

$$\int_{-\pi}^{\pi} |\sin\alpha\, T_n'(x) - n\cos\alpha\, T_n(x)|^p\, dx \le n^p \int_{-\pi}^{\pi} |T_n(x)|^p\, dx.$$

In [9], the following problem is formulated:

PROBLEM 1. Find

$$K_p = \sup_{f \in S} \frac{\|f'\|_p}{\|f\|_p} \qquad (p > 0)$$

where

$$S = \{f : f \text{ entire function exponential type } \sigma, \\ \int_{\mathbb{R}} |f|^p \, dt < \infty \text{ and } f(z) \equiv e^{i\sigma z} \ f(-z)\}.$$

It is clear that $K_p \le \sigma$. Let

$$\mathcal{T} = \{g : g \text{ even entire function of exponential type } \frac{\sigma}{2}, \\ \int_{\mathbb{R}} |g|^p \, dx < \infty, \text{ and } g(x) \in \mathbb{R} \text{ for all } x \in \mathbb{R}\}.$$

If $g \in \mathcal{T}$, then $f(z) \equiv e^{i\frac{\sigma}{2}z} g(z) \in S$, and by Theorem 1 we obtain

$$\begin{aligned} \int_{\mathbb{R}} |f'|^p \, dx &= \int_{\mathbb{R}} |g' + i\frac{\sigma}{2} g|^p \, dx \\ &\le \left(\frac{\sigma}{2}\right)^p c_p \int_{\mathbb{R}} |g|^p \, dx = \left(\frac{\sigma}{2}\right)^p c_p \int_{\mathbb{R}} |f|^p \, dx \end{aligned}$$

and so, we have $K_p^p \ge \left(\frac{\sigma}{2}\right)^p c_p$, i.e. $K_p \ge \frac{\sigma}{2} c_p^{\frac{1}{p}}$. Consequently, for $p > 0$ we obtain

$$\frac{\sigma}{2} c_p^{\frac{1}{p}} \le K_p \le \sigma.$$

THEOREM 2. *Let f be an even entire function of exponential type σ such that $\int_{\mathbb{R}} |f(x)|^p \, dx < \infty$ $(p \ge 2)$. Then the inequality*

$$\int_{\mathbb{R}} |i\sigma f + f'|^p \, dx \le 2^{p-1} \sigma^p \int_{\mathbb{R}} |f|^p \, dx$$

holds.

Proof.

$$\begin{aligned} \int_{\mathbb{R}} |i\sigma f + f'|^p \, dx &= \int_0^\infty |i\sigma f + f'|^p \, dx + \int_{-\infty}^0 |i\sigma f + f'|^p \, dx \\ &= \int_0^\infty |i\sigma f + f'|^p + |f' - i\sigma f|^p \, dx \\ &\le 2^{p-1} \int_0^\infty (|f'(x)|^p + \sigma^p |f(x)|^p) \, dx \end{aligned}$$

$$\begin{aligned}
&\text{(according to the Clarkson-McCarthy inequality)}\\
&= 2^{p-2}\int_{\mathbb{R}} (|f'(x)|^p + \sigma^p |f(x)|^p)\, dx\\
&\le 2^{p-1}\sigma^p \int_{\mathbb{R}} |f|^p\, dx\\
&\text{(according to Bernstein's inequality).} \blacksquare
\end{aligned}$$

If $p \ge 2$ and $f \in S$, then the function $g(z) \equiv e^{-i\frac{\sigma}{2}z}\, f(z)$ is exponential type $\frac{\sigma}{2}$, even and $\int_{\mathbb{R}} |g|^p\, dx = \int_{\mathbb{R}} |f|^p\, dx$. By Theorem 2 we have

$$\begin{aligned}
\int_{\mathbb{R}} |f'|^p\, dx &= \int_{\mathbb{R}} |g' + i\frac{\sigma}{2} g|^p dx\\
&\le 2^{p-1}\Big(\frac{\sigma}{2}\Big)^p \int_{\mathbb{R}} |g|^p\, dx = \frac{\sigma^p}{2}\int_{\mathbb{R}} |f|^p\, dx.
\end{aligned}$$

Therefore, for $p \ge 2$ we have $K_p^p \le \dfrac{\sigma^p}{2}$ and we obtain two sided estimation for K_p:

$$\frac{\sigma}{2}\, c_p^{\frac{1}{p}} \le K_p \le \sigma \cdot 2^{-\frac{1}{p}}.$$

If $p = 2$, then $c_2 = 2$ and $\frac{\sigma}{\sqrt{2}} \le K_2 \le \frac{\sigma}{\sqrt{2}}$, i.e.

$$K_2 = \frac{\sigma}{\sqrt{2}}.$$

(This result is obtained in [9, Theorem 3]).

Open problem. Find best possible constant A_p such that

$$\int_{-\pi}^{\pi} |inT_n(x) + T_n'(x)|^p\, dx \le n^p A_p \int_{-\pi}^{\pi} |T_n(x)|^p\, dx \qquad (p > 0)$$

holds for each polynomial T_n of the form

$$T_n(x) = \sum_{k=0}^{n} a_k \cos kx.$$

Applying theorem of Levitan-Hörmander on approximation of entire functions of exponential type σ by trigonometric polynomials, we obtain

$$\int_{\mathbb{R}} |i\sigma f(x) + f'(x)|^p\, dx \le A_p \sigma^p \int_{\mathbb{R}} |f(x)|^p\, dx$$

if $f \in L^p(\mathbb{R})$ and $f(z) \equiv f(-z)$.. Then we would also have

$$K_p \le \frac{A_p^{\frac{1}{p}}}{2}\, \sigma.$$

Conjecture. $K_p = \frac{A_p^{\frac{1}{p}}}{2}\, \sigma.$

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(received 20.09.2010; in revised form 09.05.2011; available online 01.07.2011)

University of Belgrade, Faculty of Mathematics, Studentski trg 16/IV, 11000 Beograd, Serbia.
E-mail: domi@matf.bg.ac.rs