МАТЕМАТИЧКИ ВЕСНИК
66, 3 (2014), 283–293
September 2014

оригинални научни рад
research paper

ON SOME NEW MIXED MODULAR EQUATIONS INVOLVING RAMANUJAN'S THETA-FUNCTIONS

M. S. Mahadeva Naika, S. Chandankumar and M. Harish

Abstract. In his second notebook, Ramanujan recorded altogether 23 P–Q modular equations involving his theta functions. In this paper, we establish several new mixed modular equations involving Ramanujan's theta-functions φ and ψ which are akin to those recorded in his notebook.

1. Introduction

Following Ramanujan, let $(a;q)_\infty$ denote the infinite product $\prod_{n=0}^{\infty}(1-aq^n)$ (a, q are complex numbers, $|q|<1$) and

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1,$$

denote the Ramanujan theta-function. The product representation of $f(a,b)$ follows from Jacobi's triple product identity and is given by:

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

The following definitions of theta-functions φ, ψ and f with $|q|<1$ are classical:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{1.1}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.2}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.3}$$

The ordinary or Gaussian hypergeometric function is defined by

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \quad 0 \le |z| < 1,$$

2010 Mathematics Subject Classification: 33D10, 11A55, 11F27
Keywords and phrases: Modular equations; theta-functions

where a, b, c are complex numbers such that $c \neq 0, -1, -2, \ldots$, and

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) \ \text{ for any positive integer n.}$$

Now we recall the notion of modular equation. Let $K(k)$ be the complete elliptic integral of the first kind of modulus k. Recall that

$$K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2}\varphi^2(q), \quad (0<k<1), \tag{1.4}$$

and set $K' = K(k')$, where $k' = \sqrt{1-k^2}$ is the so called complementary modulus of k. It is classical to set $q(k) = e^{-\pi K(k')/K(k)}$ so that q is one-to-one, increasing from 0 to 1.

In the same manner introduce $L_1 = K(\ell_1), L_1' = K(\ell_1')$ and suppose that the following equality

$$n_1 \frac{K'}{K} = \frac{L_1'}{L_1} \tag{1.5}$$

holds for some positive integer n_1. Then a modular equation of degree n_1 is an algebraic relation between the moduli k and ℓ_1 which is induced by (1.5). Following Ramanujan, set $\alpha = k^2$ and $\beta = \ell_1^2$. Then we say that β is of degree n_1 over α. The multiplier m is defined by

$$m = \frac{K}{L_1} = \frac{\varphi^2(q)}{\varphi^2(q^{n_1})}, \tag{1.6}$$

for $q = e^{-\pi K(k')/K(k)}$.

Let K, K', L_1, L_1', L_2, L_2', L_3 and L_3' denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$, and their complementary moduli, respectively. Let n_1, n_2 and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1 \frac{K'}{K} = \frac{L_1'}{L_1}, \quad n_2 \frac{K'}{K} = \frac{L_2'}{L_2} \quad \text{and} \quad n_3 \frac{K'}{K} = \frac{L_3'}{L_3} \tag{1.7}$$

hold. Then a "mixed" modular equation is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$ that is induced by (1.7). We say that β, γ and δ are of degrees n_1, n_2 and n_3 respectively over α. The multipliers $m = K/L_1$ and $m' = L_2/L_3$ are algebraic relation involving α, β, γ and δ.

At scattered places of his second notebook [5], Ramanujan recorded a total of nine P–Q mixed modular relations of degrees 1, 3, 5 and 15. For example, if

$$P := \frac{f(-q^3)f(-q^5)}{q^{1/3}f(-q)f(-q^{15})} \quad \text{and} \quad Q := \frac{f(-q^6)f(-q^{10})}{q^{2/3}f(-q^2)f(-q^{30})},$$

then

$$PQ + \frac{1}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 + 4. \tag{1.8}$$

The proofs of these nine P–Q mixed modular relations can be found in the book by Berndt [2, pp. 214–235]. In [3,4], S. Bhargava, C. Adiga and M. S. Mahadeva

Naika have established several new P–Q mixed modular relations with four moduli. These mixed modular equations are employed to evaluate explicit evaluations of Ramanujan-Weber class invariants, ratios of Ramanujan's theta functions and several continued fractions.

Motivated by all these works, in this article, we establish several new mixed modular equations involving Ramanujan's theta-functions.

In Section 2, we collect some identities which are useful in proofs of our main results. In Section 3, we establish several new P–Q mixed modular equations akin to those recorded by Ramanujan in his notebooks involving his theta-functions $\varphi(q)$ and $\psi(q)$.

2. Preliminary results

In this section, we collect some identities which are useful in establishing our main results.

Lemma 2.1. [1, Ch. 16, Entry 24 (ii), p. 39] *We have*

$$f^3(-q) = \varphi^2(-q)\psi(q). \tag{2.1}$$

Lemma 2.2. [1, Ch. 16, Entry 24 (iv), p. 39] *We have*

$$f^3(-q^2) = \varphi(-q)\psi^2(q). \tag{2.2}$$

Lemma 2.3. [1, Ch. 17, Entry 10 (i) and Entry 11(ii), pp. 122–123] *For $0 < \alpha < 1$, we have*

$$\varphi(q) = \sqrt{z}, \tag{2.3}$$

$$\sqrt{2}q^{1/8}\psi(-q) = \sqrt{z}\{\alpha(1-\alpha)\}^{1/8}. \tag{2.4}$$

Lemma 2.4. [1, Ch. 20, Entry 3 (xii), (xiii), pp. 352–353] *Let α, β and γ be of the first, third and ninth degrees respectively. Let m denote the multiplier relating α, β and m' be the multiplier relating γ, δ. Then*

$$\Big(\frac{\beta^2}{\alpha\gamma}\Big)^{1/4} + \Big(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\Big)^{1/4} - \Big(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\Big)^{1/4} = -3\frac{m}{m'}, \tag{2.5}$$

$$\Big(\frac{\alpha\gamma}{\beta^2}\Big)^{1/4} + \Big(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\Big)^{1/4} - \Big(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\Big)^{1/4} = \frac{m'}{m}. \tag{2.6}$$

Lemma 2.5. [1, Ch. 20, Entry 11 (viii), (ix), p. 384] *Let α, β, γ and δ be of the first, third, fifth and fifteenth degrees respectively. Let m denote the multiplier* *relating* α *and* β *and* m' *be the multiplier relating* γ *and* δ. *Then*

$$\Big(\frac{\alpha\delta}{\beta\gamma}\Big)^{1/8}+\Big(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\Big)^{1/8}-\Big(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\Big)^{1/8}=\sqrt{\frac{m'}{m}}, \tag{2.7}$$

$$\Big(\frac{\beta\gamma}{\alpha\delta}\Big)^{1/8}+\Big(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\Big)^{1/8}-\Big(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\Big)^{1/8}=-\sqrt{\frac{m}{m'}}. \tag{2.8}$$

Lemma 2.6. [3] *If* $L:=\dfrac{q^{1/6}f(-q)f(-q^9)}{f^2(-q^3)}$ *and* $M:=\dfrac{q^{1/3}f(-q^2)f(-q^{18})}{f^2(-q^6)}$, *then*

$$\frac{L^9}{M^9}+\frac{M^9}{L^9}+12\Big[\frac{L^3}{M^3}+\frac{M^3}{L^3}\Big]+27L^3M^3-\frac{1}{L^3M^3}=0. \tag{2.9}$$

Lemma 2.7. [2,3] *If* $L:=\dfrac{q^{1/3}f(-q)f(-q^{15})}{f(-q^3)f(-q^5)}$ *and* $M:=\dfrac{q^{2/3}f(-q^2)f(-q^{30})}{f(-q^6)f(-q^{10})}$, *then*

$$L^3M^3+\frac{1}{L^3M^3}-48\Big[\frac{L^3}{M^3}+\frac{M^3}{L^3}\Big]-12\Big[\frac{L^6}{M^6}+\frac{M^6}{L^6}\Big]-\Big[\frac{L^9}{M^9}+\frac{M^9}{L^9}\Big]=76. \tag{2.10}$$

3. New mixed modular equations

In this section, we establish several new mixed modular equations involving Ramanujan's theta-functions $\varphi(q)$ and $\psi(q)$.

Theorem 3.1. *If* $L:=\dfrac{\varphi(q)\varphi(q^9)}{\varphi^2(q^3)}$ *and* $M:=\dfrac{\sqrt{q}\psi(-q)\psi(-q^9)}{\psi^2(-q^3)}$, *then*

$$L^2=\frac{M^2+1}{1-3M^2}. \tag{3.1}$$

Proof. The equations (2.5) and (2.6) can be rewritten as

$$\frac{m'}{m}+\Big(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\Big)^{1/4}=1-3\frac{m}{m'}\Big(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\Big)^{1/4}. \tag{3.2}$$

Transforming the above equation (3.2) by using the equations (2.3) and (2.4), we arrive at the equation (3.1). ∎

Theorem 3.2. *If* $P:=\dfrac{\sqrt{q}\psi(-q)\psi(-q^9)}{\psi^2(-q^3)}$ *and* $Q:=\dfrac{q\psi(-q^2)\psi(-q^{18})}{\psi^2(-q^6)}$, *then*

$$\frac{P^2}{Q^2}+\frac{Q^2}{P^2}+\Big(\frac{P^2}{Q^2}+\frac{Q^2}{P^2}\Big)^2 =\Big(3PQ-\frac{1}{PQ}\Big)\Big(3PQ-\frac{1}{PQ}+\frac{3P}{Q}+\frac{3Q}{P}+\frac{P^3}{Q^3}+\frac{Q^3}{P^3}\Big)+14. \tag{3.3}$$

Proof. Using the equation (2.2) in the equation (2.9), we deduce

$$a^3P^{12} - aP^4bQ^4 + 27a^2P^8b^2Q^8 + 12a^2P^8bQ^4 + 12aP^4b^2Q^8 + b^3Q^{12} = 0, \quad (3.4)$$

where $a := a(q) := \dfrac{\varphi^2(q)\varphi^2(q^9)}{\varphi^4(q^3)}$ and $b := a(q^2)$.

Using the equation (3.1), we have

$$a := \frac{P^2+1}{1-3P^2} \quad \text{and} \quad b := \frac{Q^2+1}{1-3Q^2}. \quad (3.5)$$

Employing the equations (3.5) in the equation (3.4), we find

$$A(P,Q)B(P,Q) = 0, \quad (3.6)$$

where

$$\begin{aligned}A(P,Q) := \big(& -2P^2Q^4 - 13P^6Q^2 + 25P^6Q^4 + 9P^4Q^4 - 13P^2Q^6 - 16P^8Q^2 \\ & + 39P^8Q^4 - 6P^{10}Q^2 - 27P^6Q^6 - 18P^8Q^6 + 25P^4Q^6 - 16P^2Q^8 + 39P^4Q^8 \\ & + 9P^{10}Q^4 - 18P^6Q^8 - 27P^8Q^8 - 6P^2Q^{10} + 9P^4Q^{10} + P^2Q^2 + P^6 + Q^6 \\ & + P^{10} + 2P^8 + 2Q^8 + Q^{10} - 2P^4Q^2\big)\end{aligned}$$

and

$$\begin{aligned}B(P,Q) := \big(& -P^8 + 3P^8Q^2 - P^6Q^2 - P^6 + 9P^6Q^4 + 9P^6Q^6 - Q^6 + 9P^4Q^6 \\ & + 6P^4Q^4 - 3P^4Q^2 + 3P^2Q^8 - P^2Q^6 - 3P^2Q^4 + P^2Q^2 - Q^8\big).\end{aligned}$$

Expanding in powers of q, the first and second factor of the equation (3.6), one gets respectively,

$$A(P,Q) = -q^3\big(-1 - q + q^2 + 4q^3 - 10q^4 - 4q^5 + 37q^6 - 28q^7 - 48q^8 + \cdots\big)$$

and

$$B(P,Q) = q^5\big(4 - 12q - 28q^2 + 144q^3 - 32q^4 - 764q^5 + 1424q^6 + 1400q^7 + \cdots\big).$$

As $q \to 0$, the factor $q^{-3}B(P,Q)$ of the equation (3.6) vanishes whereas the other factor $q^{-3}A(P,Q)$ do not vanish. Hence, we arrive at the equation (3.3) for $q \in (0,1)$. By analytic continuation the equation (3.3) is true for $|q| < 1$. ∎

Theorem 3.3. *If* $P := \dfrac{\sqrt{q}\psi(q)\psi(q^9)}{\psi^2(q^3)}$ *and* $Q := \dfrac{q\psi(q^2)\psi(q^{18})}{\psi^2(q^6)}$, *then*

$$\frac{P^2}{Q^2} + \frac{Q^2}{P^2} = 4 - 3P^2 + \frac{1}{P^2}. \quad (3.7)$$

Proof. Using the equation (2.1) in the equation (2.9), we deduce

$$27a^2P^4b^2Q^4 + a^3P^6 + 12a^2P^4bQ^2 + 12aP^2b^2Q^4 + b^3Q^6 = aP^2bQ^2, \quad (3.8)$$

where $a := a(q) = \dfrac{\varphi^2(-q)\varphi^2(-q^9)}{\varphi^4(-q^3)}$ and $b := a(q^2)$.

Using the equation (3.1) after changing q by $-q$, we find

$$a=\frac{1-P^2}{1+3P^2} \quad \text{and} \quad b=\frac{1-Q^2}{1+3Q^2}. \tag{3.9}$$

Employing the equations (3.9) in the equation (3.8), we find

$$\begin{aligned}&\big(3P^2Q^2+P^2-1+Q^2\big)\big(3P^4Q^2+P^4-4P^2Q^2-Q^2+Q^4\big)\\ &\qquad\big(1-2P^2-2Q^2+P^4+44P^2Q^2+6P^4Q^2+9P^4Q^4+16PQ\\ &\qquad\quad+48P^3Q^3+6P^2Q^4+Q^4\Big)=0.\end{aligned} \tag{3.10}$$

Expanding in powers of q, factors of the equation (3.10) becomes respectively,

$$\big(-1+q+3q^2+2q^3+2q^4-2q^5-q^6-8q^7-24q^8+\cdots\big),$$
$$q^4\big(-4-8q-8q^2+12q^3+52q^4+24q^5-76q^6-120q^7+32q^8+\cdots\big)$$

and

$$\big(1-2q-5q^2+50q^3+105q^4+68q^5-82q^6-374q^7-362q^8+\cdots\big).$$

As $q\to 0$, the second factor of the equation (3.10) vanishes whereas the other factors do not vanish. Hence, we arrive at the equation (3.7) for $q\in(0,1)$. By analytic continuation the equation (3.7) is true for $|q|<1$. ■

Theorem 3.4. *If* $P:=\dfrac{\varphi(-q)\varphi(-q^9)}{\varphi^2(-q^3)}$ *and* $Q:=\dfrac{\varphi(-q^2)\varphi(-q^{18})}{\varphi^2(-q^6)}$, *then*

$$\frac{P^2}{Q^2}+\frac{Q^2}{P^2}=4-3Q^2+\frac{1}{Q^2}. \tag{3.11}$$

Proof. The proof of the equation (3.11) is similar to the proof of the equation (3.7). Hence, we omit the details. ■

Theorem 3.5. *If* $P:=\dfrac{\varphi(-q)\varphi(-q^9)}{\varphi^2(-q^3)}$ *and* $Q:=\dfrac{\varphi(-q^4)\varphi(-q^{36})}{\varphi^2(-q^{12})}$, *then*

$$\begin{aligned}&\Big(\frac{P^4}{Q^4}+\frac{Q^4}{P^4}\Big)+8\Big(\frac{P^2}{Q^2}+\frac{Q^2}{P^2}\Big)-3\Big(\frac{P^2}{Q^4}-\frac{3Q^4}{P^2}\Big)-\Big(\frac{1}{P^2Q^4}-27P^2Q^4\Big)\\ &\qquad-8\Big(\frac{1}{P^2Q^2}+9P^2Q^2\Big)-12\Big(\frac{1}{P^2}-3P^2\Big)+3\Big(9Q^4+\frac{1}{Q^4}\Big)=24.\end{aligned} \tag{3.12}$$

Proof. Using the equation (2.9) along with the equations (2.1) and (2.2), we deduce

$$27P^4u^4Q^8v^2+12P^4u^4Q^4v+12Q^8v^2P^2u^2+P^6u^6+Q^{12}v^3=P^2u^2Q^4v, \tag{3.13}$$

where $u:=\dfrac{q\psi^2(q)\psi^2(q^9)}{\psi^4(q^3)}$ and $v:=u(q^4)$.

Invoking the equation (3.1), we find

$$u := \frac{1-P^2}{3P^2+1} \text{ and } v := \frac{1-Q^2}{3Q^2+1}. \tag{3.14}$$

Using the equations (3.14) in the equation (3.13), we find

$$\begin{aligned}&\big(3Q^2P^4+P^4-2Q^2P^2-P^2+3P^2Q^4+Q^4-Q^2\big)\big(3Q^2P^2+P^2-1\\&\quad+4QP+Q^2\big)\big(3Q^2P^2+P^2-1-4QP+Q^2\big)\big(P^8+27P^6Q^8-72P^6Q^6\\&\quad+36P^6Q^4+8P^6Q^2-3P^6+27P^4Q^8-24P^4Q^4+3P^4+9P^2Q^8\\&\quad+8P^2Q^6-12P^2Q^4-8Q^2P^2-P^2+Q^8\big)=0.\end{aligned} \tag{3.15}$$

As $q \to 0$, the last factor of the equation (3.15) vanishes whereas the other factors do not vanish. Hence, we arrive at the equation (3.12) for $q \in (0,1)$. By analytic continuation the equation (3.12) is true for $|q| < 1$. ■

Theorem 3.6. *If* $P := \dfrac{\varphi(-q)\varphi(-q^9)}{\varphi^2(-q^3)}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^9)}{\varphi^2(q^3)}$, *then*

$$\left(\frac{P}{Q}+\frac{Q}{P}\right)+\left(3PQ-\frac{1}{PQ}\right)-4=0. \tag{3.16}$$

Proof. Using the equation (2.9) along with the equations (2.1) and (2.2), we deduce

$$27P^8u^2Q^4v^4+P^{12}u^3+12P^8u^2Q^2v^2+12P^4uQ^4v^4+Q^6v^6=P^4uQ^2v^2, \tag{3.17}$$

where $u := \dfrac{q\psi^2(q)\psi^2(q^9)}{\psi^4(q^3)}$ and $v := \dfrac{q\psi^2(-q)\psi^2(-q^9)}{\psi^4(-q^3)}$.

Invoking the equation (3.1), we find

$$u := \frac{1-P^2}{3P^2+1} \text{ and } v := \frac{Q^2-1}{3Q^2+1}. \tag{3.18}$$

Using the equations (3.18) in the equation (3.17), we find

$$\begin{aligned}&\big(3P^2Q^2+Q^2-1+4PQ+P^2\big)\big(3P^2Q^2+Q^2-1-4PQ+P^2\big)\big(3P^2Q^4\\&\quad+Q^4-Q^2-2P^2Q^2-P^2+3P^4Q^2+P^4\big)\big(27P^8Q^6+27P^8Q^4+9P^8Q^2\\&\quad+P^8-72P^6Q^6+8P^6Q^2+36P^4Q^6-24P^4Q^4-12P^4Q^2+8P^2Q^6\\&\quad-8P^2Q^2+Q^8-3Q^6+3Q^4-Q^2\big)=0.\end{aligned} \tag{3.19}$$

As $q \to 0$, the second factor of the equation (3.19) vanishes whereas the other factors do not vanish. Hence, we arrive at the equation (3.16) for $q \in (0,1)$. By analytic continuation the equation (3.16) is true for $|q| < 1$. ■

Theorem 3.7. *If* $P := \dfrac{\sqrt{q}\psi(q)\psi(q^9)}{\psi^2(q^3)}$ *and* $Q := \dfrac{q^2\psi(q^4)\psi(q^{36})}{\psi^2(q^{12})}$, *then*

$$\begin{aligned}&\left(\frac{P^4}{Q^4}+\frac{Q^4}{P^4}\right)+8\left(\frac{P^2}{Q^2}+\frac{Q^2}{P^2}\right)-8\left(3^2P^2Q^2+\frac{1}{P^2Q^2}\right)+3\left(\frac{3P^4}{Q^2}-\frac{Q^2}{P^4}\right)\\&\quad+\left(27P^4Q^2-\frac{1}{P^4Q^2}\right)+3\left(9P^4+\frac{1}{P^4}\right)+12\left(3Q^2-\frac{1}{Q^2}\right)-24=0.\end{aligned} \tag{3.20}$$

Proof. The proof of the equation (3.20) is similar to the proof of the equation (3.12). Hence, we omit the details. ■

THEOREM 3.8. *If* $P := \dfrac{\sqrt{q}\psi(q)\psi(q^9)}{\psi^2(q^3)}$ *and* $Q := \dfrac{\sqrt{q}\psi(-q)\psi(-q^9)}{\psi^2(-q^3)}$, *then*

$$\Big(\frac{P^2}{Q^2}+\frac{Q^2}{P^2}\Big)-\Big(3PQ+\frac{1}{PQ}\Big)\Big(\frac{P}{Q}-\frac{Q}{P}\Big)+2=0. \tag{3.21}$$

Proof. The proof of the equation (3.21) is similar to the proof of the equation (3.16). Hence, we omit the details. ■

THEOREM 3.9. *If* $L := \dfrac{\varphi(q^3)\varphi(q^5)}{\varphi(q)\varphi(q^{15})}$ *and* $M := \dfrac{\psi(-q^3)\psi(-q^5)}{q\psi(-q)\psi(-q^{15})}$, *then*

$$M=\frac{1+L}{1-L}. \tag{3.22}$$

Proof. The equations (2.7) and (2.8) can be rewritten as

$$1+\sqrt{\frac{m}{m'}}\Big(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\Big)^{1/8}=\sqrt{\frac{m}{m'}}-\frac{m}{m'}\Big(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\Big)^{1/8}. \tag{3.23}$$

Transforming the above equation (3.23) by using the equations (2.3) and (2.4), we arrive at equation (3.22). ■

THEOREM 3.10. *If* $P := \dfrac{\psi(-q^3)\psi(-q^5)}{q\psi(-q)\psi(-q^{15})}$ *and* $Q := \dfrac{\psi(-q^6)\psi(-q^{10})}{q^2\psi(-q^2)\psi(-q^{30})}$, *then*

$$\begin{aligned}\Big(\frac{P^2}{Q^2}+\frac{Q^2}{P^2}\Big)+\Big(\frac{P}{Q}+\frac{Q}{P}\Big)-\Big(PQ+\frac{1}{PQ}\Big)\\ +\Big(\sqrt{PQ}-\frac{1}{\sqrt{PQ}}\Big)\Big(\sqrt{\frac{P^3}{Q^3}}+\sqrt{\frac{Q^3}{P^3}}+\sqrt{\frac{P}{Q}}+\sqrt{\frac{Q}{P}}\Big)=0.\end{aligned} \tag{3.24}$$

Proof. Using the equation (2.2) in the equation (2.10), we find

$$\begin{aligned}a^2P^4b^2Q^4+a^4P^8b^4Q^8-a^6P^{12}-48a^4P^8b^2Q^4-12a^5P^{10}bQ^2\\ -48a^2P^4b^4Q^8-76a^3P^6b^3Q^6-b^6Q^{12}-12b^5Q^{10}aP^2=0,\end{aligned} \tag{3.25}$$

where $a := a(q) := \dfrac{\varphi(q^3)\varphi(q^5)}{\varphi(q)\varphi(q^{15})}$ and $b := a(q^2)$.

Using the equation (3.22), we have

$$a := a(q) := \frac{P-1}{P+1} \quad\text{and}\quad b := b(q) := \frac{Q-1}{Q+1}. \tag{3.26}$$

Employing the equations (3.26) in the equation (3.25), we find

$$\begin{aligned}&\big(PQ+P-1+Q\big)\big(P^2-P-2PQ+PQ^2+Q^2\big)\big(P^2Q+P^2-Q\\&\quad-2PQ+Q^2\big)\big(12P^4Q^4-P^2Q^2-5P^4Q^3-5P^4Q^2-5P^3Q^4-P^3Q^3\\&\quad-P^3Q^2-5P^2Q^4-P^2Q^3-P^6-Q^6-3P^8-3Q^8+3P^6Q^2+7P^5Q^2\\&\quad-3P^6Q+3P^2Q^6+7P^2Q^5-3PQ^6-7P^4Q^6-2P^4Q^5+5P^3Q^6\\&\quad+7P^3Q^5-12P^5Q^5-2P^5Q^4+7P^5Q^3-7P^6Q^4+5P^6Q^3+P^6Q^6\\&\quad-5P^6Q^5-5P^5Q^6+P^7Q^7-P^7Q^6+5P^7Q^5-P^6Q^7+5P^5Q^7+7P^7Q^4\\&\quad+7P^4Q^7+3Q^7+Q^9+3P^7+P^9+3P^9Q^2+P^9Q^3-3P^3Q^8-3P^3Q^7\\&\quad+P^3Q^9+3P^9Q-9P^8Q^2+3P^7Q^2-9P^8Q+9P^7Q-3P^8Q^3\\&\quad-3P^7Q^3+3Q^9P^2-9Q^8P^2+3Q^7P^2+9Q^7P-9Q^8P+3PQ^9\big)\\&\quad\big(P^4+Q^4-P^2Q-PQ^2-PQ+P^4Q-P^3Q^3+P^3Q^2+P^3Q\\&\quad+P^2Q^3+PQ^4+PQ^3-P^3-Q^3\big)=0.\end{aligned}\tag{3.27}$$

As $q\to 0$, the last factor of the equation (3.27) vanishes whereas the other factors do not vanish. Hence, we arrive at the equation (3.24) for $q\in(0,1)$. By analytic continuation the equation (3.24) is true for $|q|<1$. ∎

Theorem 3.11. *If* $P:=\dfrac{\psi(q^3)\psi(q^5)}{q\psi(q)\psi(q^{15})}$ *and* $Q:=\dfrac{\psi(q^6)\psi(q^{10})}{q^2\psi(q^2)\psi(q^{30})}$, *then*

$$\Big(\frac{P}{Q}+\frac{Q}{P}\Big)=\Big(P-\frac{1}{P}\Big)+2.\tag{3.28}$$

Proof. Using the equation (2.1) in the equation (2.10), we deduce

$$\begin{aligned}&a^2P^2b^2Q^2+a^4P^4b^4Q^4-a^6P^6-48a^4P^4b^2Q^2-12a^5P^5bQ\\&\qquad-48a^2P^2b^4Q^4-76a^3P^3b^3Q^3-b^6Q^6-12b^5Q^5aP=0,\end{aligned}\tag{3.29}$$

where $a:=a(q):=\dfrac{\varphi(-q^3)\varphi(-q^5)}{\varphi(-q)\varphi(-q^{15})}$ and $b:=a(q^2)$.

Changing q to $-q$ in the equation (3.22), we find

$$a:=\frac{1+P}{P-1}\quad\text{and}\quad b:=\frac{1+Q}{Q-1}.\tag{3.30}$$

Using the equations (3.30) in the equation (3.29), we find

$$\begin{aligned}&\big(-P+PQ-1-Q\big)\big(-P^2+P^2Q+2PQ-Q-Q^2\big)\big(P^2+P-2PQ\\&\quad+Q^2-PQ^2\big)\big(-P^4-P^3-Q^3-PQ^2-PQ^3+PQ^4-P^2Q+P^2Q^3\\&\quad-P^3Q+P^3Q^2+P^3Q^3+P^4Q-Q^4+PQ\big)\big(-3P^8-P^6-3P^7-P^9\\&\quad-P^2Q^2+P^2Q^3-5P^2Q^4+P^3Q^2-P^3Q^3+5P^3Q^4-5P^4Q^2+5P^4Q^3\\&\quad+12P^4Q^4+3QP^6+9QP^7+9QP^8+7Q^3P^5-5Q^3P^6-3Q^3P^7+3Q^3P^8\end{aligned}$$

$$
\begin{aligned}
&-7P^5Q^2+3P^6Q^2-3P^7Q^2-9P^8Q^2-3P^9Q^2+P^9Q^3+2P^5Q^4-7P^6Q^4\\
&-7P^7Q^4+3P^9Q+2P^4Q^5-7P^4Q^6-7P^4Q^7+5P^6Q^5+P^6Q^6+P^6Q^7\\
&-12P^5Q^5+5P^5Q^6+5P^5Q^7+5P^7Q^5+P^7Q^6+P^7Q^7+3PQ^6+9PQ^7\\
&+9PQ^8+7P^3Q^5-5P^3Q^6-3P^3Q^7+3P^3Q^8+3PQ^9-7P^2Q^5+3P^2Q^6\\
&-3P^2Q^7-9P^2Q^8-3P^2Q^9+P^3Q^9-3Q^8-Q^6-3Q^7-Q^9\big)=0.
\end{aligned} \tag{3.31}
$$

As $q \to 0$, the second factor of the equation (3.31) vanishes whereas the other factors do not vanish. Hence, we arrive at the equation (3.28) for $q \in (0,1)$. By analytic continuation the equation (3.28) is true for $|q| < 1$. ■

Theorem 3.12. *If* $P := \dfrac{\varphi(-q^3)\varphi(-q^5)}{\varphi(-q)\varphi(-q^{15})}$ *and* $Q := \dfrac{\varphi(-q^6)\varphi(-q^{10})}{\varphi(-q^2)\varphi(-q^{30})}$, *then*

$$
\Big(\frac{P}{Q}+\frac{Q}{P}\Big)=\Big(Q-\frac{1}{Q}\Big)+2. \tag{3.32}
$$

Proof. The proof of the equation (3.32) is similar to the proof of the equation (3.28). Hence, we omit the details. ■

Theorem 3.13. *If* $P := \dfrac{\varphi(-q^3)\varphi(-q^5)}{\varphi(-q)\varphi(-q^{15})}$ *and* $Q := \dfrac{\varphi(-q^{12})\varphi(-q^{20})}{\varphi(-q^4)\varphi(-q^{60})}$, *then*

$$
\begin{aligned}
\Big(\frac{P^2}{Q^2}+\frac{Q^2}{P^2}\Big)+3\Big(Q^2+\frac{1}{Q^2}\Big)-4\Big(PQ+\frac{1}{PQ}\Big)+3\Big(\frac{P}{Q^2}-\frac{Q^2}{P}\Big)&\\
-\Big(PQ^2+\frac{1}{PQ^2}\Big)-2\Big(P-\frac{1}{P}\Big)+4\Big(Q-\frac{1}{Q}\Big)&=0.
\end{aligned} \tag{3.33}
$$

Proof. Using the equations (2.1) and (2.2) in the equation (2.10), we deduce

$$
\begin{aligned}
&48Q^2b^2P^4a^4-Q^4b^4P^4a^4-Q^2b^2P^2a^2+48Q^4b^4P^2a^2+P^6a^6\\
&\quad+76Q^3b^3P^3a^3+12QbP^5a^5+Q^6b^6+12Q^5b^5Pa=0,
\end{aligned} \tag{3.34}
$$

where $a := a(q) := \dfrac{\psi(q^3)\psi(q^5)}{\psi(q)\psi(q^{15})}$ and $b := a(q^4)$.

Changing q to $-q$ in the equation (3.22), we find

$$
a := \frac{1+P}{P-1} \quad \text{and} \quad b := \frac{1+Q}{Q-1}. \tag{3.35}
$$

Employing the equations (3.35) in the equation (3.34), we deduce

$$
\begin{aligned}
&\big(Q^2P^2-2QP^2+P^2+2P+1-4QP+2Q-2Q^2P+Q^2\big)\big(QP^2-P^2\\
&\quad-P-Q+Q^2P-Q^2\big)\big(3Q^8-6P^4-15P^7-6P^8+45Q^8P^4+3QP^3\\
&\quad+18QP^4+45QP^5+60QP^6+45QP^7+18QP^8-Q^2P+8Q^2P^2\\
&\quad+10Q^2P^3-28Q^2P^4-68Q^2P^5-64Q^2P^6-34Q^2P^7-12Q^2P^8
\end{aligned}
$$

$$
\begin{aligned}
&+Q^3P-8Q^3P^2-12Q^3P^3+16Q^3P^4+38Q^3P^5+24Q^3P^6+4Q^3P^7\\
&-5Q^4P+3Q^4P^2-9Q^4P^3-13Q^4P^4+33Q^4P^5+17Q^4P^6-19Q^4P^7\\
&-7Q^4P^8-7Q^5P+19Q^5P^2+17Q^5P^3-33Q^5P^4-13Q^5P^5+9Q^5P^6\\
&+3Q^5P^7+5Q^5P^8+4Q^6P^2-24Q^6P^3+38Q^6P^4-16Q^6P^5-12Q^6P^6\\
&+8Q^6P^7+Q^6P^8+3QP^9-3Q^2P^9+Q^3P^9+3Q^8P^6-18Q^8P^5\\
&+15Q^9P^4-6Q^9P^5+Q^9P^6+Q^6-P^3-20P^6-15P^5-P^9+45Q^8P^2\\
&+34Q^7P^2+15Q^9P^2-64Q^7P^3+68Q^7P^4-28Q^7P^5-10Q^7P^6+8Q^7P^7\\
&+Q^7P^8-60Q^8P^3-20Q^9P^3+3Q^7+Q^9-6Q^9P-18Q^8P-12Q^7P\big)\\
&\big(P^4+Q^4+3P^2+P-4QP-4QP^2+2Q^2P-2Q^2P^3+4Q^3P^2\\
&-4Q^3P^3-3Q^4P+3Q^4P^2-Q^4P^3+3P^3\big)=0. \qquad (3.36)
\end{aligned}
$$

As $q \to 0$, the last factor of the equation (3.36) vanishes whereas the other factors do not vanish. Hence, we arrive at the equation (3.33) for $q \in (0,1)$. By analytic continuation the equation (3.33) is true for $|q| < 1$. ■

Acknowledgement. The authors are grateful to the referee for his/her valuable remarks and suggestions which considerably improved the quality of the paper.

REFERENCES

[**1**] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[**2**] B. C. Berndt, *Ramanujan's Notebooks, Part IV*, Springer-Verlag, New York, 1994.

[**3**] S. Bhargava, C. Adiga, M.S. Mahadeva Naika, *A new class of modular equations in Ramanujan's alternative theory of elliptic function of signature 4 and some new P–Q eta-function identities*, Indian J. Math. **45** (1) (2003), 23–39.

[**4**] S. Bhargava,C. Adiga, M.S. Mahadeva Naika, *A new class of modular equations akin to Ramanujan's P–Q eta-function identities and some evaluations therefrom*, Adv. Stud. Contemp. Math. **5** (1) (2002), 37–48.

[**5**] S. Ramanujan, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.

(received 14.12.2012; in revised form 24.06.2013; available online 10.09.2013)

M. S. Mahadeva Naika, M. Harish, Department of Mathematics, Bangalore University, Central College Campus, Bengaluru-560 001, Karnataka, India

E-mail: msmnaika@rediffmail.com, numharish@gmail.com

S. Chandankumar, Department of Mathematics, Acharya Institute of Graduate Studies, Soldevanahalli, Hesarghatta Main Road, Bengaluru-560 107, Karnataka, India

E-mail: chandan.s17@gmail.com