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PROXIMITY STRUCTURES AND IDEALS

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Abstract. In this paper, we present a new approach to proximity structures based on the recognition of many of the entities important in the theory of ideals. So, we give a characterization of the basic proximity using ideals. Also, we introduce the concept of g-proximities and we show that for different choice of "g" one can obtain many of the known types of generalized proximities. Also, characterizations of some types of these proximities – (g_0, h_0) – are obtained.

1. Introduction

Ideals in topological spaces were introduced by Kuratowski [6], Vaidyanathaswamy [12] and Janković and Hamlett [5]. Various classes of generalized proximities have been extensively studied by many authors including Lodato [8,9]. In [4], the authors introduced a new approach to construct generalized proximity structures based on the concept of ideal and an EF-Proximity structure. Thron [11] introduced grills to investigate proximity structures. In this paper, we present an equivalent formulation of the notion of basic proximity using ideals and study some of its properties. The concept of a basic proximity on a set and a basic proximal neighborhood of a set with respect to a basic proximity are obtained. Also we introduce the concept of g-proximity and we show that for different choice of "g" one can obtain many types of proximities.

2. Preliminaries

The purpose of this section is merely to recall known results concerning ideals and proximity spaces. For more information see [1,4–6,10–12].

Definition 2.1. [5] A nonempty collection $\mathcal{I}$ of subsets of a nonempty set X is said to be an ideal on X if it satisfies the following two conditions:

1. $A \in \mathcal{I}$ and $B \subseteq A \implies B \in \mathcal{I}$ (hereditarity),
2. $A \in \mathcal{I}$ and $B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$ (finite additivity),

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i.e., $\mathcal{I}$ is closed under finite union and subsets. $\mathfrak{I}(X)$ will denote the set of all ideals on X.

In order to exclude the trivial case where the ideal coincides with the set of all subsets of the set X, it is generally assumed that $X \notin \mathcal{I}$. In this case $\mathcal{I}$ is called a proper ideal on X.

One of the important ideals is $\mathcal{I}_A$ $(= \{B : B \in P(X), B \subseteq A\})$ (where $P(X)$ stands for the power set of X).

Definition 2.2. [10] Let δ be a binary relation on the power set $P(X)$ of a nonempty set X. For any $A, B, C \in P(X)$, consider the following axioms:

P_1: $A\delta B \Rightarrow B\delta A$,

P_2: $(A \cup B)\delta C \Leftrightarrow A\delta C$ or $B\delta C$,

P_2': $(A \cup B)\delta C \Leftrightarrow A\delta C$ or $B\delta C$ and $A\delta(B \cup C) \Leftrightarrow A\delta B$ or $A\delta C$,

P_3: $A\delta B \Rightarrow A \neq \phi$, $B \neq \phi$,

P_4: $A \cap B \neq \phi \Rightarrow A\delta B$,

P_5: $A\bar{\delta}B \Rightarrow \exists E \in P(X)$ such that $A\bar{\delta}E$ and $E^c\bar{\delta}B$ (here, and henceforth also, $\bar{\delta}$ means non-δ and $E^c = X - E$),

P_6: $\{x\}\delta\{y\} \Rightarrow x = y$,

P_7: $A\delta B$ and $\{b\}\delta C \; \forall \, b \in B \Rightarrow A\delta C$,

P_7': $\{x\}\delta B$ and $\{b\}\delta C \; \forall \, b \in B \Rightarrow \{x\}\delta C$.

Then δ is said to be :

(a) a basic proximity on X if it satisfies P_1, P_2, P_3 and P_4;

(b) an Efremovich proximity (EF-proximity) on X if it is a basic proximity and satisfies P_5;

(c) a separated proximity on X if it is an EF-proximity on X and it satisfies P_6;

(d) a Leader proximity (LE-proximity)on X if it satisfies P_2', P_3, P_4 and P_7;

(e) a Lodato proximity (LO-proximity) on X if it is an LE-proximity on X and satisfies P_1;

(f) an S-proximity on X if it is a basic proximity on X and satisfies P_6 and P_7'.

If δ is a basic proximity (resp. EF-proximity, separated proximity, LE-proximity, LO-proximity, S-proximity) on X, then the pair (X, δ) is called a basic proximity (resp. EF-proximity, separated proximity, LE-proximity, LO-proximity, S-proximity) space.

We denote by $m(X)$ the set of all basic proximities on X and we write $x\delta A$ for $\{x\}\delta A$.

Definition 2.3. [1] A binary relation δ on the power set $P(X)$ of a nonempty set X is said to be RH-proximity on X if it satisfies the following conditions:

R_1: $A\delta B \Rightarrow B\delta A$,

R_2: $(A \cup B)\delta C \Leftrightarrow A\delta C$ or $B\delta C$,

R_3: $\phi\bar{\delta}X$,

R_4: $A \neq \phi \Rightarrow A\delta A$, and

R_5: $\{x\}\bar{\delta}A \Rightarrow \exists E \in P(X)$ such that $\{x\}\bar{\delta}E$ and $E^c\bar{\delta}A$.

Lemma 2.1. [4] *For all subsets A and B of a basic proximity space (X, δ), if $A\delta B$, $A \subseteq C$ and $B \subseteq D$, then $C\delta D$.*

Lemma 2.2. [3] *For all subsets A and B of a basic proximity space (X, δ),*

(i) *if $A\delta B$, $A \subseteq C$, then $B\delta C$;*

(ii) *if $A\delta B$, $B \subseteq C$, then $A\delta C$.*

Definition 2.4. [11] A subset B of a basic proximity space (X, δ) is said to be a proximal neighborhood of a set A with respect δ if $B^c\bar{\delta}A$. The set of all proximal neighborhoods of a set A with respect to δ is denoted by $N(\delta, A)$, i.e.,

$$N(\delta, A) = \{B : B \in P(X), B^c\bar{\delta}A\}.$$

When there is no ambiguity we will write $N_\delta(A)$ for $N(\delta, A)$.

Lemma 2.3. [4] *For all subsets A and B of a basic proximity space (X, δ),*

(i) *$A \in N_\delta(B) \Leftrightarrow B^c \in N_\delta(A^c)$;*

(ii) *$N_\delta(A \cup B) = N_\delta(A) \cap N_\delta(B)$.*

Lemma 2.4. [11] *For all subsets A and B of a basic proximity space (X, δ), if $A \subseteq B$, then $N_\delta(B) \subseteq N_\delta(A)$. Also, $N_\delta(\phi) = P(X)$.*

Theorem 2.1. [11] *For all subsets A, B of a basic proximity space (X, δ), if $H \in N_\delta(A)$ and $M \in N_\delta(B)$, then $H \cup M \in N_\delta(A \cup B)$.*

Definition 2.5. [10] A subset A of a basic proximity space (X, δ) is said to be δ-closed if $x\delta A$ implies $x \in A$.

Definition 2.6. [10] Let δ_1, δ_2 be two basic proximities on a nonempty set X. We define

$$\delta_1 < \delta_2 \text{ if } A\delta_2 B \Rightarrow A\delta_1 B.$$

The above expression refers to that δ_2 is finer than δ_1, or δ_1 is coarser than δ_2.

Definition 2.7. [11] Let δ_1, δ_2 be two basic proximities on a nonempty set X. We define

$$\delta_1 \subseteq \delta_2 \text{ if } A\delta_1 B \Rightarrow A\delta_2 B.$$

Definition 2.8. [6] A mapping c: $P(X) \to P(X)$ is said to be a Čech closure operator if it satisfies the following axioms:

1. $c(\phi) = \phi$,
2. $A \subseteq c(A)\ \forall A \in P(X)$,
3. $c(A \cup B) = c(A) \cup c(B)\ \forall A, B \in P(X)$.

If in addition c satisfies the following condition

4. $c(c(A)) = c(A)\ \forall\ A \in P(X)$ ("idempotent condition"),

then c is called a Kuratowski's closure operator (or closure operator, for short).

Definition 2.9. [7] Let (X, δ_1) and (Y, δ_2) be two basic proximity spaces and $f\colon X \to Y$ be a map. Then f is called a basic-proximally continuous (BP-continuous, for short) map if $A\delta_1 B$ implies $f(A)\delta_2 f(B)$.

Theorem 2.2. [11] *Let (X, δ) be a basic proximity space. Then the operator $c_\delta : P(X) \to P(X)$ given by*

$$c_\delta(A) = \{x \in X : x\delta A\},\ \textit{for all } A \in P(X)$$

is a Čech closure operator.

Theorem 2.3. [11] *Let (X, δ) be a basic proximity space. Then*

$$c_\delta(A) = \cap\{B : B \in N_\delta(A)\}.$$

Proposition 2.1. [4] *Let (X, δ) be an EF-proximity space. Then the operator c_δ is a closure operator and the collection*

$$\tau_\delta = \{A \subseteq X : c_\delta(A^c) = A^c\}$$

is a topology on X and (X, τ_δ) is a completely regular topological space.

3. Some properties of basic proximities and ideals

Definition 3.1. Let δ be a binary relation on the power set $P(X)$ of a nonempty set X. For all $A \in P(X)$, we define

$$\delta[A] = \{B : B \in P(X), B\bar{\delta}A\}.$$

Definition 3.2. A binary relation δ on the power set $P(X)$ of a nonempty set X is said to be a basic proximity on X if it satisfies the following conditions for any $A, B, C \in P(X)$:

PI_1 : $A \in \delta[B] \Rightarrow B \in \delta[A]$,

PI_2 : $A \in \delta[C]$ and $B \in \delta[C] \Leftrightarrow A \cup B \in \delta[C]$,

PI_3 : $\phi \in \delta[A]$, for all $A \in P(X)$, and

PI_4 : $A \in \delta[B] \Rightarrow A \cap B = \phi$. δ is said to be an EF-proximity on X if it is a basic proximity on X and it satisfies the following condition:

PI_5 : $A \in \delta[B] \Rightarrow \exists H \in P(X)$ such that $A \in \delta[H]$ and $H^c \in \delta[B]$.

δ is said to be a separated proximity on X if it is an EF-proximity on X and it satisfies the following condition:

PI_6 : $x \neq y \Rightarrow \{x\} \in \delta[\{y\})]$.

For all $x \in X$, $x \in \delta[A]$ stands for $\{x\} \in \delta[A]$ and $\delta[x]$ stands for $\delta[\{x\}]$.

Lemma 3.1. *For all subsets A and B of a basic proximity space (X, δ), if $A \in \delta[B]$ and $E \subseteq B$, then $A \in \delta[E]$.*

Proof. Let $A \in \delta[B]$ and $E \subseteq B$. Assume that $A \notin \delta[E]$. Then $E\delta A$, but $E \subseteq B$, then (by Lemma 2.2(i)) $A\delta B$, i.e., $A \notin \delta[B]$, a contradiction. ∎

Lemma 3.2. *Let (X, δ) be a basic proximity space. Then*

(i) $A \subseteq B \Rightarrow \delta[B] \subseteq \delta[A]$,

(ii) $A \in \delta[B] \Rightarrow a \in \delta[B] \ \forall\, a \in A$.

Proof. (i) it is obvious by Lemma 3.1.

(ii) Let $A \in \delta[B]$ and assume that $\exists\, a \in A$ such that $a \notin \delta[B]$. Then $a\delta B$, but $\{a\} \subseteq A$, hence $A\delta B$ (by Lemma 2.2(i)), which contradicts with $A \in \delta[B]$. ∎

Proposition 3.1. *Let (X, δ) be a basic proximity space. Then*

$$\delta[A] \text{ is an ideal on } X, \ \forall\, A \in P(X).$$

Proof. Since $\phi \in \delta[A]$ (by PI_3), then $\delta[A]$ is nonempty. Let $H \in \delta[A]$ and $M \subseteq H$. Then $A \in \delta[H]$ and $M \subseteq H \Rightarrow M \in \delta[A]$ (by Lemma 3.1, PI_1). Now, let $H \in \delta[A]$ and $M \in \delta[A]$. Then $H \cup M \in \delta[A]$ (by PI_2). Hence $\delta[A]$ is an ideal on X. ∎

Lemma 3.3. *Let (X, δ) be a basic proximity space. Then the two simplest ideals on X generated by δ are $\delta[\phi] = P(X)$ and $\delta[X] = \{\phi\}$.*

Proof. Straightforward. ∎

Example 3.1. Let $X = \{a, b, c\}$ and let δ be a basic proximity defined as

$$A\delta B \Leftrightarrow A \cap B \neq \phi.$$

Then: $\delta[\phi] = P(X)$, $\delta[\{a\}] = \{\phi, \{b\}, \{c\}, \{b, c\}\}$, $\delta[\{b\}] = \{\phi, \{a\}, \{c\}, \{a, c\}\}$, $\delta[\{c\}] = \{\phi, \{a\}, \{b\}, \{a, b\}\}$, $\delta[\{a, b\}] = \{\phi, \{c\}\}$, $\delta[\{a, c\}] = \{\phi, \{b\}\}$, $\delta[\{b, c\}] = \{\phi, \{c\}\}$, $\delta[X] = \{\phi\}$, which are ideals on X.

Example 3.2. Let $X = \{a, b, c\}$ and let δ be a basic proximity defined as

$$A\delta B \Leftrightarrow A \neq \phi, \ B \neq \phi.$$

Then: $\delta[\phi] = P(X)$, $\delta[A] = \{\phi\} \ \forall A \in P(X), A \neq \phi$, which are ideals on X.

The above example shows that $A \neq B \nRightarrow \delta[A] \neq \delta[B]$.

Theorem 3.1. *A binary relation δ on the power set $P(X)$ of a nonempty set X is a basic proximity on X if and only if it satisfies the following conditions:*

I_1 : $A \in \delta[B] \Rightarrow B \in \delta[A]$,

I_2 : $\delta[A]$ *is an ideal on* $X \ \forall\, A \in P(X)$, *and*

I_3 : $\delta[A] \subseteq \mathcal{I}_{A^c}$, *where* $\mathcal{I}_{A^c} = \{B : B \in P(X), B \subseteq A^c\}$.

Proof. Suppose that δ is a basic proximity on X. Then PI_1 is equivalent to I_1, and I_2 holds (by Proposition 3.1). For I_3, let $B \in \delta[A]$. Then $A \cap B = \phi$ (by PI_4) implies $B \subseteq A^c$, so $B \in \mathcal{I}_{A^c}$. Hence $\delta[A] \subseteq \mathcal{I}_{A^c}$.

Conversely, suppose that I_1, I_2 and I_3 hold. Then I_1 is equivalent to PI_1. Since $\delta[A]$ is an ideal for all $A \in P(X)$, then PI_2 and PI_3 hold. Now, let $B \in \delta[A]$. Then $B \subseteq A^c$ (by I_3), and so $A \cap B = \phi$. Hence PI_4 holds. Consequently, δ is a basic proximity on X. ■.

Theorem 3.2. *Let (X, δ) be a basic proximity space and $A,\ B \in P(X)$. Then*

(i) $\delta[A \cup B] = \delta[A] \cap \delta[B] \subseteq \delta[A \cap B]$,

(ii) $H_1 \in \delta[A]$ *and* $H_2 \in \delta[B] \Rightarrow H_1 \cap H_2 \in \delta[A \cup B]$.

Proof. (i) Since $A,\ B \subseteq A \cup B$, then $\delta[A \cup B] \subseteq \delta[A], \delta[B]$ (by Lemma 3.2(i)), and consequently, $\delta[A \cup B] \subseteq \delta[A] \cap \delta[B]$. Let $H \notin \delta[A \cup B]$. Then $A \cup B \notin \delta[H]$ implies $A \notin \delta[H]$ or $B \notin \delta[H]$ (byPI_2). So $H \notin \delta[A]$ or $H \notin \delta[B]$ implies $H \notin \delta[A] \cap \delta[B]$. Therefore, $\delta[A \cup B] = \delta[A] \cap \delta[B]$. Now, let $H \in \delta[A] \cap \delta[B]$. Then $A,\ B \in \delta[H]$. Since $A \cap B \subseteq A, B$, then $A \cap B \in \delta[H]$ (by I_2), and so $H \in \delta[A \cap B]$. Therefore, $\delta[A] \cap \delta[B] \subseteq \delta[A \cap B]$.

(ii) Let $H_1 \in \delta[A]$ and $H_2 \in \delta[B]$. Since $H_1 \cap H_2 \subseteq H_1, H_2$, then $H_1 \cap H_2 \in \delta[A]$ and $H_1 \cap H_2 \in \delta[B] \Rightarrow H_1 \cap H_2 \in \delta[A] \cap \delta[B] = \delta[A \cup B]$. ■

Proposition 3.2. *Let $\delta_1,\ \delta_2 \in m(X)$. Then*

$$\delta_1 < \delta \text{ if and only if } \delta_1[A] \subseteq \delta_2[A], \ \forall\, A \in P(X).$$

Proof. Straightforward. ■

Corollary 3.1. *Let $\delta_1,\ \delta_2 \in m(X)$. If $\delta_1 < \delta_2$, then*

(i) $N_{\delta_1}(A) \subseteq N_{\delta_2}(A), \ \forall\, A \in P(X)$,

(ii) $c_{\delta_2}(A) \subseteq c_{\delta_1}(A), \ \forall\, A \in P(X)$.

Theorem 3.3. *Let $\delta_1,\ \delta_2 \in m(X)$. Then the following statements are equivalent:*

(1) $\delta_1[x] = \delta_2[x], \ \forall\, x \in X$,

(2) $c_{\delta_1}(A) = c_{\delta_2}(A), \ \forall\, A \in P(X)$,

(3) $N_{\delta_1}(\{x\}) = N_{\delta_2}(\{x\}), \ \forall\, x \in X$.

Proof. Straightforward. ■

Definition 3.3. Let $\delta \in m(X)$ and $A \in P(X)$. We define

$$CN_\delta(A) = \{B : B \in P(X),\ \ B \notin N_\delta(A)\}.$$

Lemma 3.4. *Let $\delta \in m(X)$, $A \in P(X)$ and $\mathcal{I} \in \mathfrak{T}(X)$. Then*

$$N_\delta(A) \cap \mathcal{I} = \phi \Leftrightarrow \mathcal{I} \subseteq CN_\delta(A).$$

Proof. Straightforward. ■

Theorem 3.4. *Let* $\delta \in m(X)$, $A \in P(X)$ *and* $\mathcal{I}_1$, $\mathcal{I}_2 \in \mathfrak{I}(X)$. *Then*

$$\mathcal{I}_1 \cap \mathcal{I}_2 \subseteq CN_\delta(A) \Rightarrow \mathcal{I}_1 \subseteq CN_\delta(A) \text{ or } \mathcal{I}_2 \subseteq CN_\delta(A).$$

Proof. If possible, suppose that $\mathcal{I}_1 \nsubseteq CN_\delta(A)$ and $\mathcal{I}_2 \nsubseteq CN_\delta(A)$. Then there exists $H_1 \in \mathcal{I}_1 \setminus CN_\delta(A)$ and $H_2 \in \mathcal{I}_2 \setminus CN_\delta(A)$. So, $H_1 \cap H_2 \in \mathcal{I}_1 \cap \mathcal{I}_2 \subseteq CN_\delta(A) \Rightarrow H_1 \cap H_2 \in CN_\delta(A)$ which implies that $H_1 \cap H_2 \notin N_\delta(A) \Rightarrow (H_1^c \cup H_2^c) \notin \delta[A] \Rightarrow H_1^c \notin \delta[A]$ or $H_2^c \notin \delta[A]$ (by I_2). Hence $H_1 \in CN_\delta(A)$ or $H_2 \in CN_\delta(A)$, a contradiction. ■

Theorem 3.5. *Let* $\mathcal{I}_1$, $\mathcal{I}_2$ *and* $\mathcal{J}$ *are ideals on a nonempty set* X. *Then* $\mathcal{J} \subseteq \mathcal{I}_1 \cup \mathcal{I}_2 \Rightarrow \mathcal{J} \subseteq \mathcal{I}_1$ *or* $\mathcal{J} \subseteq \mathcal{I}_2$.

Proof. If possible, suppose that $\mathcal{J} \nsubseteq \mathcal{I}_1$ and $\mathcal{J} \nsubseteq \mathcal{I}_2$. Then there exists $A \in \mathcal{J} \setminus \mathcal{I}_1$ and $B \in \mathcal{J} \setminus \mathcal{I}_2$, so $A \cup B \in \mathcal{J} \subseteq \mathcal{I}_1 \cup \mathcal{I}_2$. Therefore, $A \cup B \in \mathcal{I}_1$ or $A \cup B \in \mathcal{I}_2$ implies $A \in \mathcal{I}_1$ or $B \in \mathcal{I}_2$, a contradiction. ■

Definition 3.4. A mapping $g\colon m(X) \times \mathfrak{I}(X) \to \mathfrak{I}(X)$ is said to be an ideal operator on X if $\forall\, \delta \in m(X)$ and $\forall\, \mathcal{I}_1,\ \mathcal{I}_2 \in \mathfrak{I}(X)$, we have

$$g(\delta, \mathcal{I}_1) \subseteq g(\delta, \mathcal{I}_2) \text{ whenever } \mathcal{I}_1 \subseteq \mathcal{I}_2.$$

Definition 3.5. Let g be an ideal operator on X. Then a basic proximity δ on X is said to be a g-proximity if $\delta[A] \subseteq g(\delta, \delta[A])$, $\forall\, A \in P(X)$.

The family of all g-proximities is denoted by P_g.

Definition 3.6. An ideal operator g is said to be:

in class G_1 if $g(\delta, \mathcal{I}_1 \cap \mathcal{I}_2) = g(\delta, \mathcal{I}_1) \cap g(\delta, \mathcal{I}_2)$ $\forall\, \delta \in m(X)$ and $\forall\, \mathcal{I}_1,\ \mathcal{I}_2 \in \mathfrak{I}(X)$;
in class G_2 if $g(\delta, \bigcap_{\alpha \in \Lambda} \mathcal{I}_\alpha) = \bigcap_{\alpha \in \Lambda} g(\delta, \mathcal{I}_\alpha)$ $\forall\, \delta \in m(X)$ and $\forall\, \mathcal{I}_\alpha \in \mathfrak{I}(X)$;
in class T if $g(\delta_1, \mathcal{I}) = g(\delta_2, \mathcal{I})$ with $c_{\delta_1} = c_{\delta_2}$ $\forall\, \delta_1, \delta_2 \in m(X)$ and $\forall\, \mathcal{I} \in \mathfrak{I}(X)$;
in class U if $g(\delta_1, \mathcal{I}) \subseteq g(\delta_2, \mathcal{I})$ whenever $\delta_1 < \delta_2$ $\forall\, \mathcal{I} \in \mathfrak{I}(X)$;
in class E if $g(\delta, \mathcal{I}) \subseteq g(\delta, g(\delta, \mathcal{I}))$, $\forall\, \delta \in P_g$, $\forall\, \mathcal{I} \in \mathfrak{I}(X)$.

Definition 3.7. For a set X, for all $\delta \in m(X)$ and for all $\mathcal{I} \in \mathfrak{I}(X)$ we define:

$i(\delta, \mathcal{I}) = \mathcal{I}$,
$g_0(\delta, \mathcal{I}) = \{A : A \in P(X), N_\delta(A) \cap \mathcal{I} \neq \phi\}$,
$g_1(\delta, \mathcal{I}) = \{A : A \in P(X), c_\delta(A) \in \mathcal{I}\}$,
$g_2(\delta, \mathcal{I}) = \{A : A \in P(X), \{x\} \in \delta[A] \cup \mathcal{I},\ \forall\, x \in X\}$,
$h_0(\delta, \mathcal{I}) = \{A : A \in P(X), N_\delta(\{a\}) \cap \mathcal{I} \neq \phi\ \forall\, a \in A\}$,
$h_1(\delta, \mathcal{I}) = \{A : A \in P(X), c_\delta(A) \in \delta[x] \text{ with } \mathcal{I} \subseteq \delta[x]\}$.

When there is no ambiguity we will write g_i for $g_i(\delta, \mathcal{I})$ and h_i for $h_j(\delta, \mathcal{I})$, where $i = 0, 1, 2$, $j = 0, 1$.

Theorem 3.6. *For all* $\delta \in m(X)$ *and for all* $\mathcal{I} \in \mathfrak{I}(X)$ *and for* $g \in \{i, g_0, g_1, g_2, h_0, h_1\}$, *we have that* g *is an ideal operator on* X.

Proof. We prove the cases g_0 and g_2, the other cases are similar. Suppose that $\delta \in m(X)$ and $\mathcal{I} \in \mathfrak{T}(X)$. Now, since $N_\delta(\phi) \cap \mathcal{I} = P(X) \cap \mathcal{I} = \mathcal{I} \neq \phi \Rightarrow \phi \in g_0$. If $A \in g_0$ and $B \subseteq A$, then $N_\delta(A) \cap \mathcal{I} \neq \phi \Rightarrow N_\delta(B) \cap \mathcal{I} \neq \phi$ (by Lemma 2.4). Hence $B \in g_0$. If $A,\ B \in g_0$, then $N_\delta(A) \cap \mathcal{I} \neq \phi$ and $N_\delta(B) \cap \mathcal{I} \neq \phi$. So $\exists H,\ M \in \mathcal{I}$ such that $H \in N_\delta(A)$ and $M \in N_\delta(B)$ implies $H \cup M \in N_\delta(A \cup B)$ (by Theorem 2.1), and so $H \cup M \in N_\delta(A \cup B) \cap \mathcal{I}$. Consequently, $N_\delta(A \cup B) \cap \mathcal{I} \neq \phi$. Hence $A \cup B \in g_0$. Therefore, g_0 is an ideal on X. Now, let $\mathcal{I}_1 \subseteq \mathcal{I}_2$ and $H \in g_0(\delta, \mathcal{I}_1)$. Then $N_\delta(H) \cap \mathcal{I}_1 \neq \phi \Rightarrow N_\delta(H) \cap \mathcal{I}_2 \neq \phi$. So, $H \in g_0(\delta, \mathcal{I}_2)$. Hence g_0 is an ideal operator on X.

Next, since $\delta[\phi] = P(X)$, then $\{x\} \in \delta[\phi] \cup \mathcal{I},\ \forall\ x \in X \Rightarrow \phi \in g_2$. If $A \in g_2$ and $B \subseteq A$, then $\{x\} \in \delta[A] \cup \mathcal{I},\ \forall\ x \in X \Rightarrow \{x\} \in \delta[B] \cup \mathcal{I},\ \forall\ x \in X$(by Lemma 3.2(i)), and so $B \in g_2$. If $A,\ B \in g_2$, then $\{x\} \in (\delta[A] \cup \mathcal{I}) \cap (\delta[B] \cup \mathcal{I}),\ \forall\ x \in X \Rightarrow \{x\} \in (\delta[A] \cap \delta[B]) \cup \mathcal{I},\ \forall\ x \in X \Rightarrow \{x\} \in \delta[(A \cup B)] \cup \mathcal{I},\ \forall\ x \in X$ (by Theorem 3.2(i)), and so $A \cup B \in g_2$. Hence g_2 is an ideal on X. Clearly, if $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then $g_2(\delta, \mathcal{I}_1) \subseteq g_2(\delta, \mathcal{I}_2)$. Consequently, g_2 is an ideal operator on X. ■

Theorem 3.7. *For all $\delta \in m(X)$ and for all $\mathcal{I} \in \mathfrak{T}(X)$, we have $i, g_1, g_2 \in G_2 \subseteq G_1$ and $g_0, h_0 \in G_1$.*

Proof. It is clear that $G_2 \subseteq G_1$. Also, trivially $i, g_1, g_2 \in G_2$. Now, let $A \in g_0(\delta, \mathcal{I}_1 \cap \mathcal{I}_2)$. Then, $N_\delta(A) \cap (\mathcal{I}_1 \cap \mathcal{I}_2) \neq \phi \Rightarrow N_\delta(A) \cap \mathcal{I}_1 \neq \phi$ and $N_\delta(A) \cap \mathcal{I}_2 \neq \phi \Rightarrow A \in g_0(\delta, \mathcal{I}_1) \cap g_0(\delta, \mathcal{I}_2)$. Hence $g_0(\delta, \mathcal{I}_1 \cap \mathcal{I}_2) \subseteq g_0(\delta, \mathcal{I}_1) \cap g_0(\delta, \mathcal{I}_2)$. On the other hand, let $A \in g_0(\delta, \mathcal{I}_1) \cap g_0(\delta, \mathcal{I}_2)$. Then $N_\delta(A) \cap \mathcal{I}_1 \neq \phi$ and $N_\delta(A) \cap \mathcal{I}_2 \neq \phi$ imply $N_\delta(A) \cap (\mathcal{I}_1 \cap \mathcal{I}_2) \neq \phi$(by Lemma 3.4, Theorem 3.4). So $A \in g_0(\delta, \mathcal{I}_1 \cap \mathcal{I}_2)$. Hence $g_0(\delta, \mathcal{I}_1) \cap g_0(\delta, \mathcal{I}_2) \subseteq g_0(\delta, \mathcal{I}_1 \cap \mathcal{I}_2)$. Therefore, $g_0 \in G_1$.

Similarly, we can prove that $h_0 \in G_1$. ■

Theorem 3.8. *For all $\delta \in m(X)$ and for all $\mathcal{I} \in \mathfrak{T}(X)$, we have $g \in T,\ \ \forall\ g \in \{i, g_1, g_2, h_0, h_1\}$.*

Proof. It follows from Lemma 2.4 and Theorem 3.3. ■

Theorem 3.9. *For all $\delta \in m(X)$ and for all $\mathcal{I} \in \mathfrak{T}(X)$, we have $g \in U,\ \ \forall\ g \in \{i, g_0, g_1, g_2, h_0, h_1\}$.*

Proof. It follows from Proposition 3.2 and Corollary 3.1. ■

Theorem 3.10. *Let $\delta \in m(X)$. Then the following statements are equivalent:*

(1) *δ is an EF-proximity on X,*
(2) *$A \in \delta[B] \Rightarrow N_\delta(A) \cap \delta[B] \neq \phi$,*
(3) *$N_\delta(A) \cap \delta[B] = \phi \Rightarrow A \notin \delta[B]$,*
(4) *δ is a g_0-proximity, and*
(5) *$A \in N_\delta(B) \Rightarrow \exists H \in N_\delta(B)$ such that $A \in N_\delta(H)$.*

Proof. (1) $\Rightarrow$ (2): let $A \in \delta[B]$. Then, $\exists H \in P(X)$ such that $A \in \delta[H]$ and $H^c \in \delta[B]$. It follows that $H \in \delta[A]$ and $H^c \in \delta[B]$. Hence $H^c \in N_\delta(A) \cap \delta[B]$, and so $N_\delta(A) \cap \delta[B] \neq \phi$.

(2) $\Leftrightarrow$ (3): it is obvious.

(2) $\Rightarrow$ (4): let $H \in \delta[A]$. Then, $N_\delta(H) \cap \delta[A] \neq \phi \Rightarrow H \in g_0(\delta, \delta[A])$. So, $\delta[A] \subseteq g_0(\delta, \delta[A])$ and δ is a g_0-proximity.

(4) $\Rightarrow$ (2): let $A \in \delta[B]$. Then $A \in g_0(\delta, \delta[B]) \Rightarrow N_\delta(A) \cap \delta[B] \neq \phi$.

(2) $\Rightarrow$ (5): let $A \in N_\delta(B)$. Then $A^c \in \delta[B] \Rightarrow N_\delta(A^c) \cap \delta[B] \neq \phi \Rightarrow \exists M \in P(X)$ such that $M \in \delta[B]$ and $M \in N_\delta(A^c)$. Hence, by Lemma 2.3, $M^c \in N_\delta(B)$ and $A \in N_\delta(M^c)$, putting $H = M^c$. So (5) holds.

(5) $\Rightarrow$ (1): let $A \in \delta[B]$. Then $A^c \in N_\delta(B) \Rightarrow \exists H \in P(X)$ such that $H \in N_\delta(B)$ and $A^c \in N_\delta(H) \Rightarrow H^c \in \delta[B]$ and $A \in \delta[H]$. Hence δ is an EF - Proximity on X. ■

Corollary 3.2. *Let $\delta \in m(X)$. Then δ is an EF-proximity iff it is a g_0-proximity.*

Theorem 3.11. *Let $\delta \in m(X)$. If $\delta \in P_{g_1}$, then c_δ is a closure operator.*

Proof. From Theorem 2.2, it is enough to prove the idempotent property, i.e., $c_\delta(c_\delta(A)) = c_\delta(A)$. Clearly, $c_\delta(A) \subseteq c_\delta(c_\delta(A))$. Let $x \in c_\delta(c_\delta(A))$. Then, $x\delta c_\delta(A) \Rightarrow c_\delta(A) \notin \delta[x] \Rightarrow A \notin g_1(\delta, \delta[x])$. Therefore, $A \notin \delta[x]$. So, $x \in c_\delta(A)$. Consequently, $c_\delta(c_\delta(A)) \subseteq c_\delta(A)$. So, $c_\delta(c_\delta(A)) = c_\delta(A)$. Hence c_δ is a closure operator. ■

Theorem 3.12. *Let $\delta \in m(X)$. Then δ is a g_1-proximity if and only if $\forall\, B \in \delta[A] \Rightarrow c_\delta(B) \in \delta[A]$.*

Proof. Suppose that δ is a g_1-proximity and $B \in \delta[A]$. Then, $B \in g_1(\delta, \delta[A]) \Rightarrow c_\delta(B) \in \delta[A]$.

Conversely, let $B \in \delta[A]$. Then $c_\delta(B) \in \delta[A] \Rightarrow B \in g_1(\delta, \delta[A])$. So, $\delta[A] \subseteq g_1(\delta, \delta[A])$, $\forall\, A \in P(X)$. Hence δ is a g_1-proximity. ■

Theorem 3.13. *Let $\delta \in m(X)$. Then δ is an LO-proximity iff it is a g_1-proximity.*

Proof. Suppose that δ is an LO-proximity, $A \in P(X)$ and $H \notin g_1(\delta, \delta[A])$. Then $c_\delta(H) \notin \delta[A] \Rightarrow A\delta c_\delta(H)$. But, $c_\delta(H) = \{x : x\delta H\}$, then $A\delta c_\delta(H)$ and $x\delta H\ \ \forall\, x \in c_\delta(H) \Rightarrow A\delta H$, i.e., $H \notin \delta[A]$. Hence δ is a g_1-proximity.

Conversely, let $A\delta B$ and $b\delta H\ \ \forall\, b \in B$. Then $B \subseteq c_\delta(H) = \{x \in X : x\delta H\} \Rightarrow A\delta c_\delta(H)$ (by Lemma 2.2) $\Rightarrow c_\delta(H) \notin \delta[A] \Rightarrow H \notin \delta[A]$ (by Theorem 3.12),so $A\delta H$. Hence δ is an LO-proximity. ■

Theorem 3.14. *Let $\delta \in m(X)$ and $\mathcal{I} \in \mathfrak{T}(X)$. Then $g(\delta, \mathcal{I}) \subseteq \mathcal{I}\quad \forall\, g \in \{i, g_0, g_1\}$.*

Proof. Trivially, $i(\delta, \mathcal{I}) \subseteq \mathcal{I}$. Let $A \in g_0(\delta, \mathcal{I})$. Then, $N_\delta(A) \cap \mathcal{I} \neq \phi$. So, $\exists B \in P(X)$ such that $B \in N_\delta(A)$ and $B \in \mathcal{I}$. Since $B \in N_\delta(A)$, then $A \subseteq B \in$

$\mathcal{I} \Rightarrow A \in \mathcal{I}$. Hence $g_0(\delta, \mathcal{I}) \subseteq \mathcal{I}$. Next, let $A \in g_1(\delta, \mathcal{I})$. Then $c_\delta(A) \in \mathcal{I} \Rightarrow A \in \mathcal{I}$. Hence, $g_1(\delta, \mathcal{I}) \subseteq \mathcal{I}$. ■

THEOREM 3.15. *Let* $\delta \in m(X)$. *Then*

$$\delta \in P_{g_2} \Leftrightarrow (A \in \delta[B] \Rightarrow (A \in \delta[x] \text{ or } B \in \delta[x])), \ \forall x \in X.$$

Proof. Suppose that δ is a g_2-proximity and let $A \in \delta[B]$. Then, $A \in g_2(\delta, \delta[B]) \Rightarrow \{x\} \in \delta[A] \cup \delta[B], \ \forall x \in X$. It follows that $A \in \delta[x]$ or $B \in \delta[x], \ \forall x \in X$.

Conversely, let $H \in \delta[A]$. Then, $H \in \delta[x]$ or $A \in \delta[x] \ (\forall x \in X) \Rightarrow \{x\} \in \delta[H] \cup \delta[A] \ \forall x \in X$, it follows that $H \in g_2(\delta, \delta[A]), \ \forall A \in P(X))$. Hence $\delta[A] \subseteq g_2(\delta, \delta[A])$. Consequently, δ is a g_2-proximity. ■

The following definition is a reformulation of Definition 2.3.

DEFINITION 3.8. A binary relation δ on the power set $P(X)$ of a nonempty set X is said to be an RH-proximity on X if it satisfies the following conditions:

RI_1 : $A \in \delta[B] \Rightarrow B \in \delta[A]$,

RI_2 : $A \in \delta[C]$ and $B \in \delta[C] \Leftrightarrow A \cup B \in \delta[C]$,

RI_3 : $\phi \in \delta[X]$,

RI_4 : $A \in \delta[A] \Rightarrow A = \phi$, and

RI_5 : $x \in \delta[A] \Rightarrow \exists H \in P(X)$ such that $x \in \delta[H]$ and $H^c \in \delta[A]$.

THEOREM 3.16. *Let* $\delta \in m(X)$. *Then the following statements are equivalent:*

(1) $x \in \delta[A] \Rightarrow \exists H \in P(X)$ *such that* $x \in \delta[H]$ *and* $H^c \in \delta[A]$,

(2) $x \in \delta[A] \Rightarrow N_\delta(\{x\}) \cap \delta[A] \neq \phi$,

(3) $N_\delta(\{x\}) \cap \delta[A] = \phi \Rightarrow x \notin \delta[A]$,

(4) δ *is an* h_0*-proximity, and*

(5) $A \in N_\delta(\{x\}) \Rightarrow \exists B \in N_\delta(\{x\})$ *such that* $A \in N_\delta(B)$.

Proof. (1) $\Rightarrow$ (2): let $x \in \delta[A]$. Then, by (1), $\exists H \in P(X)$ such that $x \in \delta[H]$ and $H^c \in \delta[A]$. It follows that $H^c \in \delta[A]$ and $H^c \in N_\delta(\{x\})$. Hence $N_\delta(\{x\}) \cap \delta[A] \neq \phi$.

(2) $\Leftrightarrow$ (3) it is obvious.

(2) $\Rightarrow$ (4): let $B \in \delta[A]$. Implies, by Lemma 3.2 (*ii*), $b \in \delta[A] \ (\forall b \in B)$. Hence, by (2), $N_\delta(\{b\}) \cap \delta[A] \neq \phi, \ (\forall b \in B) \Rightarrow B \in h_0(\delta, \delta[A])$. Hence $\delta[A] \subseteq h_0(\delta, \delta[A])$. Consequently, δ is an h_0-proximity.

(4) $\Rightarrow$ (2): it is obvious.

(2) $\Rightarrow$ (5): let $A \in N_\delta(\{x\})$. Then, $x \in \delta[A^c] \Rightarrow N_\delta(\{x\}) \cap \delta[A^c] \neq \phi$. It follows that $\exists B \in P(X)$ such that $B \in N_\delta(\{x\})$ and $B \in \delta[A^c]$. So, $A^c \in \delta[B] \Rightarrow A \in N_\delta(B)$.

(5) $\Rightarrow$ (1): let $x \in \delta[A]$. Then, $A^c \in N_\delta(\{x\})$. By (5), $\exists H \in N_\delta(\{x\})$ such that $A^c \in N_\delta(H)$. It follows that $A \in \delta[H]$ and $H^c \in \delta[x]$, i.e. $\exists H \in P(X)$ such that $x \in \delta[H^c]$ and $H \in \delta[A]$. Hence (1) holds. ■

Corollary 3.3. *Let $\delta \in m(X)$. Then δ is an RH-proximity iff it is an h_0-proximity.*

Proof. It follows from Definition 3.8 and Theorem 3.16. ■

Theorem 3.17. *Let δ be an h_1-proximity. Then*

(1) $x \in \delta[A] \Rightarrow x \in \delta[c_\delta(A)]$.

(2) *c_δ is a closure operator.*

Proof. (1) Suppose that δ is an h_1-proximity and let $x \in \delta[A]$. Then, $A \in \delta[x] \subseteq h_1(\delta, \delta[x]) \Rightarrow A \in h_1(\delta, \delta[x]) \Rightarrow c_\delta(A) \in \delta[y]$ with $\delta[x] \subseteq \delta[y]$. But $\delta[x] \subseteq \delta[x]$, then $c_\delta(A) \in \delta[x] \Rightarrow x \in \delta[c_\delta(A)]$.

(2) From Theorem 3.2, it is enough to prove the idempotent property i.e. $c_\delta(c_\delta(A)) = c_\delta(A)\ \forall\ A \in P(X)$. Clearly, $c_\delta(A) \subseteq c_\delta(c_\delta(A))$. Let $x \in c_\delta(c_\delta(A))$. Then, $x\delta c_\delta(A) \Rightarrow x \notin \delta[c_\delta(A)]$. By (1), $x \notin \delta[A]$. Hence $x \in c_\delta(A)$. Consequently, $c_\delta(c_\delta(A)) \subseteq c_\delta(A)$. It follows that $c_\delta(A) = c_\delta(c_\delta(A))$. Hence c_δ is a closure operator. ■

Lemma 3.5. *Let δ be an S-proximity. If $A \in \delta[x]$, then $c_\delta(A) \in \delta[x]$.*

Proof. Suppose that δ is an S-proximity and let $A \in \delta[x]$. Assume that $c_\delta(A) \notin \delta[x]$. Then, $x\delta c_\delta(A)$. But $y\delta A \quad \forall\ y \in c_\delta(A)$, then $x\delta A$(by $\acute{P}_7$), i.e. $A \notin \delta[x]$, a contradiction. ■

Theorem 3.18. *Let $\delta \in m(X)$. Then δ is an S-proximity iff it is an h_1-proximity.*

Proof. Suppose that δ is an S-proximity and let $H \in \delta[A]$ with $\delta[A] \subseteq \delta[x]$. Then, $H \in \delta[x] \Rightarrow c_\delta(H) \in \delta[x]$ (by Lemma 3.5) with $\delta[A] \subseteq \delta[x] \Rightarrow H \in h_1(\delta, \delta[A])$. Hence δ is an h_1-proximity.

Conversely, suppose that δ is an h_1-proximity and let $x \notin \delta[B]$ and $b\delta H\ \forall\ b \in B$. Also, assume that $x \in \delta[H]$. Then, by Theorem 3.17(1), $x \in \delta[c_\delta(H)]$. Since $B \subseteq c_\delta(H) \Rightarrow x \in \delta[B]$ (by Lemma 3.1), a contradiction with $x \notin \delta[B]$. ■

Theorem 3.19. *For all $\delta \in m(X)$ and for all $\mathcal{I} \in \mathfrak{T}(X)$, we have $g \in E,\ \forall\ g \in \{i, g_0, g_1, g_2, h_0\}$.*

Proof. Let $\delta \in P_{g_0}$ and $A \in g_0(\delta, \mathcal{I})$. Then $N_\delta(A) \cap \mathcal{I} \neq \phi \Rightarrow \exists M \in P(X)$ such that $M \in N_\delta(A)$ and $M \in \mathcal{I}$. Since $\delta \in P_{g_0}$, then, by Theorem 3.10, there exists $H \in N_\delta(A)$ such that $M \in N_\delta(H) \Rightarrow N_\delta(H) \cap \mathcal{I} \neq \phi$. So, $H \in g_0(\delta, \mathcal{I})$. But $H \in N_\delta(A)$, thus $N_\delta(A) \cap g_0(\delta, \mathcal{I}) \neq \phi$. Hence $A \in g_0(\delta, g_0(\delta, \mathcal{I}))$. Consequently, $g_0(\delta, \mathcal{I}) \subseteq g_0(\delta, g_0(\delta, \mathcal{I}))$. It follows that $g_0 \in E$.

Next, let $\delta \in P_{g_1}$ and let $A \in g_1(\delta, \mathcal{I})$. Then $c_\delta(A) \in \mathcal{I} \Rightarrow c_\delta(A) = c_\delta(c_\delta(A) \in \mathcal{I}$ (by Theorem 3.11). So, $c_\delta(A) \in g_1(\delta, \mathcal{I})$. Hence $A \in g_1(\delta, g_1(\delta, \mathcal{I}))$. Consequently, $g_1(\delta, \mathcal{I}) \subseteq g_1(\delta, g_1(\delta, \mathcal{I}))$. It follows that $g_1 \in E$.

Now, we shall prove that $g_2 \in E$. Let $\delta \in P_{g_2}$ and let $A \notin g_2(\delta, g_2(\delta, \mathcal{I}))$. Then there exists $x \in X$ such that $\{x\} \notin \delta[A] \cup g_2(\delta, \mathcal{I})$. So, $\{x\} \notin \delta[A]$ and $\{x\} \notin g_2(\delta, \mathcal{I})$. Hence, there exists $y \in X$ such that $\{y\} \notin \delta[\{x\}] \cup \mathcal{I} \Rightarrow \{y\} \notin \mathcal{I}$, $A \notin \delta[\{x\}]$ and $\{y\} \notin \delta[\{x\}]$. Hence, by Theorem 3.15, $\{y\} \notin \delta[A] \cup \mathcal{I}$. It follows that $A \notin g_2(\delta, \mathcal{I})$. Hence $g_2(\delta, \mathcal{I}) \subseteq g_2(\delta, g_2(\delta, \mathcal{I}))$. Consequently, $g_2 \in E$.

Finally, Let $\delta \in P_{h_0}$ and let $A \in h_0(\delta, \mathcal{I})$. Then $N_\delta(a) \cap \mathcal{I} \neq \phi\ \forall a \in A \Rightarrow \exists H \in P(X)$ such that $H \in N_\delta(a)$ and $H \in \mathcal{I}$. Therefore, by Theorem 3.16.(5), there exists $B \in N_\delta(a)$ such that $H \in N_\delta(B) \Rightarrow N_\delta(B) \cap \mathcal{I} \neq \phi \Rightarrow N_\delta(b) \cap \mathcal{I} \neq \phi$, $\forall b \in B \Rightarrow B \in h_0(\delta, \mathcal{I})$. But, $B \in N_\delta(a)$, then $N_\delta(a) \cap h_0(\delta, \mathcal{I}) \neq \phi\ \forall a \in A$. It follows that $A \in h_0(\delta, h_0(\delta, \mathcal{I}))$. Hence $h_0(\delta, \mathcal{I}) \subseteq h_0(\delta, h_0(\delta, \mathcal{I}))$. Consequently, $h_0 \in E$. ■

THEOREM 3.20. *For all* $\delta \in m(X)$ *and for all* $\mathcal{I} \in \mathfrak{T}(X)$, $g_0(\delta, \mathcal{I}) = \bigcup_{A \in \mathcal{I}} \delta[A^c]$.

Proof. Straightforward. ■

THEOREM 3.21. *For all* $\delta \in m(X)$ *and for all* $\mathcal{I} \in \mathfrak{T}(X)$, *we have*

(1) $P_{g_1} \subseteq P_{g_2}$,

(2) $P_{g_1} \subseteq P_{h_1}$,

(3) $P_{g_0} \subseteq P_{h_0}$, *and*

(4) $P_{g_0} \subseteq P_{g_1}$.

Proof. (1) Let $\delta \in P_{g_1}$ and let $H \in \delta[A]$. Then, by Theorem 3.12, $c_\delta(H) \in \delta[A]$. We claim that $H \in g_2(\delta, \delta[A])$. In fact, if $H \notin g_2(\delta, \delta[A])$, then there exists $x \in X$ such that $\{x\} \notin \delta[H]$ and $\{x\} \notin \delta[A] \Rightarrow x \in c_\delta(H), \{x\} \notin \delta[A]$. But, $\{x\} \subseteq c_\delta(H)$ and $\delta[A]$ is an ideal, so $c_\delta(H) \notin \delta[A]$, a contradiction. Hence $H \in g_2(\delta, \delta[A])$. It follows that $\delta[A] \subseteq g_2(\delta, \delta[A])$. Consequently, $\delta \in P_{g_2}$. Hence, $P_{g_1} \subseteq P_{g_2}$.

(2) Let $\delta \in P_{g_1}$ and let $H \in \delta[A]$. Then, by Theorem 3.12, $c_\delta(H) \in \delta[A] \Rightarrow c_\delta(H) \in \delta[x]$ with $\delta[A] \subseteq \delta[x] \Rightarrow H \in h_1(\delta, \delta[A])$. Hence $\delta[A] \subseteq h_1(\delta, \delta[A])$. Consequently, $\delta \in P_{h_1}$. Hence, $P_{g_1} \subseteq P_{h_1}$.

(3) Let $\delta \in P_{g_0}$ and let $H \in \delta[A]$. Then, $N_\delta(H) \cap \delta[A] \neq \phi \Rightarrow N_\delta(h) \cap \delta[A] \neq \phi$, $\forall h \in H \Rightarrow H \in h_0(\delta, \delta[A])$. Hence $\delta[A] \subseteq h_0(\delta, \delta[A])$. Consequently, $\delta \in P_{h_0}$. Hence, $P_{g_0} \subseteq P_{h_0}$.

(4) Let $\delta \in P_{g_0}$ and let $H \in \delta[A]$. Then, $N_\delta(H) \cap \delta[A] \neq \phi \Rightarrow \exists M \in P(X)$ such that $M \in N_\delta(H)$ and $M \in \delta[A]$. Since $c_\delta(H) = \cap\{B : B \in N_\delta(H)\}$, then $c_\delta(H) \subseteq M \in \delta[A] \Rightarrow c_\delta(H) \in \delta[A]$ (for $\delta[A]$ is an ideal). Hence, $H \in g_1(\delta, \delta[A])$. Consequently, $\delta \in P_{g_1}$. Hence, $P_{g_0} \subseteq P_{g_1}$. ■

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