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$N(k)$-QUASI EINSTEIN MANIFOLDS SATISFYING CERTAIN CURVATURE CONDITIONS

Uday Chand De and Sahanous Mallick

Abstract. The object of the present paper is to study $N(k)$-quasi Einstein manifolds. Existence of $N(k)$-quasi Einstein manifolds are proved by two non-trivial examples. Also a physical example of an $N(k)$-quasi-Einstein manifold is given. We study an $N(k)$-quasi-Einstein manifold satisfying the curvature conditions $\tilde{Z}(\xi, X)\cdot S = 0$, $P(\xi, X)\cdot \tilde{Z} = 0$, $\tilde{Z}(\xi, X)\cdot P = 0$, $\tilde{Z}(\xi, X)\cdot C = 0$ and $P(\xi, X)\cdot C = 0$. Finally, we study Ricci-pseudosymmetric $N(k)$-quasi-Einstein manifolds.

1. Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g), $n = \dim M \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n} g \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to [4, p. 432], (1.1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor [4, pp. 432–433]. For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation:

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{1.2}$$

where a, b are smooth functions and η is a non-zero 1-form such that

$$g(X, \xi) = \eta(X), \qquad g(\xi, \xi) = \eta(\xi) = 1 \tag{1.3}$$

for all vector fields X.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold [7] if its Ricci tensor S of type $(0, 2)$ is not identically zero and

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satisfies the condition (1.2). We shall call η the associated 1-form and the unit vector field ξ is called the generator of the manifold.

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. Several authors have studied Einstein's field equations. For example, in [15], Naschie turned the tables on the theory of elementary particles and showed the expectation number of elementary particles of the standard model using Einstein's unified field equation. He also discussed possible connections between Gödel's classical solution of Einstein's field equations and E-infinity in [14]. Also quasi Einstein manifolds have some importance in the general theory of relativity. For instance, the Robertson-Walker spacetime are quasi Einstein manifolds [13]. Further, quasi Einstein manifold can be taken as a model of the perfect fluid spacetime in general relativity [9].

The study of quasi Einstein manifolds was continued by Chaki [5], Guha [16], De and Ghosh [10, 11] and many others. The notion of quasi-Einstein manifolds have been generalized in several ways by several authors. In recent papers Özgür studied super quasi-Einstein manifolds [20] and generalized quasi-Einstein manifolds [18]. Also Nagaraja [17] studied $N(k)$-mixed quasi-Einstein manifolds.

Let R denote the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$ of a Riemannian manifold M [23] is defined by

$$N(k)\colon p \mapsto N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\},$$

k being some smooth function. In a quasi Einstein manifold M, if the generator ξ belongs to some k-nullity distribution $N(k)$, then M is said to be a $N(k)$-quasi Einstein manifold [24]. In fact, k is not arbitrary as the following shows:

Lemma 2.1. [22] *In an n-dimensional $N(k)$-quasi Einstein manifold it follows that*

$$k = \frac{a+b}{n-1} \tag{1.4}$$

Now, it is immediate to note that in an n-dimensional $N(k)$-quasi-Einstein manifold [22]

$$R(X,Y)\xi = \frac{a+b}{n-1}[\eta(Y)X - \eta(X)Y], \tag{1.5}$$

which is equivalent to

$$R(X,\xi)Y = \frac{a+b}{n-1}[\eta(Y)X - g(X,Y)\xi] = -R(\xi,X)Y.$$

From (1.5) we get

$$R(\xi,X)\xi = \frac{a+b}{n-1}[\eta(X)\xi - X].$$

In [24], it was shown that an n-dimensional conformally flat quasi Einstein manifold is an $N(\frac{a+b}{n-1})$-quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is an $N(\frac{a+b}{2})$-quasi Einstein manifold. Also, in [19], Özgür cited some physical examples of $N(k)$-quasi Einstein manifolds. All these motivated us to study such a manifold.

The conformal curvature tensor play an important role in differential geometry and also in general theory of relativity. The Weyl conformal curvature tensor C of a Riemannian manifold (M^n, g) $(n > 3)$ is defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]$$

where r is the scalar curvature and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is, $g(QX,Y) = S(X,Y)$. If the dimension $n = 3$, then the conformal curvature tensor vanishes identically.

The projective curvature tensor P and the concircular curvature tensor $\tilde{Z}$ in a Riemannian manifold (M^n, g) are defined by [25]

$$P(X,Y)W = R(X,Y)W - \frac{1}{n-1}[S(Y,W)X - S(X,W)Y],$$

$$\tilde{Z}(X,Y)W = R(X,Y)W - \frac{r}{n(n-1)}[g(Y,W)X - g(X,W)Y],$$

respectively.

In [24], the authors prove that conformally flat quasi-Einstein manifolds are certain $N(k)$-quasi-Einstein manifolds. The derivation conditions $R(\xi,X)\cdot R = 0$ and $R(\xi,X)\cdot S = 0$ have been studied in [23], where R and S denote the curvature tensor and Ricci tensor respectively. Özgür and Tripathi [22] continued the study of the $N(k)$-quasi-Einstein manifold. In [22], the derivation conditions $\tilde{Z}(\xi,X)\cdot R = 0$ and $\tilde{Z}(\xi,X)\cdot\tilde{Z} = 0$ on $N(k)$-quasi-Einstein manifold were studied, where $\tilde{Z}$ is the concircular curvature tensor. Moreover in [22], for an $N(k)$-quasi-Einstein manifold it was proved that $k = \frac{a+b}{n-1}$. Özgür [19] studied the condition $R\cdot P = 0$, $P\cdot S = 0$ and $P\cdot P = 0$ for an $N(k)$-quasi-Einstein manifolds, where P denotes the projective curvature tensor and some physical examples of $N(k)$-quasi-Einstein manifolds are given. Again, in 2008, Özgür and Sular [21] studied $N(k)$-quasi-Einstein manifold satisfying $R\cdot C = 0$ and $R\cdot\tilde{C} = 0$, where C and $\tilde{C}$ represent the conformal curvature tensor and the quasi-conformal curvature tensor, respectively. This paper is a continuation of previous studies.

The paper is organized as follows: After preliminaries in Section 3, we give two examples of $N(k)$-quasi-Einstein manifolds. In the next Section we give a physical example of an $N(k)$-quasi-Einstein manifold. In Section 5, we study $N(k)$-quasi-Einstein manifold satisfying $\tilde{Z}(\xi,X)\cdot S = 0$ and Section 6 deals with $N(k)$-quasi-Einstein manifolds satisfying $P(\xi,X)\cdot\tilde{Z} = 0$. In Section 7 and Section 8, we study $N(k)$-quasi-Einstein manifolds satisfying $\tilde{Z}(\xi,X)\cdot P = 0$ and $P(\xi,X)\cdot C = 0$ respectively. In Section 9, we study $N(k)$-quasi-Einstein manifold satisfying the condition $\tilde{Z}(\xi,X)\cdot C = 0$. Finally, we study Ricci-pseudosymmetric $N(k)$-quasi-Einstein manifolds.

2. Preliminaries

From (1.2) and (1.3) it follows that

$$r = an + b \quad \text{and} \quad QX = (a+b)X,$$
$$S(X,\xi) = k(n-1)\eta(X),$$

where r is the scalar curvature and Q is the Ricci operator.

In an n-dimensional $N(k)$-quasi-Einstein manifold M, the projective curvature tensor P, the concircular curvature tensor $\tilde{Z}$ and the conformal curvature tensor C satisfy the following relations:

$$P(X,Y)\xi = 0,$$
$$P(\xi,X)Y = \frac{b}{n-1}[g(X,Y)\xi - \eta(X)\eta(Y)\xi], \tag{2.1}$$
$$\eta(P(X,Y)Z) = \frac{b}{n-1}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.2}$$
$$\tilde{Z}(X,Y)Z = [k - \frac{r}{n(n-1)}][g(Y,Z)X - g(X,Z)Y], \tag{2.3}$$
$$\tilde{Z}(\xi,Y)Z = [k - \frac{r}{n(n-1)}][g(Y,Z)\xi - \eta(Z)Y], \tag{2.4}$$
$$\begin{aligned} C(X,Y)Z = [k + \frac{r}{(n-1)(n-2)}][g(Y,Z)X - g(X,Z)Y] - \frac{1}{n-2}[(2a+b)g(Y,Z)X \\ - (2a+b)g(X,Z)Y + b\eta(Y)\eta(Z)X - b\eta(X)\eta(Z)Y], \end{aligned} \tag{2.5}$$
$$\eta(C(X,Y)Z) = 0, \tag{2.6}$$
$$C(\xi,Y)Z = -\frac{b}{n-2}[\eta(Y)\eta(Z)\xi - \eta(Z)Y], \tag{2.7}$$

for all vector fields X, Y, Z on M.

3. Examples of $N(k)$-quasi Einstein manifolds

Example 3.1. A special para-Sasakian manifold with vanishing D-concircular curvature tensor V is an $N(k)$-quasi Einstein manifold.

Let M^n be a Riemannian manifold admitting a unit concircular vector field ξ such that $\nabla_X\xi = \varepsilon(-X + \eta(X)\xi)$, $\eta(X) = g(X,\xi)$, $\varepsilon = \pm 1$, then M^n is called a special para-Sasakian manifold [1–3]. Recently Chuman [8] introduced the notion of a D-concircular curvature tensor V. V is given by the following equation

$$\begin{aligned} V(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) + \frac{r+2(n-1)}{(n-1)(n-2)}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \\ - \frac{r+n(n-1)}{(n-1)(n-2)}[g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) \\ + g(Y,W)\eta(Z)\eta(X) - g(X,W)\eta(Y)\eta(Z)], \end{aligned} \tag{3.1}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W))$ for $R(X,Y)Z$ the curvature tensor of type $(1,3)$. If $V(X,Y,Z,W)=0$, then from (3.1) it follows that

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = {} & \frac{r+2(n-1)}{(n-1)(n-2)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]\\ & + \frac{r+n(n-1)}{(n-1)(n-2)}[g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)\\ & + g(Y,W)\eta(Z)\eta(X)-g(X,W)\eta(Y)\eta(Z)]. \qquad (3.2)\end{aligned}$$

Putting $X=W=e_i$ in (3.2) where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1\le i\le n$, we get

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z),$$

where $a=\frac{r+n-1}{n-1}$ and $b=-\frac{r+n(n-1)}{n-1}$. Therefore, $\frac{a+b}{n-1}=-1$. Hence a special para-Sasakian manifold with vanishing D-concircular curvature tensor is an $N(-1)$-quasi-Einstein manifold.

EXAMPLE 3.2. Let $(\mathbb{R}^4,g)$ be a 4-dimensional Lorentzian space endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^idx^j = (1+2q)[(dx^1)^2+(dx^2)^2+(dx^3)^2-(dx^4)^2],$$

where $q=\frac{e^{x^1}}{k^2}$ and k is a non-zero constant, $(i,j=1,2,3,4)$. Then $(\mathbb{R}^4,g)$ is an $N(\frac{q(3+6q-8q^3)}{3(1+2q)^2})$-quasi-Einstein manifold.

Let us consider a Lorentzian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^idx^j = (1+2q)[(dx^1)^2+(dx^2)^2+(dx^3)^2-(dx^4)^2],$$

where $q=\frac{e^{x^1}}{k^2}$ and k is a non-zero constant, $(i,j=1,2,3,4)$. Here the signature of g is $(+,+,+,-)$ which is Lorentzian. Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are:

$$\Gamma^1_{11}=\Gamma^1_{44}=\Gamma^2_{12}=\Gamma^3_{13}=\Gamma^4_{14}=\frac{q}{1+2q},\ \ \Gamma^1_{22}=\Gamma^1_{33}=-\frac{q}{1+2q},$$
$$R_{1221}=R_{1331}=\frac{q}{1+2q},\ \ R_{1441}=-\frac{q}{1+2q},$$
$$R_{2332}=\frac{q^2}{1+2q},\ \ R_{2442}=R_{3443}=-\frac{q^2}{1+2q}$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors and their covariant derivatives are:

$$R_{11}=\frac{3q}{(1+2q)^2},\ \ R_{22}=R_{33}=\frac{q}{1+2q},\quad R_{44}=-\frac{q}{1+2q},$$
$$R_{11,1}=\frac{3q(1-2q)}{(1+2q)^3},\ \ R_{22,1}=R_{33,1}=\frac{q}{(1+2)^2},\ \ R_{44,1}=-\frac{q}{(1+2q)^2}.$$

It can be easily shown that the scalar curvature r of the resulting space $(\mathbb{R}^4,g)$ is $r=\frac{6q(1+q)}{(1+2q)^3}$, which is non-vanishing and non-constant. Now we shall show that $(\mathbb{R}^4,g)$ is an $N(k)$-quasi-Einstein manifold.

To show that the manifold under consideration is an $N(k)$-quasi-Einstein manifold, let us choose the scalar functions a, b and the 1-form η as follows:

$$a = \frac{q}{(1+2q)^2}, \qquad b = 2q(1-q), \tag{3.3}$$

$$\eta_i(x) = \begin{cases} \frac{1}{1+2q} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.2) reduces to the equations

$$R_{11} = ag_{11} + b\eta_1\eta_1, \tag{3.5}$$

$$R_{22} = ag_{22} + b\eta_2\eta_2 \tag{3.6}$$

$$R_{33} = ag_{33} + b\eta_3\eta_3, \tag{3.7}$$

$$R_{44} = ag_{44} + b\eta_4\eta_4, \tag{3.8}$$

since, for the other cases (1.2) holds trivially. By (3.3) and (3.4) we get

$$\begin{aligned} \text{R.H.S. of (3.5)} &= ag_{11} + b\eta_1\eta_1 \\ &= \frac{q}{(1+2q)^2}(1+2q) + 2q(1-q)\frac{1}{(1+2q)^2} \\ &= \frac{3q}{(1+2q)^2} = R_{11} \\ &= \text{L.H.S. of (3.5).} \end{aligned}$$

By similar argument it can be shown that (3.6)–(3.8) are also true. So, $(\mathbb{R}^4, g)$ is an $N(\frac{q(3+6q-8q^3)}{3(1+2q)^2})$-quasi-Einstein manifold.

4. A physical example of an $N(k)$-quasi-Einstein manifold

Example 4.1. This example is concerned with example of an $N(k)$-quasi-Einstein manifold in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-,+,+,+)$. The geometry of the Lorentzian manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that the Lorentzian manifold becomes a convenient choice for the study of general relativity.

Here we consider a perfect fluid $(PRS)_4$ spacetime of non-zero scalar curvature and having the basic vector field U as the timelike vector field of the fluid, that is, $g(U,U) = -1$. An n-dimensional semi-Riemannian manifold is said to be pseudo Ricci-symmetric [6] if the Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X).$$

Such a manifold is denoted by $(PRS)_n$.

For the perfect fluid spacetime, we have the Einstein equation without cosmological constant as

$$S(X,Y) - \frac{1}{2}rg(X,Y) = \kappa T(X,Y), \tag{4.1}$$

where κ is the gravitational constant, T is the energy-momentum tensor of type $(0,2)$ given by

$$T(X,Y) = (\sigma + p)B(X)B(Y) + pg(X,Y), \tag{4.2}$$

with σ and p as the energy density and isotropic pressure of the fluid respectively.

Using (4.2) in (4.1) we get

$$S(X,Y) - \frac{1}{2}rg(X,Y) = \kappa[(\sigma + p)B(X)B(Y) + pg(X,Y)]. \tag{4.3}$$

Taking a frame field and contracting (4.3) over X and Y we have

$$r = \kappa(\sigma - 3p). \tag{4.4}$$

Using (4.4) in (4.3) yields

$$S(X,Y) = \kappa[(\sigma + p)B(X)B(Y) + \frac{(\sigma - p)}{2}g(X,Y)]. \tag{4.5}$$

Putting $Y = U$ in (4.5) and since $g(U,U) = -1$, we get

$$S(X,U) = -\frac{\kappa}{2}[\sigma + 3p]B(X). \tag{4.6}$$

Again for $(PRS)_4$ spacetime [6], $S(X,U) = 0$. This condition will be satisfied by the equation (4.6) if

$$\sigma + 3p = 0 \quad as \quad \kappa \neq 0 \ \ and \ \ A(X) \neq 0. \tag{4.7}$$

Using (4.4) and (4.7) in (4.5) we see that

$$S(X,Y) = \frac{r}{3}[B(X)B(Y) + g(X,Y)].$$

Thus we can state the following:

A perfect fluid pseudo Ricci-symmetric spacetime is an $N(\frac{2r}{9})$-quasi-Einstein manifold.

5. $N(k)$-quasi Einstein manifold satisfying $\tilde{Z}(\xi, X) \cdot S = 0$

In this section we consider an n-dimensional $N(k)$-quasi-Einstein manifold M satisfying the condition

$$(\tilde{Z}(\xi, X) \cdot S)(Y,Z) = 0.$$

Putting $Z = \xi$ we get

$$S(\tilde{Z}(\xi, X)Y, \xi) + S(Y, \tilde{Z}(\xi, X)\xi) = 0. \tag{5.1}$$

Using (1.2), (2.3) and (2.4) in (5.1) we get

$$S(\tilde{Z}(\xi, X)Y, \xi) = [k - \frac{r}{n(n-1)}](a+b)[g(X,Y) - \eta(X)\eta(Y)], \tag{5.2}$$

and

$$S(Y, \tilde{Z}(\xi, X)\xi) = [k - \frac{r}{n(n-1)}][(a+b)\eta(X)\eta(Y) - S(X,Y)]. \tag{5.3}$$

Using (5.2) and (5.3) in (5.1), we obtain

$$[k-\frac{r}{n(n-1)}][(a+b)g(X,Y)-S(X,Y)]=0.$$

Therefore, either the scalar curvature of M is $kn(n-1)$ or, $S=(a+b)g$ which implies that M is an Einstein manifold. But this contradicts the definition of quasi-Einstein manifold. The converse is trivial.

Thus we can state the following:

THEOREM 5.1. *An n-dimensional $N(k)$-quasi-Einstein manifold M satisfies the condition $\tilde{Z}(\xi,X)\cdot S=0$ if and only if the scalar curvature is $kn(n-1)$.*

6. $N(k)$-quasi-Einstein manifold satisfying $P(\xi,X)\cdot\tilde{Z}=0$

In this section we consider an n-dimensional $N(k)$-quasi-Einstein manifold M satisfying the condition

$$(P(\xi,X)\cdot\tilde{Z})(Y,Z)W=0.$$

Then we have

$$P(\xi,X)\tilde{Z}(Y,Z)W-\tilde{Z}(P(\xi,X)Y,Z)W-\tilde{(}Y,P(\xi,X)Z)W-\tilde{Z}(Y,Z)P(\xi,X)W=0. \tag{6.1}$$

Using (2.1) in (6.1), we have

$$\frac{b}{n-1}[g(X,\tilde{Z}(Y,Z)W)\xi-\eta(X)\eta(\tilde{Z}(Y,Z)W)\xi-\tilde{Z}(g(X,Y)\xi-\eta(X)\eta(Y)\xi,Z)W$$
$$-\tilde{Z}(Y,g(X,Z)\xi-\eta(X)\eta(Z)\xi)W-\tilde{Z}(Y,Z)(g(X,W)\xi-\eta(X)\eta(W)\xi)]=0,$$

which implies either $b=0$, or

$$\begin{aligned}g(X,\tilde{Z}(Y,Z)W)\xi-\eta(X)\eta(\tilde{Z}(Y,Z)W)\xi-g(X,Y)\tilde{Z}(\xi,Z)W\\+\eta(X)\eta(Y)\tilde{Z}(\xi,Z)W-g(X,Z)\tilde{Z}(Y,\xi)W+\eta(X)\eta(Z)\tilde{Z}(Y,\xi)W\\-g(X,W)\tilde{Z}(Y,Z)\xi+\eta(X)\eta(W)\tilde{Z}(Y,Z)\xi=0,\end{aligned} \tag{6.2}$$

holds on M. Since $b\neq 0$, hence (6.2) holds.

Taking the inner product of both sides of (6.2) with ξ we have

$$\begin{aligned}g(X,\tilde{Z}(Y,Z)W)-\eta(X)\eta(\tilde{Z}(Y,Z)W)-g(X,Y)\eta(\tilde{Z}(\xi,Z)W)\\+\eta(X)\eta(Y)\eta(\tilde{Z}(\xi,Z)W)-g(X,Z)\eta(\tilde{Z}(Y,\xi)W)+\eta(X)\eta(Z)\eta(\tilde{Z}(Y,\xi)W)\\-g(X,W)\eta(\tilde{Z}(Y,Z)\xi)+\eta(X)\eta(W)\eta(\tilde{Z}(Y,Z)\xi)=0.\end{aligned} \tag{6.3}$$

Using (2.3) in (6.3) we obtain

$$(k-\frac{r}{n(n-1})[g(X,Z)\eta(Y)\eta(W)-g(X,Y)\eta(Z)\eta(W)]=0,$$

which gives $r=kn(n-1)$.

Thus we can state the following:

THEOREM 6.1. *In an n-dimensional $N(k)$-quasi-Einstein manifold M satisfying the condition $P(\xi,X)\cdot\tilde{Z}=0$, the scalar curvature is $kn(n-1)$.*

7. $N(k)$-quasi-Einstein manifold satisfying $\tilde{Z}(\xi, X) \cdot P = 0$

In this section we consider an n-dimensional $N(k)$-quasi-Einstein manifold M satisfying the condition

$$(\tilde{Z}(\xi, X) \cdot P)(Y, Z)W = 0.$$

Then we have

$$\tilde{Z}(\xi, X)P(Y, Z)W - P(\tilde{Z}(\xi, X)Y, Z)W - P(Y, \tilde{Z}(\xi, X)Z)W - P(Y, Z)\tilde{Z}(\xi, X)W = 0. \tag{7.1}$$

Using (2.4) in (7.1) we obtain

$$\begin{aligned}(k - \frac{r}{n(n-1)})[g(X, P(Y, Z)W)\xi - \eta(P(Y, Z)W)X - g(X, Y)P(\xi, Z)W \\ + \eta(Y)P(X, Z)W - g(X, Z)P(Y, \xi)W + \eta(Z)P(Y, X)W - g(X, W)P(Y, Z)\xi \\ + \eta(W)P(Y, Z)X] = 0,\end{aligned}$$

which implies that $r = kn(n-1)$ or

$$\begin{aligned}g(X, P(Y, Z)W)\xi - \eta(P(Y, Z)W)X - g(X, Y)P(\xi, Z)W + \eta(Y)P(X, Z)W \\ - g(X, Z)P(Y, \xi)W + \eta(Z)P(Y, X)W - g(X, W)P(Y, Z)\xi + \eta(W)P(Y, Z)X = 0.\end{aligned} \tag{7.2}$$

Taking inner product with ξ and using (2.1) and (2.2) in (7.2) we get

$$g(X, P(Y, Z)W) = \frac{b}{n-1}[g(X, Y)g(Z, W) - g(X, Z)g(Y, W)]. \tag{7.3}$$

Now using the definition of the projective curvature tensor in (7.3) and then contracting we get $b = 0$, which contradicts the definition of an $N(k)$-quasi-Einstein manifold.

Hence we can state as follows:

Theorem 7.1. *In an n-dimensional $N(k)$-quasi-Einstein manifold M, the relation $\tilde{Z}(\xi, X) \cdot P = 0$ does not hold.*

8. $N(k)$-quasi-Einstein manifold satisfying $P(\xi, X) \cdot C = 0$

In this section we consider an n-dimensional $N(k)$-quasi-Einstein manifold satisfying the condition

$$(P(\xi, X) \cdot C)(Y, Z)W = 0.$$

Then we have

$$P(\xi, X)C(Y, Z)W - C(P(\xi, X)Y, Z)W - C(Y, P(\xi, X)Z)W - C(Y, Z)P(\xi, X)W = 0. \tag{8.1}$$

Using (2.1) in (8.1) we obtain

$$\begin{aligned}\frac{b}{n-1}[g(X, C(Y, Z)W)\xi - \eta(X)\eta(C(Y, Z)W)\xi - C(g(X, Y)\xi - \eta(X)\eta(Y)\xi, Z)W \\ - C(Y, g(X, Y)\xi - \eta(X)\eta(Z)\xi)W - C(Y, Z)(g(X, W)\xi - \eta(X)\eta(W)\xi)] = 0,\end{aligned}$$

which implies either $b = 0$, or

$$g(X, C(Y,Z)W)\xi - \eta(X)\eta(C(Y,Z)W)\xi - g(X,Y)C(\xi,Z)W + \eta(X)\eta(Y)C(\xi,Z)W - g(X,Z)C(Y,\xi)W + \eta(X)\eta(Z)C(Y,\xi)W - g(X,W)C(Y,Z)\xi + \eta(X)\eta(W)C(Y,Z)\xi = 0, \quad (8.2)$$

holds on M. Since $b \neq 0$, hence (8.2) holds.

Taking inner product with ξ and using (2.5)–(2.7) in (8.2) we obtain

$$g(X, C(Y,Z)W) = 0.$$

Thus we can state the following:

Theorem 8.1. *An n-dimensional $N(k)$-quasi-Einstein manifold M satisfies the condition $P(\xi, X) \cdot C = 0$ if and only if the manifold is conformally flat.*

9. $N(k)$-quasi-Einstein manifold satisfying $\tilde{Z}(\xi, X) \cdot C = 0$

In this section we consider an n-dimensional $N(k)$-quasi-Einstein manifold satisfying the condition

$$(\tilde{Z}(\xi, X) \cdot C)(Y,Z)W = 0.$$

Then we have

$$\tilde{Z}(\xi,X)C(Y,Z)W - C(\tilde{Z}(\xi,X)Y,Z)W - C(Y,\tilde{Z}(\xi,X)Z)W - C(Y,Z)\tilde{(}\xi,X)W = 0. \quad (9.1)$$

Using (2.4) in (9.1) we obtain

$$\left(k - \frac{r}{n(n-1)}\right)[g(X, C(Y,Z)W)\xi - \eta(C(Y,Z)W)X - g(X,Y)C(\xi,Z)W + \eta(Y)C(X,Z)W - g(X,Z)C(Y,\xi)W + \eta(Z)C(Y,X)W - g(X,W)C(Y,Z)\xi + \eta(W)C(Y,Z)X] = 0, \quad (9.2)$$

which gives either $r = kn(n-1)$ or

$$g(X,C(Y,Z)W)\xi - \eta(C(Y,Z)W)X - g(X,Y)C(\xi,Z)W + \eta(Y)C(X,Z)W - g(X,Z)C(Y,\xi)W + \eta(Z)C(Y,X)W - g(X,W)C(Y,Z)\xi + \eta(W)C(Y,Z)X] = 0.$$

Taking inner product with ξ and using (2.5)–(2.7) in (9.2) we obtain

$$g(X, C(Y,Z)W) = 0.$$

Thus we can state the following:

Theorem 9.1. *An n-dimensional $N(k)$-quasi-Einstein manifold M satisfies the condition $\tilde{Z}(\xi, X) \cdot C = 0$ if and only if the manifold is conformally flat.*

10. Ricci-pseudosymmetric $N(k)$-quasi-Einstein manifolds

An n-dimensional Riemannian manifold (M^n, g) is called Ricci-pseudosymmetric [12] if the tensors $R \cdot S$ and $Q(g,S)$ are linearly dependent, where

$$(R(X,Y) \cdot S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W), \quad (10.1)$$

$$Q(g,S)(Z,W;X,Y) = -S((X \wedge Y)Z,W) - S(Z,(X \wedge Y)W), \quad (10.2)$$

and

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{10.3}$$

for vector fields X, Y, Z, W on M^n.

The condition of Ricci-pseudosymmetry is equivalent to

$$(R(X,Y) \cdot S)(Z,W) = L_s Q(g,S)(Z,W;X,Y),$$

holding on the set

$$U_s = \{x \in M : S \neq \frac{r}{n} g \text{ at } x\},$$

where L_s is some function on U_s. If $R{\cdot}S = 0$ then M^n is called Ricci-semisymmetric. Every Ricci-semisymmetric manifold is Ricci-pseudosymmetric but the converse is not true [12].

Let us assume that the manifold under consideration is Ricci-pseudosymmetric. Then with the help of (10.1)–(10.3) we can write

$$\begin{aligned} S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = L_s\{&g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &+ g(Y,W)S(X,Z) - g(X,W)S(Y,Z)\}. \end{aligned} \tag{10.4}$$

Using (1.2) and (1.4) in (10.4) we obtain

$$\begin{aligned} \left[\frac{b(a+b)}{n-1} - bL_s\right]\{&g(Y,Z)\eta(X)\eta(W) - g(X,W)\eta(Y)\eta(Z) \\ &+ g(Y,W)\eta(X)\eta(Z) - g(X,Z)\eta(Y)\eta(W)\} = 0. \end{aligned} \tag{10.5}$$

Putting $Y = Z = \xi$ in (10.5) we have

$$\left\{\frac{b(a+b)}{n-1} - bL_s\right\}[\eta(X)\eta(W) - g(X,W)] = 0. \tag{10.6}$$

Again putting $X = W = e_i$ in (10.6), where $\{e_i\}$, $(i = 1, 2, \ldots, n)$ is an orthonormal basis of the tangent space at any point of the manifold and then taking the sum for $1 \leq i \leq n$, we obtain

$$\left[\frac{b(a+b)}{n-1} - bL_s\right](1-n) = 0,$$

which implies that $L_s = \frac{a+b}{n-1}$. Thus we can state the following:

THEOREM 10.1. *A Ricci-pseudosymmetric $N(k)$-quasi-Einstein manifold is a Ricci-semisymmetric manifold if and only if $a + b = 0$.*

11. Conclusions

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. The importance of an $N(k)$-quasi-Einstein is presented in the introduction. In this paper we prove that a special para-Sasakian manifold with vanishing D-concircular curvature tensor V is an $N(k)$-quasi-Einstein manifold. Then we find a metric of a four-dimensional $N(k)$-quasi-Einstein manifold. Also we give a physical example of $N(k)$-quasi-Einstein manifolds. Moreover, we have considered $N(k)$-quasi-Einstein manifolds satisfying the curvature conditions $\tilde{Z}(\xi, X) \cdot S = 0$, $P(\xi, X) \cdot \tilde{Z} = 0$, $\tilde{Z}(\xi, X) \cdot P = 0$, $P(\xi, X) \cdot C = 0$ and $\tilde{Z}(\xi, X) \cdot C = 0$, where P, $\tilde{Z}$, C, S are projective curvature tensor, concircular curvature tensor, conformal curvature tensor and Ricci tensor respectively. Finally we prove that a Ricci-pseudosymmetric $N(k)$- quasi-Einstein manifold is a Ricci-semisymmetric manifold under certain condition.

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Uday Chand De, Department of Pure Mathematics, University of Calcutta, 35, Ballygaunge Circular Road, Kolkata 700019, West Bengal, India

E-mail: `uc_de@yahoo.com`

Sahanous Mallick, Paninala Satyapriya Roy Smriti Sikshaniketan (H.S), P.O.- Bhandarkhola, Dist-Nadia, West Bengal, India