SOME SPACES OF DOUBLE DIFFERENCE
SEQUENCES OF FUZZY NUMBERS

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Abstract. In this paper, we introduce some spaces of double difference sequences of fuzzy numbers defined by a sequence of modulus functions $F = (f_{k,l})$. We also make an effort to study some topological properties and prove some inclusion relations between these spaces.

1. Introduction and preliminaries

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [35] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [16] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. For more details about sequence spaces of fuzzy numbers (see [4], [8], [10], [15], [20], [21], [30], [31]) and references therein.

The initial works on double sequences is found in Bromwich [7]. Later on, it was studied by Hardy [11], Moricz [17], Moricz and Rhoades [18], Tripathy ([32], [33]), Başarır and Sonalcan [5] and many others. Hardy [11] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [36] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [22] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly

Cesaro summable double sequences. Subsequently, Mursaleen [19] and Mursaleen and Edely [23] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{k,l})$ into one whose core is a subset of the M-core of x. More recently, Altay and Başar [1] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively and also examined some properties of these sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(v)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Now, recently Başar and Sever [6] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. By the convergence of a double sequence we mean the convergence in the Pringsheim sense i.e. a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ see [25]. We shall write more briefly as P-convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l.

A fuzzy number is a function on the real axis, i.e., a mapping $X : \mathbb{R}^n \to [0, 1]$ which satisfies the following four conditions:

(1) X is normal, i.e., there exist an $x_0 \in \mathbb{R}^n$ such that $X(x_0) = 1$;

(2) X is fuzzy convex, i.e., for $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1, X(\lambda x + (1 - \lambda)y) \ge \min[X(x), X(y)]$;

(3) X is upper semi-continuous;

(4) the closure of $\{x \in \mathbb{R}^n : X(x) > 0\}$, denoted by $[X]^0$, is compact.

Let $C(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A$ is compact and convex $\}$. The spaces $C(\mathbb{R}^n)$ has a linear structure induced by the operations

$$A + B = \{a + b, a \in A, b \in B\}$$

and

$$\lambda A = \{\lambda a : a \in A\}$$

for $A, B \in C(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The Hausdorff distance between A and B of $C(\mathbb{R}^n)$ is defined as

$$d(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \ \sup_{b \in B} \inf_{a \in A} \|a - b\|\}$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), d)$ is a complete (non separable) metric space. For $0 < \alpha \le 1$, the α-level set

$$X^\alpha = \{x \in \mathbb{R}^n : X(x) \ge \alpha\}$$

is a nonempty compact convex, subset of $\mathbb{R}^n$, as the support $X^0 = \lim_{\alpha \to 0^+} X^\alpha$. By $L(\mathbb{R}^n)$ we denote the set of all fuzzy numbers. Define a map $\bar{d} : L(\mathbb{R}^n) \times L(\mathbb{R}^n) \to \mathbb{R}^n$ by

$$\bar{d}(X, Y) = \sup_{\alpha \in [0,1]} d(X^\alpha, Y^\alpha).$$

It is showed that $L(\mathbb{R}^n)$ is a complete metric space with the metric $\bar{d}$. For $X, Y \in L(\mathbb{R}^n)$, define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$ for any $\alpha \in [0, 1]$.

We denote by $w(f)$ the set of all double sequences $X = (X_{k,l})$ of fuzzy numbers. A sequence $X = (X_{k,l})$ of fuzzy numbers is said to be bounded if the set $\{X_{k,l} : k, l \in \mathbb{N}\}$ of fuzzy numbers is bounded. We denote by $c_\infty(f)$ the set of all bounded sequences of fuzzy numbers. The sequence $X = (X_{k,l})$ of fuzzy numbers is said to be convergent to the fuzzy number X_0, written as $\lim_{k,l} X_{k,l} = X_0$, if for every $\epsilon > 0$ there exists a positive integer $k_0 = k_0(\epsilon)$, such that $\bar{d}(X_{k,l}, X_0) < \epsilon$ for every $k > k_0$. By $c(f)$ we denote the set of all convergent sequences of fuzzy numbers. It is clear that $c(f) \subset c_\infty(f) \subset w(f)$. A metric on $L(\mathbb{R}^n)$ is said to be a translation invariant if $d(X + Z, Y + Z) = d(X, Y)$ for $X, Y, Z \in L(\mathbb{R}^n)$.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called a paranorm, if

(1) $p(x) \geq 0$ for all $x \in X$,

(2) $p(-x) = p(x)$ for all $x \in X$,

(3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,

(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called a total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [34], Theorem 10.4.2, pp. 183).

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,

(2) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,

(3) f is increasing

(4) f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus function $f(x)$ is unbounded. Subsequentially, modulus function has been discussed in ([2, 3, 13, 14, 24, 26–29]) and references therein.

The notion of difference sequence spaces was introduced by Kızmaz [12], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [9] by introducing the spaces $l_\infty(\Delta^m), c(\Delta^m)$ and $c_0(\Delta^m)$. Let w denote the set of all real or complex sequences $x = (x_k)$. Let m, n be non-negative integers, then for $Z = l_\infty$, c, c_0. We have sequence spaces

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^m x_k = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+nv}.$$

Let $X = (X_{k,l})$ be a double sequence of fuzzy numbers, $F = (f_{k,l})$ be a sequence of modulus functions and $p = (p_{k,l})$ be a bounded sequence of positive real numbers. In this paper we define the following classes of double difference sequences of fuzzy numbers:

$$c^f[F, \Delta_n^m, p, s]_0 = \left\{ X = (X_{k,l}) \in w(f) : \lim_{k,l}(kl)^{-s} \left[f_{k,l}\left(\bar{d}\left(\Delta_n^m X_{k,l}, \bar{0}\right)\right)\right]^{p_{k,l}} = 0 \right\},$$

$$c^f[F, \Delta_n^m, p, s]_1 = \left\{ X = (X_{k,l}) \in w(f) : \lim_{k,l}(kl)^{-s} \left[f_{k,l}\left(\bar{d}\left(\Delta_n^m X_{k,l}, X_0\right)\right)\right]^{p_{k,l}} = 0, \right.$$

$$\left. \text{for some } X_0 \text{ and } s \geq 0 \right\}$$

and

$$c^f[F, \Delta_n^m, p, s]_\infty = \left\{ X = (X_{k,l}) \in w(f) : \sup_{k,l}(kl)^{-s} \left[f_{k,l}\left(\bar{d}\left(\Delta_n^m X_{k,l}, \bar{0}\right)\right)\right]^{p_{k,l}} < \infty \right\},$$

If we take $F(x) = x$, we get the following classes of sequences

$$c^f[\Delta_n^m, p, s]_0 = \left\{ X = (X_{k,l}) \in w(f) : \lim_{k,l}(kl)^{-s} \left(\bar{d}\left(\Delta_n^m X_{k,l}, \bar{0}\right)\right)^{p_{k,l}} = 0 \right\},$$

$$c^f[\Delta_n^m, p, s]_1 = \left\{ X = (X_{k,l}) \in w(f) : \lim_{k,l}(kl)^{-s} \left(\bar{d}\left(\Delta_n^m X_{k,l}, X_0\right)\right)^{p_{k,l}} = 0, \right.$$

$$\left. \text{for some } X_0 \text{ and } s \geq 0 \right\}$$

and

$$c^f[\Delta_n^m, p, s]_\infty = \left\{ X = (X_{k,l}) \in w(f) : \sup_{k,l}(kl)^{-s} \left(\bar{d}\left(\Delta_n^m X_{k,l}, \bar{0}\right)\right)^{p_{k,l}} < \infty \right\}.$$

If we take $p = (p_{k,l}) = 1$, we get

$$c^f[F, \Delta_n^m, s]_0 = \left\{ X = (X_{k,l}) \in w(f) : \lim_{k,l}(kl)^{-s} \left[f_{k,l}\left(\bar{d}\left(\Delta_n^m X_{k,l}, \bar{0}\right)\right)\right] = 0 \right\},$$

$$c^f[F, \Delta_n^m, s]_1 = \left\{ X = (X_{k,l}) \in w(f) : \lim_{k,l}(kl)^{-s} \left[f_{k,l}\left(\bar{d}\left(\Delta_n^m X_{k,l}, X_0\right)\right)\right] = 0, \right.$$

$$\left. \text{for some } X_0 \text{ and } s \geq 0 \right\}$$

and

$$c^f[F, \Delta_n^m, s]_\infty = \left\{ X = (X_{k,l}) \in w(f) : \sup_{k,l}(kl)^{-s} \left[f_{k,l}\left(\bar{d}\left(\Delta_n^m X_{k,l}, \bar{0}\right)\right)\right] < \infty \right\}.$$

If we take $s = 0, F(x) = x$ and $p = (p_{k,l}) = 1$, for all k, l then, we obtain new difference sequence spaces of fuzzy numbers :

$$c^f[\Delta_n^m]_0 = \left\{ X = (X_{k,l}) \in w(f) : \lim_{k,l} \left(\bar{d}\left(\Delta_n^m X_{k,l}, \bar{0}\right)\right) = 0 \right\},$$

$$c^f[\Delta_n^m]_1 = \left\{ X = (X_{k,l}) \in w(f) : \lim_{k,l} \left(\bar{d}\left(\Delta_n^m X_{k,l}, X_0\right)\right) = 0, \text{ for some } X_0 \right\}$$

and

$$c^f[\Delta_n^m]_\infty = \left\{ X = (X_{k,l}) \in w(f) : \sup_{k,l} \left(\bar{d}\left(\Delta_n^m X_{k,l}, \bar{0}\right)\right) < \infty \right\}.$$

For a sequence of modulus functions $F = (f_{kl})$, we give the following conditions:

(1) $\sup f_{k,l}(t) < \infty$ for all $t > 0$,

(2) $\lim_{t\to\infty} \sup f_{k,l}(t) = 0$ uniformly in $k, l \geq 1$.

The following inequality will be used throughout the paper. Let $p = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} \leq \sup_{k,l} p_{k,l} = H$ and let $K = \max\{1, 2^{H-1}\}$. Then for the factorable sequences $\{a_{k,l}\}$ and $\{b_{k,l}\}$ in the complex plane, we have

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq K(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}). \tag{1.1}$$

The main purpose of this paper is to introduce some spaces of double difference sequences of fuzzy numbers defined by a sequence of modulus functions and make an effort to study some topological properties and prove inclusion relations between the above defined sequence spaces.

2. Main results

THEOREM 2.1. *If $\bar{d}$ is a translation invariant metric, then the classes of sequences $c^f[F, \Delta_n^m, p, s]_0$, $c^f[F, \Delta_n^m, p, s]_1$ and $c^f[F, \Delta_n^m, p, s]_\infty$ are closed under the operations of addition and scalar multiplication.*

Proof. Since $\bar{d}$ is a translation invariant metric, it implies that

$$\bar{d}(\Delta_n^m X_{k,l} + \Delta_n^m Y_{k,l}, X_0 + Y_0) \leq \bar{d}(\Delta_n^m X_{k,l}, X_0) + \bar{d}(\Delta_n^m Y_{k,l}, Y_0) \tag{2.1}$$

and

$$\bar{d}(\Delta_n^m \lambda X_{k,l}, \lambda X_0) \leq |\lambda| \bar{d}(\Delta_n^m X_{k,l}, X_0), \tag{2.2}$$

where λ is a scalar and $|\lambda| > 1$. We shall prove only for $c^f[F, \Delta_n^m, p, s]_1$. Suppose $X = (X_{k,l})$ and $Y = (Y_{k,l}) \in c^f[F, \Delta_n^m, p, s]_1$. Then by using inequality (1.1), we have

$$(kl)^{-s}\big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l} + \Delta_n^m Y_{k,l}, X_0 + Y_0)\big)\big]^{p_{k,l}}$$
$$\leq (kl)^{-s}\big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0) + \bar{d}(\Delta_n^m Y_{k,l}, Y_0)\big)\big]^{p_{k,l}}$$
$$\leq (kl)^{-s}\big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big) + f_{k,l}\big(\bar{d}(\Delta_n^m Y_{k,l}, Y_0)\big)\big]^{p_{k,l}}$$
$$\leq (kl)^{-s}K^H\big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big)\big]^{p_{k,l}} + (kl)^{-s}K^H\big[f_{k,l}\big(\bar{d}(\Delta_n^m Y_{k,l}, Y_0)\big)\big]^{p_{k,l}}$$

Hence $X + Y \in c^f[F, \Delta_n^m, p, s]_1$. Let $X = (X_{k,l}) \in c^f[F, \Delta_n^m, p, s]_1$. For $\lambda \in \mathbb{R}$ there exists an integer K such that $|\lambda| \leq K$. Then by equation (2.2) and the sequence of modulus functions $F = (f_{k,l})$ for all $k, l \in \mathbb{N}$, we have

$$(kl)^{-s}\big[f_{k,l}\big(\bar{d}(\lambda\Delta_n^m X_{k,l}, \lambda X_0)\big)\big]^{p_{k,l}} \leq (kl)^{-s}\big[f_{k,l}\big(\bar{d}(|\lambda|(\Delta_n^m X_{k,l}, X_0))\big)\big]^{p_{k,l}}$$
$$\leq K^H(kl)^{-s}\big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big)\big]^{p_{k,l}}.$$

This implies that $\lambda X \in c^f[F, \Delta_n^m, p, s]_1$. Similarly, we can prove for other cases. ∎

THEOREM 2.2. *Let $F = (f_{k,l})$ be a sequence of modulus functions. Then*

$$c^f[F, \Delta_n^m, p, s]_0 \subset c^f[F, \Delta_n^m, p, s]_1 \subset c^f[F, \Delta_n^m, p, s]_\infty.$$

Proof. Clearly $c^f[F, \Delta_n^m, p, s]_0 \subset c^f[F, \Delta_n^m, p, s]_1$. Let $X = (X_{k,l})$ $\in c^f[F, \Delta_n^m, p, s]_1$. Then, there is some fuzzy number X_0, such that

$$\lim_{k,l}(kl)^{-s}\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big]^{p_{k,l}} = 0, \ s \geq 0.$$

Now, by inequality (1.1), we have

$$(kl)^{-s}\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, \bar{0}\big)\big)\big]^{p_{k,l}}$$
$$\leq K.(kl)^{-s}\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big]^{p_{k,l}} + K.(kl)^{-s}\big[f_{k,l}\big(\bar{d}\big(X_0, \bar{0}\big)\big)\big]^{p_{k,l}}.$$

This implies that $X = (X_{k,l}) \in c^f[F, \Delta_n^m, p, s]_\infty$. ∎

THEOREM 2.3. *Let* $F = (f_{k,l})$ *be a sequence of modulus functions. Then,* $c^f[F, \Delta_n^m, p, s]_1$ *is a complete metric space with the metric defined by*

$$g(X,Y) = \sum_{k,l=1,1}^{m,n} d(X_{k,l}, Y_{k,l}) + \sup_{k,l}\big((kl)^{-s}\big[f_{k,l}\big(d\big(\Delta_n^m X_{k,l}, \Delta_n^m Y_{k,l}\big)\big)\big]^{p_{k,l}}\big)^{\frac{1}{p_{k,l}}}.$$

Proof. Let (X^t) be a Cauchy sequence in $c^f[F, \Delta_n^m, p, s]_1$, where $X^t = (X_{k,l}^t)$ for every $k, l, t \in \mathbb{N}$. Then

$$g(X^t, X^v) = \sum_{k,l=1,1}^{m,n} d(X_{k,l}^t, X_{k,l}^v)$$
$$+ \sup_{k,l}\big((kl)^{-s}\big[f_{k,l}\big(d\big(\Delta_n^m X_{k,l}^t, \Delta_n^m X_{k,l}^v\big)\big)\big]^{p_{k,l}}\big)^{\frac{1}{p_{k,l}}} \to 0$$

as $t, v \to \infty$, for each $k, l \in \mathbb{N}$. Hence,

$$\sum_{k,l=1,1}^{m,n} d\big(X_{k,l}^t, X_{k,l}^v\big) \to 0 \ \text{ and } \ f_{k,l}\big(d\big(\Delta_n^m X_{k,l}^t, \Delta_n^m X_{k,l}^v\big)\big) \to 0 \text{ as } t, v \to \infty,$$

for each $k, l \in \mathbb{N}$. As $(f_{k,l})$ is a sequence of modulus functions for all $k, l \in \mathbb{N}$, then

$$d\big(\Delta_n^m X_{k,l}^t, \Delta_n^m X_{k,l}^v\big) \to 0 \text{ as } t, v \to \infty, \ \text{ for each } \ k, l \in \mathbb{N}.$$

Thus we have

$$d\big(X_{k+m,l+n}^t, X_{k+m,l+n}^v\big) \leq d\big(\Delta_n^m X_{k,l}^t, \Delta_n^m X_{k,l}^v\big) \to 0 \text{ as } \ t, v \to \infty,$$

for each $k, l \in \mathbb{N}$. Therefore,

$$(X_{k,l}^t)_t = (X_{k,l}^1, X_{k,l}^2, X_{k,l}^3, \cdots)$$

is a Cauchy sequence in $L(\mathbb{R}^n)$. As $L(\mathbb{R}^n)$ is complete, it is convergent, say $\lim_t X_{k,l}^t = X_{k,l}$, for every $k, l \in \mathbb{N}$. Since (X^t) is a Cauchy sequence for each $\epsilon > 0$ there exists $n_0 = n_0(\epsilon)$ such that $g(X^t, X^v) < \epsilon$ for all $t, v \geq n_0$. So, we have

$$\lim_v \sum_{k,l=1,1}^{m,n} d\big(X_{k,l}^t, X_{k,l}^v\big) = \sum_{k,l=1,1}^{m,n} d\big(X_{k,l}^t, X_{k,l}\big) < \epsilon$$

and

$$\lim_v(kl)^{-s}\big[f_{k,l}\big(d\big(\Delta_n^m X_{k,l}^t, \Delta_n^m X_{k,l}^v\big)\big)\big]^{p_{k,l}} = (kl)^{-s}\big[f_{k,l}\big(d\big(\Delta_n^m X_{k,l}^t, \Delta_n^m X_{k,l}\big)\big)\big]^{p_{k,l}}$$
$$< \epsilon^H, \ \text{ for all } \ t \geq n_0.$$

This implies that $g(X^t, X) < 2\epsilon$ for all $t \geq n_0$, this means $X^t \to X$ as $t \to \infty$, where $X = (X_{k,l})$. As

$$(kl)^{-s}\big[f_{k,l}\big(d\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big]^{p_{k,l}} \leq 2^{p_{k,l}}\Big\{(kl)^{-s}\big)\big[f_{k,l}\big(d\big(\Delta_n^m X_{k,l}^{n_0}, X_0\big)\big)\big]^{p_{k,l}}$$
$$+ (kl)^{-s}\big)\big[f_{k,l}\big(d\big(\Delta_n^m X_{k,l}^{n_0}, \Delta_n^m X_{k,l}\big)\big)\big]^{p_{k,l}}\Big\}$$
$$\to 0 \text{ as } k, l \to \infty,$$

we obtain $X = (X_{k,l}) \in c^f[X, F, \Delta_n^m, p, s]_1$. Therefore, $c^f[X, F, \Delta_n^m, p, s]_1$ is a complete metric space. ∎

THEOREM 2.4. *Let* $\inf p_{kl} = h > 0$. *Then we have the following:*

(i) $X_{k,l} \to X_0(c^f[F, \Delta_n^m]_1)$, *implies that* $X_{k,l} \to X_0(c^f[F, \Delta_n^m, p, s]_1)$;

(ii) $X_{k,l} \to X_0(c^f[\Delta_n^m, p, s]_1)$, *implies that* $X_{k,l} \to X_0(c^f[F, \Delta_n^m, p, s]_1)$;

(iii) $\beta = \lim_{t\to\infty} \frac{f_{k,l}(t)}{t} > 0$, *implies* $c^f[\Delta_n^m, p, s] = c^f[F, \Delta_n^m, p, s]$.

Proof. (i) Suppose that $(X_{k,l}) \to X_0(c^f[F, \Delta_n^m]_1)$ (as $k, l \to \infty$). Since $(f_{k,l})$ is a modulus function for all k, l, then

$$\lim_{k,l}\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big] = f_{k,l}\big[\lim_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big] = 0.$$

Since $\inf p_{k,l} = h > 0$, then $\lim_{k,l}\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big]^h = 0$. So, for $0 < \epsilon < 1$, there exists k_0 such that for all $k > k_0$,

$$\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big]^h < \epsilon < 1,$$

and as $p_{k,l} \geq h$ for all k, l,

$$\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big]^{p_{k,l}} \leq \big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big]^h < \epsilon < 1,$$

then, we obtain $\lim_{k,l}\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big]^{p_{k,l}} = 0$. Since $(kl)^{-s}$ is bounded, we write

$$\lim_{k,l}(kl)^{-s}\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big]^{p_{k,l}} = 0.$$

Therefore, $X = (X_{k,l}) \in c^f[F, \Delta_n^m, p, s]_1$.

(ii) Let $X = (X_{k,l}) \in c^f[\Delta_n^m, p, s]_1$, so that

$$S_{k,l} = (kl)^{-s}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)^{p_{k,l}} \to 0 \text{ as } k, l \to \infty.$$

Suppose $\epsilon > 0$ and choose δ with $0 < \delta < 1$, such that $f_{k,l}(t) < \epsilon$ for $0 \leq t \leq \delta$ and for all k, l. Now, we take

$$A_1 = \big\{(k, l) \in \mathbb{N} \times \mathbb{N} : \bar{d}\big(\Delta_n^m X_{k,l}, X_0\big) \leq \delta\big\}$$

and

$$A_2 = \big\{(k, l) \in \mathbb{N} \times \mathbb{N} : \bar{d}\big(\Delta_n^m X_{k,l}, X_0\big) > \delta\big\}.$$

For $\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big) > \delta$,

$$\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big) < \bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\delta^{-1} < 1 + \big[|\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\delta^{-1}|\big],$$

where $(k, l) \in A_2$ and $[|t|]$ denotes the integer part of t. By using properties of modulus function and for $\bar{d}(\Delta_n^m X_{k,l}, X_0) > \delta$, we have

$$f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big) \leq \big(1 + [|\bar{d}(\Delta_n^m X_{k,l}, X_0)\delta^{-1}|]\big) f_{k,l}(1)$$
$$\leq 2 f_{k,l}(1) \bar{d}(\Delta_n^m X_{k,l}, X_0)\delta^{-1}.$$

For $\bar{d}(\Delta_n^m X_{k,l}, X_0) \leq \delta$, $f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big) < \epsilon$, where $(k, l) \in A_1$. Hence

$$(kl)^{-s}\big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big)\big]^{p_{k,l}} = (kl)^{-s})\big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}^{n_0}, X_0)\big)\big]^{p_{k,l}} |(k, l) \in A_1$$
$$+ (kl)^{-s}\big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}^{n_0}, \Delta_n^m X_{k,l})\big)\big]^{p_{k,l}} |(k, l) \in A_2.$$

Then

$$(kl)^{-s}\big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big)\big]^{p_{k,l}} \leq (kl)^{-s}\epsilon^H + \big[2f_{k,l}(1)\delta^{-1}\big]^H S_{k,l} \to 0$$

as $k, l \to \infty$. Thus $X = (X_{k,l}) \in c^f[F, \Delta_n^m, p, s]_1$.

(*iii*) In (*ii*), it was shown that $c^f[\Delta_n^m, p, s]_1 \subset c^f[F, \Delta_n^m, p, s]_1$. We have to only show that $c^f[\Delta_n^m, p, s]_1 \supset c^f[F, \Delta_n^m, p, s]$. For any modulus function, the existence of positive limit given with β in Maddox [13, Proposition 1]. Now $\beta > 0$ and let $X = (X_{k,l}) \in c^f[F, \Delta_n^m, p, s]_1$. Since $\beta > 0$, for every $t > 0$, we have $f_{k,l}(t) \geq \beta t$ for all $k, l \in \mathbb{N}$. From this inequality, it is seen that $X = (X_{k,l}) \in c^f[\Delta_n^m, p, s]_1$. This completes the proof. ∎

THEOREM 2.5. *Let* $F = (f_{k,l})$ *and* $G = (g_{k,l})$ *be two sequences of modulus functions and* $s_1, s_2 \geq 0$. *Then we have the following:*

(i) $c^f[F, \Delta_n^m, p, s]_1 \cap c^f[G, \Delta_n^m, p, s]_1 \subset c^f[F \cap G, \Delta_n^m, p, s]_1$;

(ii) $s_1 \leq s_2$ *implies* $c^f[F, \Delta_n^m, p, s_1]_1 \subset c^f[F, \Delta_n^m, p, s_2]_1$.

Proof. (*i*) Let $X = (X_{k,l}) \in c^f[F, \Delta_n^m, p, s]_1 \cap c^f[G, \Delta_n^m, p, s]_1$. From (1.1) we have

$$\big[(f_{k,l} + g_{k,l})\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big)\big]^{p_{k,l}}$$
$$= \big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big) + g_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big)\big]^{p_{k,l}}$$
$$\leq K\big\{\big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big)\big]^{p_{k,l}} + \big[g_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big)\big]^{p_{k,l}}\big\}.$$

Since $(kl)^{-s}$ is bounded, write

$$(kl)^{-s}\big[(f_{k,l} + g_{k,l})\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big)\big]^{p_{k,l}} \leq K(kl)^{-s}\big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big)\big]^{p_{k,l}}$$
$$+ K(kl)^{-s}\big[g_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big)\big]^{p_{k,l}}$$

Thus $X = (X_{k,l}) \in c^f[F \cap G, \Delta_n^m, p, s]_1$. This completes the proof of (i).

(*ii*) Let $s_1 \leq s_2$. Then $(kl)^{-s_2} \leq (kl)^{-s_1}$ for all $k, l \in \mathbb{N}$. As

$$(kl)^{-s_2}\big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big)\big]^{p_{k,l}} \leq (kl)^{-s_1}\big[f_{k,l}\big(\bar{d}(\Delta_n^m X_{k,l}, X_0)\big)\big]^{p_{k,l}},$$

this implies that

$$c^f[F, \Delta_n^m, p, s_1]_1 \subset c^f[F, \Delta_n^m, p, s_2]_1. \quad ∎$$

THEOREM 2.6. *Let $F = (f_{k,l})$ be a sequence of modulus functions. Then we have the following:*

(i) let $0 < \inf_{k,l} p_{k,l} \leq p_{k,l} \leq 1$, then

$$c^f[F, \Delta_n^m, p, s]_1 \subset c^f[F, \Delta_n^m, s]_1;$$

(ii) let $1 \leq p_{k,l} \leq \sup_{k,l} p_{k,l} < \infty$, then

$$c^f[F, \Delta_n^m, s]_1 \subset c^f[F, \Delta_n^m, p, s]_1,$$

(iii) let $0 \leq p_{k,l} \leq q_{k,l}$ and $(\frac{q_{k,l}}{p_{k,l}})$ be bounded, then

$$c^f[F, \Delta_n^m, q, s]_1 \subset c^f[F, \Delta_n^m, p, s]_1.$$

Proof. (*i*) Let $X = (X_{k,l}) \in c^f[F, \Delta_n^m, p, s]_1$, since $0 < \inf_{k,l} p_{k,l} \leq 1$, we obtain

$$(kl)^{-s}\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big] \leq (kl)^{-s}\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big]^{p_{k,l}}$$

for all k, l and hence we obtain $X = (X_{k,l}) \in c^f[F, \Delta_n^m, s]_1$.

(*ii*) Let $1 \leq p_{k,l} \leq \sup_{k,l} p_{k,l} < \infty$ for each k, l and $X = (X_{k,l}) \in c^f[F, \Delta_n^m, s]_1$. Then, for each $0 < \epsilon < 1$, there exists a positive integer k_0, such that

$$(kl)^{-s}\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big] \leq \epsilon,$$

for all $k \geq k_0$. This implies that

$$(kl)^{-s}\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big]^{p_{k,l}} \leq (kl)^{-s}\big[f_{k,l}\big(\bar{d}\big(\Delta_n^m X_{k,l}, X_0\big)\big)\big].$$

Thus $X = (X_{k,l}) \in c^f[F, \Delta_n^m, p, s]_1$.

(*iii*) The proof is easy so we omit it. ∎

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