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MAJORIZATION PROBLEM FOR A SUBCLASS OF p-VALENTLY ANALYTIC FUNCTIONS DEFINED BY THE WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTION

S. P. Goyal and Sanjay Kumar Bansal

Abstract. In this paper we investigate the majorization problem for a subclass of p-valently analytic functions involving the Wright generalized hypergeometric function. Some useful consequences of the main result are mentioned and relevance with some of the earlier results are also pointed out.

1. Introduction

Let f and g be analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C},\ 0 \leq |z| < 1\}.$$

We say that $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$ [16] and write $f(z) \ll g(z) \ \ (z \in \mathbb{U})$, if there exists a function φ, analytic in $\mathbb{U}$ such that

$$|\varphi(z)| \leq 1 \text{ and } f(z) = \varphi(z)g(z)\ (z \in \mathbb{U}). \tag{1.1}$$

Note that majorization is closely related to the concept of quasi-subordination between analytic functions [22].

Further, f is said to be subordinate to g in $\mathbb{U}$, if there exists a Schwarz function $w(z)$ which is analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. We denote this subordination by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

In particular, if $f(z)$ is univalent in $\mathbb{U}$, we have the following equivalence (see [18])

$$f(z) \prec g(z)\ (z \in \mathbb{U}) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

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2. The class $S^q_{p,l,s}[\alpha_1, A_1, A, B; \gamma]$

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N}), \tag{2.1}$$

which are analytic and multivalent in the open unit disk $\mathbb{U}$. In particular if $p = 1$, then $\mathcal{A}_1 = \mathcal{A}$.

For the functions $f_j \in \mathcal{A}_p$ given by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k, \quad (j = 1, 2;\ p \in \mathbb{N}),$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z).$$

Let $l, s \in \mathbb{N}$. For positive real parameters $\alpha_i, A_i; \beta_j, B_j$ $(i = 1, \ldots, l; j = 1, \ldots, s)$, with

$$1 + \sum_{j=1}^{s} B_j - \sum_{i=1}^{l} A_i \geq 0,$$

the Fox-Wright function ${}_l\psi_s$ is defined by (see [24])

$${}_l\psi_s[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,s}; z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{l} \Gamma(\alpha_j + nA_j) z^n}{\prod_{j=1}^{s} \Gamma(\beta_j + nB_j) n!} \quad (z \in \mathbb{U}).$$

In particular, when $A_i = B_j = 1$ $(i = 1, \ldots, l; j = 1, \ldots, s)$, we have the following relationship:

$${}_lF_s(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_s; z) = \Omega\ {}_l\psi_s[(\alpha_1, 1)_{1,l}; (\beta_j, 1)_{1,s}; z] \quad (l \leq s+1; z \in \mathbb{U}),$$

where

$$\Omega := \frac{\Gamma(\beta_1) \ldots \Gamma(\beta_s)}{\Gamma(\alpha_1) \ldots (\alpha_l)}.$$

The Fox-Wright generalized hypergeometric function has been used in many papers on geometric function theory [see e.g. [3–5,8,9]).

Corresponding to the function ϕ_p defined by

$$\phi_p[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,s}; z] = \Omega z^p\ \ {}_l\psi_s[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,s}; z] \quad (z \in \mathbb{U}),$$

Dziok and Raina [8] considered a linear operator

$$\theta_p[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_s, B_s)] : \mathcal{A}_p \longrightarrow \mathcal{A}_p$$

defined by the following Hadamard product

$$\theta_{p,l,s}(\alpha_1, A_1) f(z) := \phi_p[(\alpha_j, A_j)_{1,l}; (b_j, \beta_j)_{1,s}; z] * f(z).$$

If $f \in \mathcal{A}_p$ is given by the equation (2.1), then we have

$$\theta_{p,l,s}(\alpha_1, A_1) f(z) = z^p + \Omega \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{l} \Gamma(\alpha_j + nA_j)}{\prod_{j=1}^{s} \Gamma(\beta_j + nB_j) n!} a_{n+p} z^{n+p} \quad (z \in \mathbb{U}).$$

In particular, for $A_i = B_j = 1 (i = 1, \ldots, l; j = 1, \ldots, s)$, we get the linear operator

$$\mathcal{H}_{p,l,s}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_s)f(z) = \mathcal{H}_{p,l,s}(\alpha_1)f(z)$$
$$= z^p + \sum_{n=1}^{\infty} \frac{\Pi_{j=1}^{l}(\alpha_j)_n}{\Pi_{j=1}^{s}(\beta_j)_n n!} a_{n+p} z^{n+p} \quad (z \in \mathbb{U}),$$

studied by Dziok and Srivastava [10]. It should be remarked that the linear operator $H_{p,l,s}(\alpha_1)$ is a generalization of many other linear operators considered earlier. In particular, for $f(z) \in \mathcal{A}_p$, we have the following observations:

(i) $H_{p,2,1}(a, 1; c)f(z) = L_p(a; c)f(z)$ $(a \in \mathbb{R};\ c \in \mathbb{R} \setminus \mathbb{Z}_0^-)$, where $L_p(a; c)$ is the linear operator studied earlier by Saitoh [21]. It yields another operator $L(a, c)f(z)$ introduced by Carlson and Shaffer [6] for p=1.

(ii) $H_{p,2,1}(n+p, 1; 1)f(z) = D^{n+p-1}f(z)$ $(n \in \mathbb{N};\ n > -p)$, the linear operator studied by Goel and Sohi [11]. In the case $p = 1$, we get $D^n f(z)$, the well-known Ruscheweyh derivative [20] of $f(z) \in \mathcal{A}$.

(iii) $H_{p,2,1}(c, \lambda + p; a)f(z) = I_p^{\lambda}(a, c)f(z)$ $(a, n \in \mathbb{N} \setminus \mathbb{Z}_0^-,\ \lambda > -p)$, where I_p^{λ} is the linear operator studied earlier by Cho, Kwon and Srivastava [7].

(iv) $H_{p,2,1}(1, p+1; n+p)f(z) = I_{n,p}f(z)$ $(n \in \mathbb{Z};\ n > -p)$, where I_{n,p^-} is the extended integral operator considerd by Liu and Noor [15].

(v) $H_{p,2,1}(p+1, 1; p+1-\lambda)f(z) = \Omega_z^{\lambda,p}f(z)$ $(-\infty < \lambda < p+1)$, where $\Omega_z^{\lambda,p}$ is the extended fractional differintegral operator studied by Patel and Mishra [19].

It is easy to verify the following three-term recurrence relation for the operator $\theta_{p,l,s}$:

$$z\left(\theta_{p,l,s}(\alpha_1, A_1)f(z)\right)^{(q+1)} = \frac{\alpha_1}{A_1}\left(\theta_{p,l,s}(\alpha_1 + 1, A_1)f(z)\right)^{(q)}$$
$$-\left(\frac{\alpha_1}{A_1} - p + q\right)\left(\theta_{p,l,s}(\alpha_1, A_1)f(z)\right)^{(q)} \quad (p \in \mathbb{N},\ q \in \mathbb{N} \cup \{0\},\ p > q). \tag{2.2}$$

Using of the operator $\theta_{p,l,s}\{(\alpha_1, A_1)\}$, we now introduce the following subclass of functions $f \in \mathcal{A}_p$:

Definition 1. A function $f(z) \in \mathcal{A}_p$ is said to be in the class $S_{p,l,s}^{q}[\alpha_1, A_1, A, B; \gamma]$ of p-valently analytic functions of complex order $\gamma \neq 0$ in $\mathbb{U}$ if and only if

$$\frac{z(\theta_{p,l,s}(\alpha_1, A_1)f(z))^{(q+1)}}{(\theta_{p,l,s}(\alpha_1, A_1)f(z))^{(q)}} - p + q \prec \gamma \frac{(A-B)z}{1+Bz}, \tag{2.3}$$

$(z \in \mathbb{U}, -1 \leq B < A \leq 1, \alpha_i, A_i, B_j, \beta_j > 0,\ (i = 1, \ldots, l; j = 1, \ldots, s),\ p \in \mathbb{N}, q \in \mathbb{N}_0, p > q$ and $\gamma \in \mathbb{C}^* = \mathbb{C} - \{0\})$.

Also, $T_{p,l,s}^{q}(\alpha_1, A_1; \gamma) = S_{p,l,s}^{q}(\alpha_1, A_1, 1, -1; \gamma)$, where $T_{p,l,s}^{q}(\alpha_1, A_1; \gamma)$ denote the class of functions $f \in \mathcal{A}_p$ satisfying the following inequality.

$$\operatorname{Re}\left\{\frac{1}{\gamma}\left(\frac{z(\theta_{p,l,s}(\alpha_1, A_1)f(z))^{(q+1)}}{(\theta_{p,l,s}(\alpha_1, A_1)f(z))^{(q)}} - p + q\right)\right\} > -1.$$

Obviously we have the following relationships:

(i) $T^q_{p,1,0}(1;\gamma) = S^q_p(\gamma)$;

(ii) $T^0_{1,1,0}(1;\gamma) = S(\gamma)(\gamma \in \mathbb{C}^*)$ (see [17] and [23]);

(iii) $T^0_{1,1,0}(1;1-\alpha) = S^*(\alpha)(0 \le \alpha < 1)$.

Further we observe that:

(i) For $q = 0, l = s+1, \alpha_1 = \beta_1 = p, A_1 = B_1 = 1, \alpha_i = A_i = \beta_j = B_j = 1$ $(i = 2,3,\ldots,s+1;\ j = 2,3,\ldots,s)$, our class $T^q_{p,l,s}(\alpha_1, A_1, ;\gamma)$ reduces to the class $S_p(\gamma)$ $(\gamma \in \mathbb{C}^*)$ of p-valently starlike functions of order γ in $\mathbb{U}$, where

$$S_p(\gamma) = \left\{ f(z) \in \mathcal{A}_p \ :\ \mathrm{Re}\left(1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - p\right)\right) > 0,\ p \in \mathbb{N},\ \gamma \in \mathbb{C}^* \right\}.$$

(ii) For $q = 0, l = s+1, \alpha_1 = p+1, \beta_1 = p, A_1 = B_1 = 1, \alpha_i = A_i = \beta_j = B_j = 1$ $(i = 2,3,\ldots,s+1;\ j = 2,3,\ldots,s)$, $T^q_{p,l,s}(\alpha_1, A_1, ;\gamma)$ reduces to the class $K_p(\gamma)$ $(\gamma \in \mathbb{C}^*)$ of p-valently convex functions of order γ in $\mathbb{U}$, where

$$K_p(\gamma) = \left\{ f(z) \in \mathcal{A}_p \ :\ \mathrm{Re}\left(1 + \frac{1}{\gamma}\left(1 + \frac{zf''(z)}{f'(z)} - p\right)\right) > 0,\ p \in \mathbb{N}, \gamma \in \mathbb{C}^* \right\}.$$

We shall require the following lemma

Lemma 1. [1] *Let $\gamma \in \mathbb{C}^*$ and $f \in K^q_p(\gamma)$. Then*

$$K^q_p(\gamma) \subset S^q_p(\frac{1}{2}\gamma) \qquad (\gamma \in \mathbb{C}^*).$$

Altintas et al. [1] investigated the majorization problem for the class $S(\gamma)(\gamma \in \mathbb{C}^*)$. Macgregor [16] investigated the same problem for the class $S^* \equiv S^*(0)$, while Goyal and Goswami [14] and Goyal, Bansal and Goswami [13], Goswami and Wang [12] have investigated the majorization problem for certain subclasses of analytic functions defined by derivatives and Saitoh operators. In this paper we investigate majorization problem for the class $S^q_{p,l,s}[\alpha_1, A_1, A, B;\gamma]$ which is an extension of all the aforementioned and related subclasses. We also give some special cases of our main result.

3. Majorization problem for the class $S^q_{p,l,s}[\alpha_1, A_1, A, B;\gamma]$

We shall assume throughout the paper that $-1 \le B < A \le 1$, $\gamma \in \mathbb{C}^*$; $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$, $\alpha_i, A_i, \beta_j, B_j > 0$, $(i = 1,\ldots,l;\ j = 1,\ldots,s)$ and $p > q$.

Theorem 1. *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in S^q_{p,l,s}[\alpha_1, A_1, A, B;\gamma]$ and $\frac{\alpha_1}{A_1} > |\gamma(A-B) + \frac{\alpha_1}{A_1}B|$. If*

$$(\theta_{p,l,s}(\alpha_1, A_1)f(z))^{(q)} \ll (\theta_{p,l,s}(\alpha_1, A_1)g(z))^{(q)} \ \ (z \in \mathbb{U}), \tag{3.1}$$

then

$$|(\theta_{p,l,s}(\alpha_1+1, A_1)f(z))^{(q)}| \le |(\theta_{p,l,s}(\alpha_1+1, A_1)g(z))^{(q)}| \ \textit{for}\ |z| \le r_0,$$

where $r_0 = r_0(\gamma, \alpha_1, A_1, A, B)$ is the smallest positive root of the equation

$$r^3\left|\frac{\alpha_1}{A_1}B+\gamma(A-B)\right|-\left(\frac{\alpha_1}{A_1}+2|B|\right)r^2-\left[\left|\gamma(A-B)+\frac{\alpha_1}{A_1}B\right|+2\right]r+\frac{\alpha_1}{A_1}=0. \quad (3.2)$$

Proof. Since $g \in S^q_{p,l,s}[\alpha_1, A_1, A, B; \gamma]$, we find from (2.3) that

$$\frac{z(\theta_{p,l,s}(\alpha_1, A_1)g(z))^{(q+1)}}{(\theta_{p,l,s}(\alpha_1, A_1)g(z))^{(q)}} - p + q = \frac{\gamma(A-B)\omega(z)}{1+B\omega(z)}, \quad (3.3)$$

where $\omega(z) = c_1 z + c_2 z^2 + \ldots, \omega \in \mathcal{P}, \mathcal{P}$ denotes the well known class of the bounded analytic functions in $\mathbb{U}$ (see [18]) and satisfies the conditions

$$\omega(0) = 0, \text{ and } |\omega(z)| \le |z| \ (z \in \mathbb{U}). \quad (3.4)$$

Using (2.2) and (3.4) in (3.3), we get

$$|\,(\theta_{p,l,s}(\alpha_1, A_1)g(z))^{(q)}\,| \le \frac{\frac{\alpha_1}{A_1}[1+|B||z|]}{\frac{\alpha_1}{A_1} - |\frac{\alpha_1}{A_1}B + (A-B)\gamma||z|}|\,(\theta_{p,l,s}(\alpha_1+1, A_1)g(z))^{(q)}\,|, \quad (3.5)$$

provided that $\frac{\alpha_1}{A_1} > |\frac{\alpha_1}{A_1}B + \gamma(A-B)|$ and $z \in \mathbb{U}$.

Next, since $(\theta_{p,l,s}(\alpha_1, A_1)f(z))^{(q)}$ is majorized by $(\theta_{p,l,s}(\alpha_1, A_1)g(z))^{(q)}$ in the unit disk $\mathbb{U}$, we have from (1.1) that

$$(\theta_{p,l,s}(\alpha_1, A_1)f(z))^{(q)} = \varphi(z)\,(\theta_{p,l,s}(\alpha_1, A_1)g(z))^{(q)}\,, \quad (3.6)$$

where $|\phi(z)| \le 1$.

Differentiating (3.6) with respect to 'z' and multiplying by 'z', we get

$$\begin{aligned} &z(\theta_{p,l,s}(\alpha_1, A_1)f(z))^{(q+1)} \\ &\qquad = z\varphi'(z)\,(\theta_{p,l,s}(\alpha_1, A_1)g(z))^{(q)} + z\varphi(z)(\theta_{p,l,s}(\alpha_1, A_1)g(z))^{(q+1)}, \end{aligned}$$

which on using (2.2) once again, yields

$$\begin{aligned} &(\theta_{p,l,s}(\alpha_1+1, A_1)f(z))^{(q)} \\ &\qquad = \frac{A_1}{\alpha_1}z\varphi'(z)\,(\theta_{p,l,s}(\alpha_1, A_1)g(z))^{(q)} + \varphi(z)\,(\theta_{p,l,s}(\alpha_1+1, A_1)g(z))^{(q)}. \quad (3.7) \end{aligned}$$

Thus, noting that $\varphi \in \mathcal{P}$ satisfies the inequality (see, e.g. Nehari [18])

$$|\varphi'(z)| \le \frac{1-|\varphi(z)|^2}{1-|z|^2} \quad (z \in \mathbb{U}), \quad (3.8)$$

and making use of (3.5) and (3.8) in (3.7), we get

$$\begin{aligned} &|\,(\theta_{p,l,s}(\alpha_1+1, A_1)f(z))^{(q)}\,| \le \left[|\varphi(z)| + \left(\frac{1-|\varphi(z)|^2}{1-|z|^2}\right)\right. \\ &\qquad \left.\left(\frac{|z|(1+|B||z|)}{\frac{\alpha_1}{A_1} - |\frac{\alpha_1}{A_1}B + (A-B)\gamma||z|}\right)\right]|\,(\theta_{p,l,s}(\alpha_1+1, A_1)g(z))^{(q)}\,|, \end{aligned}$$

which upon setting $|z| = r$ and $|\varphi(z)| = \rho$ $(0 \le \rho \le 1)$ leads to the inequality

$$| (\theta_{p,l,s}(\alpha_1+1, A_1) f(z))^{(q)} | \le \frac{\upsilon(\rho)}{(1-r^2)[\frac{\alpha_1}{A_1} - |\frac{\alpha_1}{A_1}B + (A-B)\gamma|r]} | (\theta_{p,l,s}(\alpha_1+1, A_1) g(z))^{(q)} |,$$

where

$$\upsilon(\rho) = -r(1+|B|r)\rho^2 + (1-r^2)\rho \left[\frac{\alpha_1}{A_1} - |\frac{\alpha_1}{A_1}B + (A-B)\gamma)|r\right] + r(1+|B|r).$$

takes its maximum value at $\rho = 1$ with $r_0 = r_0(\gamma, \alpha_1, A_1, A, B)$ is the smallest positive root of the equation (3.2). Furthermore, if $0 \le \sigma \le r_0$, then the function $\chi(\rho)$ defined by

$$\chi(\rho) = -\sigma(1+|B|\sigma)\rho^2 + (1-\sigma^2)\Big[\frac{\alpha_1}{A_1} - |\frac{\alpha_1}{A_1}B + (A-B)\gamma)|\sigma\Big] + \sigma(1+|B|\sigma) \quad (3.9)$$

is an increasing function on the interval $0 \le \rho \le 1$, so that

$$\chi(\rho) \le \chi(1) = (1-\sigma^2)\Big[\frac{\alpha_1}{A_1} - |\frac{\alpha_1}{A_1}B + (A-B)\gamma)|\sigma\Big] \;\; (0 \le \rho \le 1;\; 0 \le \sigma \le r_0).$$

Hence, upon setting $\rho = 1$ in (3.9), we conclude that (3.1) of Theorem 1 holds true for $|z| \le r_0 = r_0(\gamma, \alpha_1, A_1, A, B)$ where $r_0(\gamma, \alpha_1, A_1, A, B)$ is the smallest positive root of the equation (3.2). In fact, as one can see easily, in any case, either $\left|\frac{\alpha_1}{A_1}B + (A-B)\gamma\right| \neq 0$, or if it is equal to zero, (3.2) has a unique root in the interval $(0, 1)$ and this is the smallest positive root of equation (3.2). This completes the proof of the theorem. ■

4. Special cases

Setting $A_i = B_j = 1$, $(i = 1, \dots, l;\ j = 1, \dots, s)$ in Theorem 1, we get the following result:

Corollary 1. *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in S^q_{p,l,s}(\alpha_1, A, B; \gamma)$ and $\alpha_1 > |\gamma(A-B) - \alpha_1 B|$. If $(\mathcal{H}_{p,l,s}(\alpha_1)f(z))^{(q)} \ll (\mathcal{H}_{p,l,s}(\alpha_1)g(z))^{(q)}$, $z \in \mathbb{U}$, then*

$$|(\mathcal{H}_{p,l,s}(\alpha_1+1)f(z))^{(q)}| \le |(\mathcal{H}_{p,l,s}(\alpha_1+1)g(z))^{(q)}| \text{ for } |z| \le r_1,$$

where $r_1 = r_1(\gamma, \alpha_1, A, B)$ is the smallest positive root of the equation

$$r_1^3|\gamma(A-B) + \alpha_1 B| - (\alpha_1 + 2|B|)r_1^2 - \Big[|\gamma(A-B) + \alpha_1 B| + 2\Big]r_1 + \alpha_1 = 0.$$

Setting $A = 1$ and $B = -1$, in Theorem 1, we get

Corollary 2. *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in T^q_{p,l,s}(\alpha_1, A_1, \gamma)$ and $\frac{\alpha_1}{A_1} > \left|2\gamma - \frac{\alpha_1}{A_1}\right|$. If $(\theta_{p,l,s}(\alpha_1, A_1)f(z))^{(q)} \ll (\theta_{p,l,s}(\alpha_1, A_1)g(z))^{(q)}$ in $\mathbb{U}$, then*

$$|(\theta_{p,l,s}(\alpha_1+1, A_1)f(z))^{(q)}| \;\le\; |(\theta_{p,l,s}(\alpha_1+1, A_1)g(z))^{(q)}| for |z| \le r_2, \quad (4.1)$$

where

$$r_2 = r_2(\gamma, \alpha_1, A_1) = \begin{cases} \frac{k-\sqrt{k^2-4\frac{\alpha_1}{A_1}\left|2\gamma-\frac{\alpha_1}{A_1}\right|}}{2\left|2\gamma-\frac{\alpha_1}{A_1}\right|}, & \textit{if } 2\gamma \neq \frac{\alpha_1}{A_1}, \\ \frac{\alpha_1}{\alpha_1+2A_1}, & \textit{if } 2\gamma = \frac{\alpha_1}{A_1} \end{cases}$$

$(k = 2 + \frac{\alpha_1}{A_1} + |2\gamma - \frac{\alpha_1}{A_1}|;\ \gamma \in \mathbb{C}^*)$.

Remark 1. The expression under the square root in (4.1) is positive, since

$$\begin{aligned} k^2 - 4\frac{\alpha_1}{A_1}\left|2\gamma - \frac{\alpha_1}{A_1}\right| &= \left(\frac{\alpha_1}{A_1} - \left|2\gamma - \frac{\alpha_1}{A_1}\right|\right)^2 + 4 + \frac{4\alpha_1}{A_1} + 4\left|2\gamma - \frac{\alpha_1}{A_1}\right| \\ &> \left(\frac{\alpha_1}{A_1} - \left|2\gamma - \frac{\alpha_1}{A_1}\right|\right)^2 + 4 + 8\left|2\gamma - \frac{\alpha_1}{A_1}\right| > 0. \end{aligned}$$

Further, putting $l = s+1$, $\alpha_1 = \beta_1 = p$, $A_1 = B_1 = 1$, $\alpha_i = A_i = \beta_j = B_j = 1$, $(i = 2, \dots, s+1; j = 2, \dots, s)$, in Corollary 2, we get

Corollary 3. [1] *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in S_p^q$. If $(f(z))^{(q)} \ll (g(z))^{(q)}$ in $\mathbb{U}$, then*

$$|(f(z))^{(q+1)}| \leq |(g(z))^{(q+1)}| \textit{ for } |z| \leq r_3,$$

where

$$r_3 = r_3(\gamma, p, q) = \frac{k - \sqrt{k^2 - 4p|2\gamma - p + q|}}{2|2\gamma - p + q|} \tag{4.2}$$

$(k = 2 + p - q + |2\gamma - p + q|;$ *and* $p \in \mathbb{N},\ q \in \mathbb{N}_0,\ \gamma \in \mathbb{C}^*)$.

Putting $q = 0$ in Corollary 3, we obtain

Corollary 4. *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in S_p(\gamma)$. If $f(z) \ll g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leq |g'(z)| \textit{ for } |z| \leq r_4,$$

where

$$r_4 = r_4(\gamma, p) = \frac{k - \sqrt{k^2 - 4p|2\gamma - p|}}{2|2\gamma - p|} \tag{4.3}$$

$(k = 2 + p + |2\gamma - p|;$ *and* $p \in \mathbb{N},\ \gamma \in \mathbb{C}^*)$.

Putting $q = 0$, $l = s+1$, $\beta_1 = p$, $\alpha_1 = p+1$, $A_1 = B_1 = 1$, $\alpha_i = A_i = B_j = \beta_j = 1$ $(i = 2, 3, \dots, s+1;\ j = 2, 3, \dots, s)$ in Corollary 2, with the aid of Lemma 1, we get the following result.

Corollary 5. *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in K_p(\gamma)$. If $f(z) \ll g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leq |g'(z)| for |z| \leq r_5,$$

where

$$r_5 = r_5(\gamma, p) = \frac{k - \sqrt{k^2 - 4p|\gamma - p|}}{2|\gamma - p|} \tag{4.4}$$

$(k = 2 + p + |\gamma - p|;$ *and* $p \in \mathbb{N},\ \gamma \in \mathbb{C}^*)$.

Further, putting $l = 2$, $s = 1$, $\alpha_1 = \alpha$, $\alpha_2 = 1$, $\beta_1 = \beta$, $A_1 = A_2 = B_1 = 1$ in Corollary 2, we get

Corollary 6. *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in T^q_{p,2,1}(\alpha_1, 1, \beta; \gamma)$ and $\alpha \geq |2\gamma - \alpha|$. If $(L_p(\alpha,\beta)f(z))^{(q)} \ll (L_p(\alpha,\beta)g(z))^{(q)}$ in $\mathbb{U}$, then*

$$|(L_p(\alpha+1,\beta)f(z))^{(q)}| \leq |(L_p(\alpha+1,\beta)g(z))^{(q)}| for |z| \leq r_6,$$

where

$$r_6 = r_6(\gamma,\alpha) = \begin{cases} \frac{k-\sqrt{k^2-4\alpha|2\gamma-\alpha|}}{2|2\gamma-\alpha|}, & if\ 2\gamma \neq \alpha, \\ \frac{\alpha}{\alpha+2} & , if\ 2\gamma = \alpha \end{cases} \tag{4.5}$$

$(k = 2 + \alpha + |2\gamma - \alpha|$ *and* $\gamma \in \mathbb{C} \setminus \{0\})$.

This is a known result obtained recently by Goyal, Bansal and Goswami [13].

Further, putting $\alpha = 1$, we get a known result obtained by Altinas et al. [2], which contains another known result obtained by MacGregor [16], when $\gamma = 1$. Also, putting $\alpha = p + 1$ and $\beta = p - \lambda + 1$ in Corollary 6, we get a known result obtained by Goyal and Goswami [14].

Remark 2. In view of Remark 1 mentioned with Corollary 1, it can be proved easily that the expressions under the square roots occurring in (4.2)–(4.5) are positive.

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Department of Mathematics, University of Rajasthan, Jaipur-302004, India

E-mail: `somprg@gmail.com`

Department of Mathematics, Global Institute of Technology, Sitapura, Jaipur 302022, India

E-mail: `bansalindian@gmail.com`